Semidiscrete Finite Element Galerkin Approximations to the Equations of Motion Arising in the Oldroyd Model

Amiya K. Pani * Jin Yun Yuan †

Abstract

In this paper, a semidiscrete finite element Galerkin method for the equations of motion arising in the two dimensional Oldroyd model of viscoelastic fluids with zero forcing function is analysed. Some new *a priori* bounds for the exact solutions are derived under realistically assumed conditions on the data. Moreover, the longtime behaviour of the solution is established. By introducing Stokes-Volterra projection, optimal error bounds for the velocity in $L^\infty(\mathbf{L^2})$ as well as in $L^\infty(\mathbf{H}^1)$-norms and for the pressure in $L^\infty(L^2)$-norm are derived which are valid uniformly in time $t > 0$.

Key words. Viscoelastic fluids, Oldroyd model, *a priori* bounds, exponential decay, finite element method, semidiscrete scheme, linearized oldroyd model, Stokes Volterra projection, optimal error estimates, uniform convergence in time.

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1. Introduction. The motion of an incompressible fluid in a bounded domain Ω in R^2 is described by the following system of partial differential equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\sigma + \nabla p = \mathbf{F}(\mathbf{x},\mathbf{t}), \;\; \mathbf{x}\in\boldsymbol{\Omega}, \;\; \mathbf{t} > \mathbf{0},$$

$$\nabla\cdot\mathbf{u} = 0, \;\; x\in\Omega, \, t>0,$$

with appropriate initial and boundary conditions. Here, $\sigma = (\sigma_{ik})$ denotes the stress tensor with $tr\sigma = 0$, $\mathbf{u}$ represents the velocity vector, p is the pressure of the fluid and $\mathbf{F}$ is the external force. The defining relation between the stress tensor σ and the tensor of deformation velocities $\mathbf{D} = (\mathbf{D}_{ik}) = \frac{1}{2}(\mathbf{u}_{ix_k} + \mathbf{u}_{kx_i})$, called the equation of state or sometimes the rheological equation establishes the type of fluids under consideration. For example, when $\sigma = 2\nu\mathbf{D}$ (using Newton's law) with ν the kinematic coefficient of viscosity, we obtain Newton's model of incompressible viscous fluid and the corresponding system is widely known as the Navier-Stokes equations. This has been a basic model for describing the flow at moderate velocities of the majority of the incompressible viscous fluids encountered in practice. However, models (of viscoelastic fluids) have been proposed in the mid-twentieth century which take into account the prehistory of the flow and are not subject to the Newtonian flow. One such model, proposed by J. G. Oldroyd (ref. [22]) is called Oldroyd model. In this case, the defining relation has a special form like

$$(1+\lambda\frac{\partial}{\partial t})\sigma = 2\nu(1+\kappa\nu^{-1}\frac{\partial}{\partial t})\mathbf{D},$$

*Department of Mathematics, Industrial Mathematics Group, Indian Institute of Technology, Bombay, Powai, Mumbai 400076, India. Email Address: akp@math.iitb.ac.in

†Department of Mathematics, Federal University of Paraná, Curitiba, Centro Politécnico, Cx.P: 19081, CEP: 81531-990,PR, Brazil.

where λ, ν, κ are positive constants with $(\nu - \kappa\lambda^{-1}) > 0$. Now the equation of motion arising from the Oldroyd's model gives rise to the following integro-differential equation :

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - \int_0^t \beta(t-\tau) \Delta \mathbf{u}(x,\tau)\, d\tau + \nabla p = \mathbf{f}(x,t), \quad x \in \Omega, \; t > 0, \tag{1.1}$$

and incompressibility condition

$$\nabla \cdot \mathbf{u} = 0, \;\; x \in \Omega, \, t > 0, \tag{1.2}$$

with initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \;\; \text{in } \Omega, \qquad \mathbf{u} = 0, \;\; \text{on } \partial\Omega, \; t \geq 0. \tag{1.3}$$

Here, Ω is a bounded domain in two dimensional Euclidean space R^2 with boundary $\partial\Omega$, $\mu = 2\kappa\lambda^{-1} > 0$ and the kernel $\beta(t) = \gamma \exp(-\delta t)$, where $\gamma = 2\lambda^{-1}(\nu - \kappa\lambda^{-1})$ and $\delta = \lambda^{-1}$. For details of the physical background and its mathematical modelling, we refer to [13], [22] and [23].

Throughout this paper, we assume that the nonhomogeneous term $\mathbf{f} = 0$. In fact, assuming conservative force, the function $\mathbf{f}$ can be absorbed in the pressure term.

Based on the analysis of Ladyzenskaya [15] for the solvability of the Navier Stokes equations, Oskolkov [23] proved the global existence of unique 'almost' classical solution in finite time interval for the initial and boundary value problem (1.1)–(1.3). The invesigations on solvability were further continued by the co-workers of Oskolkov, see [14] and Agranovich and Sobolevskii [1] under various sufficient conditions. In these articles, the regularity results are proved under the assumption of some nonlocal compatibility conditions on the data at $t = 0$, which are either hard to verify or difficult to meet in practice. In the present paper, we have obtain some new *a priori* bounds for the solution under realistically assumed conditions on the data. In the Oldroyd fluid, the stresses after instantaneous cessation of the motion decay like $\exp(-\lambda^{-1}t)$, while the velocities of the flow after instantaneous removal of the stresses die out like $\exp(-\kappa^{-1}t)$. Therefore, it is of interest to discuss the behaviour of the solution as $t \mapsto \infty$. Recently, Sobolevskii [24] discussed the long time behaviour of the solution under some stabilizing conditions on the nonhomogeneous forcing function using a combination of energy arguments and semigroup theoretic approach. When the forcing function is zero, we have derived, in Sections 2, the exponential decay properties for the exact solution using only energy arguments.

For the earlier results on the numerical approximations to the solutions of the problem (1.1)–(1.3), we refer to [2] and [5]. Akhmatov and Oskolkov [2] discussed stable finite difference schemes for approximating the solutions of (1.1)–(1.3) without any order of convergence. Cannon *et al.* [5] proposed a modified nonlinear Galerkin scheme for a periodic problem using spectral Galerkin procedure and established the rates of convergence keeping time variable continuous.

The approach of the present article is influenced by the earlier results of Heywood and Rannacher [9]. In [9], the semidiscrete error estimates were derived without making assumptions about the solution regularity that would depend on the nonlocal compatibility conditions on the data at $t = 0$. Because of the exponential growth of the error constants with respect to time, the error estimates are virtually meaningless for large values of t. Subsequently, assuming stability of the exact solution, Heywood and Rannacher [10]-[11] proved that the error in the discrete approximation would remain small uniformly in time as $t \mapsto \infty$. For higher order elements, optimal semidiscrete error estimates were derived in [11]. Finally, the completely discrete scheme was obtained using Crank-Nicolson method in time and *a priori* error bounds were extended uniformly in time for approximation of an exponentially stable solution, see [12].

There is hardly any literature devoted to the analysis of the finite element Galerkin methods for the problem (1.1)–(1.3), and hence, the present investigation is a step towards achieving this objective. Therefore, in this paper, we address ourselves to the finite element Galerkin approximations to the system of equations (1.1)–(1.3) under realistically assumed regularity conditions on the exact solution. It is to be noted that the system (1.1)–(1.3) can be thought of an integral perturbation of the Navier Stokes equations. Therefore, we would like to investigate '*how far the results on finite element analysis for the Navier-Stokes equations (ref. [9]–[12]) can be carried over to the present case*'. More precisely, our emphasis is to bringout the role played by the integral term.

The main results of the present article consist of

(i) proving new regularity results for the solution which are valid for all time $t > 0$ with out nonlocal compatibility conditions and establishing the exponential decay property for the exact solution.

(ii) the derivation of *a priori* error estimates for the linearized problem which are uniformly bounded for all time t by using duality arguments.

(iii) an introduction of Stokes-Volterra projection.

(iv) obtaining optimal error estimates for the semidiscrete Galerkin approximations to the velocity in $L^\infty(\mathbf{L^2})$-norm and to the pressure in $L^\infty(L^2)$-norm.

For the proof of (i), we have made use of exponential weights for the derivation of the new regularity results for large enough time, while for the behaviour of the solutions at $t = 0$, we have applied the weight $\tau^*(t) = \min(1, t)$. These weights also become crucial in the proof of (ii). Compared to the Navier-Stokes equations, there are difficulties in introducing these weights in the analysis of the present problem due to the presence of the integral term. We note that the smoothing property proved via energy argument in [21] is useful for deriving the regularity results for the present problem without nonlocal compatibility assumptions on the data at $t = 0$. In order to derive optimal error estimates for the velocity in (iv), we first split the error by using a Galerkin approximation to a linearised Oldroyd model and then introduce a Stokes-Volterra projection. While it is possible to avoid Stokes-Volterra projection by appealing to Stokes projection alone, but its introduction makes analysis simpler. For the major part of this article, special care has been taken (see, Lemmas 4.1–4.3 and 5.3–5.4) to avoid the use of standard Gronwall's Lemma. We note that in the context of Navier-Stokes equations, Okamoto [19] has derived similar convergence analysis using semigroup theoretic arguments.

The remaining part of this paper is organised as follows. While in Section 2, we discuss *a priori* bounds for the exact solutions, in Section 3, we describe the semidiscrete Galerkin approximations. Section 4 is devoted to the optimal error estimates for the velocity. In Section 5, we derive the optimal error estimate for the pressure. Finally, we coclude the paper with a summary and possible extensions in Section 6.

2. Preliminaries and A Priori Bounds. For our subsequent use, we denote by boldface letters the R^2- valued function space such as

$$\mathbf{H}_0^1 = [H_0^1(\Omega)]^2, \quad \mathbf{L}^2 = [L^2(\Omega)]^2 \text{ and } \quad \mathbf{H}^m = [H^m(\Omega)]^2,$$

where $H^m(\Omega)$ is the standard Hilbert Sobolev space of order m. Note that $\mathbf{H}_0^1$ is equipped with a norm

$$\|\nabla\mathbf{v}\| = \left(\sum_{i,j=1}^{2}(\partial_j v_i, \partial_j v_i)\right)^{1/2} = \left(\sum_{i=1}^{2}(\nabla v_i, \nabla v_i)\right)^{1/2}.$$

Further, we introduce some more function spaces which are well suited for our problem:

$$\begin{aligned} \mathbf{J}_1 &= \{\boldsymbol{\phi} \in \mathbf{H}_0^1 : \nabla\cdot\boldsymbol{\phi} = 0\} \\ \mathbf{J} = \{\boldsymbol{\phi} \in \mathbf{L}^2 : \nabla\cdot\boldsymbol{\phi} &= 0 \text{ in } \Omega, \boldsymbol{\phi}\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ holds weakly}\}, \end{aligned}$$

where $\mathbf{n}$ is the outward normal to the boundary $\partial\Omega$ and $\boldsymbol{\phi}\cdot\mathbf{n}|_{\partial\Omega} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$, see [25]. Let H^m/R be the quotient space consisting of equivalence classes of elements of H^m differing by constants, with norm $\|p\|_{H^m/R} = \|p + c\|_m$. For any Banach space X, let $L^p(0,T;X)$ denote the space of measurable X- valued functions $\boldsymbol{\phi}$ on $(0,T)$ such that

$$\int_0^T \|\boldsymbol{\phi}(t)\|_X^p dt < \infty \quad \text{if } 1 \le p < \infty,$$

and for $p = \infty$

$$ess \sup_{0<t<T} \|\boldsymbol{\phi}(t)\|_X < \infty \quad \text{if } p = \infty.$$

Further, let P be the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{J}$.

Through out this paper, we make the following assumptions, which will be used for our subsequent analysis.

(**A1**). For $\mathbf{g} \in \mathbf{L^2}$, let the unique pair of solutions $\{\mathbf{v} \in \mathbf{J}_1, q \in L^2/R\}$ for the steady state Stokes problem

$$\begin{aligned} -\Delta\mathbf{v} + \nabla q = \mathbf{g}, \\ \nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0 \end{aligned}$$

satisfy the following regularity result

$$\|\mathbf{v}\|_2 + \|q\|_{H^1/R} \le C\|\mathbf{g}\|. \tag{2.1}$$

Setting

$$-\tilde{\Delta} = -P\Delta : \mathbf{J}_1 \cap \mathbf{H}^2 \subset \mathbf{J} \to \mathbf{J}$$

as the Stokes operator, the condition (**A1**) implies

$$\begin{aligned} \|\mathbf{v}\|_2 &\le C\|\tilde{\Delta}\mathbf{v}\| \quad \forall\mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2, \\ \|\mathbf{v}\|^2 &\le \lambda_1^{-1}\|\nabla\mathbf{v}\|^2 \ \ \forall\mathbf{v} \in \mathbf{J}_1, \quad \|\nabla\mathbf{v}\|^2 \le \lambda_1^{-1}\|\tilde{\Delta}\mathbf{v}\|^2 \ \ \forall\mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2, \end{aligned}$$

where λ_1 is the least positive eigenvalue of the Stokes operator $-\tilde{\Delta}$.

(**A2**). The initial velocity $\mathbf{u}_0(x)$ satisfies for some constants M_1, and M_2

$$\begin{aligned} (i) \quad & \mathbf{u}_0 \in \mathbf{J}_1 \quad \text{with} \quad \|\mathbf{u}_0\|_1 \le M_1, \\ (ii) \quad & \mathbf{u}_0 \in \mathbf{H}^2 \cap \mathbf{J}_1 \quad \text{with} \quad \|\mathbf{u}_0\|_2 \le M_2. \end{aligned}$$

Before going into the details, let us introduce the weak formulation of (1.1)–(1.3). Find a pair of functions $\{\mathbf{u}(t), p(t)\} \in \mathbf{H}^1 \times L^2,\;\; t>0$, such that

$$
\begin{aligned}
(\mathbf{u}_t, \boldsymbol{\phi}) + \mu(\nabla \mathbf{u}, \nabla \boldsymbol{\phi}) \;\;&+\;\; (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}) + \int_0^t \beta(t-s)(\nabla\mathbf{u}(s), \nabla\boldsymbol{\phi})\, ds \\
&= \;\; (p, \nabla\cdot\boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \mathbf{H}_0^1 \\
(\nabla\cdot\mathbf{u}, \chi) \;\;&=\;\; 0 \quad \forall \chi \in L^2.
\end{aligned} \tag{2.2}
$$

Equivalently, find $\mathbf{u}(.,t) \in \mathbf{J}_1$ such that

$$
\begin{aligned}
(\mathbf{u}_t, \boldsymbol{\phi}) + \mu(\nabla \mathbf{u}, \nabla \boldsymbol{\phi}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}) \;\;&+\;\; \int_0^t \beta(t-s)(\nabla\mathbf{u}(s), \nabla\boldsymbol{\phi})\, ds = 0 \\
& \forall \boldsymbol{\phi} \in \mathbf{J}_1, \;\; t>0.
\end{aligned} \tag{2.3}
$$

Now, we derive some regularity results for the solutions $\{\mathbf{u}, p\}$ that are required to prove the optimal error estimates in the later sections.

For our subsequent analysis, we use the positive property (see, [18] for a definition) of the kernel β associated with the integral operator in (1.1). This can be seen as a consequence of the following lemma. For a proof, we refer the reader to Sobolevskii ([24], p.1601), McLean and Thomeé [18].

Lemma 2.1 *For arbitrary $\alpha > 0$, $t^* > 0$ and $\phi \in L^2(0,t^*)$, the following positive definite property holds*

$$
\int_0^{t^*} \left(\int_0^t \exp\left[-\alpha(t-s)\right]\phi(s)\, ds \right) \phi(t)\, dt \geq 0.
$$

In order to deal with the integral term, from time to time, we appeal to the following Lemma.

Lemma 2.2 *Let $g \in L^1(0,t^*)$ and $\phi \in L^2(0,t^*)$ for some $t^* > 0$. Then the following estimate holds*

$$
\left(\int_0^{t^*} \left(\int_0^s g(s-\tau)\phi(\tau)\, d\tau \right)^2 ds \right)^{1/2} \leq \left(\int_0^{t^*} |g(s)|\, ds \right) \left(\int_0^{t^*} |\phi(s)|^2\, ds \right)^{1/2}.
$$

With change of variable and change of integrals, it is easy to check the validity of the above result.

Below, we discuss some *a priori* bounds for the solution $\mathbf{u}$ of (2.3).

