CENTRED SCHEMES FOR NONLINEAR HYPERBOLIC EQUATIONS

GUI-QIANG CHEN AND ELEUTERIO F. TORO

ABSTRACT. A hierarchy of centred (non-upwind) schemes is identified for solving hyperbolic equations. The bottom of the hierarchy is the classical Lax-Friedrichs scheme, which is the least accurate, and the top of the hierarchy is the FORCE scheme, which is the optimal scheme in the family. The FORCE scheme is optimal in the sense that it is monotone, has the optimal stability condition for explicit methods, and has the smallest numerical viscosity. It is shown that the FORCE scheme is consistent with the entropy inequality, that is, the limit functions of the FORCE approximate solutions are entropy solutions. The convergence of the FORCE scheme is also established for the isentropic Euler equations and the shallow water equations. Some related centred schemes are also surveyed and discussed.

1. INTRODUCTION

We are concerned with numerical methods of the centred type (non-upwind) for computing solutions to hyperbolic equations both in conservative and non-conservative (or primitive) form. Up-to-date background information on the development of numerical methods in the last two to three decades is found, for example, in the textbooks [19], [31], [53], [54]. The most accurate methods for solving hyperbolic equations are upwind (or upstream) schemes. These require the explicit provision of wave propagation information, which is normally achieved via local solutions of the Riemann problem, approximate or exact. For most known hyperbolic systems, the exact or approximate solutions of the Riemann problem are available, although in some cases this may be very expensive to evaluate. There are systems of equations that are exceedingly complicated and for which, so far, the solutions of the Riemann problem are not, to our knowledge, available. In such circumstances, there is no option but to adopt a centred approach, in which no explicit information regarding wave propagation is used in the scheme, apart from stability constraints via a Courant (or CFL) condition, for which at least the eigenvalues of the system must be known, even if it is only numerically. But this information must in any case be available, as knowledge on the nature of the eigenvalues would inform us on the character of the equations being solved, hyperbolic or otherwise.

In this paper we identify a hierarchy of centred (non-upwind) schemes for solving hyperbolic equations in both conservative and in non-conservative form. At the bottom of the hierarchy lies the classic Lax-Friedrichs scheme, which is the least

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accurate of them all, having the largest numerical viscosity. At the top of the hierarchy of centred methods is the FORCE scheme [51]. This is the most accurate of all three-point centred methods within the considered family. Any attempts to reduce the numerical viscosity will result in a scheme that depends on wave propagation information, that is upwinding. We show that the FORCE scheme is optimal in the sense that it is the most accurate of all centred methods that (i) are monotone and (ii) have the optimal stability condition for explicit methods of Courant number unity. We establish the connection between the FORCE scheme and the first-order version of the *Staggered Lax-Friedrichs* scheme of Tadmor and collaborators [36], [24], [25], [35], [26], [27], and [39].

The FORCE flux can be used to construct very simple and general upwind numerical fluxes [55], in a multi-stage predictor-corrector fashion. Various other extensions of the first-order monotone FORCE scheme are possible. Second-order extensions are possible by following the TVD approach [20], [21], [37], and [56]. Other approaches for constructing higher order extensions of the FORCE flux include the ENO/WENO approach [22], [23], [43], [44], [45]; the ADER approach [58], [62], [42], [47], [49], and discontinuous Galerkin finite element methods [12], [13], [63]. FORCE-based schemes can also be extended to solve multi-dimensional problems in a straightforward way following established approaches [12], [13], [46], [53], [57], [63], [65].

In Section 2, we review the FORCE approach for solving hyperbolic equations both in conservative and non-conservative form, construct and analyse a family of three-point schemes and show that the FORCE scheme is the best of all centred three-point schemes in the family. We also survey some of recent efforts to extend the FORCE schemes to higher order numerical methods and to dealing with multidimensional problems. In Section 3, we show that the FORCE scheme is consistent with the entropy inequality and is convergent for the Euler equations for elasticity. Then we show the convergence and entropy-consistency of the FORCE scheme for the isentropic Euler equations in fluid dynamics in Section 4. In Section 5, we show the convergence and entropy-consistency of the fractional-step FORCE scheme for hyperbolic systems of conservation laws through a concrete model, the shallow water equations. Conclusions are drawn in Section 6.

2. The FORCE Approach for Hyperbolic Systems

2.1. **Background.** We are interested in numerical schemes for solving hyperbolic partial differential equations. In differential conservation-law form, these read

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}\left(\mathbf{Q}\right) = \mathbf{0}, \tag{2.1}$$

in which $\mathbf{Q}$ is the vector of conserved variables and $\mathbf{F} = \mathbf{F}(\mathbf{Q})$ is the vector of fluxes. In the presence of discontinuous solutions, one uses the integral form of (2.1), which is obtained, for example, by integrating (2.1) in a control volume $\mathbf{V} = [x_L, x_R] \times [t_B, t_T]$ in the $x - t$ plane, leading to

$$\int_{x_L}^{x_R} \mathbf{Q}(x, t_T) dx = \int_{x_L}^{x_R} \mathbf{Q}(x, t_B) dx - \left(\int_{t_B}^{t_T} \mathbf{F}\left(\mathbf{Q}(x_R, t)\right) dt - \int_{t_B}^{t^T} \mathbf{F}\left(\mathbf{Q}(x_L, t)\right) dt \right). \tag{2.2}$$

We can also consider the non-conservative form of the equations, namely,

$$\partial_t \mathbf{W} + \mathbf{A}\left(\mathbf{W}\right) \partial_x \mathbf{W} = \mathbf{0}, \tag{2.3}$$

in which $\mathbf{W}$ is any vector of representative variables of the system, usually called *primitive variables*, and $\mathbf{A} = \mathbf{A}(\mathbf{W})$ is the coefficient matrix. Assuming a local linearisation in (2.3), with a constant matrix $\tilde{\mathbf{A}}_{LR}$, followed by integration of (2.3) in the control volume $\mathbf{V}$ gives

(2.4)
$$\int_{x_L}^{x_R} \mathbf{Q}(x,t_T)dx = \int_{x_L}^{x_R} \mathbf{Q}(x,t_B)dx - \tilde{\mathbf{A}}_{LR}\left(\int_{t_B}^{t_T} \mathbf{Q}(x_R,t)dt - \int_{t_B}^{t_T} \mathbf{Q}(x_L,t)dt\right).$$

The methods studied in this paper have their origin in the staggered-grid version of the Random Choice Method (RCM) of Glimm [18]. A deterministic interpretation of the stochastic steps of RCM leads to a centred (non-upwind) numerical approach that is applicable to hyperbolic systems, both in conservative and in non-conservative form. RCM makes use of local solutions of Riemann problems $RP(\mathbf{Q}_L, \mathbf{Q}_R)$, where $\mathbf{Q}_L$ and $\mathbf{Q}_R$ denote the two constant states defining the initial condition for the conventional Riemann problem for system (2.1) or (2.3).

