Finite element analysis of a current density - electric field formulation of Bean's model for superconductivity

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Abstract

We study a current density-electric field formulation of Bean's model for the experimental set-up of a infinitely long cylindrical superconductor subject to a transverse magnetic field. We introduce a finite element approximation of the model and prove an error between the exact solution and the approximate solution for the current density of order $(h + \Delta t)^{1/2}$. Numerical simulations for a variety of given source currents are presented.

1 Introduction

In this paper we consider a critical state model for type-II superconductors formulated in terms of the current density and the electric field intensity. The physical setting is that of an infinitely long cylinder of type-II superconducting material subject to an applied transverse magnetic field. We take the cylindrical superconductor to occupy the region $D = \Omega \times \mathbb{R}$, where Ω is a bounded simply connected domain in $\mathbb{R}^2$ that denotes the cross section of the superconductor. In this set-up the current density $\mathbf{J} = (0, 0, J(\underline{x}, t))$ and the electric field intensity $\mathbf{E} = (0, 0, E(\underline{x}, t))$ lie parallel to the axis of the cylinder. Surrounding the superconductor we have copper windings, with cross section region denoted by $\Omega_w = \cup_{i=1}^{k} \Omega_{w_i}$, where each Ω_{w_i} is a simply connected bounded domain in $\mathbb{R}^2$, see Figure 1. In this region we apply a given source current $\mathbf{J}_s = (0, 0, J_s(\underline{x}, t))$.
An evolutionary variational inequality formulation of the model involving the current density J was derived and analysed by Prigozhin in [10, 11, 12]. In these works a numerical method was developed and computations presented. Engineering applications of this approach, relating to the modelling of superconducting induction motors, may be found in [2, 3].
In a recent paper [7] we gave a finite element approximation of the model and proved error estimates between the exact solution and the approximate solution for the current density and the magnetic field. As observed by Bossavit, [4], Bean's critical state model can be formulated as a degenerate Stefan problem. In this paper we study a Stefan problem involving the current density and the electric field equivalent to the variational inequality. We formulate the model in Section 2 and state the relationship between solutions of the model and the unique solution of the variational inequality studied in [7]. In Sections 2.1 and 3 respectively we consider continuous in time and fully discrete finite element approximations of the model and we show an error estimate between

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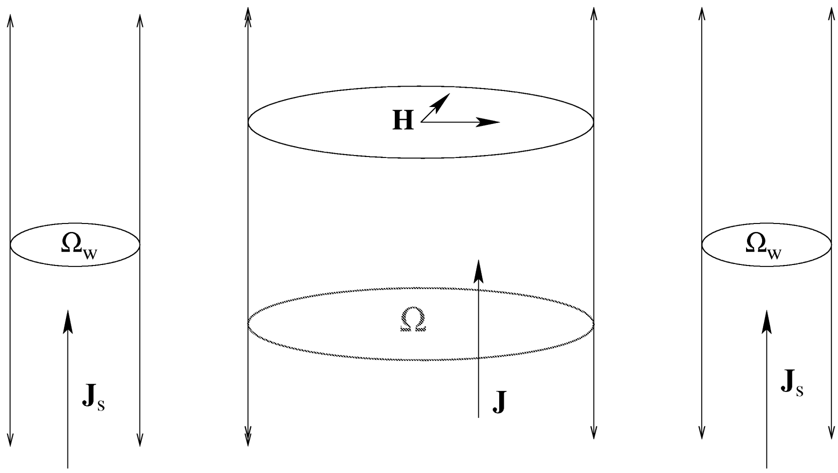

Figure 1: Infinitely long superconducting cylinder and copper windings.

the exact solution of the model and the solution of the fully discrete model. We observe that the discretizations of the variational inequality and Stefan problems are equivalent. The error bound in this paper is for a practical fully discrete scheme involving numerical integration in the non-linear term and this differs from the fully discrete discretization analysed in [7]. In Section 4 we present a Gauss Siedel iteration to solve the fully discrete approximation and we show the convergence of this iteration. We conclude with Section 5 where we present some numerical results.

2 Formulation of the model

Inside the superconductor the electric field intensity is related to the current density by a critical state form of Ohm's law

$$E = \beta(J),$$

where $\beta(\cdot)$ is the is the multi-valued maximal monotone mapping defined by:- for $r \in [-J_c, J_c]$,

$$\beta(r) = \begin{cases} (-\infty, 0] & \text{if} \quad r = -J_c \\ 0 & \text{if} \quad |r| < J_c \\ [0, \infty) & \text{if} \quad r = J_c \end{cases}$$

with J_c being the critical current magnitude.
The current inside the superconductor Ω and the copper windings Ω_w, satisfies

$$|J(\underline{x}, t)| \leq J_c \ \text{ for a.e. } \underline{x} \in \Omega, \quad \int_\Omega J \ d\underline{x} = \int_{\Omega_w} J_s \ d\underline{x} = 0.$$

We use the eddy current form of Maxwell's equations given by

$$\partial_t \mathbf{B} + \text{curl } \mathbf{E} = \mathbf{0} \qquad \text{in} \quad \mathbb{R}^3 \times [0, T] \tag{2.1}$$

$$\text{curl } \mathbf{H} = \mathbf{J}\chi_\Omega + \mathbf{J}_s \chi_{\Omega_w} \qquad \text{in} \quad \mathbb{R}^3 \times [0, T] \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{in} \quad \mathbb{R}^3 \times [0, T], \tag{2.3}$$

where $\mathbf{B}$ is the magnetic flux density. We assume a linear constitutive law $\mathbf{B} = \mu \mathbf{H}$ where the permeability μ is piecewise constant in space, taking different values in Ω, Ω_w and $\mathbb{R}^2 \backslash (\overline{\Omega} \cup \overline{\Omega}_w)$. By taking the curl of (2.1) it is easy to see that

$$\partial_t J \chi_\Omega - \text{div} \left(\frac{1}{\mu} \nabla E \right) = -\partial_t J_s \chi_{\Omega_w}, \tag{2.4}$$

holds in the sense of distributions on the space-time cylinder $\mathbb{R}^2 \times (0,T)$ where

$$|J(\underline{x},t)| \leq J_c \qquad \text{for a.e.} \quad (\underline{x},t) \in \Omega \times (0,T) \tag{2.5}$$
$$J(\underline{x},t) = 0 \qquad \text{for a.e.} \quad (\underline{x},t) \notin \overline{\Omega} \times (0,T) \tag{2.6}$$

and $J_s(\underline{x},t)$ is given such that

$$J_s(\underline{x},t) = 0 \quad \text{for a.e.} \quad (\underline{x},t) \notin \overline{\Omega}_w \times (0,T). \tag{2.7}$$

Inside the superconductor we have

$$E(\underline{x},t) \in \beta(J(\underline{x},t)) \qquad \text{for a.e. } \underline{x} \in \Omega,\ t \in (0,T]. \tag{2.8}$$

This system has the initial condition

$$J(\underline{x},0) = J_0(\underline{x}) \qquad \underline{x} \in \Omega,$$

where $J_0(\underline{x}) = 0$ for $\underline{x} \notin \overline{\Omega}$ and the boundary condition

$$\nabla E \sim 0 \quad \text{as } \underline{x} \sim \infty.$$

We suppose that

$$J_s \in H^1(0,T;L^2(\mathbb{R}^2)), \quad J_s(\underline{x},\cdot) = 0 \quad \text{for a.e. } \underline{x} \in \mathbb{R}^2 \backslash \overline{\Omega}_w, \quad \int_{\Omega_w} J_s(\cdot,t) = 0$$

and we seek a weak solution defined in the following way:-
(P) Find $J \in L^\infty(\mathbb{R}^2 \times (0,T))$ and $E \in L^2(0,T;H^1_{loc}(\mathbb{R}^2))$ such that

$$\int_0^T \int_{\mathbb{R}^2} \left(-J\partial_t \eta + \frac{1}{\mu} \nabla E \cdot \nabla \eta \right) d\underline{x}dt = -\int_0^T \int_{\Omega_w} \partial_t J_s \eta d\underline{x}dt + \int_\Omega J_0(\underline{x})\eta(\underline{x},0)d\underline{x} \tag{2.9}$$

for all

$$\eta \in \mathcal{J} := \{\eta \in H^1(0,T;L^2(\mathbb{R}^2)) : \ \nabla\eta \in L^2(0,T;\mathbb{R}^2),\ \eta(\cdot,T) = 0\}$$

where

$$|J(\underline{x},t)| \leq J_c \quad \text{for a.e. } (\underline{x},t) \in \Omega \times (0,T)$$
$$J(\underline{x},t) = 0 \quad \text{for a.e. } (\underline{x},t) \notin \overline{\Omega} \times (0,T)$$

and

$$\int_\Omega E(\eta - J)d\underline{x} \leq 0 \quad \text{for a.e. } t \in (0,T),\ \forall\ \eta \in K$$