Lemma 2.3 *Let $0 < \alpha < \min(\delta, \lambda_1\mu)$, and let the assumption* (**A2**) *hold. Then, the solution* $\mathbf{u}$ *of (2.3) satisfies*

$$
\|\mathbf{u}(t)\|^2 + (\mu - \frac{\alpha}{\lambda_1})e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\|\nabla\mathbf{u}(\tau)\|^2\, d\tau \;\;\leq\;\; e^{-2\alpha t}\|\mathbf{u}_0\|^2, \quad t>0.
$$

Proof. Setting $\hat{\mathbf{u}}(t) = e^{\alpha t}\mathbf{u}(t)$ for some $\alpha \geq 0$, we rewrite (2.3) as

$$
\begin{aligned}
(\hat{\mathbf{u}}_t, \boldsymbol{\phi}) - \alpha(\hat{\mathbf{u}}, \boldsymbol{\phi}) + e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \boldsymbol{\phi}) + \mu(\nabla\hat{\mathbf{u}}, \nabla\boldsymbol{\phi}) \;\;&+\;\; \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}(\nabla\hat{\mathbf{u}}(\tau), \nabla\boldsymbol{\phi})\, d\tau \\
&= \;\; 0 \quad \forall \boldsymbol{\phi} \in \mathbf{J}_1.
\end{aligned} \tag{2.4}
$$

Choose $\boldsymbol{\phi} = \hat{\mathbf{u}}$ in (2.4). Since $(\hat{\mathbf{u}} \cdot \nabla\hat{\mathbf{u}}, \hat{\mathbf{u}}) = 0$ and $\|\hat{\mathbf{u}}\|^2 \leq \lambda_1^{-1}\|\nabla\hat{\mathbf{u}}\|^2$, the following estimate holds :

$$\frac{d}{dt}\|\hat{\mathbf{u}}\|^2 + 2(\mu - \frac{\alpha}{\lambda_1})\|\nabla\hat{\mathbf{u}}\|^2 + 2\int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}(\nabla\hat{\mathbf{u}}(\tau), \nabla\hat{\mathbf{u}}(t))d\tau \leq 0. \tag{2.5}$$

After integrating with respect to time, the third term on the left hand side of (2.5) becomes nonnegative, provided $\delta > \alpha \geq 0$ and the second term on the left hand side is also nonnegative if $\alpha < \lambda_1\mu$. With $0 < \alpha < \min(\delta, \lambda_1\mu)$, we complete the rest of the proof. □

Lemma 2.4 *Let* $0 \leq \alpha < \min(\delta, \lambda_1\mu)$ *and let the assumption* (**A2**) hold. then there is a positive constant $K = K(\delta, \mu, \gamma, \lambda_1, M_1)$ such that for all $t > 0$

$$\|\nabla\mathbf{u}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\|\tilde{\Delta}\mathbf{u}(\tau)\|^2\, d\tau \leq Ke^{-2\alpha t}, \qquad t > 0.$$

Proof. Using the Stokes operator $\tilde{\Delta}$, we rewrite (2.4) as

$$\begin{aligned}(\hat{\mathbf{u}}_t, \boldsymbol{\phi}) - \alpha(\hat{\mathbf{u}}, \boldsymbol{\phi}) - \mu(\tilde{\Delta}\hat{\mathbf{u}}, \boldsymbol{\phi}) \;&-\; \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}(\tilde{\Delta}\hat{\mathbf{u}}(\tau), \boldsymbol{\phi})\, d\tau \\ &= -e^{-\alpha t}(\hat{\mathbf{u}} \cdot \nabla\hat{\mathbf{u}}, \boldsymbol{\phi}).\end{aligned} \tag{2.6}$$

With $\boldsymbol{\phi} = -\tilde{\Delta}\hat{\mathbf{u}}$ in (2.6), we note that

$$-(\hat{\mathbf{u}}_t, \tilde{\Delta}\hat{\mathbf{u}}) = \frac{1}{2}\frac{d}{dt}\|\nabla\hat{\mathbf{u}}\|^2.$$

Thus,

$$\begin{aligned}\frac{d}{dt}\|\nabla\hat{\mathbf{u}}\|^2 \;&+\; 2\mu\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 + 2\int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}(\tilde{\Delta}\hat{\mathbf{u}}(\tau), \tilde{\Delta}\hat{\mathbf{u}}(t))\, d\tau \\ &= -2\alpha(\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}}) + 2e^{-\alpha t}(\hat{\mathbf{u}} \cdot \nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}}).\end{aligned}$$

On integration with respect to time and using Lemma 2.1 with definition of β, it follows for $0 < \alpha < \min(\delta, \lambda_1\mu)$ that

$$\begin{aligned}\|\nabla\hat{\mathbf{u}}(t)\|^2 \;&+\; 2\mu\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|^2 d\tau \leq \|\nabla\mathbf{u}_0\|^2 - 2\alpha\int_0^t (\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}})\, d\tau \\ &+\; 2\int_0^t e^{-\alpha\tau}(\hat{\mathbf{u}} \cdot \nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}})\, d\tau = \|\nabla\mathbf{u}_0\|^2 + I_1 + I_2.\end{aligned} \tag{2.7}$$

To estimate $|I_1|$, we apply Cauchy-Schwarz inequality and Poincaré inequality. With $ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$, $a, b \geq 0$, $\epsilon > 0$, we obtain from Lemma 2.3

$$\begin{aligned}|I_1| \;&\leq\; C(\lambda_1, \alpha, \mu, \epsilon)\int_0^t \|\nabla\hat{\mathbf{u}}(\tau)\|^2 d\tau + \epsilon\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|^2 d\tau \\ &\leq\; C(\alpha, \mu, \lambda_1, \epsilon)\|\mathbf{u}_0\|^2 + \epsilon\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|^2 d\tau.\end{aligned}$$

To estimate of I_2 term in (2.7), we note that repeated use of Hölder's inequality yields

$$|(\hat{\mathbf{u}} \cdot \nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}})| \leq \|\hat{\mathbf{u}}\|_{L^4(\Omega)}\|\nabla\hat{\mathbf{u}}\|_{L^4(\Omega)}\|\tilde{\Delta}\hat{\mathbf{u}}\|.$$

By Sobolev inequality, see Temam [25],

$$\|\phi\|_{L^4(\Omega)} \le C\|\phi\|^{\frac{1}{2}}\|\nabla\phi\|^{\frac{1}{2}}, \quad \phi \in \mathbf{H}^1(\Omega),$$

and hence,

$$|I_2| \le C\int_0^t e^{-\alpha\tau}\|\hat{\mathbf{u}}\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac{3}{2}}\,d\tau.$$

An appeal to the Young's inequality $ab \le \frac{a^p}{\epsilon^{p/q}} + \frac{\epsilon b^q}{q}$, $a, b \ge 0$, $t > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with Lemma 2.3 yields

$$|I_2| \le C(\epsilon)\int_0^t e^{-4\alpha\tau}\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^4\,d\tau + \epsilon\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}\|^2 d\tau.$$

Altogether, we have from (2.7)

$$\|\nabla\hat{\mathbf{u}}(t)\|^2 + 2(\mu-\epsilon)\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|^2\,d\tau \le C(\alpha,\mu,\lambda_1,\epsilon)\|\nabla\mathbf{u}_0\|^2 + C(\epsilon)\int_0^t e^{-4\alpha\tau}\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^4\,d\tau.$$

With $\epsilon = \frac{\mu}{2}$, apply Gronwall's inequality to obtain

$$\|\nabla\hat{\mathbf{u}}(t)\|^2 + \mu\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|^2\,d\tau \le C(\alpha,\mu,\lambda_1)\|\nabla\mathbf{u}_0\|^2 \exp\{C(\mu)\int_0^t e^{-4\alpha\tau}\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^2 d\tau\}.$$

Using Lemma 2.3, we bound the

$$\int_0^t e^{-4\alpha\tau}\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^2 d\tau \le \|\mathbf{u}_0\|^2\int_0^t \|\nabla\hat{\mathbf{u}}\|^2\,d\tau \le C\|\mathbf{u}_0\|^4,$$

and hence, the result follows. This completes the rest of the proof. □

Remark 2.1 The *a priori* bounds of the above two Lemmas are useful for proving existence of global strong solutions to (1.1)–(1.3) by using the Faedo-Galerkin method, see Temam [25], Ladyzhenskaya [15] for similar analysis in case of Navier Stokes equation. Since the existence analysis is a routine one, we refrain from discussing this for (1.1)–(1.3), see Pani [20].

Theorem 2.1 *Let the assumptions* (**A1**) and (**A2**) hold. Then, there is a constant $K = K(M_1, M_2, \lambda_1, \mu, \delta, \gamma)$ such that for $0 < \alpha < \min(\delta, \lambda_1\mu)$ the following estimates hold:

$$\|\mathbf{u}(t)\|_2^2 + \|\mathbf{u}_t(t)\|^2 + \|p(t)\|_{H^1/R}^2 + \int_0^t e^{2\alpha s}\|\mathbf{u}_t\|_1^2\,ds \le Ke^{-2\alpha t}, \quad t > 0. \tag{2.8}$$

Proof. Set $\phi = e^{2\alpha t}\mathbf{u}_t$ in (2.3) and rewrite it as

$$\begin{aligned} e^{2\alpha t}\|\mathbf{u}_t\|^2 &= \mu(\tilde{\Delta}\hat{\mathbf{u}}, e^{\alpha t}\mathbf{u}_t) - e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, e^{\alpha t}\mathbf{u}_t) \\ &+ \int_0^t \beta(t-s)e^{\alpha(t-s)}(\tilde{\Delta}\hat{\mathbf{u}}(s), e^{\alpha t}\mathbf{u}_t)\,ds. \end{aligned}$$

Using both Sobolev imbedding Theorem and Sobolev inequality, the second term on the right hand side can be evaluated as

$$\begin{aligned} |e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, e^{\alpha t}\mathbf{u}_t)| &\le e^{-\alpha t}\|\hat{\mathbf{u}}\|_{L^4}\|\nabla\hat{\mathbf{u}}\|_{L^4}\|e^{\alpha t}\mathbf{u}_t\| \\ &\le C(\epsilon)\left(\|\mathbf{u}_0\|^2 e^{-4\alpha t}\|\nabla\hat{\mathbf{u}}\|^4 + \|\tilde{\Delta}\hat{\mathbf{u}}\|^2\right) + \epsilon e^{2\alpha t}\|\mathbf{u}_t\|^2. \end{aligned}$$

For the remaining two terms, we apply Cauchy-Schwarz inequality with Young's inequality. A use of Lemma 2.4 yields

$$
\begin{aligned}
(1-3\epsilon)\int_0^t e^{2\alpha s}\|\mathbf{u}_t(s)\|^2\,ds &\le C(\mu,\epsilon)\Big[\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2\,ds + \int_0^t e^{-4\alpha s}\|\nabla\hat{\mathbf{u}}(s)\|^2\,ds\Big] \\
&+ C(\mu,\epsilon)\int_0^t \Big(\int_0^s \beta(s-\tau)e^{\alpha(s-\tau)}\|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|\,d\tau\Big)^2\,ds. \qquad (2.9)
\end{aligned}
$$

For last term on the right hand side of (2.9), we use the definition of β, Hölder's inequality and Lemma 2.2 to find that

$$
\begin{aligned}
I &= \int_0^t \Big(\int_0^s \beta(s-\tau)e^{\alpha(s-\tau)}\|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|\,d\tau\Big)^2\,ds \\
&= \gamma^2\int_0^t \Big(\int_0^s e^{-(\delta-\alpha)(s-\tau)}\|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|\,d\tau\Big)^2\,ds \\
&\le \gamma^2\int_0^t \Big(\int_0^s e^{-(\delta-\alpha)(s-\tau)}\,d\tau\Big)\Big(\int_0^s e^{-(\delta-\alpha)(s-\tau)}\|\Delta\hat{\mathbf{u}}(\tau)\|^2\,d\tau\Big)\,ds \\
&\le \frac{\gamma^2}{(\delta-\alpha)}\int_0^t\int_0^s e^{-(\delta-\alpha)(s-\tau)}\|\Delta\hat{\mathbf{u}}(\tau)\|^2\,d\tau\,ds.
\end{aligned}
$$

Using change of variables, we obtain

$$
I \le \frac{\gamma^2}{(\delta-\alpha)}\int_0^t\int_0^s e^{-(\delta-\alpha)\tau}\|\tilde{\Delta}\hat{\mathbf{u}}(s-\tau)\|^2\,d\tau\,ds.
$$

Now, changing the order of integration, we arrive at

$$
\begin{aligned}
I &\le \frac{\gamma^2}{(\delta-\alpha)}\int_0^t e^{-(\delta-\alpha)\tau}\Big(\int_\tau^t \|\tilde{\Delta}\hat{\mathbf{u}}(s-\tau)\|^2\,ds\Big)\,d\tau \\
&\le \frac{\gamma^2}{(\delta-\alpha)}\int_0^t e^{-(\delta-\alpha)(t-\tau)}\Big(\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2\,ds\Big)\,d\tau \\
&\le \Big(\frac{\gamma}{\delta-\alpha}\Big)^2\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2\,ds. \qquad (2.10)
\end{aligned}
$$

Altogether with $\epsilon=\frac{1}{6}$, we obtain using Lemma 2.4

$$
\begin{aligned}
(2.11)\qquad \int_0^t e^{2\alpha s}\|\mathbf{u}_t\|^2\,ds &\le C(\alpha,\delta,\gamma)\Big[\int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}\|^2\,ds + \int_0^t e^{2\alpha s}\|\nabla\mathbf{u}\|^4\,ds\Big] \\
&\le C(\alpha,\delta,\gamma,M_1).
\end{aligned}
$$

On differentiating the equation (2.6) (with $\alpha=0$) with respect to time, we obtain

$$
(2.12)\qquad \mathbf{u}_{tt}-\mu\tilde{\Delta}\mathbf{u}_t-\beta(0)\tilde{\Delta}\mathbf{u}-\int_0^t \beta_t(t-s)\tilde{\Delta}\mathbf{u}(s)\,ds = -(\mathbf{u}_t\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}_t).
$$

Using again the form of β, we have $\beta_t(t-s)=-\beta_s(t-s)$. Integration by parts in time now yields

$$
\begin{aligned}
-\int_0^t \beta_t(t-s)\tilde{\Delta}\mathbf{u}(s)\,ds &= \int_0^t \beta_s(t-s)\tilde{\Delta}\mathbf{u}(s)\,ds \\
&= \beta(0)\tilde{\Delta}\mathbf{u}(t)-\beta(t)\tilde{\Delta}\mathbf{u}_0-\int_0^t \beta(t-s)\tilde{\Delta}\mathbf{u}_s(s)\,ds.
\end{aligned}
$$

Thus, forming an inner product between (2.12) and $e^{2\alpha t}\mathbf{u}_t$, we rewrite the resulting equation as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(e^{2\alpha t}\|\mathbf{u}_t\|^2) &- \alpha e^{2\alpha t}\|\mathbf{u}_t\|^2 + \mu e^{2\alpha t}\|\nabla \mathbf{u}_t\|^2 + \gamma\int_0^t e^{-(\delta-\alpha)(t-s)}(\nabla e^{\alpha s}\mathbf{u}_s, \nabla e^{\alpha t}\mathbf{u}_t)\, ds \\ &= \gamma e^{-(\delta-\alpha)t}(\tilde{\Delta}\mathbf{u}_0, e^{\alpha t}\mathbf{u}_t) - e^{-\alpha t}(e^{\alpha t}\mathbf{u}_t\cdot\nabla\hat{\mathbf{u}} + \hat{\mathbf{u}}\cdot\nabla e^{\alpha t}\mathbf{u}_t, e^{\alpha t}\mathbf{u}_t).\end{aligned}$$

Observe that $(\hat{\mathbf{u}}\cdot\nabla e^{\alpha t}\mathbf{u}_t, e^{\alpha t}\mathbf{u}_t) = 0$. Integrate with respect to time the resulting equation and use Lemma 2.1 to obtain

$$\begin{aligned}(2.13)\quad e^{2\alpha t}\|\mathbf{u}_t\|^2 &+ 2\mu\int_0^t e^{2\alpha s}\|\nabla \mathbf{u}_t(s)\|^2\, ds \le \|\mathbf{u}_t(0)\|^2 + 2\alpha\int_0^t e^{2\alpha s}\|u_t(s)\|^2\, ds \\ &+ 2\int_0^t e^{-\alpha s}(e^{\alpha s}\mathbf{u}_t\cdot\nabla\hat{\mathbf{u}}, e^{\alpha s}\mathbf{u}_t)\, ds + 2\gamma\|\Delta\mathbf{u}_0\|\int_0^t e^{-(\delta-\alpha)s}\|e^{\alpha s}\mathbf{u}_t\|\, ds.\end{aligned}$$

For the third term on the right hand side of (2.13), use of Sobolev imbedding Theorem and Sobolev inequality now yields

$$2\int_0^t e^{-\alpha s}|(e^{\alpha s}\mathbf{u}_t\cdot\nabla\hat{\mathbf{u}}, e^{\alpha s}\mathbf{u}_t)|\, ds \le C(\mu)\sup_{0\le s\le t}\|\nabla\mathbf{u}(s)\|\int_0^t e^{2\alpha s}\|\mathbf{u}_t\|^2\, ds + \mu\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_t\|^2\, ds.$$