2.2. **Review of RCM on a Staggered Grid.** We first briefly review RCM. The staggered grid version of the RCM to solve (2.1) updates $\mathbf{Q}_i^n$ to a new value $\mathbf{Q}_i^{n+1}$ in two steps, as follows:

Step 1. Solve the Riemann problems $RP(\mathbf{Q}_{i-1}^n, \mathbf{Q}_i^n)$ and $RP(\mathbf{Q}_i^n, \mathbf{Q}_{i+1}^n)$ to find respective solutions

$$\hat{\mathbf{Q}}_{i-\frac{1}{2}}^{n+\frac{1}{2}}(x,t), \quad \hat{\mathbf{Q}}_{i+\frac{1}{2}}^{n+\frac{1}{2}}(x,t). \tag{2.5}$$

Random sample these solutions at a stable time $\Delta t^{n+\frac{1}{2}}$ to find the values

$$\mathbf{Q}_{i-\frac{1}{2}}^{n+\frac{1}{2}} = \hat{\mathbf{Q}}_{i-\frac{1}{2}}^{n+\frac{1}{2}}(\theta^n \Delta x, \Delta t^{n+\frac{1}{2}}), \qquad \mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \hat{\mathbf{Q}}_{i+\frac{1}{2}}^{n+\frac{1}{2}}(\theta^n \Delta x, \Delta t^{n+\frac{1}{2}}) . \tag{2.6}$$

Step 2. Solve the Riemann problem $RP(\mathbf{Q}_{i-\frac{1}{2}}^{n+\frac{1}{2}}, \mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}})$ to find the solution $\hat{\mathbf{Q}}_i^{n+1}(x,t)$ and random sample it, at a stable time Δt^{n+1}, to obtain $\mathbf{Q}_i^{n+1}$, namely,

$$\mathbf{Q}_i^{n+1} = \hat{\mathbf{Q}}_i^{n+1}(\theta^{n+1}\Delta x, \Delta t^{n+1}) . \tag{2.7}$$

The time steps $\Delta t^{n+\frac{1}{2}}$ and Δt^{n+1} must be chosen according to the usual stability restriction for the RCM and need not be the same. The symbol θ_n denotes a member of a sequence of pseudo-random numbers with some particular properties. See Figure 1. For full details of RCM, see Chapter 7 of [53].

2.3. **FORCE Schemes for Conservative Systems.** In the FORCE approach [51], [56], we replace the stochastic quantities (2.6) by the deterministic quantities

$$\mathbf{Q}_{i-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{\Delta x}\int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} \hat{\mathbf{Q}}_{i-\frac{1}{2}}^{n+\frac{1}{2}}(x, \frac{\Delta t}{2})\, dx, \tag{2.8}$$

and

$$\mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{\Delta x}\int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} \hat{\mathbf{Q}}_{i+\frac{1}{2}}(x, \frac{\Delta t}{2})\, dx , \tag{2.9}$$

where it has been assumed that $\Delta t^{n+\frac{1}{2}} = \Delta t^{n+1} = \frac{1}{2}\Delta t$. Then we apply the integral form of the conservation laws (2.2) to (2.8) and (2.9), in the control volumes $\mathbf{V}_{i-\frac{1}{2}} =$

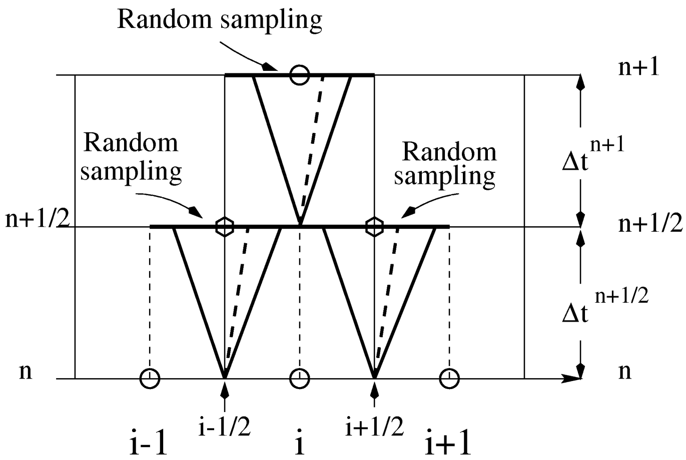

FIGURE 1. Illustration of the Random Choice Method on a staggered grid.

$[x_{i-\frac{1}{2}}-\frac{1}{2}\Delta x, x_{i-\frac{1}{2}}+\frac{1}{2}\Delta x]\times[0,\frac{1}{2}\Delta t]$ and $\mathbf{V}_{i+\frac{1}{2}} = [x_{i+\frac{1}{2}}-\frac{1}{2}\Delta x, x_{i+\frac{1}{2}}+\frac{1}{2}\Delta x]\times[0,\frac{1}{2}\Delta t]$, respectively. We obtain

$$\mathbf{Q}_{i-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{Q}_{i-1}^{n}+\mathbf{Q}_{i}^{n}\right) - \frac{\Delta t}{2\Delta x}\left(\mathbf{F}_{i}^{n}-\mathbf{F}_{i-1}^{n}\right), \tag{2.10}$$

and

$$\mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{Q}_{i}^{n}+\mathbf{Q}_{i+1}^{n}) - \frac{\Delta t}{2\Delta x}\left(\mathbf{F}_{i+1}^{n}-\mathbf{F}_{i}^{n}\right), \tag{2.11}$$

where $\mathbf{F}_{k}^{n} = \mathbf{F}(\mathbf{Q}_{k}^{n})$. Denoting by $\hat{\mathbf{Q}}_i(x,t)$ the solution of the Riemann problem $RP(\mathbf{Q}_{i-\frac{1}{2}}^{n+\frac{1}{2}}, \mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}})$, we now define an integral average $\mathbf{Q}_i^{n+1}$ of $\hat{\mathbf{Q}}_i(x,t)$ at the complete time step Δt:

$$\mathbf{Q}_{i}^{n+1} = \frac{1}{\Delta x}\int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} \hat{\mathbf{Q}}_i(x,\frac{1}{2}\Delta t)\,dx. \tag{2.12}$$