with

$$K := \{\eta \in L^2(\Omega) : |\eta| \leq J_c\}.$$

2.1 Reduction to a bounded domain

It is convenient to work on a bounded domain B_R which is a ball of radius R such that $\overline{\Omega} \cup \overline{\Omega}_w \subset B_R$ and μ is constant outside B_R. We observe that for v being harmonic outside B_R and $\nabla v \in L^2(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \frac{1}{\mu} \nabla v \cdot \nabla \eta = \int_{B_R} \frac{1}{\mu} \nabla v \cdot \nabla \eta + \int_{\partial B_R} \frac{1}{\mu} \mathcal{B}(v)\eta \qquad \forall\ \eta \in H^1(\mathbb{R}^2)$$

where $\mathcal{B} : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is the Dirichlet to Neumann map,

$$\mathcal{B}(v)\Big|_{\partial B_R} := \sum_{k=1}^{\infty} \frac{1}{\pi R} \int_0^{2\pi} \frac{\partial v_\gamma}{\partial \phi} \sin k(\phi - \theta) d\phi$$

where v_γ is the trace of v on ∂B_R. It is useful to introduce the bilinear forms:-

$$\begin{aligned}(\xi,\eta) &:= \int_{B_R} \xi\eta, \quad a(\xi,\eta) := \left(\frac{1}{\mu}\nabla\xi, \nabla\eta\right), \quad (\xi,\eta)_\omega := \int_\omega \xi\eta \\ b(\xi,\eta) &:= \int_{\partial B_R} \frac{1}{\mu}\mathcal{B}(\xi)\eta, \quad A\left(\xi,\eta\right) := a(\xi,\eta) + b(\xi,\eta).\end{aligned}$$

We set

$$\fint_\omega \eta = \frac{1}{|\omega|}\int_\omega \eta d\underline{x}$$

and

$$\begin{aligned}L^2_\Omega(B_R) &:= \{\eta \in L^2(B_R) : \eta = 0 \text{ for a.e. } \underline{x} \notin \overline{\Omega}\} \\ L^2_0(B_R) &:= \left\{\eta \in L^2(B_R) : \fint_{B_R} \eta = 0\right\} \\ L^2_{0,\Omega}(B_R) &:= \{\eta \in L^2_0(B_R) : \eta = 0 \text{ for a.e. } \underline{x} \notin \overline{\Omega}\} \\ L^2_{0,\Omega_w}(B_R) &:= \{\eta \in L^2_0(B_R) : \eta = 0 \text{ for a.e. } \underline{x} \notin \overline{\Omega}_w\} \\ K_\Omega &:= \{\eta \in L^2_\Omega(B_R) : |\eta| \le J_c \ \text{ on } \Omega\} \\ K_{0,\Omega} &:= \{\eta \in L^2_{0,\Omega}(B_R) : |\eta| \le J_c \ \text{ on } \Omega\}.\end{aligned}$$

The problem $(\mathbf{P})$ may be rewritten as:-
$(\mathbf{P_R})$ Find $J \in L^\infty(0,T;K_{0,\Omega})$ and $E \in L^2(0,T;H^1(B_R))$ such that

$$\int_0^T [(-J, \partial_t\xi) + A\left(E,\xi\right)]dt = \int_0^T (-\partial_t J_s, \xi)dt + (J_0, \xi(\cdot,0)) \tag{2.10}$$

for all

$$\xi \in \mathcal{J}_R := \{\xi \in H^1(0,T;L^2(B_R)) \cap L^2(0,T;H^1(B_R)), \ \xi(\cdot,T) = 0\}$$

and

$$(E, \eta - J) \le 0 \quad \forall\, \eta \in K_\Omega, \ \text{ for a.e. } t \in (0,T). \tag{2.11}$$

Also useful for the analysis is the Green's operator

$$G : L^2_0(B_R) \to V := \left\{\eta \in L^2_{loc}(\mathbb{R}^2) : \nabla\xi \in L^2(\mathbb{R}^2), \ \fint_{B_R} \xi = 0\right\}$$

defined as the unique solution of

$$\int_{\mathbb{R}^2} \frac{1}{\mu}\nabla G\eta \cdot \nabla\xi d\underline{x} = \int_{B_R} \eta\xi d\underline{x} \quad \forall\, \xi \in V. \tag{2.12}$$

This can be formulated on B_R by noting that G can be extended as the solution operator of:-
For $\eta \in \mathcal{F} := (H^1_e(B_R))'$ where

$$H^1_e(B_R) := \left\{\xi \in H^1(B_R) : \fint_{B_R} \xi = 0\right\}$$

find $G\eta \in H^1_e(B_R)$ such that

$$A(G\eta, \xi) = \langle \eta, \xi \rangle \quad \forall\, \xi \in H^1_e(B_R) \tag{2.13}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^1_e(B_R)$ and $(H^1_e(B_R))'$. Note that:-

$$\langle \eta, \xi \rangle = (\eta, \xi) \qquad \forall\, \eta \in L^2_0(B_R).$$

We define the semi-norm and the norm

$$|\eta|_A^2 := A(\eta,\eta) \qquad \forall\, \eta \in H^1(B_R)$$
$$||\eta||_{\mathcal{F}} = ||\eta||_{A^{-1}} := |G\eta|_A \qquad \forall\, \eta \in \mathcal{F}$$

and we set

$$|\eta|_{0,\omega} := ||\eta||_{L^2(\omega)},\ \ |\eta|_{1,\omega} = ||\nabla\eta||_{L^2(\omega)},\ \ ||\eta||_{1,\omega} = ||\eta||_{H^1(\omega)}.$$

From [9] we have that $A(\cdot,\cdot)$ is continuous with respect to the $H^1(B_R)$ norm

$$|A(\xi,\eta)| \le C||\xi||_{1,B_R}||\eta||_{1,B_R}$$

and also (note $A(\eta,1) = 0,\ \ \eta \in H^1(B_R)$)

$$(\xi,\eta) = A(G\xi,\eta) \le |G\xi|_A|\eta|_A = ||\xi||_{A^{-1}}|\eta|_A \ \ \forall\, \xi \in L^2_0(B_R),\ \eta \in H^1(B_R).$$

Henceforth for convenience of notation we set $f := -\partial_t J_s \in C([0,T]; L^2_{0,\Omega_w}(B_R))$.

We introduce the following problem:-
($\mathbf{Q_R}$): Find J such that

$$J \in L^\infty(0,T;K_{0,\Omega}) \cap C([0,T;\mathcal{F}]), \qquad \partial_t J \in L^2(0,T;\mathcal{F}) \tag{2.14}$$

satisfying for any $\tau \in [0,T]$

$$\int_0^\tau (\partial_t GJ, \eta - J)dt \ge \int_0^\tau (Gf, \eta - J)dt \quad \forall\, \eta \in L^2(0,T;K_{0,\Omega}) \tag{2.15}$$

$$J|_{t=0} = J_0.$$

Proposition 2.1 *Let $J_s \in H^1(0,T;\mathcal{F})$ and $J_0 \in K_{0,\Omega}$. Then there exists a solution (J,E) of* ($\mathbf{P_R}$) *and $J \in H^1(0,T;\mathcal{F})$. Also J is unique and E is unique up to an additive function of time. Furthermore J is the unique solution of* ($\mathbf{Q_R}$).

Proof: This follows by standard methods. For example using estimates for time discretizations of the kind proved in Proposition 3.1 and then passing to the limit yields existence.

Finite element approximation

2.2 Notation

In this section we consider a finite element approximation of ($\mathbf{P_R}$). We make the following assumptions on the partitioning:
Let T^h be a partitioning of B_R into disjoint open elements $\kappa \in T^h$ such that $\cup_{\kappa\in T^h}\overline{\kappa} = \overline{B}_R$. Furthermore if $\bar{\kappa} \cap (\partial\Omega \cup \partial\Omega_w \cup \partial B_R)$ is non empty then the intersection consists of either one vertex of κ or one curved edge of κ. There exist subsets $T^h_\omega \subset T^h$ such that $\cup_{\kappa\in T^h_\omega}\bar{\kappa} = \bar{\omega}$ with $\omega = B_R,\ \Omega$ or Ω_w.
Associated with T^h are the finite element spaces

$$S^h(\equiv S^h_{B_R}) := \Big\{\eta \in C(\overline{B_R}) : \eta|_\kappa \ \text{ is linear } \forall\, \kappa \in T^h\Big\},$$
$$S^h_0 := \Big\{\eta \in S^h : (\eta,1)^h = 0\}$$
$$S^h_\Omega := \{\eta \in L^2(B_R) : \eta \in C(\overline{\Omega}) : \eta|_\kappa \ \text{ is linear } \forall\, \kappa \in T^h_\Omega,\ \eta|_{B_R\setminus\Omega} = 0\Big\},$$
$$S^h_{\Omega_w} := \{\eta \in L^2(B_R) : \eta \in C(\overline{\Omega_w}) : \eta|_\kappa \ \text{ is linear } \forall\, \kappa \in T^h_{\Omega_w},\ \eta|_{B_R\setminus\Omega_w} = 0\Big\},$$