Substituting the above inequality in (2.13), it now follows that

$$\begin{aligned}e^{2\alpha t}\|\mathbf{u}_t\|^2 &+ \mu\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_t(s)\|^2\, ds \\ &\le C(\delta,\alpha,\gamma,\mu,M_1)\left[\|\mathbf{u}_t(0)\|^2 + \|\Delta\mathbf{u}_0\|^2 + \int_0^t e^{2\alpha s}\|\mathbf{u}_t(s)\|^2\, ds\right].\end{aligned}$$

Note that $\|\mathbf{u}_t(0)\| \le C(\|\tilde{\Delta}\mathbf{u}_0\| + \|\mathbf{f}_0\|)$. Using (2.11), we find that

$$(2.14)\qquad \|\mathbf{u}_t\|^2 + \mu e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_t(s)\|^2\, ds \le C(\delta,\alpha,\mu,M_2)e^{-2\alpha t}.$$

To estimate $\|\tilde{\Delta}\mathbf{u}(t)\|$, choose $\phi = -\tilde{\Delta}\hat{\mathbf{u}}$ in (2.6) and rewrite as

$$\begin{aligned}(2.15)\qquad \mu\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 &\le e^{\alpha t}\|\mathbf{u}_t\|\|\Delta\hat{\mathbf{u}}\| + e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}}) \\ &+ \int_0^t \beta(t-s)e^{\alpha(t-s)}\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|\|\tilde{\Delta}\hat{\mathbf{u}}(t)\|\, ds.\end{aligned}$$

The first two terms on the right hand side of (2.15) are bounded by

$$\le C(\epsilon)\left[e^{2\alpha t}\|\mathbf{u}_t\|^2 + e^{-4\alpha t}\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^4\right] + 2\epsilon e^{2\alpha t}\|\tilde{\Delta}\mathbf{u}\|^2.$$

Using Hölder's inequality the third term on the right hand side of (2.15) is bounded by

$$C(\gamma,\alpha,\delta,\epsilon)\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2\, ds + \epsilon\|\tilde{\Delta}\hat{\mathbf{u}}\|^2.$$

With $\epsilon = \frac{\mu}{6}$, multiply both sides by $e^{-2\alpha t}$. Then apply Lemma 2.3 and (2.14) to obtain

$$\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 \le C(\|\mathbf{u}_0\|_2) + C(\gamma,\alpha,\delta,\mu)\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2\, ds.$$

A use of Lemma 2.4 yields

$$\|\tilde{\Delta}\mathbf{u}(t)\|^2 \le K(\gamma,\alpha,\delta,M_2)e^{-2\alpha t}.$$

From (**A1**), we have $\|\mathbf{v}\|_2 \le \|\tilde{\Delta}\mathbf{v}\|$ and thus, using the equation (1.1) we obtain the desired estimate for the pressure term. This completes the rest of the proof. □

Theorem 2.2 *Under the assumptions of the Theorem 2.1 , there is a positive constant K such that the pair of solutions $\{\mathbf{u}, p\}$ satisfies the following estimates for $0 < \alpha < \min(\delta, \lambda\mu)$:*

$$\tau^*(t)\|\mathbf{u}_t\|_1^2 \le Ke^{-2\alpha t}, \quad t > 0, \tag{2.16}$$

where $\tau^*(t) = \min(t, 1)$. Moreover,

$$\int_0^t \sigma(s)(\|\mathbf{u}_t\|_2^2 + \|\mathbf{u}_{tt}\|^2 + \|p_t\|_{H^1/R}^2)\,ds \le K, \quad t > 0, \tag{2.17}$$

where $\sigma(t) = \tau^*(t)e^{2\alpha t}$.

Proof. For the first estimate (2.16), form an L^2-innerproduct between (2.12) and $-\sigma(t)\tilde{\Delta}\mathbf{u}_t$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\sigma(t)\|\nabla\mathbf{u}_t\|^2) &+ \mu\sigma(t)\|\tilde{\Delta}\mathbf{u}_t\|^2 = -\gamma\sigma(t)(\tilde{\Delta}\mathbf{u}, \tilde{\Delta}\mathbf{u}_t) + \frac{1}{2}\sigma_t\|\nabla\mathbf{u}_t\|^2 \\ &- \sigma(t)\int_0^t \beta_t(t-s)(\tilde{\Delta}\mathbf{u}(s), \tilde{\Delta}\mathbf{u}_t)\,ds + \sigma(t)(\mathbf{u}_t\cdot\nabla\mathbf{u}, \tilde{\Delta}\mathbf{u}_t) \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{2.18}$$

For I_1, we obtain in a standard way

$$|I_1| \le C(\epsilon, \gamma)\sigma(t)\|\tilde{\Delta}\mathbf{u}\|^2 + \frac{\epsilon}{2}\sigma\|\tilde{\Delta}\mathbf{u}_t\|^2. \tag{2.19}$$

Since $\sigma_t = \tau_t^* e^{2\alpha t} + 2\alpha\tau^* e^{2\alpha t}$ with τ^*, $\tau_t^* \le 1$, we find that

$$|I_2| \le C(\alpha)e^{2\alpha t}\|\nabla\mathbf{u}_t\|^2. \tag{2.20}$$

Using the Sobolev imbedding theorem, the Sobolev inequality and Young's inequality, the nonlinear term in I_4 is bounded by

$$\begin{aligned}|I_4| &\le C(\epsilon)\sigma(t)\|\nabla\mathbf{u}_t\|^2(\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\| + \|\nabla\mathbf{u}\|^2) + \frac{\epsilon}{2}\sigma(t)\|\tilde{\Delta}\mathbf{u}_t\|^2 \\ &\le C(\sup_{t>0}\|\tilde{\Delta}\mathbf{u}\|^2)\sigma(t)\|\nabla\mathbf{u}_t\|^2 + \frac{\epsilon}{2}\sigma(t)\|\tilde{\Delta}\mathbf{u}_t\|^2.\end{aligned} \tag{2.21}$$

For I_3, we have with $\beta_t(t-s) = -\frac{1}{\delta}\beta(t-s)$,

$$|I_3| \le \frac{\gamma^2}{2\epsilon\delta^2}\tau^*(\int_0^t e^{-(\delta-\alpha)(t-s)}\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|\,ds)^2 + \frac{\epsilon}{2}\sigma(t)\|\tilde{\Delta}\mathbf{u}_t\|^2,$$

and hence, integrating with respect to time and using the estimate (2.10) for the I term, we obtain

$$\begin{aligned}\int_0^t |I_3|\,ds &\le \frac{\gamma^2}{2\epsilon\delta^2}I + \frac{\epsilon}{2}\int_0^t \sigma(s)\|\tilde{\Delta}\mathbf{u}_t\|^2\,ds \\ &\le C(\gamma, \delta, \alpha, \epsilon)\int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}(s)\|^2\,ds + \frac{\epsilon}{2}\int_0^t \sigma(s)\|\tilde{\Delta}\mathbf{u}_t(s)\|^2\,ds.\end{aligned} \tag{2.22}$$

Multiply (2.18) by 2 and integrate with respect to time. Substitute (2.19)–(2.22) in the resulting equation. With $\epsilon = \mu/3$, we find that

$$\begin{aligned}\sigma(t)\|\nabla\mathbf{u}_t\|^2 &+ \mu\int_0^t \sigma(s)\|\tilde{\Delta}\mathbf{u}_t\|^2\,ds \\ &\le C(\|\tilde{\Delta}\mathbf{u}_0\|^2)\int_0^t e^{2\alpha s}\left(\|\nabla\mathbf{u}_t\|^2 + \|\tilde{\Delta}\mathbf{u}(s)\|^2\right)ds.\end{aligned} \tag{2.23}$$

From the assumption (**A1**)–(**A2**) and Theorem 2.1, we complete the proof of (2.16) and the first estimate of (2.17).

Now, form an inner product between (2.12) and $\sigma(t)\mathbf{u}_{tt}$. Then proceed, similarly, as in the estimate of (2.11) to obtain

$$\int_0^t \sigma(s)\|\mathbf{u}_{tt}\|^2\,ds \le C(\gamma,\alpha,\delta)\int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}\|^2\,ds + C(\sup_{t>0}\|\tilde{\Delta}\mathbf{u}(t)\|)\int_0^t \sigma(s)\|\tilde{\Delta}\mathbf{u}_t\|^2\,ds,$$

and the required estimate for $\mathbf{u}_{tt}$ now follows.

The estimate of $\tilde{\Delta}\mathbf{u}_t$ can be obtained in a similar way as in the estimate of $\mathbf{u}_t$ leading to a bound for $\tilde{\Delta}\mathbf{u}$. Again for the pressure, we differentiate the equation (1.1) with respect to time and with $\mathbf{f}=0$ and use *a priori* bounds for $\mathbf{u}_{tt}$ and $\tilde{\Delta}\mathbf{u}_t$ to complete the rest of the proof. □

In the next section, we shall also use the *a priori* bounds of the following unsteady Stokes-Volterra problem:

$$\text{(2.24)}\qquad \mathbf{v}_t - \mu\Delta\mathbf{v} + \nabla p - \int_0^t \beta(t-s)\Delta\mathbf{v}(s)\,ds = \mathbf{g},\quad x\in\Omega, t>0$$

$$\text{(2.25)}\qquad \nabla\cdot\mathbf{v} = 0,\quad x\in\Omega, t>0$$

$$\text{(2.26)}\qquad \mathbf{v}|_{t=0}=0,\qquad \mathbf{v}|_{\partial\Omega}=0.$$

With appropriate modification of the arguments leading to the estimates of Theorem 2.2, we easily derive the following estimates.

Corollary 2.1 *For $0<\alpha<\min(\delta,\mu\lambda_1)$, and $\mathbf{g}\in L^2(0,\infty;\mathbf{L}^2)$, the following estimate holds*

$$\int_0^t e^{\alpha\tau}\{\|\mathbf{v}(\tau)\|_2^2 + \|\mathbf{v}_t(\tau)\|^2 + \|p\|^2_{H^1/R}\}\,d\tau \le C\left(\frac{\lambda_1^2}{(\mu\lambda_1-\alpha)^2}\right)\int_0^t e^{2\alpha s}\|\mathbf{g}(s)\|^2\,ds.$$

Proof. Note that

$$(\hat{\mathbf{v}}_t,\phi) - \alpha(\hat{\mathbf{v}},\phi) - \mu(\tilde{\Delta}\hat{\mathbf{v}},\phi) - \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}(\tilde{\Delta}\hat{\mathbf{v}}(\tau),\phi)\,d\tau = (\hat{\mathbf{g}},\phi).$$

As in (2.6), choose $\phi = -\tilde{\Delta}\hat{\mathbf{v}}$ and obtain

$$\frac{d}{dt}\|\nabla\hat{\mathbf{v}}\|^2 + (\mu-\alpha\lambda_1^{-1})\|\tilde{\Delta}\hat{\mathbf{v}}\|^2 + 2\int_0^t \gamma e^{-(\delta-\alpha)(t-s)}(\tilde{\Delta}\hat{\mathbf{v}}(s),\tilde{\Delta}\hat{\mathbf{v}})\,ds \le \frac{1}{\mu-\alpha\lambda_1^{-1}}\|\hat{\mathbf{g}}\|^2.$$

Integrating with respect to time, the third term on the left hand side becomes nonnegative for $0<\alpha<\min(\delta,\mu\lambda_1)$, and hence,

$$e^{2\alpha t}\|\nabla\mathbf{v}(t)\|^2 + (\mu-\alpha\lambda_1^{-1})\int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{v}(s)\|^2\,ds \le \frac{\lambda_1}{\mu\lambda_1-\alpha}\int_0^t e^{2\alpha s}\|\mathbf{g}(s)\|^2\,ds.$$

Proceed exactly as in the proof of theorem 2.2 to complete the rest of the proof. □

3. Semidiscrete Galerkin Approximations. From now on, we denote h with $0<h<1$ by a real positive discretization parameter tending to zero. Let $\mathbf{H}_h$ and L_h, $0<h<1$ be two family of finite dimensional subspaces of $\mathbf{H}_0^1$ and L^2, respectively, approximating velocity vector and the pressure. Assume that the following approximation properties are satisfied for the spaces $\mathbf{H}_h$ and L_h:

($\mathbf{B_1}$) For each $\mathbf{w} \in \mathbf{H}_0^1 \cap \mathbf{H}^2$ and $q \in H^1/R$ there exist approximations $i_h v \in \mathbf{H}_h$ and $j_h q \in L_h$ such that

$$\|\mathbf{w} - i_h \mathbf{w}\| + h\|\nabla(\mathbf{w} - i_h \mathbf{w})\| \leq K_0 h^2 \|\mathbf{w}\|_2, \quad \|q - j_h q\|_{L^2/R} \leq K_0 h \|q\|_{H^1/R}.$$

Further, suppose that the following inverse hypothesis holds for $\mathbf{w}_h \in \mathbf{H}_h$

$$\|\nabla \mathbf{w}_h\| \leq K_0 h^{-1} \|\mathbf{w}_h\|.$$

For defining the Galerkin approximations, set for $\mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1$,

$$a(\mathbf{v}, \boldsymbol{\phi}) = (\nabla \mathbf{v}, \nabla \boldsymbol{\phi})$$

and

$$b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = \frac{1}{2}(\mathbf{v} \cdot \nabla \mathbf{w}, \boldsymbol{\phi}) - \frac{1}{2}(\mathbf{v} \cdot \nabla \boldsymbol{\phi}, \mathbf{w}).$$

Note that the operator $b(\cdot,\cdot,\cdot)$ preserves the antisymmetric properties of the original nonlinear term that is

$$b(\mathbf{v}_h, \mathbf{w}_h, \mathbf{w}_h) = 0 \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h.$$

The discrete analogue of the weak formulation (2.2) now reads as: Find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$\begin{aligned} (\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \mu a(\mathbf{u}_h, \boldsymbol{\phi}_h) &+ b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - (p_h, \nabla \cdot \boldsymbol{\phi}_h) \\ &= -\int_0^t \beta(t-s) a(\mathbf{u}_h(s), \boldsymbol{\phi}_h)\, ds \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \\ (\nabla \cdot \mathbf{u}_h, \chi_h) &= 0 \quad \forall \chi_h \in L_h \end{aligned} \tag{3.1}$$

where $\mathbf{u}_{0h} \in \mathbf{H}_h$ is a suitable approximation of $\mathbf{u}_0 \in \mathbf{J}_1$.

In order to consider a discrete space analogous to $\mathbf{J}_1$, we impose the discrete incompressibility condition on $\mathbf{H}_h$ and call it as $\mathbf{J}_h$. Thus, we define $\mathbf{J}_h$ as

$$\mathbf{J}_h = \{v_h \in \mathbf{H}_h : (\chi_h, \nabla_h \cdot v_h) = 0 \quad \forall \chi_h \in L_h\}.$$

Note that the space $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$. With $\mathbf{J}_h$ as above, we now introduce the Galerkin formulation : Find $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$\begin{aligned} (\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \mu a(\mathbf{u}_h, \boldsymbol{\phi}_h) &+ \int_0^t \beta(t-s) a(\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \\ &= -b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \end{aligned} \tag{3.2}$$

Since $\mathbf{J}_h$ is finite dimensional, the problem (3.2) leads to a system of nonlinear integro-differential equations. A use of Picard's theorem now yields existence of a unique local solution in an interval $[0, t^*)$, for some $t^* > 0$. For continuation of solution beyond t^*, we need to establish an $L^\infty(\mathbf{L^2})$ bound for the approximate solution $\mathbf{u}_h$. Setting $\boldsymbol{\phi}_h = \mathbf{u}_h$ in (3.2), we obtain as in (2.5)

$$\frac{d}{dt}\|\mathbf{u}_h\|^2 + 2\mu\|\nabla \mathbf{u}_h\|^2 + 2\int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{u}_h(t))\, ds = 0.$$

On integration with respect to the temporal variable t and using the positive property of the kernel β, we find that

$$\|\mathbf{u}_h(t)\|^2 \leq C(\mu)\|\mathbf{u}_{0h}\|^2 \leq C \quad \forall t \geq 0,$$

provided $\|\mathbf{u}_{0h}\| \leq C\|\mathbf{u}_0\|$. This is indeed true, which we shall see later on. This shows the global existence of the Galerkin approximation $\mathbf{u}_h$ for all $t > 0$.