This is the deterministic version of (2.7) which, by virtue of the integral form of the conservation laws, (2.2) becomes

$$\mathbf{Q}_{i}^{n+1} = \frac{1}{2}\left(\mathbf{Q}_{i-\frac{1}{2}}^{n+\frac{1}{2}}+\mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right) - \frac{\Delta t}{2\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}}^{LW}-\mathbf{F}_{i-\frac{1}{2}}^{LW}\right), \tag{2.13}$$

where $\mathbf{F}_{i-\frac{1}{2}}^{\mathrm{LW}} = \mathbf{F}(\mathbf{Q}_{i-\frac{1}{2}}^{n+\frac{1}{2}})$ and $\mathbf{F}_{i+\frac{1}{2}}^{\mathrm{LW}} = \mathbf{F}(\mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}})$. As a matter of fact $\mathbf{F}_{i+\frac{1}{2}}^{\mathrm{LW}}$ is the two-step Lax-Wendroff flux. In conservation form, the scheme (2.13) reads

$$\mathbf{Q}_{i}^{n+1} = \mathbf{Q}_{i}^{n} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}}-\mathbf{F}_{i-\frac{1}{2}}\right), \tag{2.14}$$

with intercell numerical flux

$$\mathbf{F}_{i+\frac{1}{2}}^{\mathrm{force}} = \frac{1}{4}\left(\mathbf{F}_{i}^{n}+2\mathbf{F}_{i+\frac{1}{2}}^{LW}+\mathbf{F}_{i+1}^{n}-\frac{\Delta x}{\Delta t}\left(\mathbf{Q}_{i+1}^{n}-\mathbf{Q}_{i}^{n}\right)\right). \tag{2.15}$$

A surprising outcome is that the intercell flux (2.15) is in fact the arithmetic mean of the fluxes for the two-step Lax-Wendroff and Lax–Friedrichs schemes, namely,

$$\mathbf{F}^{\text{force}}_{i+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{F}^{\text{LW}}_{i+\frac{1}{2}} + \mathbf{F}^{\text{LF}}_{i+\frac{1}{2}}\right). \tag{2.16}$$

The resulting scheme (2.14)–(2.15) is in conservative form in a non-staggered grid, it is monotone and has the linear stability condition $|c| \leq 1$, where c is the Courant number, as discussed in Section 3, and the scheme is, in a sense to be defined, optimal. At this point, it is also interesting to discuss the connection between the FORCE scheme (2.14)–(2.15) and the first-order version of the *Staggered Lax-Friedrichs* scheme of Tadmor and collaborators [36], [24], [25], [35], [26], [27], [39]. The first-order version of their scheme is precisely the two-step, or staggered, scheme (2.11), (2.13), as obtained from a deterministic interpretation of the staggered-grid random choice method of Glimm [18]. The staggered-grid scheme (2.11), (2.13) advances the solution in two steps and has the stability restriction $|c| \leq 1/2$ at each step. In the next section, we review non-conservative versions of the FORCE scheme.

2.4. **FORCE-Type Non-Conservative Schemes.** The FORCE approach can also be applied to a non-conservative system of the form (2.3). This allows the construction of centred, non-conservative (or primitive) schemes. For details, see [59]. The stochastic quantities (2.6) are replaced by the integral averages:

$$\mathbf{W}^{n+\frac{1}{2}}_{i-\frac{1}{2}} = \frac{1}{\Delta x}\int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} \hat{\mathbf{W}}^{n+\frac{1}{2}}_{i-\frac{1}{2}}(x, \frac{\Delta t}{2})\, dx, \tag{2.17}$$

and

$$\mathbf{W}^{n+\frac{1}{2}}_{i+\frac{1}{2}} = \frac{1}{\Delta x}\int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} \hat{\mathbf{W}}^{n+\frac{1}{2}}_{i+\frac{1}{2}}(x, \frac{\Delta t}{2})\, dx. \tag{2.18}$$

Then we apply the integral form (2.4) of the locally linearised version of the non-conservative equations (2.3) on control volumes $\mathbf{V}_{i-\frac{1}{2}} = [x_{i-\frac{1}{2}} - \frac{1}{2}\Delta x, x_{i-\frac{1}{2}} + \frac{1}{2}\Delta x] \times [0, \frac{1}{2}\Delta t]$ and $\mathbf{V}_{i+\frac{1}{2}} = [x_{i+\frac{1}{2}} - \frac{1}{2}\Delta x, x_{i+\frac{1}{2}} + \frac{1}{2}\Delta x] \times [0, \frac{1}{2}\Delta t]$. The result is

$$\mathbf{W}^{n+\frac{1}{2}}_{i-\frac{1}{2}} = \frac{1}{2}(\mathbf{W}^n_{i-1} + \mathbf{W}^n_i) - \frac{1}{2}\frac{\Delta t}{\Delta x}\tilde{\mathbf{A}}_{\mathbf{i}-\frac{1}{2}}\left(\mathbf{W}^n_i - \mathbf{W}^n_{i-1}\right), \tag{2.19}$$

and

$$\mathbf{W}^{n+\frac{1}{2}}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{W}^n_i + \mathbf{W}^n_{i+1}) - \frac{1}{2}\frac{\Delta t}{\Delta x}\tilde{\mathbf{A}}_{\mathbf{i}+\frac{1}{2}}\left(\mathbf{W}^n_{i+1} - \mathbf{W}^n_i\right). \tag{2.20}$$

Denoting by $\hat{\mathbf{W}}_i(x,t)$ the solution of the Riemann problem $RP(\mathbf{W}^{n+\frac{1}{2}}_{i-\frac{1}{2}}, \mathbf{W}^{n+\frac{1}{2}}_{i+\frac{1}{2}})$, we then define an average $\mathbf{W}^{n+1}_i$ of $\hat{\mathbf{W}}_i(x,t)$, namely,

$$\mathbf{W}^{n+1}_i = \frac{1}{\Delta x}\int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} \hat{\mathbf{W}}_i(x, \frac{1}{2}\Delta t)\, dx. \tag{2.21}$$