$$\begin{aligned}
S^h_{0,\Omega} &:= \left\{\eta \in S^h_\Omega : \fint_{B_R} \eta = 0\right\} \\
S^h_{0,\Omega_w} &:= \left\{\eta \in S^h_{\Omega_w} : \fint_{B_R} \eta = 0\right\} \\
K^h_\Omega &:= \{\eta \in S^h_\Omega : |\eta| \le J_c\} \\
K^h_{0,\Omega} &:= \{\eta \in S^h_{0,\Omega} : |\eta| \le J_c\}
\end{aligned}$$

where

$$(\eta_1, \eta_2)^h := \sum_{\kappa \in T^h_\omega} \int_\kappa \Pi^h(\eta_1\eta_2) d\underline{x}, \quad \eta_i \in S^h_\omega := S^h_{B_R}, S^h_\Omega, S^h_{\Omega_w}$$

and $\Pi^h : C(\overline{B_R}) \to S^h$ is the standard piecewise linear interpolant. Observe that $S^h_0 \subset H^1_e(B_R)$ since

$$(\eta, 1)^h = (\eta, 1) \quad \forall\, \eta \in S^h.$$

For $\xi, \eta \in S^h_\omega$ we define

$$I^h(\xi, \eta) := (\xi, \eta) - (\xi, \eta)^h$$

and we note the well-known result

$$|I^h(\xi, \eta)| = |(\xi, \eta) - (\xi, \eta)^h| \le Ch^2 |\xi|_{1,\omega} |\eta|_{1,\omega} \le Ch|\xi|_{1,\omega}|\eta|_{0,\omega}. \tag{2.1}$$

Analogous to (2.13) it is convenient to introduce the operator $G^h : L^2_0(B_R) \to S^h_0$ such that for any $\xi \in L^2_0(B_R)$, $G^h\xi \in S^h_0$ is the unique solution of :-

$$A(G^h\xi, \psi) = (\xi, \psi) \quad \forall\, \psi \in S^h_0. \tag{2.2}$$

For $\eta \in L^2_0(B_R)$ we set

$$\|\eta\|_{A^{-h}} := \left\|G^h\eta\right\|_A = (\eta, G^h\eta)^{1/2} \quad \forall \eta \in L^2_0(B_R)$$

and we note that

$$(G^h\xi, \psi) = (\xi, G^h\psi) \quad \forall\, \xi, \psi \in S^h_0 \tag{2.3}$$

and that

$$(\xi, \psi) \;\le\; C\, \|\xi\|_{A^{-h}}\, |\psi|_A \quad \forall\, \psi \in S^h_0,\ \xi \in L^2_0(B_R). \tag{2.4}$$

Standard finite element estimates, see [9], yield

$$\left|(G - G^h)\eta\right|_{0,B_R} + h\left|(G - G^h)\eta\right|_{1,B_R} \;\le\; Ch^2|\eta|_{0,B_R} \qquad \forall\, \eta \in L^2_0(B_R). \tag{2.5}$$

It is easy to see that

$$|G^h\eta|_A \le |G\eta|_A \quad \forall\, \eta \in L^2_0(B_R) \tag{2.6}$$

and since

$$|\eta|^2_{0,B_R} = A(G\eta, \eta) \le C|G\eta|_A||\eta||_{1,B_R} \quad \forall\, \eta \in S^h_0$$

a standard inverse inequality yields

$$|\eta|_{0,B_R} \le Ch^{-1}|G\eta|_A \quad \forall\, \eta \in S^h_0, \tag{2.7}$$

which implies using the error bound (2.5)

$$|G\eta|_A \le C|G^h\eta|_A \quad \forall\, \psi \in S^h_0. \tag{2.8}$$

Defining the projection operator $Q^h : L^2(B_R) \to S^h$ by

$$\left(Q^h \eta, \psi\right)^h = (\eta, \psi) \quad \forall\, \psi \in S^h \tag{2.9}$$

we note that if $\xi \in L^2_0(B_R)$ then $Q^h\xi \in S^h_0 \subset H^1_e(B_R)$ and if $\xi \in K_{0,\Omega}$ then $Q^h\xi \in K^h_{0,\Omega}$. Furthermore

$$|Q^h\xi|_{0,B_R} \le C|\xi|_{0,B_R}. \tag{2.10}$$

Following [1] we have

$$|G(\xi - Q^h\xi)|_A \le Ch|\xi|_{0,B_R} \quad \forall\, \xi \in L^2_0(B_R). \tag{2.11}$$

2.3 Continuous in time discretization

We now introduce a continuous in time finite element approximation of $(\mathbf{P_R})$:

For $J^0_h \in K^h_{0,\Omega}$ and $f_h(\cdot, t) \in S^h_{0,\Omega_w}$ satisfying

$$\int_0^T \left|G^h f_h\right|^2_A \le C \tag{2.12}$$

we have the following:

$(\mathbf{P^h_R})$ Find for $t \in (0, T]$, $J_h(\cdot, t) \in K^h_{0,\Omega}$ and $E_h(\cdot, t) \in S^h$ such that

$$(\partial_t J_h, \psi) + A(E_h, \psi) = (f_h, \psi) \quad \forall\, \psi \in S^h \tag{2.13}$$

$$J_h(\underline{x}, 0) = J^h_0(\underline{x}) \qquad \forall\, \underline{x} \in \Omega \tag{2.14}$$

$$(E_h(\cdot, t), \eta - J_h(\cdot, t)) \le 0 \quad \forall\, \eta \in K^h_\Omega,\ \forall\, t \in (0, T]. \tag{2.15}$$

For $\chi \in S^h_0$, setting $\psi = G^h\chi$ in (2.13) and noting (2.15) yields the following variational formulation of $(\mathbf{P^h_R})$:

$(\mathbf{Q^h_R})$ Find $J_h \in L^\infty(0, T; K^h_{0,\Omega})$ such that

$$\left(\partial_t G^h J_h, \chi - J_h\right) \ge \left(G^h f_h, \chi - J_h\right) \quad \forall\, \chi \in K^h_{0,\Omega}. \tag{2.16}$$

Proposition 2.2 *There exists a solution (J_h, E_h) of $(\mathbf{P^h_R})$ such that*

$$\int_0^T \left|\partial_t G^h J_h\right|^2_A dt + \int_0^T |E_h|^2_A dt \le C.$$

Also J_h is unique and E_h is unique up to an additive function of time. Furthermore J_h is the unique solution of $(\mathbf{Q^h_R})$.

Proof: This follows by standard results. For example, existence can be proved using the ideas of discretization in time and using estimates of the type proved in Proposition 3.1.

For the forthcoming error analysis we require $|f|_{0,B_R} \le C$ and

$$\int_0^T ||f_h - f||^2_{A^{-1}} dt \le Ch. \tag{2.17}$$

Lemma 2.1 *For $f_h(\cdot, t) \in S^h_{0,\Omega_w}$ satisfying (2.17) the unique solutions of $(\mathbf{P_R})$ and $(\mathbf{P^h_R})$ satisfy*

$$||J - J_h||_{L^\infty(0,T;A^{-1})} \le C(T)h^{1/2}. \tag{2.18}$$

Proof: It is convenient to define

$$\begin{aligned}\mathcal{E}(t) &:= \Big(\partial_t G^h(J-J_h), J-J_h\Big) \\ &= (\partial_t(G^h-G)J, J-J_h) + (\partial_t GJ, J-J_h) - (\partial_t G^h J_h, J-J_h), \end{aligned} \tag{2.19}$$

setting $\xi = G(J-J_h)$ in (2.10) and $\psi = G^h(J-J_h)$ in (2.13) and noting (2.13), (2.2) and (2.3) we have

$$\begin{aligned}\mathcal{E}(t) &= (\partial_t(G^h-G)J, J-J_h) + (f, G(J-J_h)) - (f_h, G^h(J-J_h)) \\ &\quad +(E_h - E, J-J_h).\end{aligned}$$

From (2.5), (2.7) and Young's inequality we have

$$\begin{aligned}(\partial_t(G^h-G)J, J-J_h) &\leq |\partial_t(G^h-G)J|_A||J-J_h||_{A^{-1}} \\ &\leq Ch^2|\partial_t J|^2_{0,B_R} + \frac{1}{2}\|J-J_h\|^2_{A^{-1}} \\ &\leq Ch\,\|\partial_t J\|^2_{A^{-1}} + \frac{1}{2}||J-J_h||^2_{A^{-1}}.\end{aligned} \tag{2.20}$$