Once, we compute $\mathbf{u}_h(t) \in J_h$, the approximation $p_h(t) \in L_h$ to the pressure $p(t)$ can be found out by solving the following system :

$$\begin{aligned}(p_h, \nabla \cdot \boldsymbol{\phi}_h) &= (\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \mu a(\mathbf{u}_h, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \\ &+ b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - (\mathbf{f}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h. \end{aligned} \tag{3.3}$$

For the solvability of the above system (3.3), we note that the right hand side defines a linear functional ℓ on $\mathbf{H}_h$, i.e., $\boldsymbol{\phi}_h \mapsto \ell(\boldsymbol{\phi}_h)$. By construction $\ell(\boldsymbol{\phi}_h) = 0$ for all $\boldsymbol{\phi} \in \mathbf{J}_h$. It is now easy to check that this condition implies existence of $p_h \in L_h$, see [7]. Uniqueness is obtained on the quotient space L_h/N_h, where

$$N_h = \{q_h \in L_h : (q_h, \nabla_h \cdot \boldsymbol{\phi}_h) = 0, \forall \boldsymbol{\phi}_h \in \mathbf{H}_h\}.$$

The norm on L_h/N_h is given by

$$\|q_h\|_{L^2/N_h} = \inf_{\chi_h \in N_h} \|q_h + \chi_h\|.$$

For continuous dependence of the discrete pressure $p_h(t) \in L_h/N_h$ on the discrete velocity $u_h(t) \in \mathbf{J}_h$, we assume the following discrete inf-sup (LBB) condition for the finite dimensional spaces $\mathbf{H}_h$ and L_h:

(**B2**) For every $q_h \in L_h$, there exist a non-trivial function $\boldsymbol{\phi}_h \in \mathbf{H}_h$ and a positive constant K_0 such that

$$|(q_h, \nabla_h \cdot \boldsymbol{\phi}_h)| \geq K_0 \|\nabla \boldsymbol{\phi}_h\| \|q_h\|_{L^2/N_h}.$$

As a consequence of conditions (**B1**)–(**B2**), we have the following properties of the L^2 projection $P_h : \mathbf{L^2} \mapsto \mathbf{J_h}$. For a proof, see [7],[9]. For $\boldsymbol{\phi} \in \mathbf{J}_h$, we note that

$$\|\boldsymbol{\phi} - P_h \boldsymbol{\phi}\| + h\|\nabla P_h \boldsymbol{\phi}\| \leq Ch\|\nabla \boldsymbol{\phi}\|, \tag{3.4}$$

and for $\boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$

$$\|\boldsymbol{\phi} - P_h \boldsymbol{\phi}\| + h\|\nabla(\boldsymbol{\phi} - P_h \boldsymbol{\phi})\| \leq Ch^2\|\tilde{\Delta} \boldsymbol{\phi}\|. \tag{3.5}$$

We may define the discrete operator $\Delta_h : \mathbf{H}_h \mapsto \mathbf{H}_h$ through the bilinear form $a(\cdot,\cdot)$ as

$$a(\mathbf{v}_h, \boldsymbol{\phi}_h) = (-\Delta_h \mathbf{v}_h, \boldsymbol{\phi}) \quad \forall \mathbf{v}_h, \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{3.6}$$

Set the discrete analogue of the Stokes operator $\tilde{\Delta} = P\Delta$ as $\tilde{\Delta}_h = P_h \Delta_h$.

With a use of Sobolev imbedding Theorem with Sobolev inequality and the $\mathbf{L^2}$-projection, it is a routine calculation to to derive the following estimates: Let (**A1**),(**B1**) and (**B2**) be satisfied. Then, there exists a positive constant C such that

$$|b(\mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\xi})| + |b(\boldsymbol{\phi}, \mathbf{v}, \boldsymbol{\xi})| \leq C\|\nabla \mathbf{v}\|^{1/2}\|\tilde{\Delta}\mathbf{v}\|^{1/2}\|\nabla \boldsymbol{\phi}\|\ \|\xi\|, \tag{3.7}$$

for $\mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2$ and $\boldsymbol{\phi}, \boldsymbol{\xi}$ *in* $\mathbf{H}_h$, and

$$|b(\boldsymbol{\phi}_h, \boldsymbol{\xi}, \chi)| \leq C\|\nabla \boldsymbol{\phi}_h\|^{1/2}\|\tilde{\Delta}_h \boldsymbol{\phi}_h\|^{1/2}\|\nabla \boldsymbol{\xi}\|\ \|\chi\| \tag{3.8}$$

for $\boldsymbol{\phi}_h \in \mathbf{J}_h$ and $\boldsymbol{\xi}, \chi \in \mathbf{H}_h$.

We conclude this section by citing some examples of the subspaces $\mathbf{H}_h$ and L_h satisfying the assumptions (**B1**) and (**B2**). Let Ω be a convex polygon in $\mathbb{R}^2$. Let $\{\mathcal{T}_h\}$ be a family of finite decomposition of the domain Ω into 2-simplexes K with diameter h_K. Let $h = \max_{K \in \mathcal{T}_h} h_K$. Further, assume that this family of triangulations is regular and satisfies the quasi-uniformity condition, see Ciarlet [6]. For any nonnegative integer r, let $P_r(K)$ denote the space of all polynomials of degree less than or equal to r. Now, we present two examples of the finite dimensional spaces, which satisfy the assumptions $\mathbf{B_1}$ and $\mathbf{B_2}$.

Example 3.1. (Bercovier-Pironneau [3])

$$\begin{aligned} \mathbf{H}_h &= \{v_h \in (C^0(\bar{\Omega}))^2 \cap \mathbf{H}_0^1 : v_h|_K \in (P_1(K))^2 \quad \forall K \in \mathcal{T}_{h/2}\} \\ L_h &= \{q_h \in (C^0(\bar{\Omega})) \cap \in L^2(\Omega) : q_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\}, \end{aligned}$$

where $\mathcal{T}_{h/2}$ is obtained by dividing each triangle of $\mathcal{T}_h$ in four triangles.

Example 3.2. (Girault-Raviart [7])

$$\begin{aligned} \mathbf{H}_h &= \{v_h \in (C^0(\bar{\Omega}))^d \cap \mathbf{H}_0^1 : v_h|_K \in (P_2(K))^2 \quad \forall K \in \mathcal{T}_h\} \\ L_h &= \{q_h \in L^2(\Omega) : q_h|_K \in P_0(K) \quad \forall K \in \mathcal{T}_h\} \end{aligned}$$

For other conforming finite element spaces, we refer to Girault-Raviart [7] and Brezzi-Fortin [4].

4. Error Estimates for the Velocity. Since $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$, the weak solution $\mathbf{u}$ satisfies

$$\begin{aligned} (\mathbf{u}_t, \boldsymbol{\phi}_h) + \mu a(\mathbf{u}, \boldsymbol{\phi}_h) &+ \int_0^t \beta(t-s) a(\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds = -b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) \\ &+ (p, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi} \in \mathbf{J}_h. \end{aligned} \tag{4.1}$$

In this section, we discuss optimal error estimates for the error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$. Below, we first dissociate the nonlinearity by introducing an intermediate solution $\mathbf{v}_h$. Let $\mathbf{v}_h$ be a finite element Galerkin approximation to a linearized Oldroyd equation satisfisfying

$$\begin{aligned} (\mathbf{v}_{ht}, \boldsymbol{\phi}_h) &+ \mu a(\mathbf{v}_h, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\mathbf{v}_h(s), \boldsymbol{\phi}_h)\, ds \\ &= -b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi} \in \mathbf{J}_h. \end{aligned} \tag{4.2}$$

Now, we split the error $\mathbf{e}$ as

$$\mathbf{e} := \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathbf{v}_h) + (\mathbf{v}_h - \mathbf{u}_h) = \boldsymbol{\xi} + \boldsymbol{\eta}.$$

Note that $\boldsymbol{\xi}$ is the error committed by approximating a linearized Oldroyd equation and $\boldsymbol{\eta}$ represents the error due to the presence of non-linearity in the equation.

Below, we derive some estimates for $\boldsymbol{\xi}$. Subtracting (4.2) from (4.1), we write the equation in $\boldsymbol{\xi}$ as

$$(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \mu a(\boldsymbol{\xi}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\boldsymbol{\xi}(s), \boldsymbol{\phi}_h)\, ds = (p, \nabla \cdot \boldsymbol{\phi}_h), \quad \boldsymbol{\phi}_h \in J_h. \tag{4.3}$$

Lemma 4.1 *Let $\mathbf{v}_h(t) \in J_h$ be a solution of (4.2) with initial condition $\mathbf{v}_h(0) = P_h\mathbf{u}_0$. Then, $\boldsymbol{\xi}$ satisfies*

$$\int_0^t e^{2\alpha\tau}\|\boldsymbol{\xi}(\tau)\|^2\,d\tau \le Ch^4\int_0^t e^{2\alpha s}(\|\tilde{\Delta}\mathbf{u}(s)\|^2 + \|\nabla p\|^2)\,ds, \quad t > 0.$$

Proof. Using $\hat{\boldsymbol{\xi}} = e^{\alpha t}\boldsymbol{\xi}$, rewrite (4.3) as

$$\begin{aligned}(\hat{\boldsymbol{\xi}}_t, \boldsymbol{\phi}_h) - \alpha(\hat{\boldsymbol{\xi}}, \boldsymbol{\phi}_h) + \mu a(\hat{\boldsymbol{\xi}}, \boldsymbol{\phi}_h) \quad &+ \quad \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}a(\hat{\boldsymbol{\xi}}(\tau), \boldsymbol{\phi}_h)\,d\tau \\ &= \quad (\hat{p}, \nabla\cdot\boldsymbol{\phi}_h), \qquad \boldsymbol{\phi}_h \in J_h.\end{aligned} \tag{4.4}$$

Choosing $\boldsymbol{\phi}_h = P_h\hat{\boldsymbol{\xi}}$ in (4.4), we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\hat{\boldsymbol{\xi}}\|^2 \quad &+ \quad (\mu - \alpha\lambda_1^{-1})\|\nabla\hat{\boldsymbol{\xi}}\|^2 + \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}a(\hat{\boldsymbol{\xi}}(\tau), \hat{\boldsymbol{\xi}})\,d\tau \\ &= \quad (\hat{\boldsymbol{\xi}}_t, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) + \mu a(\hat{\boldsymbol{\xi}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) - \alpha(\hat{\boldsymbol{\xi}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) \\ &+ \quad \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}a(\hat{\boldsymbol{\xi}}(\tau), \hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\,d\tau + (\hat{p}, \nabla\cdot P_h\hat{\boldsymbol{\xi}}).\end{aligned} \tag{4.5}$$

We use approximation property (**B1**) with discrete incompressibility condition and H_0^1-stability of the L^2-projection P_h to obtain

$$|(\hat{p}, \nabla\cdot P_h\hat{\boldsymbol{\xi}})| = |(\hat{p} - j_h\hat{p}, \nabla\cdot P_h\hat{\boldsymbol{\xi}})| \le Ch\|\nabla\hat{p}\|\|\nabla\hat{\boldsymbol{\xi}}\|,$$

and

$$|a(\hat{\boldsymbol{\xi}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}})| \le Ch\|\nabla\hat{\boldsymbol{\xi}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|.$$

With $(\hat{\boldsymbol{\xi}}_t, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) = \frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{u}} - P_h\hat{\mathbf{u}}\|^2$, we first integrate (4.5) with respect to time. Then an application of Lemma 2.1 yields

$$\begin{aligned}\|\hat{\boldsymbol{\xi}}(t)\|^2 \quad &+ \quad 2(\mu - \alpha\lambda_1^{-1})\int_0^t \|\nabla\hat{\boldsymbol{\xi}}(s)\|^2\,ds \le \|\hat{\mathbf{u}}(t) - P_h\hat{\mathbf{u}}(t)\|^2 - \|\mathbf{u}(0) - P_h\mathbf{u}(0)\|^2 \\ &+ \quad C(\epsilon)h^2\int_0^t (\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2 + \|\nabla\hat{p}(s)\|^2)\,ds + \epsilon\int_0^t \|\nabla\hat{\boldsymbol{\xi}}(s)\|^2\,ds \\ &+ \quad Ch\int_0^t \left(\int_0^s \beta(s-\tau)e^{\alpha(s-\tau)}\|\nabla\hat{\boldsymbol{\xi}}(\tau)\|\,d\tau\right)\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|\,ds.\end{aligned} \tag{4.6}$$

Using Hölder's inequality, we estimate the last term I_1 on the right hand side of (4.6) as

$$I_1 \le Ch\left(\int_0^t \left(\int_0^s \beta(s-\tau)e^{\alpha(s-\tau)}\|\nabla\hat{\boldsymbol{\xi}}(\tau)\|\,d\tau\right)^2 ds\right)^{1/2}\left(\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2\,ds\right)^{1/2}.$$

We now recall the estimates of the I-term in Theorem 2.1 and use the definition of β to bound it by

$$\begin{aligned}I_1 \quad &\le \quad C(\gamma,\alpha,\delta)h\left(\int_0^t \|\nabla\hat{\boldsymbol{\xi}}(\tau)\|^2\,d\tau\right)^{1/2}\left(\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2\,ds\right)^{1/2} \\ &\le \quad \epsilon\int_0^t \|\nabla\hat{\boldsymbol{\xi}}(\tau)\|^2\,d\tau + C(\epsilon,\gamma,\alpha,\delta)h^2\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2\,ds.\end{aligned}$$

Setting $\epsilon = (\mu - \alpha\lambda_1^{-1})/2$ in (4.6) and using Theorem 2.1, it follows that

$$\int_0^t e^{2\alpha s}\|\nabla \boldsymbol{\xi}(s)\|^2\,ds \le C\left(\frac{\lambda_1}{\mu\lambda_1-\alpha}\right)^2 h^2\int_0^t (\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2 + \|\nabla\hat{p}\|^2)\,ds. \tag{4.7}$$

To estimate L^2-error, we use the following duality argument: For fixed $h > 0$ and $t > 0$, let $w(\tau) \in \mathbf{J}_1$, $q(\tau) \in L^2/R$ be the unique solution of the backward problem

$$\begin{cases} \mathbf{w}_\tau + \mu\Delta\mathbf{w} + \int_\tau^t \beta(s-\tau)\Delta\mathbf{w}(s)\,ds - \nabla q = e^{2\alpha\tau}\boldsymbol{\xi}, \quad 0\le\tau\le t, \\ \mathbf{w}(t) = 0. \end{cases} \tag{4.8}$$

With a change of variable $t \to t-\tau$, set $\mathbf{w}^*(\tau) = \mathbf{w}(t-\tau)$. Then $\mathbf{w}^*(\tau)$ satisfies the forward linear unsteady Stokes Volterra problem (2.24)–(2.26) in section 2. Thus, we obtain the following a priori estimate

$$\int_0^t e^{-2\alpha\tau}\{\|\Delta\mathbf{w}\|^2 + \|\mathbf{w}_\tau\|^2 + \|\nabla q\|^2\}\,d\tau \le C\left(\frac{\lambda_1}{\mu\lambda_1-\alpha}\right)^2\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|^2\,ds. \tag{4.9}$$

For some similar arguments, see Pani and Sinha [21] and Larsson *et al.* [16].