Application of the integral form (2.4) of the linearised version of (2.3) gives

$$\mathbf{W}^{n+1}_i = \frac{1}{2}(\mathbf{W}^{n+\frac{1}{2}}_{i-\frac{1}{2}} + \mathbf{W}^{n+\frac{1}{2}}_{i+\frac{1}{2}}) - \frac{1}{2}\frac{\Delta t}{\Delta x}\tilde{\mathbf{A}}_{\mathbf{i}}\left(\mathbf{W}^{n+\frac{1}{2}}_{i+\frac{1}{2}} - \mathbf{W}^{n+\frac{1}{2}}_{i-\frac{1}{2}}\right). \tag{2.22}$$

In the linearisation of (2.3), the constant coefficient matrices $\tilde{\mathbf{A}}_{i+\frac{1}{2}}$ in (2.20) and $\tilde{\mathbf{A}}_i$ in (2.22) are respectively given as

$$\tilde{\mathbf{A}}_{\mathbf{i}+\frac{1}{2}} = \mathbf{A}\left(\frac{1}{2}\left(\mathbf{W}_{i-1}^n + \mathbf{W}_{i+1}^n\right)\right), \quad \tilde{\mathbf{A}}_i = \mathbf{A}\left(\frac{1}{2}\left(\mathbf{W}_{i-\frac{1}{2}}^{n+\frac{1}{2}} + \mathbf{W}_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right)\right). \tag{2.23}$$

2.5. **Some Basic Properties of the FORCE Scheme.** For the purpose of the analysis of the FORCE scheme, we now consider the model linear advection equation

$$\partial_t q + \partial_x f(q) = 0 , \quad f(q) = \lambda q , \tag{2.24}$$

where λ is a constant wave propagation speed. The FORCE approach, when applied to (2.24), gives the three-point scheme

$$q_i^{n+1} = b_{-1} q_{i-1}^n + b_0 q_i^n + b_1 q_{i+1}^n , \tag{2.25}$$

with coefficients given as

$$b_{-1} = \frac{1}{4}(1+c)^2 , \; b_0 = \frac{1}{2}(1-c^2) , \; b_1 = \frac{1}{4}(1-c)^2 , \tag{2.26}$$

which is a convex combination: $b_{-1} + b_0 + b_1 = 1, b_j \geq 0, j = -1, 0, 1$.

Proposition (Stability, Monotonicity, and Modified Equation). The FORCE scheme (2.25)–(2.26)

- is conditionally stable, with the stability condition:

$$0 \leq | c | \leq 1 , \quad c = \frac{\Delta t \lambda}{\Delta x} : \text{Courant Number}; \tag{2.27}$$

- is monotone;
- has the modified equation:

$$q_t + \lambda q_x = \alpha_{\text{force}} q_{xx} \qquad \alpha_{\text{force}} = \frac{1}{4}\lambda \Delta x \left(\frac{1-c^2}{c}\right) = \frac{1}{2}\alpha_{\text{lf}} , \tag{2.28}$$

  where α_{lf} is the coefficient of artificial viscosity for the Lax–Friedrichs scheme.

The proof can be found in [56]. In addition, we note that, for a non-linear system (2.1), it is obvious that the FORCE flux (2.15) is consistent. A more general analysis of the FORCE scheme is carried out in Sections 3 and 4.

2.6. **A Numerical Map for Three-Point Schemes.** It is instructive to consider the family of three-point schemes of the form (2.25) with general coefficients b_{-1}, b_0, and b_1 for solving the linear advection equation (2.24). We consider the conservative version of the schemes

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta t}\left(f_{i+\frac{1}{2}}^{rus} - f_{i-\frac{1}{2}}^{rus}\right), \tag{2.29}$$

with an *upwind* numerical flux of the Rusanov type [41]

$$f_{i+\frac{1}{2}}^{rus} = \frac{1}{2}(f_i^n + f_{i+1}^n) - \frac{1}{2}S^+(f_{i+1}^n - f_i^n), \tag{2.30}$$

in which S^+ is an estimate for λ, the exact wave propagation speed in the differential equation (2.24). This flux may be expressed as

$$f_{i+\frac{1}{2}} = \frac{1}{2}(1+\Gamma)f_i^n + \frac{1}{2}(1-\Gamma)f_{i+1}^n , \quad \Gamma = \frac{S^+}{\lambda} , \tag{2.31}$$

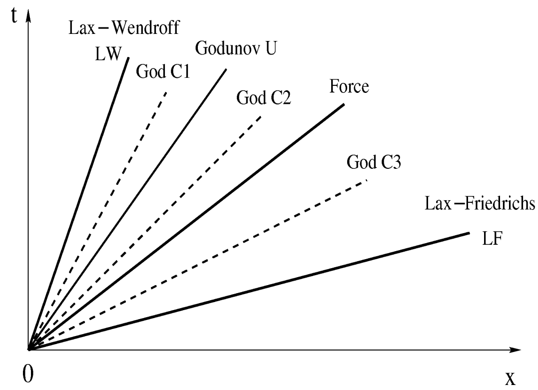

FIGURE 2. Numerical map for explicit, stable three-point schemes. All schemes lie in the wedge $LWOLF$ between the Lax–Wendroff and the Lax–Friedrichs schemes.

If the estimate is actually exact, $\Gamma = 1$, then we reproduce the first-order Godunov upwind method, with the highest level of upwinding consistent with monotonicity. Other levels of upwinding will result from other (possibly inaccurate) estimates of S^+ and thus of the ratio Γ. Various familiar schemes may be reproduced by appropriate choices of Γ. There are two extreme values consistent with stability, and in general the following must be satisfied:

$$\Gamma_{LW} \equiv c \le \Gamma \le \Gamma_{LF} \equiv \frac{1}{c} \,. \tag{2.32}$$

The upper bound $\Gamma_{LF} \equiv 1/c$ reproduces the Lax-Friedrichs scheme and the lower bound $\Gamma_{LW} \equiv c$ reproduces the Lax-Wendroff method.

Fig. 2 shows a *numerical map* in the x-t plane of admissible three-point schemes (2.25). They all lie in the wedge $LWOLF$ between the Lax-Wendroff and the Lax-Friedrichs schemes. In particular, the Godunov upwind method, denoted by Godunov U, obviously, lies between the extreme schemes.