Using (2.11), (2.15) and (2.11) we have

$$\begin{aligned}(E_h - E, J-J_h) &\leq (E_h, J-J_h) \\ &= (E_h, J-Q^hJ) + (E_h, Q^hJ - J_h) \\ &\leq (E_h, J-Q^hJ) \\ &\leq |E_h|_A||J-Q^hJ||_{A^{-1}} \\ &\leq Ch|E_h|_A|J|_{0,B_R}.\end{aligned} \tag{2.21}$$

Lastly from (2.5), (2.6) and Young's inequality we have

$$\begin{aligned}(f, G(J-J_h)) - (f_h, G^h(J-J_h)) &= (f-f_h, G^h(J-J_h)) + (f, (G-G^h)(J-J_h)) \\ &\leq ||f-f_h||_{A^{-1}}|G^h(J-J_h)|_A + |f|_{0,B_R}|(G-G^h)(J-J_h)|_{0,B_R} \\ &\leq \frac{1}{2}||f-f_h||^2_{A^{-1}} + \frac{1}{2}|G^h(J-J_h)|^2_A + Ch^2|f|_{0,B_R}|J-J_h|_{0,B_R} \\ &\leq \frac{1}{2}||f-f_h||^2_{A^{-1}} + \frac{1}{2}||J-J_h||^2_{A^{-1}} + Ch^2(|J|_{0,B_R} + |J_h|_{0,B_R}).\end{aligned} \tag{2.22}$$

From (2.19)-(2.22) together with Propositions 2.1 and 2.2 we have

$$\frac{1}{2}\frac{d}{dt}||J-J_h||^2_{A^{-h}} = \mathcal{E}(t) \leq Ch + ||J-J_h||^2_{A^{-1}} + \frac{1}{2}||f-f_h||^2_{A^{-1}}.$$

Noting (2.8), using Grönwall's inequality and (2.17) yields the required result. □

3 Fully discrete model

In this section we consider a fully discrete discretization of $(\mathbf{P_R})$. We set $N\Delta t = T$, $t_n := n\Delta t$ for $n = 0 \to N$, $f_h^n \in S^h_{0,\Omega_w}$ to be an approximation to $f(\cdot, t_n)$ and for any $g_h \in S^h$ we define

$$\delta_t g_h^n = \frac{g_h^n - g_h^{n-1}}{\Delta t}.$$

We introduce the operator $\hat{G}^h : (S^h_{0,\Omega}, S^h_{0,\Omega_w}) \to S^h_0$ such that for any $\xi \in (S^h_{0,\Omega}, S^h_{0,\Omega_w})$, $\hat{G}^h\xi \in S^h_0$ is the unique solution of :-

$$A(\hat{G}^h\xi, \psi) = (\xi, \psi)^h \quad \forall\, \psi \in S^h_0. \tag{3.1}$$

We set

$$||\eta||^2_{\hat{A}^{-h}} := |\hat{G}^h \eta|^2_A = (\hat{G}^h \eta, \eta)^h \qquad \forall\, \eta \in S^h_0$$

and we note using (2.1) that

$$|(G^h - \hat{G}^h)\eta|_A \le Ch|\eta|_{0,B_R} \qquad \forall\, \eta \in S^h_0 \tag{3.2}$$

and hence from (2.6) we have that

$$||\eta||^2_{\hat{A}^{-h}} \le ||\eta||^2_{A^{-1}} + Ch^2|\eta|^2_{0,B_R} \qquad \forall\, \eta \in S^h_0. \tag{3.3}$$

Furthermore from (2.2) and (3.1) we note that

$$(\xi, G^h \psi)^h = (\hat{G}^h \xi, \psi) \qquad \forall\, \xi, \psi \in S^h_0. \tag{3.4}$$

We consider the following fully discrete discretization of $(\mathbf{P_R})$:

$(\mathbf{P_R^{h,\Delta t}})$ Find $\{J^n_h, E^n_h\} \in K^h_{0,\Omega} \times S^h$ such that

$$(\delta_t J^n_h, \psi)^h + A(E^n_h, \psi) = (f^n_h, \psi)^h \quad \forall\, \psi \in S^h \tag{3.5}$$

$$J^0_h(\underline{x}) = J^h_0(\underline{x}) \quad \forall\, \underline{x} \in \Omega \tag{3.6}$$

$$(E^n_h, \eta - J^n_h)^h \le 0 \quad \forall\, \eta \in K^h_\Omega. \tag{3.7}$$

For $\chi \in S^h_0$, setting $\psi = \hat{G}^h \chi$ in (3.5) and noting (3.7) yields the following variational formulation of $(\mathbf{P_R^{h,\Delta t}})$:

$(\mathbf{Q_R^{h,\Delta t}})$ Find $J^n_h \in K^h_{0,\Omega}$ such that

$$\left(\hat{G}^h(\delta_t J^n_h), \chi - J^n_h\right)^h \ge \left(\hat{G}^h f^n_h, \chi - J^n_h\right)^h \quad \forall\, \chi \in K^h_{0,\Omega}. \tag{3.8}$$

Proposition 3.1 *Let $\Delta t = Ch$ and $\sum_{n=1}^N \Delta t |f^n_h|^2_h \le C$. Then there exists a solution pair $\{J^n_h, E^n_h\}_{n\ge 0}$ to $(\mathbf{P_R^{h,\Delta t}})$ such that*

$$\sum_{n=1}^N \Delta t\, ||\delta_t J^n_h||^2_{\hat{A}^{-h}} + h \sum_{n=1}^N \Delta t |\delta_t J^n_h|^2_{0,B_R} + \Delta t \sum_{n=1}^N |E^n_h|^2_A \le C. \tag{3.9}$$

Also J^n_h is unique and E^n_h is unique up to an additive constant. Furthermore $\{J^n_h\}_{n\ge 0}$ is the unique solution of $(\mathbf{Q_R^{h,\Delta t}})$.

Proof: Let I^Ω be the index set of triangle vertices $\underline{x}_i \in \overline{\Omega}$. Let $E^n_i := E^n_h(\underline{x}_i)$ and $J^n_i := J^n_h(\underline{x}_i)$ for $i \in I^\Omega$. It is easy to see that (3.7) is equivalent to

$$|J^n_i| \le J_c \ \text{ and } \ E^n_i(\psi - J^n_i) \le 0 \quad \forall\, |\psi| \le J_c. \tag{3.10a}$$

Elementary calculations yield the equivalence of (3.10a) with

$$J_c(|\psi| - |E^n_i|) \ge J^n_i(\psi - E^n_i) \quad \forall\, |\psi| \le J_c \tag{3.10b}$$

and also the equivalence with

$$J^n_i \in J_c \text{sign} E^n_i. \tag{3.10c}$$

It follows that (3.5) and (3.7) are equivalent to :-

$$J_c \int_\Omega \Pi^h(|\psi| - |E^n_h|) d\underline{x} + \Delta t A(E^n_h, \psi - E^n_h) \ge (J^{n-1}_h + \Delta t f^n_h, \psi - E^n_h)^h \qquad \forall\, \psi \in S^h.$$

This is a necessary condition for E_h^n to be a solution of the minimization problem:-

$$\mathcal{F}(E_h^n) := \min_{\psi \in S^h} \mathcal{F}(\psi)$$

$$\mathcal{F}(\psi) := J_c \int_\Omega \Pi^h(|\psi|)d\underline{x} + \frac{\Delta t}{2} A(\psi,\psi) - (J_h^{n-1} + \Delta t f_h^n, \psi)^h.$$

Since $\mathcal{F}$ is continuous and bounded below (using the fact that $A(\cdot,\cdot)$ is positive definite on S_0^h) there exists a minimiser E_h^n and hence by equation (3.5) there also exists a $J_h^n \in S^h$. Furthermore by the above equivalences it follows that for $J_h^n \in S_{0,\Omega}^h$ we have also $J_h^n \in K_{0,\Omega}^h$ and that (3.7) holds. Hence we have existence of a solution pair $\{J_h^n, E_h^n\}$.
Suppose $\{J_h^n, E_h^n\}$ and $\{\tilde{J}_h^n, \tilde{E}_h^n\}$ are two separate solution pairs. It follows from (3.5) and (3.7) that

$$(J_h^n - \tilde{J}_h^n, \psi)^h + A(E_h^n - \tilde{E}_h^n, \psi) = 0 \quad \forall\, \psi \in S^h$$

and

$$(E_h^n - \tilde{E}_h^n, J_h^n - \tilde{J}_h^n) \geq 0.$$

This immediately implies that J_h^n is unique and that E_h^n is unique up to an additive constant. Furthermore it follows from (3.1) and (3.5) that

$$E_h^n = \hat{G}^h(f_h^n - \delta_t J_h^n) + \lambda_h^n$$

for a scalar λ_h^n. By considering (3.7) for $\eta \in K_{0,\Omega}^h$ we obtain (3.8) which implies that J_h^n is the unique solution of $(\mathbf{Q}_\mathbf{R}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$.
Taking $\chi = J_h^{n-1}$ in (3.8) we obtain

$$|\hat{G}^h \delta_t J_h^n|_A^2 \leq (\hat{G}^h f_h^n, \delta_t J_h^n)^h$$

and so

$$|\delta_t J_h^n|_{\hat{A}^{-h}} \leq |f_h^n|_{\hat{A}^{-h}}.$$

Setting $\tilde{J}_h^n \in S^h$ to be the interpolant of J_h^n we observe that, see [6]

$$A(E_h^n, \tilde{J}_h^n) = \int_{B_R} \frac{1}{\mu} \nabla E_h^n \nabla \tilde{J}_h^n \geq 0$$

and hence it follows from (3.5) that

$$\begin{aligned} |\tilde{J}_h^n|_h^2 - |\tilde{J}_h^{n-1}|_h^2 + |\tilde{J}_h^n - \tilde{J}_h^{n-1}|_h^2 &\leq \Delta t (f_h^n, \tilde{J}_h^n)^h \\ &\leq C \Delta t |f_h^n|_h. \end{aligned}$$

By elementary calculations we have that the required bounds hold. □.