Form L^2-innerproduct between (4.8) and $\boldsymbol{\xi}$ to find that

$$e^{2\alpha\tau}\|\boldsymbol{\xi}\|^2 = (\boldsymbol{\xi}, \mathbf{w}_\tau) - \mu a(\boldsymbol{\xi}, \mathbf{w}) - \int_\tau^t \beta(s-\tau)a(\boldsymbol{\xi}(s), \mathbf{w})\,ds + (q, \nabla\cdot\boldsymbol{\xi}).$$

With t replaced by τ and $\boldsymbol{\phi}_h = P_h\mathbf{w}$ in (4.3), we obtain using weak form of (4.8)

$$\begin{aligned} e^{2\alpha\tau}\|\boldsymbol{\xi}\|^2 &= \frac{d}{d\tau}(\boldsymbol{\xi}, \mathbf{w}) - (\boldsymbol{\xi}_\tau, \mathbf{w} - P_h\mathbf{w}) - \mu a(\boldsymbol{\xi}, \mathbf{w} - P_h\mathbf{w}) - (p, \nabla\cdot P_h\mathbf{w}) \\ &+ (q, \nabla\cdot\boldsymbol{\xi}) + \left[\int_0^\tau \beta(\tau-s)a(\boldsymbol{\xi}(s), P_h\mathbf{w})\,ds - \int_\tau^t \beta(s-\tau)a(\boldsymbol{\xi}(s), \mathbf{w})\right]. \end{aligned} \tag{4.10}$$

We rewrite the last term on the right hand side of (4.10),i.e., the integral terms as

$$\begin{aligned} -\int_\tau^t \beta(s-\tau)a(\boldsymbol{\xi}(s), \mathbf{w} - P_h\mathbf{w})\,ds \;&+\; \int_0^\tau \beta(\tau-s)a(\boldsymbol{\xi}(s), P_h\mathbf{w})\,ds \\ &-\; \int_\tau^t \beta(s-\tau)a(\boldsymbol{\xi}(s), P_h\mathbf{w})\,ds. \end{aligned} \tag{4.11}$$

From the definition of P_h, the second term on the right hand side of (4.10) becomes

$$(\boldsymbol{\xi}_\tau, \mathbf{w} - P_h\mathbf{w}) = \frac{d}{d\tau}(\boldsymbol{\xi}, \mathbf{w} - P_h\mathbf{w}) - (\mathbf{u} - P_h\mathbf{u}, \mathbf{w}_\tau). \tag{4.12}$$

Using weak discrete incompressibility condition for the fourth and fifth terms on the right hand side of (4.10), we arrive at

$$-(p, \nabla\cdot P_h\mathbf{w}) = -(p - j_h p, \nabla\cdot P_h\mathbf{w}) = (p - j_h p, \nabla\cdot(\mathbf{w} - P_h\mathbf{w})), \tag{4.13}$$

and

$$(q, \nabla\cdot\boldsymbol{\xi}) = (q - j_h q, \nabla\cdot\boldsymbol{\xi}). \tag{4.14}$$

Substitute (4.11)–(4.14) in (4.10) and integrate with respect to τ from 0 to t. Note that the last two terms in (4.11) cancel each other using change of variables. Thus, we now arrive at

$$\begin{aligned}
\int_0^t e^{2\alpha\tau}\|\boldsymbol{\xi}(\tau)\|^2\,d\tau \ &\leq\ (\boldsymbol{\xi}(t),P_h\mathbf{w}(t))-(\boldsymbol{\xi}(0),P_h\mathbf{w}(0))+\int_0^t \big(\|\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}\|e^{-\alpha\tau}\|\mathbf{w}_\tau\| \\
&+\ \|\nabla\hat{\boldsymbol{\xi}}\|\,\|e^{-\alpha\tau}(\mathbf{w}-P_h\mathbf{w})\|\big)\,d\tau+Ch\int_0^t \big(\|\nabla\hat{p}\|\,\|e^{-\alpha\tau}\nabla(\mathbf{w}-P_h\mathbf{w})\| \\
&+\ e^{-\alpha\tau}\|\nabla q\|\,\|\nabla\hat{\boldsymbol{\xi}}\|\big)\,d\tau \\
&+\ \int_0^t\int_\tau^t \beta(s-\tau)e^{\alpha(\tau-s)}\|\nabla\hat{\boldsymbol{\xi}}\|\,\|e^{-\alpha\tau}(\mathbf{w}-P_h\mathbf{w})\|\,ds\,d\tau.
\end{aligned} \tag{4.15}$$

The first two terms on the right hand side of (4.15) become zero because of $\mathbf{w}(t)=0$ and $\mathbf{w}_h(0)=P_h\mathbf{u}_0$. For the last term on the right hand side of (4.15), we again appeal to the estimates of I in Theorem 2.1. Now a use of young's inequality yields

$$\begin{aligned}
\int_0^t e^{2\alpha\tau}\|\boldsymbol{\xi}(\tau)\|^2\,d\tau \ &\leq\ \epsilon\int_0^t e^{-2\alpha\tau}\left(\|\Delta\mathbf{w}\|^2+\|\mathbf{w}_\tau\|^2+\|\nabla q\|^2\right)d\tau \\
&+\ C(\epsilon)\left(h^2\int_0^t e^{2\alpha\tau}\|\nabla\boldsymbol{\xi}(\tau)\|^2\,d\tau+h^4\int_0^t e^{2\alpha\tau}(\|\nabla p\|^2+\|\tilde{\Delta}\mathbf{u}\|^2)\,d\tau\right).
\end{aligned}$$

Using the regularity result (4.9), we choose ϵ so that $(1-C\epsilon\left(\frac{\lambda_1}{\mu\lambda_1-\alpha}\right)^2)=\frac{1}{2}$. We then apply Theorem 2.1 with estimate (4.7) to complete the rest of the proof. □

For optimal error estimates of $\boldsymbol{\xi}$ in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms, we again introduce the following auxiliary projection $V_h\mathbf{u}:[0,\infty)\to J_h$ satisfying

$$\mu a(\mathbf{u}-V_h\mathbf{u},\boldsymbol{\phi}_h)+\int_0^t\beta(t-s)a(\mathbf{u}(s)-V_h\mathbf{u}(s),\boldsymbol{\phi}_h)\,ds=(p,\nabla\cdot\boldsymbol{\phi}_h)\quad\forall\boldsymbol{\phi}_h\in\mathbf{J}_h \tag{4.16}$$

and call it as Stokes-Volterra projection. The form of the above projection is motivated by Ritz-Volterra projection, see Lin *et al.* [17] for parabolic integro-differential equations and by the elliptic projection introduced by Wheeler [26] for parabolic initial and boundary value problems.

With $V_h\mathbf{u}$ defined as above, we now decompose $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi}:=(\mathbf{u}-V_h\mathbf{u})+(V_h\mathbf{u}-\mathbf{v}_h)=\boldsymbol{\zeta}+\boldsymbol{\theta}.$$

First of all, we derive optimal error bounds for the error $\boldsymbol{\zeta}$.

Lemma 4.2 *Assume that the assumptions* (**A1**), (**B1**) and (**B2**) are satisfied. Then there is a positive constant C such that

$$\begin{aligned}
\|(\mathbf{u}-V_h\mathbf{u})(t)\|^2 \ &+\ h^2\|\nabla(\mathbf{u}-V_h\mathbf{u})(t)\|^2\leq Ch^4\Big[\|\tilde{\Delta}\mathbf{u}\|^2+\|\nabla p\|^2 \\
&+\ e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\tilde{\Delta}\mathbf{u}\|^2+\|\nabla p\|^2)\,ds\Big]\leq Kh^4e^{-2\alpha t}.
\end{aligned}$$

Moreover, the error in the time derivative satisfies

$$\begin{aligned}
\|(\mathbf{u}-V_h\mathbf{u})_t(t)\|^2 \ &+\ h^2\|\nabla(\mathbf{u}-V_h\mathbf{u})_t(t)\|^2\leq Ch^4\left[\sum_{j=0}^1\left(\|\frac{\partial^j}{\partial t^j}(\tilde{\Delta}\mathbf{u})\|^2+\|\frac{\partial^j}{\partial t^j}\nabla p\|^2\right)\right. \\
&+\ \left. e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\tilde{\Delta}\mathbf{u}\|^2+\|\nabla p\|^2)\,ds\right].
\end{aligned}$$

Proof. With $\boldsymbol{\zeta} = \mathbf{u} - V_h\mathbf{u}$, rewrite (4.16) as

$$\mu a(\hat{\boldsymbol{\zeta}}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s)e^{\alpha(t-s)} a(\hat{\boldsymbol{\zeta}}(s), \boldsymbol{\phi}_h)\, ds = (\hat{p}, \nabla\cdot\boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{4.17}$$

Choose $\boldsymbol{\phi}_h = P_h\hat{\boldsymbol{\zeta}}$ in (4.17) and use discrete incompressibility condition to obtain

$$\begin{aligned} \mu\|\nabla\hat{\boldsymbol{\zeta}}\|^2 &+ \int_0^t \beta(t-s)e^{\alpha(t-s)} a(\hat{\boldsymbol{\zeta}}(s), \hat{\boldsymbol{\zeta}})\, ds = \mu a(\hat{\boldsymbol{\zeta}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) \\ &+ \int_0^t \beta(t-s)e^{\alpha(t-s)} a(\hat{\boldsymbol{\zeta}}(s), \hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\, ds + (\hat{p} - j_h\hat{p}, \nabla\cdot P_h\hat{\boldsymbol{\zeta}}). \end{aligned} \tag{4.18}$$

Integrate (4.18) with respect to time and use the positivity property for $\beta(t-s)e^{\alpha(t-s)}$ as $0 < \alpha < \delta$. Applying $\mathbf{H}_0^1$-stability of $\mathbf{L^2}$ projection P_h, estimate (3.4) with approximation property (**B1**) and Young's inequality, we arrive at

$$\begin{aligned} \int_0^t \|\nabla\hat{\boldsymbol{\zeta}}\|^2\, ds &\leq C(\mu,\gamma,\delta,\epsilon)h^2 \left[\int_0^t \left(\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2 + \|\nabla\hat{p}(s)\|^2\right) ds\right] \\ &+ (\frac{\delta-\alpha}{\gamma})^2\epsilon \int_0^t \left(\int_0^s \beta(s-\tau)e^{\alpha(t-s)}\|\nabla\hat{\boldsymbol{\zeta}}(\tau)\|\, d\tau\right)^2 ds + \epsilon\int_0^t \|\nabla\hat{\boldsymbol{\zeta}}(s)\|^2\, ds. \end{aligned} \tag{4.19}$$

For the second term, i.e., the double integral term on the right handside of (4.19), we bound it following Theorem 2.1 by

$$\leq \epsilon\int_0^t \|\nabla\hat{\boldsymbol{\zeta}}(s)\|^2\, ds.$$

With $\epsilon = 1/4$, we altogether obtain from (4.19)

$$\int_0^t \|\nabla\hat{\boldsymbol{\zeta}}\|^2\, ds \leq Ch^2\left[\int_0^t \left(\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2 + \|\nabla\hat{p}(s)\|^2\right) ds\right]. \tag{4.20}$$

From (4.18), we now easily derive

$$\|\nabla\hat{\boldsymbol{\zeta}}\|^2 \leq C(\mu)h^2\left(\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 + \|\nabla\hat{p}\|^2\right) + C(\mu,\gamma,\delta,\alpha)\int_0^t \|\nabla\hat{\boldsymbol{\zeta}}(s)\|^2\, ds.$$

Using (4.20) and Theorem 2.2, we, therefore, obtain

$$\begin{aligned} \|\nabla\boldsymbol{\zeta}\|^2 &\leq C(\mu,\gamma,\delta,\alpha))h^2\Big[\|\tilde{\Delta}\mathbf{u}\|^2 + \|\nabla p\|^2 \\ &+ e^{-2\alpha t}\int_0^t (\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2 + \|\nabla\hat{p}(s)\|^2)\, ds\Big] \leq Kh^2e^{-2\alpha t}. \end{aligned} \tag{4.21}$$

For L^2 estimate, we recall the Aubin-Nitsche duality argument. Let $\{\mathbf{w}, q\}$ be a pair of unique solution of the following steady state Stokes system:

$$-\mu\Delta\mathbf{w} + \nabla q = \hat{\boldsymbol{\zeta}} \text{ in } \Omega, \tag{4.22}$$
$$\nabla\cdot\mathbf{w} = 0 \text{ in } \Omega, \tag{4.23}$$
$$\mathbf{w}|_{\partial\Omega} = 0. \tag{4.24}$$

From assumption (**A1**), the above pair satisfies the following regularity result

$$\|\mathbf{w}\|_2 + \|q\|_{H^1/R} \leq C\|\hat{\boldsymbol{\zeta}}\|.$$

Form L^2- inner product between (4.22) and $\hat{\boldsymbol{\zeta}}$. Now use of (4.17) with ϕ_h replaced by $P_h\mathbf{w}$ yields

$$\begin{aligned} \|\hat{\boldsymbol{\zeta}}(t)\|^2 &= \mu a(\hat{\boldsymbol{\zeta}}, \mathbf{w}-P_h\mathbf{w}) - (\nabla\cdot\hat{\boldsymbol{\zeta}}, q) + (\hat{p}-j_h\hat{p}, \nabla\cdot(P_h\mathbf{w}-\mathbf{w})) \\ &+ \int_0^t \beta(t-s)e^{\alpha(t-s)} a(\hat{\boldsymbol{\zeta}}(s), \mathbf{w}-P_h\mathbf{w})\,ds - \int_0^t \beta(t-s)e^{\alpha(t-s)} a(\hat{\boldsymbol{\zeta}}(s), \mathbf{w})\,ds. \end{aligned} \tag{4.25}$$

For deriving (4.25), we have used (4.23)–(4.24). Rewrite the second term on the right hand side of (4.25) as

$$\begin{aligned} -(\nabla\cdot\hat{\boldsymbol{\zeta}}, q) &= -(\nabla\cdot(\hat{\boldsymbol{\zeta}}-P_h\hat{\boldsymbol{\zeta}}), q) - (\nabla\cdot P_h\hat{\boldsymbol{\zeta}}, q-j_hq) \\ &= (\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}, \nabla q) - (\nabla\cdot P_h\hat{\boldsymbol{\zeta}}, q-j_hq), \end{aligned} \tag{4.26}$$

and using (4.22), the last term on the right hand side of (4.25) is now written as

$$\begin{aligned} -\int_0^t \beta(t-s)e^{\alpha(t-s)} a(\hat{\boldsymbol{\zeta}}(s), \mathbf{w})\,ds &= -\frac{1}{\mu}\int_0^t \beta(t-s)e^{\alpha(t-s)}(\hat{\boldsymbol{\zeta}}(s), -\mu\Delta\mathbf{w}(t))\,ds \\ &= \frac{1}{\mu}\int_0^t \beta(t-s)e^{\alpha(t-s)}(\hat{\boldsymbol{\zeta}}(s), \nabla q-\hat{\boldsymbol{\zeta}})\,ds \\ &= -\frac{1}{\mu}\int_0^t \beta(t-s)e^{\alpha(t-s)}(\hat{\boldsymbol{\zeta}}(s), \hat{\boldsymbol{\zeta}}(t))\,ds \\ &- \frac{1}{\mu}\int_0^t \beta(t-s)e^{\alpha(t-s)}(\nabla\cdot\hat{\boldsymbol{\zeta}}(s), q)\,ds. \end{aligned} \tag{4.27}$$

Substituting (4.26)–(4.27) in (4.25), we find that

$$\begin{aligned} \|\hat{\boldsymbol{\zeta}}\|^2 &+ \frac{1}{\mu}\int_0^t \beta(t-s)e^{\alpha(t-s)}(\hat{\boldsymbol{\zeta}}(s), \hat{\boldsymbol{\zeta}}(t))\,ds = \mu a(\hat{\boldsymbol{\zeta}}, \mathbf{w}-P_h\mathbf{w}) \\ &+ \int_0^t \beta(t-s)e^{\alpha(t-s)} a(\hat{\boldsymbol{\zeta}}(s), \mathbf{w}-P_h\mathbf{w})\,ds + (\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}, \nabla q) \\ &- (\nabla\cdot P_h\hat{\boldsymbol{\zeta}}, q-j_hq) + (\hat{p}-j_h\hat{p}, \nabla\cdot(P_h\mathbf{w}-\mathbf{w})) \\ &+ \frac{1}{\mu}\int_0^t \beta(t-s)e^{\alpha(t-s)}\Big[(\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}, \nabla q) - (\nabla\cdot P_h\hat{\boldsymbol{\zeta}}(s), q-j_hq)\Big]\,ds. \end{aligned} \tag{4.28}$$

Integrating (4.28) from 0 to t, apply Hölder's inequality and use Lemmas 2.1-2.2. Finally, apply the approximation property with estimates for the $\mathbf{L^2}$-projection and employ regularity result to obtain

$$\int_0^t \|\hat{\boldsymbol{\zeta}}(s)\|^2\,ds \le Ch^4\int_0^t \left(\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2 + \|\nabla\hat{p}(s)\|^2\right)\,ds. \tag{4.29}$$

From (4.28), it is easy to note that

$$\begin{aligned} \|\hat{\boldsymbol{\zeta}}(t)\|^2 &\le C(\mu,\gamma,\delta,\alpha)\left[h^4\left(\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 + \|\nabla\hat{p}\|^2 + \int_0^t \left(\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2 + \|\nabla\hat{p}(s)\|^2\right)\,ds\right)\right. \\ &+ \left. h^2\left(\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + \int_0^t \|\nabla\hat{\boldsymbol{\zeta}}(s)\|^2\,ds\right)\right] + C\int_0^t \|\hat{\boldsymbol{\zeta}}(s)\|^2\,ds. \end{aligned}$$

From (4.20)–(4.21) and (4.29), we obtain the desired estimate for $\|\boldsymbol{\zeta}\|$.