The FORCE scheme is obtained when

$$\Gamma_{force} \equiv \frac{1+c^2}{2c} \tag{2.33}$$

and is located in a fixed position in the wedge $LWOLF$. The first-order Godunov centred scheme corresponds to the choice

$$\Gamma_{GodC} \equiv \frac{1-2c^2}{(1-c)(1+c)} \tag{2.34}$$

and does not have a fixed position in the wedge of Fig. 2, its position depends on the Courant number c. Recall that GodC has the restricted linearised stability condition

$$0 \le |c| \le \frac{1}{2}\sqrt{2} \,. \tag{2.35}$$

which is entirely consistent with the condition $\Gamma \ge 0$ and Fig. 2. Note also that for $0 \le c \le \frac{1}{2}$ the Godunov centred scheme lies between the Lax-Wendroff and the Godunov upwind scheme and that within this range the GodC scheme is *not monotone*. For $\frac{1}{2} \le c \le \frac{1}{3}\sqrt{3}$, God C lies between the Godunov upwind method

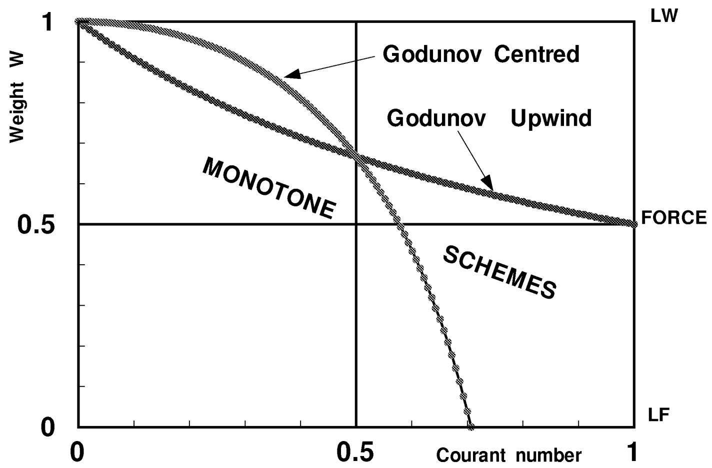

FIGURE 3. Three-point schemes expressed as convex averages of the Lax-Friedrichs scheme (bottom) and the Lax-Wendroff scheme (top). The Godunov upwind scheme divides the square into a region of monotone schemes (bottom part) and non-monotone schemes (top part). The FORCE scheme is the optimal average with constant weight $\omega = 1/2$ independent of c.

and the FORCE scheme. Finally, for $\frac{1}{3}\sqrt{3} \le c \le \frac{1}{2}\sqrt{2}$, God C lies between the FORCE and the Lax-Friedrichs schemes.

Any three-point scheme (2.25) has a choice of Γ and thus an explicit or implicit choice of wave speed S^+. Large values of Γ, and thus of S^+, result in more diffusive methods; schemes with lower values of Γ are less diffusive but at the cost of losing monotonicity.

2.7. **FORCE: The Optimal Centred Scheme.** We start by considering the class of all three point schemes contained in the wedge $LWOLF$ of Fig. 2. These can conveniently be interpreted as convex averages of the two boundary schemes, namely, the Lax-Wendroff method on the left and the Lax-Friedrichs method on the right, that is,

$$f^{tps}_{i+\frac{1}{2}} = \omega f^{lw}_{i+\frac{1}{2}} + (1-\omega) f^{lf}_{i+\frac{1}{2}} \ , \tag{2.36}$$

where the superscripts lw and lf identify the Lax-Wendroff and Lax-Friedrichs fluxes, respectively. Manipulations of (2.25) give

$$f^{tps}_{i+\frac{1}{2}} = \frac{1+c}{2c}\left((c-1)\omega+1\right)\lambda q^n_i + \frac{1-c}{2c}\left((c+1)\omega-1\right)\lambda q^n_{i+1} \ . \tag{2.37}$$

Insertion of the numerical fluxes into the conservative formula gives the three-point scheme

$$q^{n+1}_i = b_{-1} q^n_{i-1} + b_0 q^n_i + b_1 q^n_{i+1} \ , \tag{2.38}$$

with coefficients given as
(2.39)

$$b_{-1} = \frac{1+c}{2}\left((c-1)\omega+1\right), \quad b_0 = (1-c)(1+c)\omega, \quad b_1 = -\frac{1-c}{2c}\left((c+1)\omega-1\right).$$

Monotonicity, Stability, and Modified Equation. Monotonicity is assured from positivity (non-negativity) of all coefficients b_k in the scheme (2.25). First note that

$$b_{-1} = \frac{1+c}{2}\left((c-1)\omega + 1\right) \geq 0 \tag{2.40}$$

implies the condition

$$\omega \leq \frac{1}{1-c}, \tag{2.41}$$

and

$$b_0 = (1-c)(1+c) \geq 0 \tag{2.42}$$

implies the condition

$$\omega \geq 0. \tag{2.43}$$

Finally, the condition

$$b_1 = -\frac{1-c}{2c}\left((c+1)\omega - 1\right) \geq 0 \tag{2.44}$$

implies

$$\omega \leq \frac{1}{1+c}. \tag{2.45}$$

In summary, we have that the monotone schemes satisfy

$$0 \leq \omega \leq \omega_{max} \equiv \frac{1}{1+|c|}\ , \quad \frac{1}{2} \leq \omega_{max} \leq 1\ . \tag{2.46}$$

Fig. 3 shows the unit square in the c-ω plane containing all three-point schemes that can be written as a convex average between the Lax-Friedrichs scheme (bottom) and the Lax-Wendroff (top). The bottom boundary $\omega = 0$ corresponds to the Lax-Friedrichs method (monotone) and the top line $\omega = 1$ corresponds to the Lax-Wendroff method (non-monotone). The curve $\omega_{max}(c) = \frac{1}{1+|c|}$ corresponds to the first-order Godunov upwind method and divides the complete family of three-point schemes into two classes: the class of monotone schemes, which corresponds to values of ω below $\omega_{max}(c)$ and the family of non-monotone schemes corresponding to values of ω above $\omega_{max}(c)$. The first-order Godunov centred scheme corresponds to the choice $\omega_{max}(c) = \frac{2c^2-1}{1-c^2}$. In Fig. 3, we verify that this scheme is non-monotone in the range $0 \leq |c| \leq \frac{1}{2}$, it is monotone for the range $\frac{1}{2} \leq |c| \leq \frac{1}{2}\sqrt{2}$ and unstable for $|c| > \frac{1}{2}\sqrt{2}$.