Before we derive an error bound on the solutions of $(\mathbf{P}_\mathbf{R}^\mathbf{h})$ and $(\mathbf{P}_\mathbf{R}^{\mathbf{h},\boldsymbol{\Delta}\mathbf{t}})$ we introduce some useful notation. For $n \geq 1$ we set

$$J_{h,\Delta t}(t) := \frac{t - t_{n-1}}{\Delta t} J_h^n + \frac{t_n - t}{\Delta t} J_h^{n-1}, \quad f_{h,\Delta t}(t) := \frac{t - t_{n-1}}{\Delta t} f_h^n + \frac{t_n - t}{\Delta t} f_h^{n-1} \quad \forall\, t \in [t_{n-1}, t_n], \quad (3.11)$$

and

$$\hat{J}_{h,\Delta t}(t) := J_h^n, \quad \hat{f}_{h,\Delta t}(t) := f_h^n \quad \forall\, t \in (t_{n-1}, t_n]. \quad (3.12)$$

From (3.11) and (3.12) it follows that for a.e. $t \in (0,T)$

$$J_{h,\Delta t} - \hat{J}_{h,\Delta t} = -(t_n - t) \partial_t J_{h,\Delta t}. \quad (3.13)$$

For the forthcoming error analysis we require that

$$\int_0^T ||f_h - \hat{f}_{h,\Delta t}||_{\hat{A}^{-h}}^2 dt \leq C \Delta t. \quad (3.14)$$

Lemma 3.1 *For $\Delta t = Ch$ the unique solutions of* ($\mathbf{P_R^h}$) *and* ($\mathbf{P_R^{h,\Delta t}}$) *satisfy*

$$\|J_h - J_{h,\Delta t}\|_{A^{-1}} \le C(T)(h+\Delta t)^{1/2}. \tag{3.15}$$

Proof: Setting $\chi = J_{h,\Delta t}$ in (2.16) and noting (3.4) we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|J_h - J_{h,\Delta t}\|^2_{A^{-h}} &= \Big(\partial_t G^h J_h, J_h - J_{h,\Delta t}\Big) - \Big(\partial_t G^h J_{h,\Delta t}, J_h - J_{h,\Delta t}\Big)\\
&\le \Big(G^h f_h, J_h - J_{h,\Delta t}\Big) - \Big(\partial_t G^h J_{h,\Delta t}, J_h - J_{h,\Delta t}\Big)\\
&= I^h(G^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}) + \Big(G^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}\Big)^h\\
&= I^h(G^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}) + \Big(f_h - \partial_t J_{h,\Delta t}, \hat{G}^h(J_h - J_{h,\Delta t})\Big)\\
&= I^h(G^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}) + I^h(f_h - \partial_t J_{h,\Delta t}, \hat{G}^h(J_h - J_{h,\Delta t}))\\
&\quad + \Big(\hat{G}^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}\Big)^h.
\end{aligned} \tag{3.16}$$

Setting $\chi = J_h$ in (3.8) we have

$$\begin{aligned}
\Big(\partial_t \hat{G}^h J_{h,\Delta t}\,,\, J_h - J_{h,\Delta t}\Big)^h &= \Big(\partial_t \hat{G}^h J_{h,\Delta t}, J_h - \hat{J}_{h,\Delta t}\Big)^h + \Big(\partial_t \hat{G}^h J_{h,\Delta t}, \hat{J}_{h,\Delta t} - J_{h,\Delta t}\Big)^h\\
&\le \Big(\hat{G}^h \hat{f}_{h,\Delta t}, J_h - \hat{J}_{h,\Delta t}\Big)^h + \Big(\partial_t \hat{G}^h J_{h,\Delta t}, \hat{J}_{h,\Delta t} - J_{h,\Delta t}\Big)^h\\
&= \Big(\hat{G}^h \hat{f}_{h,\Delta t}, J_h - J_{h,\Delta t}\Big)^h + \Big(\hat{G}^h(\partial_t J_{h,\Delta t} - f_h), \hat{J}_{h,\Delta t} - J_{h,\Delta t}\Big)^h.
\end{aligned} \tag{3.17}$$

From (3.16), (3.17), (2.1) and Propositions 2.2 and 3.1 we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||J_h - J_{h,\Delta t}||^2_{A^{-h}} &\le I^h(G^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}) + I^h(f_h - \partial_t J_{h,\Delta t}, \hat{G}^h(J_h - J_{h,\Delta t}))\\
&\quad + \Big(\hat{G}^h(f_h - \hat{f}_{h,\Delta t}), J_h - J_{h,\Delta t}\Big)^h - \Big(\hat{G}^h(\partial_t J_{h,\Delta t} - f_h), \hat{J}_{h,\Delta t} - J_{h,\Delta t}\Big)^h\\
&\le Ch\Big|G^h(f_h - \partial_t J_{h,\Delta t})\Big|_{1,B_R} |J_h - J_{h,\Delta t}|_{0,B_R} + Ch|f_h - \partial_t J_{h,\Delta t}|_{0,B_R}\Big|\hat{G}^h(J_h - J_{h,\Delta t})\Big|_{1,B_R}\\
&\quad + \Big\|f_h - \hat{f}_{h,\Delta t}\Big\|_{\hat{A}^{-h}} \|J_h - J_{h,\Delta t}\|_{A^{-1}} + |\partial_t J_{h,\Delta t} - f_h|_{\hat{A}^{-h}}\Big|\hat{J}_{h,\Delta t} - J_{h,\Delta t}\Big|_{A^{-1}}\\
&\le Ch + Ch|f_h - \partial_t J_{h,\Delta t}|_{0,B_R}|J_h - J_{h,\Delta t}|_{\hat{A}^{-h}} + \Big\|f_h - \hat{f}_{h,\Delta t}\Big\|_{\hat{A}^{-h}} \|J_h - J_{h,\Delta t}\|_{A^{-1}}\\
&\quad + |\partial_t J_{h,\Delta t} - f_h|_{\hat{A}^{-h}}\Big|\hat{J}_{h,\Delta t} - J_{h,\Delta t}\Big|_{A^{-1}}.
\end{aligned} \tag{3.18}$$

Using (3.18), (2.8), (3.3), (3.13) and Young's inequality we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|J_h - J_{h,\Delta t}\|^2_{A^{-1}} &\le Ch + Ch^2|f_h - \partial_t J_{h,\Delta t}|^2_{0,B_R} + C\|J_h - J_{h,\Delta t}\|^2_{A^{-1}} + C\Big\|f_h - \hat{f}_{h,\Delta t}\Big\|^2_{\hat{A}^{-h}}\\
&\quad + C\Delta t|\partial_t J_{h,\Delta t} - f_h|_{\hat{A}^{-h}}|\partial_t J_{h,\Delta t}|_{A^{-1}}.
\end{aligned}$$

The result follows using a Grönwall inequality, (3.9) and (3.14). □

Finally from Lemmas 2.1 and 3.1 we have our main result.