For the time derivative, differentiate (4.16) with respect to t to find that

$$\mu a(\boldsymbol{\zeta}_t,\boldsymbol{\phi}_h)+\beta(0)a(\boldsymbol{\zeta}(t),\boldsymbol{\phi}_h) + \int_0^t \beta_t(t-s)a(\boldsymbol{\zeta}(s),\boldsymbol{\phi}_h)\,ds = (p_t,\nabla\cdot\boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h\in J_h. \tag{4.30}$$

With $\beta_t(t-s)=-\delta\beta(t-s)$, choose $\boldsymbol{\phi}_h=e^{2\alpha t}P_h\boldsymbol{\zeta}_t$ and proceed as in the case of the estimate of $\|\nabla\hat{\zeta}(t)\|$ to obtain

$$\begin{aligned}\mu\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|^2 &= \mu a(e^{\alpha t}\boldsymbol{\zeta}_t,e^{\alpha t}(\mathbf{u}_t-P_h\mathbf{u}_t))-\frac{1}{\gamma}a(\hat{\boldsymbol{\zeta}},P_h(e^{\alpha t}\boldsymbol{\zeta}_t))\\ &+ \delta\int_0^t\beta(t-s)e^{\alpha(t-s)}a(\hat{\boldsymbol{\zeta}}(s),P_h(e^{\alpha t}\boldsymbol{\zeta}_t))\,ds+(e^{\alpha t}p_t-j_h(e^{\alpha t}p_t),\nabla\cdot P_h(e^{\alpha t}\boldsymbol{\zeta}_t)).\end{aligned} \tag{4.31}$$

Again using $\mathbf{H}_0^1$-stability of P_h and approximation properties in (4.31), we arrive at

$$\begin{aligned}\|\nabla(e^{\alpha t}\boldsymbol{\zeta}_t(t))\|^2 &\le Ch^2e^{2\alpha t}\sum_{j=0}^{1}\left(\|\frac{\partial^j}{\partial t^j}(\tilde{\Delta}\mathbf{u})\|^2+\|\frac{\partial^j}{\partial t^j}\nabla p\|^2\right)\\ &+ Ch^2\int_0^t e^{2\alpha s}(\|\tilde{\Delta}\mathbf{u}\|^2+\|\nabla p\|^2)\,ds.\end{aligned} \tag{4.32}$$

This completes the proof of the estimate $\|\nabla\boldsymbol{\zeta}_t\|$.

Finally, for the estimation of $\boldsymbol{\zeta}_t$ in $\mathbf{L^2}$-norm, we again appeal to the Aubin-Nitsche duality argument. In (4.22), consider $e^{\alpha t}\boldsymbol{\zeta}_t$ in stead of $\hat{\boldsymbol{\zeta}}$ and now form $\mathbf{L^2}$-innerproduct with $e^{\alpha t}\boldsymbol{\zeta}_t$ to obtain

$$\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2=\mu a(e^{\alpha t}\boldsymbol{\zeta}_t,\mathbf{w}-P_h\mathbf{w})-(e^{\alpha t}\nabla\cdot\boldsymbol{\zeta}_t,q)+\mu a(e^{\alpha t}\boldsymbol{\zeta}_t,P_h\mathbf{w}). \tag{4.33}$$

From (4.30) with $\boldsymbol{\phi}_h=e^{\alpha t}P_h\mathbf{w}$, it now follows in a similar manner as in the $\mathbf{L^2}$-estimate of $\boldsymbol{\zeta}$ that

$$\begin{aligned}\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 &= \mu a(e^{\alpha t}\boldsymbol{\zeta}_t,\mathbf{w}-P_h\mathbf{w})+(e^{\alpha t}(\mathbf{u}_t-P_h\mathbf{u}_t),\nabla q)-(\nabla\cdot P_h(e^{\alpha t}\boldsymbol{\zeta}_t),q-j_hq)\\ &+ (e^{\alpha t}(p_t-j_hp_t),\nabla\cdot(P_h\mathbf{w}-\mathbf{w}))-\beta(0)a(\hat{\boldsymbol{\zeta}},P_h\mathbf{w}-\mathbf{w})+\beta(0)(\hat{\boldsymbol{\zeta}},-\Delta\mathbf{w})\\ &+ \delta\int_0^t\beta(t-s)e^{\alpha(t-s)}a(\hat{\boldsymbol{\zeta}}(s),P_h\mathbf{w}-\mathbf{w})\,ds-\int_0^t\beta(t-s)e^{\alpha(t-s)}(\hat{\boldsymbol{\zeta}}(s),-\Delta\mathbf{w})\,ds.\end{aligned}$$

Using Cauchy-Schwarz inequality, properties of P_h and regularity condition, we arrive at

$$\begin{aligned}\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 &\le C(\mu,\gamma,\delta)h^2\left(\|\nabla(e^{\alpha t}\boldsymbol{\zeta}_t)\|^2+\|e^{\alpha t}(p_t-j_hp_t)\|^2+\|\nabla\hat{\boldsymbol{\zeta}}\|^2+\int_0^t\|\nabla\hat{\boldsymbol{\zeta}}(s)\|^2\,ds\right)\\ &+ C\left(\|e^{\alpha t}(\mathbf{u}_t-j_h\mathbf{u}_t)\|^2+\|\hat{\boldsymbol{\zeta}}\|^2+\int_0^t\|\hat{\boldsymbol{\zeta}}(s)\|^2\,ds\right).\end{aligned} \tag{4.34}$$

On substituting (4.21), (4.29), (4.32) and the estimates of $\hat{\boldsymbol{\zeta}}$ with approximation properties in (4.34), we obtain the required result for $\boldsymbol{\zeta}_t$ in $\mathbf{L^2}$-norm. This completes the rest of the proof. □

Now we are in a position to estimate $\boldsymbol{\xi}$ in $L^\infty(\mathbf{L^2})$ and $L^\infty(\mathbf{H}_0^1)$-norms.

Since $\boldsymbol{\xi}=\boldsymbol{\zeta}+\boldsymbol{\theta}$ and the estimates of $\boldsymbol{\zeta}$ are known from the Lemma 4.2, it is sufficient to estimate $\boldsymbol{\theta}$. From (4.3) and (4.16), the equation in $\boldsymbol{\theta}$ becomes

$$(\boldsymbol{\theta}_t,\boldsymbol{\phi}_h)+\mu a(\boldsymbol{\theta},\boldsymbol{\phi}_h) + \int_0^t\beta(t-s)a(\boldsymbol{\theta}(s),\boldsymbol{\phi}_h)\,ds = -(\boldsymbol{\zeta}_t,\boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h\in\mathbf{J}_h. \tag{4.35}$$

Note that for the estimation of $\|\boldsymbol{\zeta}_t\|$ in (4.35), it is essential to introduce $\sigma(t)$ term so that we can avoid nonlocal compatibility conditions. However, a direct use of $\sigma(t)$ as in Heywood and Rannacher [9] leads to an application of Gronwall's Lemma and this is mainly due to the presence of the integral term in (4.35). As in Pani and Sinha [21], we first introduce

$$\tilde{\boldsymbol{\theta}}(t) = \int_0^t \boldsymbol{\theta}(s)\,ds,$$

and derive an improved estimate for

$$\int_0^t e^{\alpha t}\|\nabla\tilde{\boldsymbol{\theta}}(s)\|^2\,ds.$$

This, in turn, helps us to introduce $\sigma(t)$ and hence, we can estimate $\boldsymbol{\theta}$ without using Gronwall's Lemma.

Lemma 4.3 *There is a positive constant K such that $\boldsymbol{\xi}$ satisfies for $t > 0$ the following estimate*

$$\begin{aligned}\|\boldsymbol{\xi}(t)\|^2 + h^2\|\nabla\boldsymbol{\xi}(t)\|^2 \;&\le\; Ch^4\left[\frac{e^{-2\alpha t}}{\tau^*}\int_0^t e^{2\alpha s}\left(\|\tilde{\Delta}\mathbf{u}\|^2 + \|\nabla p\|^2\right)ds\right.\\ &+\; \left.\sigma^{-1}\int_0^t \sigma(s)\left(\|\tilde{\Delta}\mathbf{u}_t\|^2 + \|\nabla p_t\|^2\right)ds\right] \le Kh^4e^{-2\alpha t}.\end{aligned}$$

Proof. With $\tilde{\boldsymbol{\theta}} = \int_0^t \boldsymbol{\theta}(s)\,ds$, we integrate (4.35) with respect to time from 0 to t to obtain

$$\begin{aligned}(\boldsymbol{\theta},\boldsymbol{\phi}_h) + \mu a(\tilde{\boldsymbol{\theta}},\boldsymbol{\phi}_h) \;&+\; \int_0^t\int_0^s \beta(s-\tau)a(\boldsymbol{\theta}(\tau),\boldsymbol{\phi}_h)\,d\tau\,ds\\ &=\; -(\boldsymbol{\zeta},\boldsymbol{\phi}_h) + (\mathbf{u}_0 - P_h\mathbf{u}_0,\boldsymbol{\phi}_h), \quad \boldsymbol{\phi}_h \in \mathbf{J}_h.\end{aligned} \tag{4.36}$$

Since P_h is the $\mathbf{L^2}$- projection, the last term on the right hand side vanishes. For the third term on the left hand side of (4.36), use $\boldsymbol{\theta}(\tau)$ as $\tilde{\boldsymbol{\theta}}_\tau$ and intergrate by parts with respect to τ. Since $\beta_\tau(s-\tau) = -\beta_s(s-\tau)$, we now arrive at

$$\begin{aligned}\int_0^t\int_0^s \beta(s-\tau)a(\boldsymbol{\theta}(\tau),\boldsymbol{\phi}_h)\,d\tau\,ds \;&=\; \int_0^t\int_0^s \beta(s-\tau)a(\tilde{\boldsymbol{\theta}}_\tau(\tau),\boldsymbol{\phi}_h)\,d\tau\,ds\\ &=\; \int_0^t \beta(0)a(\tilde{\boldsymbol{\theta}}(s),\boldsymbol{\phi}_h)\,ds - \int_0^t\int_0^s \beta_\tau(s-\tau)a(\tilde{\boldsymbol{\theta}}(\tau),\boldsymbol{\phi}_h)\,d\tau\,ds\\ &=\; \int_0^t \frac{d}{ds}\left(\int_0^s \beta(s-\tau)a(\tilde{\boldsymbol{\theta}}(\tau),\boldsymbol{\phi}_h)\,d\tau\right)ds\\ &=\; \int_0^t \beta(t-\tau)a(\tilde{\boldsymbol{\theta}}(\tau),\boldsymbol{\phi}_h)\,d\tau.\end{aligned}$$

Thus, the equation (4.36) becomes

$$\begin{aligned}(\boldsymbol{\theta},\boldsymbol{\phi}_h) + \mu a(\tilde{\boldsymbol{\theta}},\boldsymbol{\phi}_h) \;&+\; \int_0^t \beta(t-\tau)a(\tilde{\boldsymbol{\theta}}(\tau),\boldsymbol{\phi}_h)\,d\tau\,ds\\ &=\; -(\boldsymbol{\zeta},\boldsymbol{\phi}_h), \quad \boldsymbol{\phi}_h \in \mathbf{J}_h.\end{aligned} \tag{4.37}$$

Choose $\boldsymbol{\phi}_h = e^{2\alpha t}\tilde{\boldsymbol{\theta}}$ in (4.37) and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(e^{2\alpha t}\|\tilde{\boldsymbol{\theta}}\|^2\right) \;&-\; \alpha e^{2\alpha t}\|\tilde{\boldsymbol{\theta}}\|^2 + \mu e^{2\alpha t}\|\nabla\tilde{\boldsymbol{\theta}}\|^2\\ &+\; \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}a(e^{\alpha\tau}\tilde{\boldsymbol{\theta}}(\tau),e^{\alpha t}\tilde{\boldsymbol{\theta}})\,d\tau = -(\hat{\boldsymbol{\zeta}},e^{\alpha t}\tilde{\boldsymbol{\theta}}).\end{aligned} \tag{4.38}$$

On integrating (4.38) with respect to t, we use $\|\tilde{\boldsymbol\theta}\|^2 \leq \frac{1}{\lambda_1}\|\nabla\tilde{\boldsymbol\theta}\|^2$ and positivity property to find that

$$
\begin{aligned}
e^{2\alpha t}\|\tilde{\boldsymbol\theta}(t)\|^2 &+ (\mu - \frac{\alpha}{\lambda_1})\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol\theta}(s)\|^2\,ds \leq \frac{1}{(\mu\lambda_1-\alpha)}\int_0^t \|\hat{\boldsymbol\zeta}(s)\|^2\,ds \\
&\leq Ch^4\int_0^t e^{2\alpha s}\left(\|\tilde{\Delta}\mathbf{u}(s)\|^2 + \|\nabla p(s)\|^2\right)\,ds. \qquad (4.39)
\end{aligned}
$$

Now, choosing $\boldsymbol\phi_h = \sigma(t)\boldsymbol\theta$ in (4.36), it now follows that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\sigma(t)\|\boldsymbol\theta\|^2\right) &+ \mu\sigma(t)\|\nabla\boldsymbol\theta\|^2 = -\sigma(t)(\boldsymbol\zeta_t,\boldsymbol\theta) \\
&+ \frac{1}{2}\sigma_t(t)\|\boldsymbol\theta\|^2 - \sigma(t)\int_0^t \beta(t-\tau)a(\boldsymbol\theta(\tau),\boldsymbol\theta)\,d\tau \qquad (4.40)\\
&\leq \frac{\sigma^2(t)}{2\sigma_t(t)}\|\boldsymbol\zeta_t\|^2 + \sigma_t(t)\|\boldsymbol\theta(t)\|^2 - \sigma(t)\int_0^t \beta(t-\tau)a(\boldsymbol\theta(\tau),\boldsymbol\theta)\,d\tau.
\end{aligned}
$$

To estimate the third term say I_1 on the right hand side of (4.40), use again the form $\boldsymbol\theta(\tau) = \tilde{\boldsymbol\theta}_\tau$ and then integrate by parts with respect to τ. With $\beta_\tau(t-\tau) = \delta\beta(t-\tau)$, we now obtain

$$
I_1 = -\gamma\sigma(t)a(\tilde{\boldsymbol\theta}(t),\boldsymbol\theta(t)) + \delta\sigma(t)\int_0^t \beta(t-\tau)a(\tilde{\boldsymbol\theta}(\tau),\boldsymbol\theta(t))\,d\tau.
$$

Here, we have also used $\beta_s(t-s) = \delta\beta(t-s)$. On integration of $|I_1|$ with respect to t, we find that

$$
\begin{aligned}
\int_0^t |I_1(s)|\,ds &\leq C(\mu,\delta,\gamma)\left[\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol\theta}(s)\|^2\,ds + \int_0^t \sigma(s)\left(\int_0^s \beta(s-\tau)\|\nabla\tilde{\boldsymbol\theta}(\tau)\|\,d\tau\right)^2 ds\right] \\
&+ \frac{\mu}{2}\int_0^t \sigma(s)\|\nabla\boldsymbol\theta(s)\|^2\,ds,
\end{aligned}
$$

and hence, using Lemma 2.2, we now obtain

$$
\int_0^t |I_1(s)|\,ds \leq C(\mu,\delta,\gamma)\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol\theta}(s)\|^2\,ds + \frac{\mu}{2}\int_0^t \sigma(s)\|\nabla\boldsymbol\theta(s)\|^2\,ds.
$$

On integrating (4.40) with respect to time and substituting the above estimate in the resulting equation, it follows that

$$
\begin{aligned}
\sigma(t)\|\boldsymbol\theta(t)\|^2 &+ \mu\int_0^t \sigma(s)\|\nabla\boldsymbol\theta(s)\|^2\,ds \leq C\int_0^t \frac{\sigma^2(s)}{2\sigma_s(s)}\|\boldsymbol\zeta_t\|^2\,ds \\
&+ C\left[\int_0^t \sigma_s(s)\|\boldsymbol\theta(s)\|^2\,ds + \int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol\theta}(s)\|^2\,ds\right] \\
&\leq C\left[\int_0^t \frac{\sigma^2(s)}{2\sigma_s(s)}\|\boldsymbol\zeta_t\|^2\,ds + \int_0^t \left(\|\hat{\boldsymbol\xi}\|^2 + \|\hat{\boldsymbol\zeta}\|^2\right)\,ds\right] \\
&+ C\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol\theta}(s)\|^2\,ds.
\end{aligned}
$$

Since $\frac{\sigma^2(s)}{2\sigma_s(s)} \leq \frac{\tau^*(s)}{4\alpha}\sigma(s)$, we use Lemmas 4.1–4.2, the estimate (4.29) and (4.39), to obtain

$$
\begin{aligned}
\|\boldsymbol\theta(t)\|^2 &\leq C\frac{h^4}{\tau^*}e^{-2\alpha t}\int_0^t e^{2\alpha s}\left(\|\tilde{\Delta}\mathbf{u}\|^2 + \|\nabla p\|^2\right)\,ds \qquad (4.41)\\
&+ Ch^4\sigma^{-1}(t)\int_0^t \tau^*(s)\sigma(s)\left(\|\tilde{\Delta}\mathbf{u}_t\|^2 + \|\nabla p_t\|^2\right)\,ds.
\end{aligned}
$$

For $0 < t \leq 1$, we use Theorem 2.1 to bound the first term on the right hand side of (4.41) and for the second term, we note that using Theorem 2.2

$$\begin{aligned} \sigma^{-1}(t)\int_0^t \tau^*(s)\sigma(s)\left(\|\tilde{\Delta}\mathbf{u}_t\|^2+\|\nabla p_t\|^2\right)ds &\leq t^{-1}e^{-2\alpha t}\int_0^t s\sigma(s)\left(\|\tilde{\Delta}\mathbf{u}_t\|^2+\|\nabla p_t\|^2\right)ds \\ &\leq e^{-2\alpha t}\int_0^t \sigma(s)\left(\|\tilde{\Delta}\mathbf{u}_t\|^2+\|\nabla p_t\|^2\right)ds \leq Ke^{-2\alpha t}. \end{aligned}$$

When $t > 1$, $\tau^*(t) = 1$ and we use Lemma 2.4 and Theorem 2.2. Thus, we obtain

$$\|\boldsymbol{\theta}(t)\|^2 \leq Kh^4 e^{-2\alpha t}.$$

Use of triange inequality with Lemma 4.2 completes the rest of the proof. □

Below, we discuss the main theorem of this section.