The uniform boundedness for scalar conservation laws follows from the monotonicity:

$$|q_i^{n+1}| \leq \max_{i-1 \leq j \leq i+1} |q_j^0|.$$

Standard analysis shows that the modified equation for the schemes under study has the form

$$q_t + \lambda q_x = \alpha_{tps} q_{xx}\ , \tag{2.47}$$

with coefficient of numerical viscosity given by

$$\alpha_{\rm tps} = \frac{\lambda \Delta x}{2c}(1-c^2)(1-\omega). \tag{2.48}$$

2.8. **Extended FORCE Schemes for Hyperbolic Systems.** Here we discuss possible extensions of the FORCE schemes.

2.8.1. *Source Terms.* We consider a hyperbolic system of conservation laws:

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{S}(\mathbf{Q}, x) \ , \tag{2.49}$$

where $\mathbf{S}(\mathbf{Q}, x)$ is continuous with respect to $(\mathbf{Q}, x)$. Then we can construct a fractional-step FORCE scheme to compute solutions of (2.49).

In the first step we use the FORCE scheme for the homogeneous case:

$$\mathbf{Q}_i^* = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}}^{force} - \mathbf{F}_{i-\frac{1}{2}}^{force}\right) \ ,$$

and for the second step we solve an ODE system:

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^* + \Delta t \mathbf{S}(Q_i^*, x_i)$$

Combining them together we obtain the fractional step FORCE scheme:

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}}^{force} - \mathbf{F}_{i-\frac{1}{2}}^{force}\right) + \Delta t \mathbf{S}(Q_i^*, x_i).$$

In Section 4 we will show the convergence of the fractional-step FORCE scheme through the system of shallow water equations.

2.8.2. *High-Order Extensions of the FORCE Flux.* The simplest approach to extend the first-order monotone FORCE scheme (2.14)–(2.15) to second order of accuracy is by following the TVD approach [20], [21], [37]. In [56], two ways of constructing TVD centred schemes based on the FORCE flux are presented. The first approach is the so-called *flux-limiter* approach, whereby use is made of a lower-order (monotone) flux as the building block and a higher-order flux. The FORCE and the Lax-Wendroff fluxes are chosen as the lower-order and higher-order fluxes, respectively. A flux limiter is derived from within a TVD region which is obtained after appropriate TVD conditions for *centred* schemes were constructed. The resulting scheme is called **FLIC**. Another way of constructing second-order TVD extensions of the FORCE flux is via the MUSCL approach and *slope limiters*. The resulting scheme is called **SLIC**. Other approaches for constructing higher order extensions of the FORCE flux include the ENO/WENO approach [22], [23], [43], [44], [45]; the ADER approach [58], [62], [49], [42], [47], and discontinuous Galerkin finite element methods [12], [13], [63]. Further centred-based high-order schemes can be found in [29], [33], [34], [36], [24], [25], [35], [26], [27], [66], [50], [39], and the references cited therein.

2.8.3. *Multi-Dimensional Problems.* The most straightforward way of extending schemes based on the FORCE flux to multidimensional problems is by operator splitting or methods of fractional steps [46], [65]. Details of this approach for any one-dimensional scheme are given in [53], which works reasonably well for structured meshes in which mesh distortions are not large. Another more attractive method of extending FORCE-type schemes is via an unsplit finite volume method. For details, see [57]. The discontinuous Galerkin finite element method [12], [13], [63] would be another way of extending the FORCE flux, both to higher order of accuracy and to multi-dimensional problems. Other centred-based schemes for multidimensional problems can be found in [29], [33], [34], [36], [24], [25], [35], [26], [27], [66], [39], and the references cited therein.

2.8.4. *Upwind Fluxes via Centred FORCE Fluxes.* A very simple and general approach was proposed in [55] to construct upwind numerical fluxes, whereby centred fluxes are utilised in a multi-stage predictor-corrector fashion. A particularly successful scheme, constructed on the basis of the centred FORCE flux, is summarised here. Consider any hyperbolic system of the form (2.1) and given two states $\mathbf{Q}_i^n$ and $\mathbf{Q}_{i+1}^n$ to the left and right of the interface at $x = x_{i+\frac{1}{2}}$, an upwind numerical flux is computed in the following manner. The scheme has essentially two steps and is started by setting $l = 1$, $\mathbf{Q}_i^{(1)} = \mathbf{Q}_i^n$ and $\mathbf{Q}_{i+1}^{(1)} = \mathbf{Q}_{i+1}^n$. Then we do

(1) Flux evaluation:

$$\mathbf{F}_i^{(l)} = \mathbf{F}(\mathbf{Q}_i^{(l)}) \ , \ \ \mathbf{F}_{i+1}^{(l)} = \mathbf{F}(\mathbf{Q}_{i+1}^{(l)}) \ ,$$

$$\mathbf{Q}_{i+\frac{1}{2}}^{(l)} = \frac{1}{2}(\mathbf{Q}_i^{(l)} + \mathbf{Q}_{i+1}^{(l)}) - \frac{1}{2}\frac{\Delta t}{\Delta x}(\mathbf{F}_{i+1}^{(l)} - \mathbf{F}_i^{(l)}) \ , \ \ \mathbf{F}_M^{(l)} = \mathbf{F}(\mathbf{Q}_{i+\frac{1}{2}}^{(l)}) \ ,$$

$$\mathbf{F}_{i+\frac{1}{2}}^{(l)} = \frac{1}{4}\left(\mathbf{F}_i^{(l)} + 2\mathbf{F}_M^{(l)} + \mathbf{F}_{i+1}^{(l)} - \frac{\Delta x}{\Delta t}\left(\mathbf{Q}_{i+1}^{(l)} - \mathbf{Q}_i^{(l)}\right)\right) ;$$

(2) Open Riemann fan:

$$\mathbf{Q}_i^{(l+1)} = \mathbf{Q}_i^{(l)} - \frac{\Delta t}{\Delta x}(\mathbf{F}_{i+\frac{1}{2}}^{(l)} - \mathbf{F}_i^{(l)}) \ , \ \ \mathbf{Q}_{i+1}^{(l+1)} = \mathbf{Q}_{i+1}^{(l)} - \frac{\Delta t}{\Delta x}(\mathbf{F}_{i+1}^{(l)} - \mathbf{F}_{i+\frac{1}{2}}^{(l)}) ;$$

(3) Go to Step 1.