Theorem 3.1 *The unique solutions of* ($\mathbf{P_R^{h,\Delta t}}$) *and* ($\mathbf{P}$) *satisfy*

$$\|J - J_{h,\Delta t}\|_{L^\infty(0,T;A^{-1})} \le C(T)(h+\Delta t)^{1/2}.$$

4 The Gauss-Seidel iteration

It is easy to see that the fully discrete scheme ($\mathbf{P}_{\mathbf{R}}^{\mathbf{h},\mathbf{\Delta t}}$) yields the algebraic problem:-
Find $(\mathbf{J},\mathbf{E}) \in \mathbb{R}^N \times \mathbb{R}^N$ such that

$$\begin{aligned} M\mathbf{J} + A\mathbf{E} - \mathbf{b} &= \mathbf{0}, \\ J_i = 0, & \qquad i \notin I^\Omega, \\ J_i \in J_c \,\text{sign} E_i, & \qquad i \in I^\Omega. \end{aligned}$$

Here $\mathbf{J}$ and $\mathbf{E}$ are the nodal values of J_h^n and E_h^n at the vertices of the triangulation according to some ordering. We denote by I^Ω the set of vertices on $\overline{\Omega}$ and set $J_i = 0$ for all $i \notin I^\Omega$. The diagonal mass matrix M^Ω is defined by

$$M_{ii}^\Omega = \begin{cases} \int_\Omega \chi_i d\underline{x} & i \in I^\Omega \\ 0 & i \notin I^\Omega, \end{cases}$$

where χ_i is the basis function associated with node i. A is the symmetric positive semi-definite matrix defined by

$$\boldsymbol{\xi}^T A \boldsymbol{\psi} = A(\boldsymbol{\xi}, \boldsymbol{\psi}) \qquad \xi, \psi \in S^h$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$ are the nodal values of ξ and ψ. It follows that

$$A\mathbf{e} = 0,$$

and

$$\boldsymbol{\xi}^T A \boldsymbol{\xi} \geq C_A ||\boldsymbol{\xi}||^2 \quad \forall\, \boldsymbol{\xi} \;\text{ such that } \boldsymbol{\xi}^T \mathbf{e} = 0$$

where $\{\mathbf{e}\}_j = 1$ for all j. The right-hand side $\mathbf{b}$ is defined by

$$\mathbf{b}^T \boldsymbol{\psi} = (J_h^{n-1} + \Delta t f_h^n, \psi)^h \qquad \psi \in S^h$$

and

$$\mathbf{b}^T \mathbf{e} = 0,$$

since $J_h^{n-1} \in S_{0,\Omega}^h$ and $f_h^n \in S_{0,\Omega_w}^h$. We set $|\mathbf{v}| := (|v_1|, |v_2|, \ldots, |v_N|)^T$ and $\mathbf{v_p} := \mathbf{v} - \frac{1}{N}\mathbf{e}^T \mathbf{v}\mathbf{e}$.
In order to solve this problem we set out a version of the Gauss-Seidel iteration formulated by Elliott, [6], for the enthalpy method for the Stefan problem.

Gauss Seidel Iteration
Given $\mathbf{E}^0$, for $k \geq 1$, $\{\mathbf{E}^k, \mathbf{J}^k\}$ are defined by:-
For $i = 1 \to N$, (J_i^{k+1}, E_i^{k+1}) are the unique solutions of

$$(A\mathbf{E}^{i-1,k+1} - \mathbf{b})_i + A_{ii}(E_i^{k+1} - E_i^k) + M_{ii} J_i^{k+1} = 0 \tag{4.19}$$

$$J_i^{k+1} = 0 \qquad i \notin I^\Omega \tag{4.20}$$

$$J_i^{k+1} \in J_c \text{sign} E_i^{k+1} \qquad i \in I^\Omega, \tag{4.21}$$

where

$$\mathbf{E}^{i,k+1} := (E_1^{k+1}, E_2^{k+1}, \ldots, E_i^{k+1}, E_{i+1}^k, \ldots, E_N^k)^T \qquad i = 0 \to N.$$

As noted in the proof of existence in Proposition 3.1, this problem is associated with energy minimization.
We set

$$\begin{aligned} \mathcal{F}(\mathbf{E}) &:= J_c (M^\Omega \mathbf{e})^T |\mathbf{E}| + \frac{1}{2} \mathbf{E}^T A \mathbf{E} - \mathbf{b}^T \mathbf{E} \\ &= J_c (M^\Omega \mathbf{e})^T |\mathbf{E}| + \frac{1}{2} \mathbf{E}_\mathbf{p}^T A \mathbf{E}_\mathbf{p} - \mathbf{b}^T \mathbf{E}_\mathbf{p} \\ &\geq J_c \mathbf{e}^T M^\Omega |\mathbf{E}| + \frac{1}{2} C_{A,b} ||\mathbf{E}_\mathbf{p}||^2 - \hat{C}_{A,b}. \end{aligned}$$

Hence $\mathcal{F}(\mathbf{E})$ is bounded below and $||\mathbf{E}|| \leq C(\mathcal{F}(\mathbf{E}), A, \mathbf{b})$.
We define

$$\mathcal{F}_i^k(z) := \mathcal{F}(E_1^{k+1}, \ldots, E_{i-1}^{k+1}, z, E_{i+1}^k, \ldots, E_N^k).$$

Clearly,

$$\mathcal{F}_i^k(E_i^{k+1}) = \mathcal{F}(\mathbf{E}^{i,k+1}) \quad \text{and} \quad \mathcal{F}_i^k(E_i^k) = \mathcal{F}(\mathbf{E}^{i-1,k+1}).$$

Furthermore

$$\begin{aligned}\sum_{i=1}^{N} \left(\mathcal{F}_i^k(E_i^{k+1}) - \mathcal{F}_i^k(E_i^k)\right) &= \mathcal{F}(\mathbf{E}^{N,k+1}) - \mathcal{F}(\mathbf{E}^{0,k+1}) \\ &= \mathcal{F}(\mathbf{E}^{k+1}) - \mathcal{F}(\mathbf{E}^k). \qquad (4.22)\end{aligned}$$

Lemma 4.1 *The above iteration satisfies*

$$\mathcal{F}(\mathbf{E}^{k+1}) - \mathcal{F}(\mathbf{E}^k) \leq -C_A \left\|\mathbf{E}^{k+1} - \mathbf{E}^k\right\|^2$$

and

$$\left\|\mathbf{J}^k\right\|_\infty \leq J_c,$$

for all $k \geq 0$.

Proof:
A straightforward calculation gives

$$\begin{aligned}\delta_i^k := \mathcal{F}_i^k(E_i^{k+1}) - \mathcal{F}_i^k(E_i^k) &= \frac{1}{2}A_{ii}(E_i^{k+1} - E_i^k)^2 + (A\mathbf{E}^{i-1,k+1} - \mathbf{b})_i(E_i^{k+1} - E_i^k) \\ &\quad + J_c M_{ii}(|E_i^{k+1}| - |E_i^k|).\end{aligned}$$

From (4.19) we have

$$\delta_i^k = -\frac{1}{2}A_{ii}(E_i^{k+1} - E_i^k)^2 + M_{ii}\left(J_c|E_i^{k+1}| - J_i^{k+1}E_i^{k+1} + J_i^{k+1}E_i^k - J_c|E_i^k|\right).$$

Since $E_i^{k+1} = 0$ if $J_i^{k+1} = 0$ from (4.20) and (4.21) we have

$$J_c|E_i^{k+1}| - J_i^{k+1}E_i^{k+1} = 0 \quad \text{and} \quad J_i^{k+1}E_i^k - J_c|E_i^k| \leq 0 \qquad \text{for } i = 1 \to N.$$

Noting that $A_{ii} > 0$ for $i = 1 \to N$ and using (4.22) we have

$$\sum_{i=1}^{N} \delta_i^k = \mathcal{F}(\mathbf{E}^{k+1}) - \mathcal{F}(\mathbf{E}^k) \leq -C_A \left\|\mathbf{E}^{k+1} - \mathbf{E}^k\right\|^2.$$

The bound on $\mathbf{J}^k$ follows directly from (4.20) and (4.21). □

Theorem 4.1 *The Gauss-Seidel iteration is globally convergent.*

Proof: By Lemma 4.1 we have

$$\mathcal{F}(\mathbf{E}^k) + C_A \sum_{l=0}^{k-1} ||\mathbf{E}^{l+1} - \mathbf{E}^l||^2 \leq \mathcal{F}(\mathbf{E}^0).$$

Hence for $k \geq 1$,

$$||\mathbf{E}^k|| \leq C, \quad \max_{i \in I^\Omega} |J_i^k| \leq C, \quad J_i^k = 0 \;\; i \notin I^\Omega, \quad \sum_{l=0}^{k-1} ||\mathbf{E}^{l+1} - \mathbf{E}^l||^2 \leq C,$$

where the constants C depend on $\mathbf{E}^0$. It follows that there is a subsequence labelled $\{\mathbf{E}^{k_p}\}$ such that as $k_p \to \infty$

$$\mathbf{E}^{k_p} \to \mathbf{E}^*, \quad \mathbf{E}^{k_p+1} - \mathbf{E}^{k_p} \to 0, \quad \mathbf{J}^{k_p} \to \mathbf{J}^*.$$

Clearly

$$J_i^* = 0 \quad i \notin I^\Omega, \quad |J_i^*| \le J_c \quad i \in I^\Omega, \quad \mathbf{e}^T M \mathbf{J}^* = 0.$$