Theorem 4.1 *Let Ω be a convex polygon, and let the assumptions* **(A1)**–**(A2)** *and* **(B1)**–**(B2)** *be satisfied. Further, let the discrete initial velocity $\mathbf{u}_{0h} \in \mathbf{J}_h$ satisfy*

$$\|\mathbf{u}_0 - \mathbf{u}_{0h}\| \leq Ch^2\|\mathbf{u}_0\|_2.$$

Then, there exists a positive constant K which depends on $\delta, \gamma, \lambda_1, M_1$, and M_2 such that for all time $t > 0$, the following estimate holds:

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\| + h\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\| \leq Kh^2 e^{-\alpha t}.$$

Proof. Since $e = \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathbf{v_h}) + (\mathbf{v_h} - \mathbf{u}_h) = \boldsymbol{\xi} + \boldsymbol{\eta}$ and the estimate of $\boldsymbol{\xi}$ is known from Lemma 4.3, it is enough to estimate $\|\boldsymbol{\eta}\|$. From (3.2) and (4.2), the equation in $\boldsymbol{\eta}$ becomes

$$(\boldsymbol{\eta}_t, \boldsymbol{\phi}_h) + \mu a(\boldsymbol{\eta}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s)a(\boldsymbol{\eta}(s), \boldsymbol{\phi}_h)\,ds = b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h), \;\; \boldsymbol{\phi}_h \in \mathbf{J}_h.$$

Choose $\boldsymbol{\phi}_h = e^{2\alpha t}\boldsymbol{\eta}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{\boldsymbol{\eta}}\|^2 - \alpha\|\hat{\boldsymbol{\eta}}\|^2 + \mu\|\nabla\hat{\boldsymbol{\eta}}\|^2 + \int_0^t \beta(t-s)e^{\alpha(t-s)}a(\hat{\boldsymbol{\eta}}(s), \hat{\boldsymbol{\eta}}(s))\,ds = e^{\alpha t}\Lambda_h(\hat{\boldsymbol{\eta}}), \tag{4.42}$$

where

$$\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h).$$

Note that

$$e^{\alpha t}\Lambda_h(\hat{\boldsymbol{\eta}}) = -e^{-\alpha t}[b(\hat{\mathbf{e}}, \hat{\mathbf{v}}_h, \hat{\boldsymbol{\eta}}) + b(\hat{\mathbf{u}}, \hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}})].$$

Using Hölder's inequality, discrete Sobolev inequality

$$\|\boldsymbol{\phi}_h\|_{\mathbf{L}^4} \leq C\|\nabla\boldsymbol{\phi}_h\|, \;\; \boldsymbol{\phi} \in \mathbf{H}_h$$

and estimate (3.7), we obtain

$$e^{\alpha t}|\Lambda_h(\hat{\boldsymbol{\eta}})| \leq C\|\hat{\mathbf{e}}\|\|\nabla\hat{\boldsymbol{\eta}}\|(\|\mathbf{v_h}\|_{\mathbf{L}^\infty} + \|\nabla\mathbf{v_h}\|_{\mathbf{L}^4}) + C\|\nabla\mathbf{u}\|^{1/2}\|\tilde{\Delta}\mathbf{u}\|^{1/2}\|\nabla\hat{\boldsymbol{\eta}}\|\|\hat{\boldsymbol{\xi}}\|.$$

As in Heywood and Rannacher [9], using inverse hypothesis and approximation properties, it is easy to check that

$$\|\mathbf{v_h}\|_{\mathbf{L}^\infty} + \|\nabla\mathbf{v_h}\|_{\mathbf{L}^4} \leq C\|\nabla\mathbf{u}\|^{1/2}\|\tilde{\Delta}\mathbf{u}\|^{1/2} + Kh^{1/2}.$$

Since $\mathbf{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$, we arrive at

$$
\begin{aligned}
e^{\alpha t}|\Lambda_h(\hat{\boldsymbol{\eta}})| &\le (C\|\nabla \mathbf{u}\|^{1/2}\|\tilde{\Delta}\mathbf{u}\|^{1/2} + Kh^{1/2})(\|\hat{\boldsymbol{\xi}}\| + \|\hat{\boldsymbol{\eta}}\|)\|\nabla\hat{\boldsymbol{\eta}}\| \\
&\le \frac{1}{\mu}(C\|\nabla \mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + Kh)(\|\hat{\boldsymbol{\xi}}\|^2 + \|\hat{\boldsymbol{\eta}}\|^2) + \frac{\mu}{4}\|\nabla\hat{\boldsymbol{\eta}}\|^2 \\
&\le \frac{1}{\mu}(C\|\nabla \mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\|\|\hat{\boldsymbol{\eta}}\|^2 + K\|\hat{\boldsymbol{\xi}}\|^2 + (\frac{\mu}{4} + \frac{K_1 h}{\lambda_1})\|\nabla\hat{\boldsymbol{\eta}}\|^2.
\end{aligned}
\tag{4.43}
$$

Here, K_1 depends on K and the constant in the Poincaré inequality. Substitute (4.43) in (4.42) to find that

$$
\begin{aligned}
\frac{d}{dt}\|\hat{\boldsymbol{\eta}}\|^2 + \left(2\mu - \frac{\alpha}{\lambda_1} - \frac{\mu}{2} - \frac{2K_1 h}{\lambda_1}\right)\|\nabla\hat{\boldsymbol{\eta}}\|^2 &+ 2\int_0^t \beta(t-s)e^{\alpha(t-s)}a(\hat{\boldsymbol{\eta}}(s), \hat{\boldsymbol{\eta}}(t))\,ds \\
&\le \frac{C}{\mu}\|\nabla \mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\|\;\|\hat{\boldsymbol{\eta}}\|^2 + K\|\hat{\boldsymbol{\xi}}\|^2.
\end{aligned}
\tag{4.44}
$$

Choose $h_0 > 0$ so that for $0 < h \le h_0$, $(\frac{\mu}{2} - \frac{2K_1 h}{\lambda_1}) > 0$. Since $\lambda_1\mu > \alpha$, we have $(2\mu - \frac{\alpha}{\lambda_1} - \frac{\mu}{2} - \frac{2K_1 h}{\lambda_1}) > 0$. On integrating (4.44) with respect to time from 0 to t and using positivity of the integral operator, we obtain

$$
\|\hat{\boldsymbol{\eta}}\|^2 + \int_0^t \|\nabla\hat{\boldsymbol{\eta}}\|^2\,ds \le C[\|\boldsymbol{\eta}(0)\|^2 + \int_0^t \|\hat{\boldsymbol{\xi}}\|^2\,ds] + C(\mu)\int_0^t \|\nabla \mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\|\|\hat{\boldsymbol{\eta}}\|^2\,ds.
$$

Note that

$$
\|\boldsymbol{\eta}(0)\| = \|P_h\mathbf{u}_0 - \mathbf{u}_{0h}\| \le \|P_h\mathbf{u}_0 - \mathbf{u}_0\| + \|\mathbf{u}_0 - \mathbf{u}_{0h}\| \le Ch^2\|u_0\|_2.
$$

Using Lemma 4.1, an application of Gronwall's Lemma yields

$$
\|\hat{\boldsymbol{\eta}}\|^2 + \int_0^t \|\nabla\hat{\boldsymbol{\eta}}\|^2\,ds \le Kh^4 \exp\left[\frac{c}{\mu}\int_0^t \|\nabla \mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\|\,ds\right].
\tag{4.45}
$$

Now the integral term on the right hand side of (4.45) is bounded using Lemma 2.4 and Theorem 2.1. Hence, we obtain

$$
\|\boldsymbol{\eta}\|^2 + e^{-2\alpha t}\int_0^t \|\nabla\hat{\boldsymbol{\eta}}\|^2\,ds \le Kh^4 e^{-2\alpha t}.
\tag{4.46}
$$

Since $\boldsymbol{\eta} \in \mathbf{J}_h$, we use inverse hypothesis to obtain an estimate for $\|\nabla\boldsymbol{\eta}\|$. A use of triangle inequality with Lemma 4.3 completes the rest of the proof. □

5. Error Estimate for the Pressure. In this section, we derive optimal error estimates for the Galerkin approximation p_h of the pressure p. The main theorem of this section is as follows.

Theorem 5.1 *Let the hypotheses of Theorem 4.1 hold. Then, there exists a constant $K = K(\mu, \delta, \gamma, \lambda_1, M_1, M_2)$ such that for all $t > 0$*

$$
\|(p - p_h)(t)\|_{\mathbf{L^2/N_h}} \le K\frac{h}{(\tau^*)^{1/2}}e^{-\alpha t}.
$$

Below, we prove this theorem with the help of a series of Lemmas.

From (**B2**), we note that

$$(5.1)\ \|(p-p_h)(t)\|_{\mathbf{L^2/N_h}} \le C\|p-j_hp\|_{\mathbf{L^2/N_h}} + C\sup\left\{\frac{(p-p_h,\nabla\cdot\phi_h)}{\|\nabla\phi_h\|},\ \phi_h\in\mathbf{H}_h/\{0\}\right\}.$$

Since the estimate of the first term on the right hand side of (5.1) follows from the approximation property, it is sufficient to estimate the second term.

From (3.3) and (4.1), we find that

$$(p-p_h,\nabla\cdot\phi_h) = (\mathbf{e}_t,\phi_h) + \mu a(\mathbf{e},\phi_h) + \int_0^t \beta(t-s)a(\mathbf{e}(s),\phi_h)\,ds - \Lambda_h(\phi_h),\ \forall\phi_h\in\mathbf{H}_h,$$

where

$$-\Lambda_h(\phi_h) = b(\mathbf{u},\mathbf{u},\phi_h) - b(\mathbf{u}_h,\mathbf{u}_h,\phi_h) = -b(\mathbf{e},\mathbf{e},\phi_h) + b(\mathbf{u},\mathbf{e},\phi_h) + b(\mathbf{e},\mathbf{u},\phi_h).$$

Using Hölder's inequality, Sobolev inequality and the boundedness of

$$\|\mathbf{e}\|_{\mathbf{L}^4} \le C\|\nabla\mathbf{e}\| \le C$$

from Theorem 4.1, the following eatimate

$$|\Lambda_h(\phi_h)| \le C(1+\|\mathbf{e}\|_{\mathbf{L}^4})\|\nabla\mathbf{e}\|\|\nabla\phi_h\| \le C\|\nabla\mathbf{e}\|\|\nabla\phi_h\|.$$

Thus,

$$(p-p_h,\nabla\cdot\phi_h) \le \left[C|||\mathbf{e}_t|||_{-1;h} + \mu\|\nabla\mathbf{e}\| + \int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|\,ds + C\|\nabla\mathbf{e}\|\right]\|\nabla\phi_h\|,$$

where

$$|||\mathbf{g}|||_{-1;h} = \sup\left\{\frac{<\mathbf{g},\phi_h>}{\|\nabla\phi_h\|},\ \phi_h\in\mathbf{H}_h,\ \phi_h\neq 0\right\}.$$

Altogether, we obtain the following result.

Lemma 5.1 *The semi-discrete Galerkin approximation p_h of the pressure satisfies for all $t>0$*

$$(5.2)\qquad \|(p-p_h)(t)\|_{\mathbf{L^2/N_h}} \le C\left[|||\mathbf{e}_t|||_{-1;h} + \|\nabla\mathbf{e}\| + \int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|\,ds\right].$$

From Theorem 4.1, the estimate $\|\nabla\mathbf{e}\|$ is known. Therefore, we need to estimate $|||\mathbf{e}_t|||_{-1;h}$ in (5.2) in order to complete the proof of Theorem 5.1. Since,

$$|||\mathbf{e}_t|||_{-1;h} \le c\|\mathbf{e}_t\|,$$

we now concentrate on deriving a bound for $\|\mathbf{e}_t\|$.

Note that from Theorem 4.1, the discrete velocity $\mathbf{u}_h$ satisfies

$$\|\nabla\mathbf{u}_h(t)\| \le K, \quad \text{for } t>0.$$

Now, we obtain some *a priori* bounds for the discrete solution $\mathbf{u}_h$ for our subsequent use. Using the definition of the discrete Stokes operator $\tilde{\Delta}_h$, see (3.6), we proceed along the lines of proof of Theorem 2.2 to derive the following bounds for $\mathbf{u}_h$.

Lemma 5.2 *The semi-discrete Galerkin approximation* $\mathbf{u}_h$ *for the velocity satisfies for all* $t > 0$

$$\|\tilde{\Delta}_h \mathbf{u}_h(t)\|^2 + \|\mathbf{u}_{ht}(t)\|^2 + \int_0^t e^{2\alpha\tau} \|\nabla \mathbf{u}_{ht}\|^2 \, d\tau \le K. \tag{5.3}$$

Although, it is possible to obtain exponential decay property for the discrete velocity $\mathbf{u}_h$ following the lines of proof of Lemmas 2.3-2.4 and Theorems 2.1-2.2, we need only boundedness property for our subsequent use.

Lemma 5.3 *The error* $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ *in the velocity satisfies for all* $t > 0$

$$\int_0^t e^{2\alpha s} \|\mathbf{e}_t\|^2 \, ds \le K h^2. \tag{5.4}$$

Proof. For the estimate (5.4), we first split the integral on left for $t > h^2$ as:

$$\begin{aligned} \int_0^t e^{2\alpha s} \|\mathbf{e}_t\|^2 \, ds &= \int_0^{h^2} e^{2\alpha s} \|\mathbf{e}_t\|^2 \, ds + \int_{h^2}^t e^{2\alpha s} \|\mathbf{e}_t\|^2 \, ds \\ &= I_1 + I_2. \end{aligned} \tag{5.5}$$

Note that using estimate (5.3), we obtain below a bound for I_1

$$I_1 \le C h^2 e^{2\alpha h^2} \left(\|\mathbf{u}_t\|^2 + \|\mathbf{u}_{ht}\|^2 \right) \le K h^2.$$

Thus, it is enough to estimate I_2 on the right hand side of (5.5). Using Stokes-Volterra projection $V_h \mathbf{u}$ of $\mathbf{u}$ with $\boldsymbol{\rho} = V_h \mathbf{u} - \mathbf{u}_h$, we write

$$\mathbf{e} := \mathbf{u} - \mathbf{u}_h = \boldsymbol{\rho} + \boldsymbol{\zeta}.$$

Now, the equation in $\boldsymbol{\rho}$ becomes

$$(\boldsymbol{\rho}_t, \boldsymbol{\phi}_h) + \mu a(\boldsymbol{\rho}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\boldsymbol{\rho}(s), \boldsymbol{\phi}_h) \, ds = (\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h), \ \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{5.6}$$

where

$$\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) = b(\mathbf{e}, \mathbf{e}, \boldsymbol{\phi}_h) - b(\mathbf{e}, \mathbf{u}, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{e}, \boldsymbol{\phi}_h).$$

Note that using definition of $b(\cdot,\cdot,\cdot)$, bounds for $\mathbf{u}$ and (3.7), we obtain

$$|b(\mathbf{e}, \mathbf{u}, \boldsymbol{\phi}_h)| + |b(\mathbf{u}, \mathbf{e}, \boldsymbol{\phi}_h)| \le C \|\nabla \mathbf{e}\| \|\boldsymbol{\phi}_h\|. \tag{5.7}$$

Further, using Hölder's inequality and discrete Sobolev inequality

$$\|\boldsymbol{\phi}_h\|_{\mathbf{L}^4} \le C \|\nabla \boldsymbol{\phi}_h\|, \ \boldsymbol{\phi}_h \in \mathbf{H}_h,$$

it now follows from the inverse hypothesis that

$$|b(\mathbf{e}, \mathbf{e}, \boldsymbol{\phi}_h)| \le C \|\mathbf{e}\|_{\mathbf{L}^4} \|\nabla \mathbf{e}\| \|\nabla \boldsymbol{\phi}_h\| \le C h^{-1} \|\mathbf{e}\|_{\mathbf{L}^4} \|\nabla \mathbf{e}\| \|\boldsymbol{\phi}_h\|. \tag{5.8}$$