The procedure is stopped at the end of Step 1 if the desired number of stages k has been reached. Practical experience suggests that a number of $k = 3$ stages gives numerical results that are comparable with those from the most accurate of fluxes, namely, the first-order Glodunov upwind flux used in conjunction with the exact Riemann solver. Concerning efficiency, it is found that, for the one-dimensional Euler equations for ideal gases, such a scheme is comparable to typical existing approximate Riemann solvers, such as Roe's solver [40] or the HLLC solver [60], for example, but much more efficient than the Osher-Solomon Riemann solver [38]. The advantage of the multi-stage predictor-corrector solver is its simplicity and generality and will be fully realised when solving very complex hyperbolic systems such as those arising in multi-phase flows, magnetohydrodynamics, and general relativity.

Another approach is to combine the FORCE schemes with entropy flux splittings developed in [7] and [8] so that upwind fluxes can directly be derived from the centred FORCE fluxes for the splitting systems, since the centred FORCE fluxes for the splitting systems are automatically upwind fluxes which share the same physical entropy functions.

3. Convergence and Entropy-Consistency of the FORCE Scheme for Strictly Hyperbolic Systems

Now we illustrate in this section and Sections 4–5 how the convergence and entropy-consistency of the FORCE scheme can be achieved via compactness arguments and numerical entropy dissipation for the scheme for hyperbolic systems of conservation laws.

Consider the Cauchy problem for a strictly hyperbolic system of conservation laws:

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = 0, \qquad \mathbf{Q} \in \mathbb{R}^m, \tag{3.1}$$

with initial data

$$\mathbf{Q}|_{t=0} = \mathbf{Q}_0(x), \tag{3.2}$$

endowed with a strictly convex entropy $\eta_*(\mathbf{Q})$ and corresponding entropy flux $q_*(\mathbf{Q})$, in some physical region $\mathcal{V} \subset \mathbb{R}^m$.

3.1. **FORCE Approximate Solutions.** The FORCE scheme is a three-point scheme and provides a simple digital way to calculate entropy solutions avoiding the use of Riemann solvers. On the other hand, this scheme can be interpreted through Riemann solutions as explained in Section 2. Now we use this interpretation to construct approximate solutions for hyperbolic systems of conservation laws.

As every difference scheme, the FORCE scheme satisfies the property of propagation with finite speed, which is an advantage over the vanishing viscosity method: the convergence result applies without assumption on the decay of initial data at infinity. We now construct the family of FORCE approximate solutions $\mathbf{Q}^\ell(x,t)$, similar to these for the Glimm scheme. The FORCE scheme can be interpreted to be based on a regular partition of the half-plane $t \geq 0$ defined by $t_n = n\,\Delta t$, $x_i = i\,\ell$ for $n \in \mathcal{N}_+$, $i \in \mathcal{N}$, where Δt and ℓ are the sizes of time-step and space-step, respectively. It is assumed that the ratio $\Delta t/\ell$ is constant and satisfies the Courant-Friedrichs-Lewy (CFL) stability condition:

$$\frac{\Delta t}{\ell}\|\lambda_j(\mathbf{Q}\ell)\|_{L^\infty} \leq 1,$$

where $\lambda_j(\mathbf{Q})$ are the eigenvalues of $\nabla\mathbf{F}(\mathbf{Q})$, that is, the CFL number is 1.

Each strip $\{(x,t)\,:\, t_n \leq t < t_{n+1}\}$ is divided into two substrips, $\{(x,t)\,:\, t_n \leq t < t_{n+1/2}\}$ and $\{(x,t)\,:\, t_{n+1/2} \leq t < t_{n+1}\}$.

In the first strip $\{(x,t)\,:\, 0 \leq t < t_1\}$, we can construct the approximate solutions as follows:

In the first substrip $\{(x,t)\,:\, x_i < x < x_{i+1},\ 0 \leq t < t_{1/2},\ i \in \mathcal{N}\}$, we define $\mathbf{Q}^\ell(x,t)$ by solving a sequence of Riemann problems for (3.1) corresponding to the Riemann data:

$$\mathbf{Q}^\ell(x,0) = \begin{cases} \mathbf{Q}_i^0, & x < x_{i+1/2}, \\ \mathbf{Q}_{i+1}^0, & x > x_{i+1/2}, \end{cases}$$

with

$$\mathbf{Q}_i^0 = \frac{1}{\ell}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}_0(x)\,dx.$$

We set

$$\mathbf{Q}_{i+1/2}^{1/2} = \frac{1}{\ell}\int_{x_i}^{x_{i+1}} \mathbf{Q}^\ell(x, t_{1/2}-0)\,dx.$$

In the second substrip $\{(x,t)\,:\, x_i < x < x_{i+1},\ t_{1/2} \leq t < t_1,\ i \in \mathcal{N}\}$, we define $\mathbf{Q}^\ell(x,t)$ by solving a sequence of Riemann problems for (3.1) corresponding to the Riemann data:

$$\mathbf{Q}^\ell(x,0) = \begin{cases} \mathbf{Q}_{i-1/2}^{1/2}, & x < x_i, \\ \mathbf{Q}_{i+1/2}^{1/2}, & x > x_i. \end{cases}$$

Then the FORCE scheme at $t = t_1$ is

$$\mathbf{Q}_i^1 = \frac{1}{\ell}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}^\ell(x, t_1 - 0)\,dx.$$

When $\mathbf{Q}^\ell(x,t)$ is defined for $t < t_n$, we set

$$\mathbf{Q}_i^n = \frac{1}{\ell}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}^\ell(x, t_n - 0)\, dx.$$

In the substrip $\{(x,t) \,:\, x_i < x < x_{i+1}, t_n < t < t_{n+1/2}, i \in \mathcal{N}\}$, we define $\mathbf{Q}^\ell(x,t)$ by solving the Riemann problems with the data:

$$\mathbf{Q}^\ell(x,t_n) = \begin{cases} \mathbf{Q}_i^n, & x < x_i, \\ \mathbf{Q}_{i+1}^n, & x > x_i. \end{cases}$$