Observe that

$$A\mathbf{E}^{i-1,k_p+1} = A\mathbf{E}^{k_p} + A(\mathbf{E}^{i-1,k_p+1} - \mathbf{E}^{k_p}).$$

Since $||\mathbf{E}^{k_p+1} - \mathbf{E}^{k_p}|| \to 0$ it then follows by passing to the limit in (4.19) for $k = k_p$,

$$A\mathbf{E}^* - \mathbf{b} + M\mathbf{J}^* = 0.$$

From the equivalence of (3.10a)-(3.10c) we have

$$(\mathbf{E}^{k_p})^T M^\Omega(\boldsymbol{\eta} - \mathbf{J}^{k_p}) \le 0 \quad \forall\, \boldsymbol{\eta},\ |\eta_i| \le J_c,$$

and passing to the limit we have

$$(\mathbf{E}^*)^T M^\Omega(\boldsymbol{\eta} - \mathbf{J}^*) \le 0 \quad \forall\, \boldsymbol{\eta},\ |\eta_i| \le J_c.$$

Hence $\mathbf{J}^*, \mathbf{E}^*$ solve our problem and since $\mathbf{J}^* = \mathbf{J}$ is unique, the whole sequence $\{\mathbf{J}^k\}$ converges to $\mathbf{J}$. □

5 Numerical results

In this section we report on numerical computations associated with a particular geometric configuration. We suppose that Ω is the interior of a circle of radius 0.5 that is set in an annular region Ω_I with inner radius 0.55 and outer radius 1. Contained in Ω_I are 12 symmetrically arranged components Ω_{w_i} of Ω_w. Each Ω_{w_i} is a section of an annular region with inner radius 0.55 and outer radius 0.8 sub-tending an angle $\pi/12$, see Figure 2. This geometric configuration can be used to model superconducting induction motors by viewing Ω_w as the copper windings set in an annular iron region Ω_I with a thin air gap separating Ω and Ω_I.

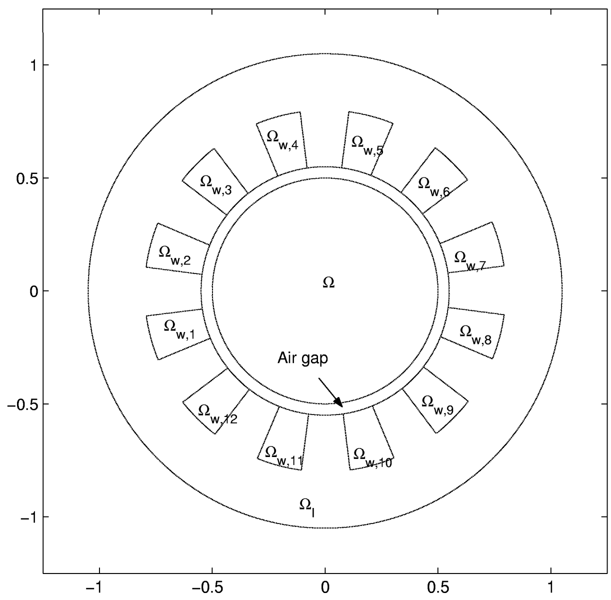

Figure 2: Geometric configuration

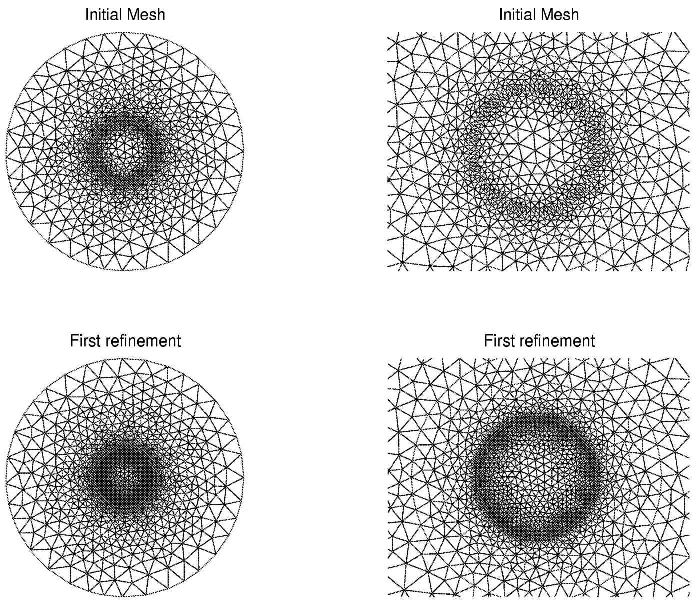

Figure 3: Initial mesh and first refinement in the superconductor.

The applied source current J_s is given by

$$J_s|_{\Omega_{w,n+1}\cup\Omega_{w,n+2}}(t) = \min(5t, 1)\cos(4t + n\pi/3),$$

for $n = 0, 2, 4$, and

$$J_s|_{\Omega_{w,n+1}\cup\Omega_{w,n+2}}(t) = -\min(5t, 1)\cos(4t + n\pi/3),$$

for $n = 6, 8, 10$. In all computations B_R has radius 2, and the critical current density $J_c = 1$.

5.1 Constant magnetic permeability

Some computations were performed for $\mu = 1$ everywhere in order to test the rate of convergence. Since an exact solution is not known the results on coarser meshes are compared with the solution on a fine mesh with a mesh size $h_{\max} \leq 1/128$. Typical meshes are shown in Figure 3.

	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
$h_{ave} \approx 1/8,\ \Delta t = 1/25$	0.0292	0.0278	0.0304	0.0319
$h_{ave} \approx 1/16,\ \Delta t = 1/50$	0.0160	0.0159	0.0166	0.0176
$h_{ave} \approx 1/32,\ \Delta t = 1/100$	0.0066	0.0075	0.0079	0.0080

Table 1: $H^{-1}(\Omega)$ errors for current density.

5.2 Piecewise constant permeability

In order to simulate the high magnetic permeability in the annular iron region Ω_I we set

$$\mu = \begin{cases} 1 & \text{in } \mathbb{R}^2\backslash\overline{\Omega}_1 \\ 10^3 & \text{in } \Omega_I. \end{cases}$$

In Figures 6 and 7 we can clearly see the effect on the amount of current in the superconductor and the insulation of the electric field as μ in the iron increases, as expected.

Note, a larger current density in the superconductor leads to a stronger magnetic field and thus a much more powerful motor.

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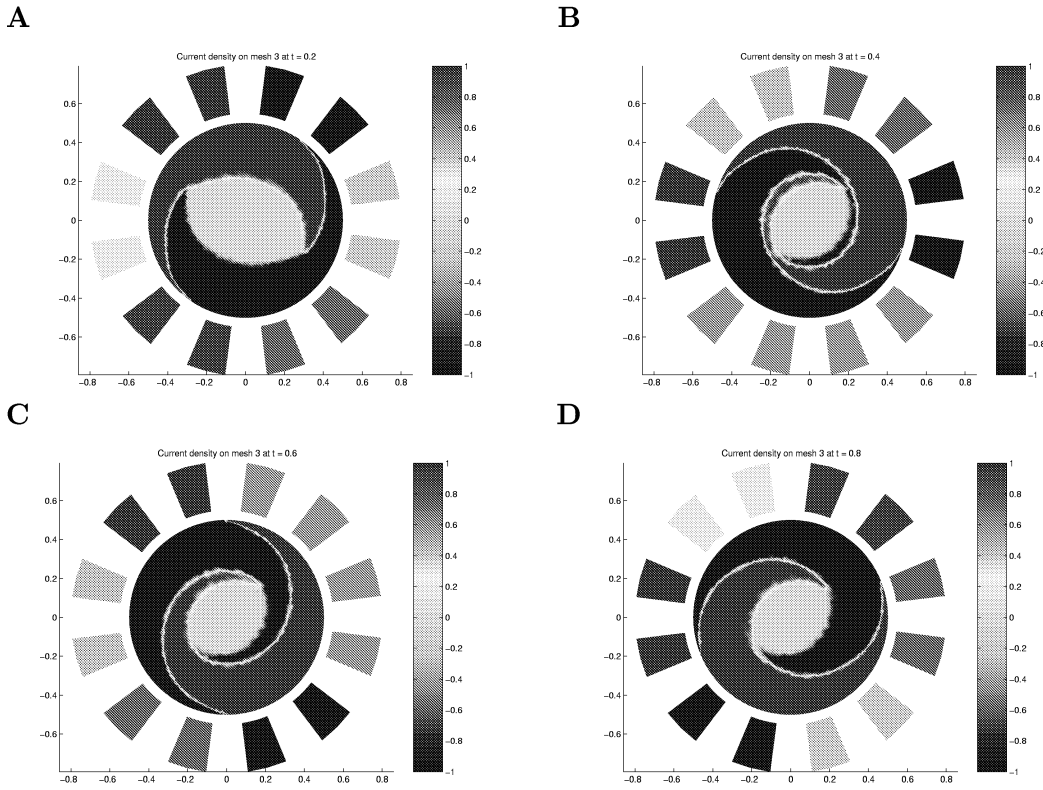