Choose $\boldsymbol{\phi}_h = \boldsymbol{\rho}_t$ in (5.6), and obtain

$$2\|\boldsymbol{\rho}_t\|^2 + \mu \frac{d}{dt} \|\nabla \boldsymbol{\rho}\|^2 = -2 \int_0^t \beta(t-s) a(\boldsymbol{\rho}(s), \boldsymbol{\rho}_t) \, ds + 2(\boldsymbol{\zeta}_t, \boldsymbol{\rho}_t) + 2\Lambda_h(\boldsymbol{\rho}_t). \tag{5.9}$$

Using $\beta_t(t-s) = -\delta\beta(t-s)$, we find that

$$2\int_0^t \beta(t-s)a(\boldsymbol{\rho}(s),\boldsymbol{\rho}_t)\,ds = 2\frac{d}{dt}\left(\int_0^t \beta(t-s)a(\boldsymbol{\rho}(s),\boldsymbol{\rho})\,ds\right) + 2\delta\int_0^t \beta(t-s)a(\boldsymbol{\rho}(s),\boldsymbol{\rho})\,ds = -A(t). \tag{5.10}$$

Substituting (5.10) in (5.9), multiply the resulting equation by $e^{2\alpha t}$. Integrate with respect to time from h^2 to t and obtain

$$\begin{aligned} 2\int_{h^2}^t e^{2\alpha s}\|\boldsymbol{\rho}_t(s)\|^2\,ds &+ \mu e^{2\alpha t}\|\nabla\boldsymbol{\rho}(t)\|^2 = \mu e^{2\alpha h^2}\|\nabla\boldsymbol{\rho}(h)\|^2 + 2\alpha\mu\int_0^t \|\nabla\hat{\boldsymbol{\rho}}(s)\|^2\,ds \\ &+ \int_{h^2}^t e^{2\alpha s}A(s)\,ds + 2\int_{h^2}^t e^{2\alpha s}(\boldsymbol{\zeta}_t,\boldsymbol{\rho}_t)\,ds + 2\int_{h^2}^t e^{2\alpha s}\Lambda_h(\boldsymbol{\rho}_t)(s)\,ds. \end{aligned} \tag{5.11}$$

For the integral term involving A, we first write it using integration by parts as

$$\begin{aligned} \int_{h^2}^t e^{2\alpha s}A(s)\,ds &= -2\int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}a(\hat{\boldsymbol{\rho}}(\tau),\hat{\boldsymbol{\rho}}(t))\,d\tau + 2\int_0^{h^2} \beta(h^2-\tau)e^{\alpha(h^2-\tau)}a(\hat{\boldsymbol{\rho}}(\tau),\hat{\boldsymbol{\rho}}(h^2))\,d\tau \\ &+ 2\delta(2\alpha-\delta)\int_{h^2}^t\int_0^s \beta(s-\tau)e^{\alpha(s-\tau)}a(\hat{\boldsymbol{\rho}}(\tau),\hat{\boldsymbol{\rho}}(s))\,d\tau\,ds. \end{aligned}$$

Thus using Lemma 2.2, we find that

$$\begin{aligned} \int_{h^2}^t e^{2\alpha s}A(s)\,ds &\leq \frac{2\gamma^2}{\mu(\delta-\alpha)}\int_0^t \|\nabla\hat{\boldsymbol{\rho}}(t)\|^2\,d\tau + \frac{\mu}{2}\|\nabla\hat{\boldsymbol{\rho}}(t)\|^2 + \frac{\gamma^2}{(\delta-\alpha)}\int_0^{h^2}\|\nabla\hat{\boldsymbol{\rho}}(\tau)\|^2\,d\tau \\ &+ \|\nabla\hat{\boldsymbol{\rho}}(h^2)\|^2 + (\frac{\gamma^2}{(\delta-\alpha)^2}+1)\int_0^t \|\nabla\hat{\boldsymbol{\rho}}(s)\|^2\,ds \\ &\leq C(\gamma,\delta,\alpha)\int_0^t \|\nabla\hat{\boldsymbol{\rho}}\|^2\,ds + \|\nabla\hat{\boldsymbol{\rho}}(h^2)\|^2 + \frac{\mu}{2}\|\nabla\hat{\boldsymbol{\rho}}(t)\|^2. \end{aligned} \tag{5.12}$$

From (5.7)-(5.8), we obtain

$$2\Lambda_h(\boldsymbol{\rho}_t) \leq C(1+h^{-2}\|\nabla\mathbf{e}\|^2)\|\nabla\mathbf{e}\|^2 + \frac{1}{2}\|\boldsymbol{\rho}_t\|^2. \tag{5.13}$$

using Cauchy-Schwarz inequality, we find for the fourth term on the right hand side of (5.11) that

$$2(\boldsymbol{\zeta}_t,\boldsymbol{\rho}_t) \leq 2\|\boldsymbol{\zeta}_t\|^2 + \frac{1}{2}\|\boldsymbol{\rho}_t\|^2. \tag{5.14}$$

Substituting (5.12)–(5.14) in (5.11), and using Theorem 4.1, we obtain

$$\begin{aligned} \int_{h^2}^t e^{2\alpha s}\|\boldsymbol{\rho}_t(s)\|^2\,ds &+ \frac{\mu}{2}e^{2\alpha t}\|\nabla\boldsymbol{\rho}(t)\|^2 \leq C(\gamma,\delta,\alpha)\left(e^{2\alpha h^2}\max_{0\leq t\leq h^2}\|\nabla\boldsymbol{\rho}(t)\|^2 + \int_0^t \|\nabla\hat{\boldsymbol{\rho}}(s)\|^2\,ds\right) \\ &+ C\int_{h^2}^t e^{2\alpha s}\|\nabla\mathbf{e}\|^2\,ds + \int_{h^2}^t e^{2\alpha s}\|\boldsymbol{\zeta}_t(s)\|^2\,ds. \end{aligned} \tag{5.15}$$

Since $\boldsymbol{\rho} = \mathbf{e} - \boldsymbol{\zeta}$, we use estimate (4.20) to bound the first and second term on the right hand side of (5.15) as

$$\leq Ke^{2\alpha h^2}h^2 + Ce^{2\alpha h}\sup_{0\leq s\leq h^2}\|\nabla\mathbf{e}(s)\|^2 + C\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}\|^2\,ds.$$

Note that using $\mathbf{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$ with estimates (4.7) and (4.46), we obtain

$$\int_0^t e^{2\alpha s}\|\nabla \mathbf{e}\|^2\, ds \le C\left(\int_0^t \|\nabla \hat{\boldsymbol{\xi}}\|^2\, ds + \int_0^t \|\nabla \hat{\boldsymbol{\eta}}\|^2\, ds\right) \le Kh^2.$$

Similarly, we estimate the second term on the right hand side of (5.15). For the third term on the right hand side of (5.15), we use Lemma 4.2 and Theorem 2.2 to arrive at

$$\int_{h^2}^t e^{2\alpha s}\|\boldsymbol{\zeta}_t\|^2\, ds = h^{-2}\int_{h^2}^t h^2 e^{2\alpha s}\|\boldsymbol{\zeta}_t\|^2\, ds \le h^{-2}\int_{h^2}^t \sigma(s)\|\boldsymbol{\zeta}_t\|^2\, ds \le Kh^2. \tag{5.16}$$

Altogether, we now obtain

$$\int_{h^2}^t e^{2\alpha s}\|\boldsymbol{\rho}_t(s)\|^2\, ds + \frac{\mu}{2} e^{2\alpha t}\|\nabla \boldsymbol{\rho}(t)\|^2 \le Kh^2. \tag{5.17}$$

For I_2, an application of triangle inequality with estimate (5.17) yields

$$I_2 = \int_{h^2}^t e^{2\alpha s}\|\mathbf{e}_t\|^2\, ds \le Kh^2.$$

On combining the estimates of I_1 and I_2 in (5.5), we complete the rest of the proof. □

Lemma 5.4 *The error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ in approximating the velocity satisfies for all $t > 0$*

$$\|\mathbf{e}_t(t)\| \le K\frac{h}{(\tau^*)^{1/2}} e^{-\alpha t},$$

where $\tau^* = \min(t, 1)$.

Proof. From (3.3) and (4.1), we write the equation in $\mathbf{e}$ as

$$(\mathbf{e}_t, \boldsymbol{\phi}_h) + \mu a(\mathbf{e}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\mathbf{e}(s), \boldsymbol{\phi}_h)\, ds = \Lambda_h(\boldsymbol{\phi}_h) + (p, \nabla\cdot\boldsymbol{\phi}_h),\;\; \boldsymbol{\phi}_h \in \mathbf{H}_h.$$

On differentiating with respect to time, we obtain

$$\begin{aligned}(\mathbf{e}_{tt}, \boldsymbol{\phi}_h) + \mu a(\mathbf{e}_t, \boldsymbol{\phi}_h) &= -\beta(0) a(\mathbf{e}, \boldsymbol{\phi}_h) - \int_0^t \beta_t(t-s) a(\mathbf{e}(s), \boldsymbol{\phi}_h)\, ds \\ &+ \Lambda_{ht}(\boldsymbol{\phi}_h) + (p_t, \nabla\cdot\boldsymbol{\phi}_h),\end{aligned} \tag{5.18}$$

where

$$\Lambda_{ht}(\boldsymbol{\phi}_h) = \left(b(\mathbf{u}_{ht}, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}_t, \mathbf{u}, \boldsymbol{\phi})\right) + \left(b(\mathbf{u}_h, \mathbf{u}_{ht}, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}_t, \boldsymbol{\phi}_h)\right).$$

Choosing $\boldsymbol{\phi}_h = P_h\mathbf{e}_t$ in (5.18), we now make use of the $\mathbf{H}_0^1$ stability property of $\mathbf{L^2}-$projection P_h. To estimate $\Lambda_{ht}(P_h\mathbf{e}_t)$, we apply Sobolev inequality along with its discrete version and (3.7)–(3.8) to obtain

$$|b(\mathbf{e}_t, \mathbf{u}, P_h\mathbf{e}_t)| + |b(\mathbf{u}, \mathbf{e}_t, P_h\mathbf{u}_t)| \le K\|\mathbf{e}_t\|\|\nabla \mathbf{e}_t\|,$$

and

$$|b(\mathbf{u}_h, \mathbf{e}, P_h\mathbf{e}_t)| + |b(\mathbf{e}, \mathbf{u}_{ht}, P_h\mathbf{e}_t)| \le C\|\nabla\mathbf{e}\|\|\nabla\mathbf{e}_t\|\|\nabla\mathbf{u}_{ht}\|.$$

Hence,

$$|\Lambda_{ht}(P_h\mathbf{e}_t)| \le (K\|\mathbf{e}_t\| + C\|\nabla\mathbf{u}_{ht}\|\|\nabla\mathbf{e}\|)\,\|\nabla\mathbf{e}_t\|.$$

In order to estimate $(p_t, \nabla \cdot P_h \mathbf{e}_t)$, a use of the discrete incompressible condition yields

$$|(p_t, \nabla \cdot P_h \mathbf{e}_t)| = |(p_t - j_h p_t, \nabla \cdot P_h \mathbf{e}_t)| \leq \|p_t - j_h p_t\| \|\nabla \mathbf{e}_t\|.$$

Now we apply standard kickback argument to absorb $\|\nabla \mathbf{e}_t\|$ term. Altogether, we find that

$$\begin{aligned} \frac{d}{dt}\|\mathbf{e}_t\|^2 + \mu\|\nabla \mathbf{e}_t\|^2 &\leq \frac{d}{dt}\|\mathbf{u}_t - P_h \mathbf{u}_t\|^2 + C(\mu,\gamma)\left(\|\nabla(\mathbf{u}_t - P_h \mathbf{u}_t)\|^2 + \|\nabla \mathbf{e}\|^2 + \|p_t - j_h p_t\|^2\right) \\ &+ C(\mu,\delta)\left(\int_0^t \beta(t-s)\|\nabla \mathbf{e}(s)\|\,ds\right)^2 + K\|\mathbf{e}_t\|^2 + C\|\nabla \mathbf{u}_{ht}\|^2\|\nabla \mathbf{e}\|^2. \end{aligned} \tag{5.19}$$

Multiply (5.19) by $\sigma(t)$, where $\sigma(t) = \tau^*(t)e^{2\alpha t}$ and integrate the resulting inequalities with respect to time. From Lemma 5.3, it follows that $\sigma(t)\|\mathbf{e}_t(t)\|^2 \mapsto 0$ as $t \mapsto 0$. Thus, we derive

$$\begin{aligned} \sigma(t)\|\mathbf{e}_t\|^2 &+ \mu\int_0^t \sigma(s)\|\nabla \mathbf{e}_t\|^2\,ds \leq \sigma(t)\|\mathbf{u}_t - P_h \mathbf{u}_t\|^2 + C\int_0^t e^{2\alpha s}\|\mathbf{e}_t\|^2\,ds \\ &+ C(\mu,\gamma)\int_0^t \sigma(s)\left(\|\nabla(\mathbf{u}_t - P_h \mathbf{u}_t)\|^2 + \|\nabla \mathbf{e}\|^2 + \|p_t - j_h p_t\|^2\right)\,ds \\ &+ C\int_0^t \sigma(s)\|\nabla \mathbf{u}_{ht}\|^2\,\|\nabla \mathbf{e}\|^2\,ds + C\int_0^t \tau^*(s)\left(e^{\alpha s}\int_0^s \beta(s-\tau)\|\nabla \mathbf{e}(\tau)\|\,d\tau\right)^2\,ds. \end{aligned} \tag{5.20}$$

Using Theorem 4.1 and the estimate (5.3), the last two terms on the right hand side of (5.20) are bounded by

$$\begin{aligned} &\leq C\sup_{(0,t)}\|\nabla \mathbf{e}\|^2\Big(\int_0^t \sigma(s)\|\nabla \mathbf{u}_{ht}\|^2\,ds\Big) + C\int_0^t e^{-(\delta-\alpha)(t-s)}\|\nabla \mathbf{e}\|^2 \\ &\leq Kh^2\Big(\int_0^t \sigma(s)\|\nabla \mathbf{u}_{ht}\|^2\,ds\Big) + Kh^2. \end{aligned}$$

Apply approximation property for j_h and estimate (3.4) for P_h to obtain

$$\begin{aligned} \sigma(t)\|\mathbf{e}_t\|^2 &\leq Ch^2\sigma(t)\|\nabla \mathbf{u}_t\|^2 + C\int_0^t e^{2\alpha s}\|\mathbf{e}_t\|^2\,ds + Ch^2\int_0^t \sigma(s)\left(\|\tilde{\Delta}\mathbf{u}_t\|^2 + \|\nabla p_t\|^2\right)\,ds \\ &+ Kh^2\left(\int_0^t \sigma(s)\|\nabla \mathbf{u}_{ht}\|^2\,ds\right) + Kh^2. \end{aligned}$$

By Lemma 5.2-5.3 and Theorem 2.2, we find that

$$\sigma(t)\|\mathbf{e}_t\|^2 \leq Kh^2,$$

and this completes the rest of the proof. □

Proof of Theorem 5.1. Using Lemma 5.4 with approximation property for j_h, we now complete the error estimate for the pressure. □

6. Conclusion. In this article, We have derived optimal error estimates for the velocity in $L^\infty(\mathbf{L^2})$ as we as in $L^\infty(\mathbf{H}^1)$-norms and for the pressure in $L^\infty(L^2)$-norm which are valid uniformly for all $t > 0$. It is not difficult to extend these results for 3-D problem with smallness assumption on the initial data. Moreover, as in Heywood and Rannacher [9], we can use nonconforming elements to derive the results of this paper. Since the estimates are quite cumbersom and do not differ much in their analysis from [9], we refrain from presenting them here.

After the completion of this paper, we came across an article by He *et al.* [8] and found that they derived local optimal error estimates for the velocity in $L^\infty(\mathbf{H}^1)$-norm and for the pressure in $L^\infty(L^2)$-norm for the problem (1.1)–(1.3) with $\mathbf{u_0} \in \mathbf{J_1}$, $\mathbf{f} \in L^\infty(0,\infty;\mathbf{L^2})$ and $(\tau^*)^{1/2}\mathbf{f}_t \in L^\infty(0,\infty;\mathbf{L^2})$. We note that they obtained nonoptimal error estimate for the velocity in $L^\infty(\mathbf{L^2})$. However, when $\mathbf{f}=0$, it is possible to establish, using the present analysis, optimal error estimates for the velocity in $L^\infty(\mathbf{L^2})$ as we as in $L^\infty(\mathbf{H}^1)$-norms and for the pressure in $L^\infty(L^2)$-norm with $\mathbf{u_0} \in \mathbf{J_1}$. The possible changes that we have to make are in the proof of Theorem 2.1 for computing the estimates of $\|\mathbf{u}_t\|^2$ by introducing $\sigma(t)$ in stead of $e^{2\alpha t}$ and the places where we have used $\sigma(t)$ replacing them by $\sigma_1(t) = \tau^*\sigma(t)$. Similar changes should be done throughout the paper to completing the proof.

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