Figure 4: Current Density with $\mu = 1$

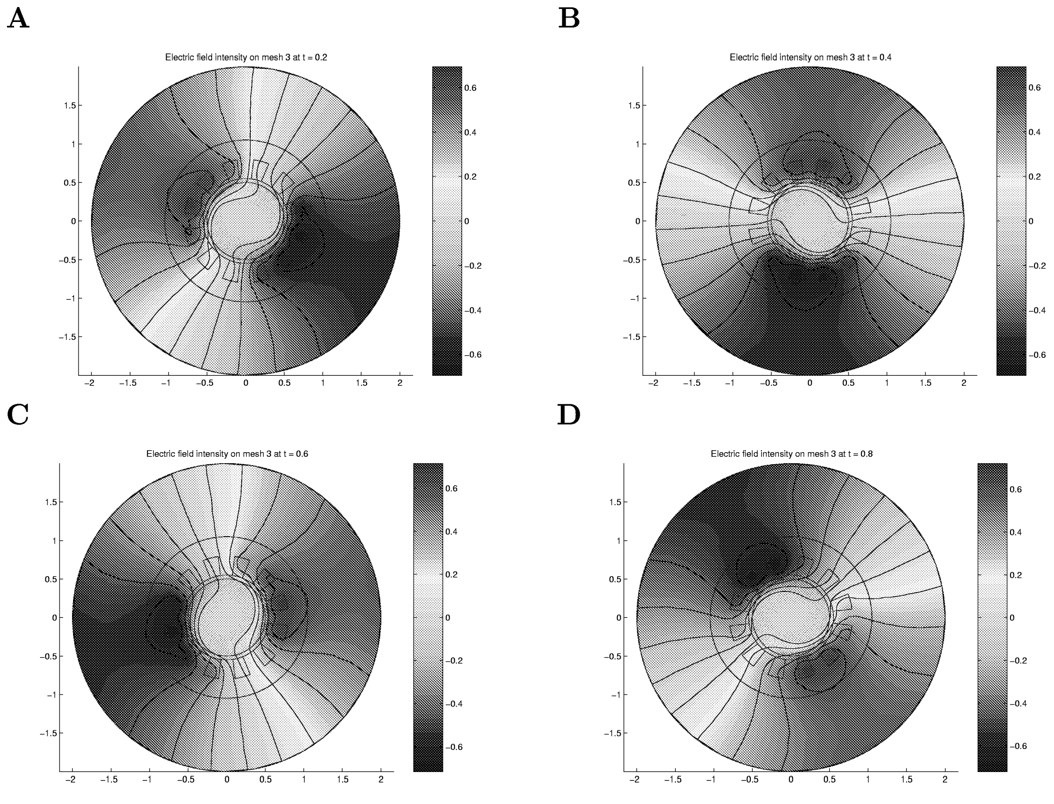

Figure 5: Electric field intensity with $\mu = 1$

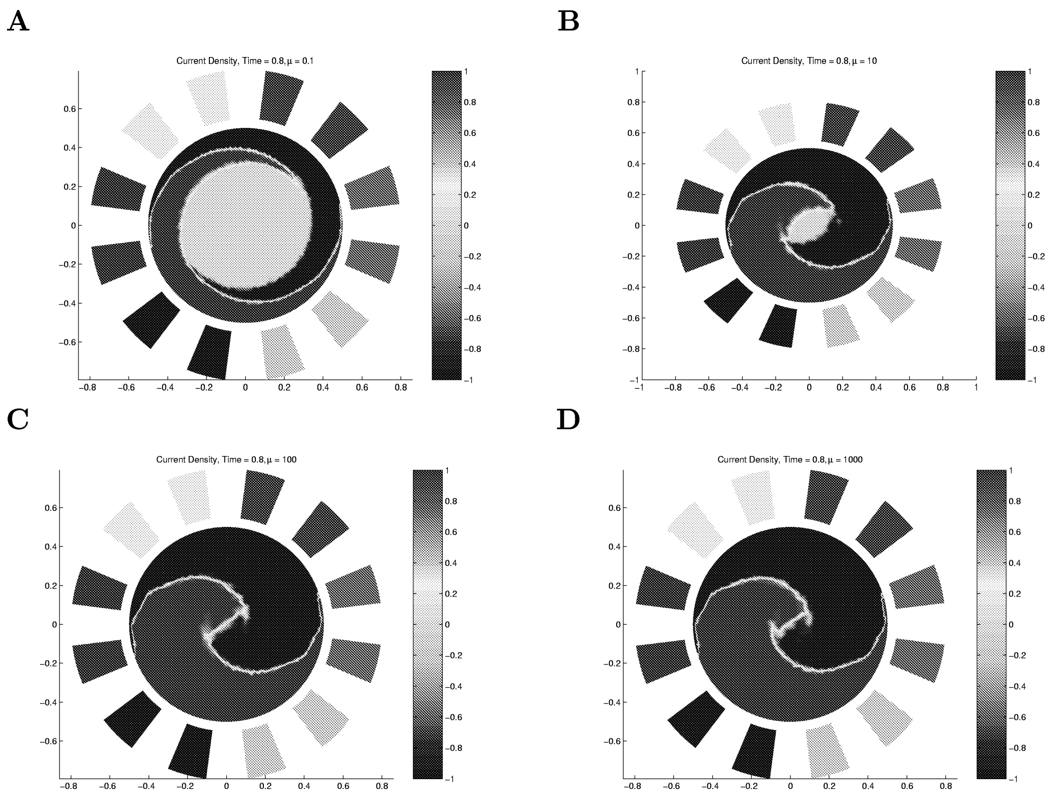

Figure 6: Current Density at time $t = 0.8$ with $\mu = 0.1, 10, 100$ and 1000

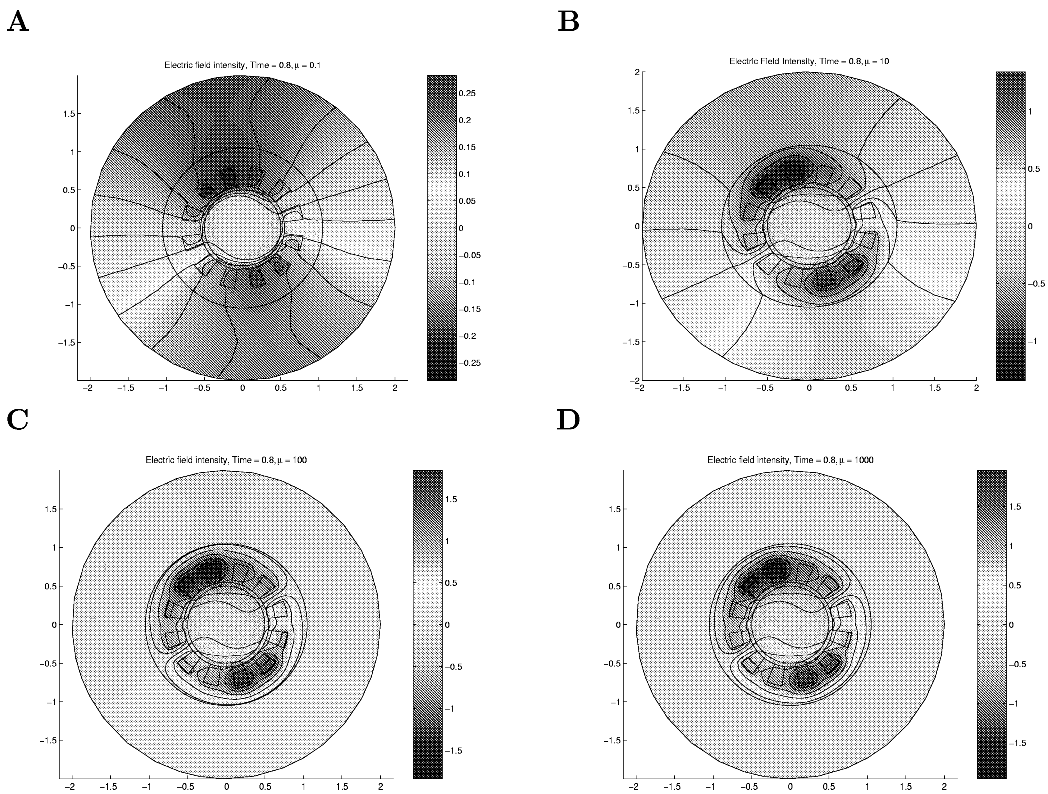

Figure 7: Electric field intensity at time $t = 0.8$ with $\mu = 0.1, 10, 100$ and 1000