We set

$$\mathbf{Q}_{i+1/2}^{n+1/2} = \frac{1}{\ell}\int_{x_i}^{x_{i+1}} \mathbf{Q}^\ell(x, t_{n+1/2} - 0)\, dx.$$

In the second substrip $\{(x,t) \,:\, x_i < x < x_{i+1},\ t_{n+1/2} \le t < t_{n+1},\ i \in \mathcal{N}\}$, we define $\mathbf{Q}^\ell(x,t)$ by solving a sequence of Riemann problems for (4.1) corresponding to the Riemann data:

$$\mathbf{Q}^\ell(x,t_{n+1}) = \begin{cases} \mathbf{Q}_{i-1/2}^{n+1/2}, & x < x_i, \\ \mathbf{Q}_{i+1/2}^{n+1/2}, & x > x_i. \end{cases}$$

Then the FORCE scheme at $t = t_n$ is

$$\mathbf{Q}_i^{n+1} = \frac{1}{\ell}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}^\ell(x, t_{n+1} - 0)\, dx.$$

This completes the construction of the FORCE approximate solutions $\mathbf{Q}^\ell(x,t)$ for which $\{\mathbf{Q}_i^n\}$ is the FORCE scheme interpreted via Riemann solutions.

3.2. **Estimates of Numerical Dissipation.** Without loss of generality, we assume $\eta_*(0) = 0$ and $\eta_*(\mathbf{Q}) \ge 0$; otherwise, we can use the following entropy pair:

$$\tilde{\eta}_*(\mathbf{Q}) = \eta_*(\mathbf{Q}) - \eta_*(0) - \nabla\eta_*(0)\mathbf{Q}, \quad \tilde{q}_*(\mathbf{Q}) = q_*(\mathbf{Q}) - q_*(0) - \nabla\eta_*(0)(\mathbf{F}(\mathbf{Q}) - \mathbf{F}(0)),$$

instead of (η_*, q_*).

Consider the entropy dissipation measures $\partial_t\eta_*(\mathbf{Q}^\ell) + \partial_x q_*(\mathbf{Q}^\ell)$ associated with the convex entropy pair (η_*, q_*). Since each $\mathbf{Q}^\ell(x,t)$ has compact support, we may use $\varphi(x,t) = 1$ as a test function and then use the Gauss-Green formula in the strip $\mathbb{R} \times [0,T)$ with $T \equiv K\Delta t$ for some integer K. Using $\eta_*(\mathbf{Q}) \ge 0$, one has

(3.3)
$$\sum_{i,n}\int_{x_{i-\frac12}}^{x_{i+\frac12}} \left(\eta_*(\mathbf{Q}_-^n) - \eta_*(\mathbf{Q}_i^n)\right)\, dx + \sum_{i,n}\int_{x_i}^{x_{i+1}} \left(\eta_*(\mathbf{Q}_-^{n+\frac12}) - \eta_*(\mathbf{Q}_{i+\frac12}^{n+\frac12})\right)\, dx$$
$$+ \int_0^T \sum_{\text{shocks } x(t)} \left(x'(t)\ [\eta_*](t) - [q_*](t)\right) dt \le \int_{\mathbb{R}} \eta_*(\mathbf{Q}_0(x))\, dx,$$

while

$$\begin{aligned}
&\sum_{i,n}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\eta_*(\mathbf{Q}_-^n)-\eta_*(\mathbf{Q}_i^n)\right)\,dx \\
&=\sum_{i,n}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_0^1 (\mathbf{Q}_-^n-\mathbf{Q}_i^n)\nabla^2\eta_*(\mathbf{Q}_i^n+\tau(\mathbf{Q}_-^n-\mathbf{Q}_i^n))(\mathbf{Q}_-^n-\mathbf{Q}_i^n)^\top\,(1-\tau)\,d\tau dx \\
&\ge 0,
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&\sum_{i,n}\int_{x_i}^{x_{i+1}} \Big(\eta_*(\mathbf{Q}_-^{n+\frac{1}{2}})-\eta_*(\mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}})\Big)\,dx \\
&=\sum_{i,n}\int_{x_i}^{x_{i+1}}\int_0^1 (\mathbf{Q}_-^{n+\frac{1}{2}}-\mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}})\nabla^2\eta_*(\mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}}+\tau(\mathbf{Q}_-^{n+\frac{1}{2}}-\mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}})) \\
&\qquad\qquad \times(\mathbf{Q}_-^{n+\frac{1}{2}}-\mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}})^\top\,(1-\tau)\,d\tau dx \ge 0,
\end{aligned} \tag{3.5}$$

where $\mathbf{Q}_-^n=\mathbf{Q}^\ell(x,t_n-0)$ and $\mathbf{Q}_-^{n+\frac{1}{2}}=\mathbf{Q}^\ell(x,t_{n+\frac{1}{2}}-0)$.

Note that the entropy inequality, $x'(t)\,[\eta_*](t)-[q_*](t)\ge 0$, is satisfied for shocks. On the other hand, η_* is convex in the conservative variables $\mathbf{Q}$. Estimates (3.3)–(3.4) yield

$$\int_0^T \sum_{\text{shocks } x(t)} \Big(x'(t)[\eta_*](t)-[q_*](t)\Big)\,dt\,\le\,C, \tag{3.6}$$

$$\sum_{i,n}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Big(|\mathbf{Q}_-^n-\mathbf{Q}_i^n|^2+|\mathbf{Q}_-^{n+\frac{1}{2}}-\mathbf{Q}_{i+\frac{1}{2}}^{n+\frac{1}{2}}|^2\Big)\,dx\le C. \tag{3.7}$$

3.3. **Consistency with the Lax Entropy Inequality.** Now we check here that the limit of the FORCE approximate solutions $\mathbf{Q}^\ell(x,t)$ is actually an entropy solution of the Cauchy problem (3.1)–(3.2).

Theorem 3.1. *Let $\mathbf{Q}^\ell(x,t)$ be the FORCE approximate solutions which are uniformly bounded and converge strongly almost everywhere to a limit $\mathbf{Q}(x,t)$. Then $\mathbf{Q}(x,t)$ is an entropy solution of the Cauchy problem* (3.1)–(3.2).

Proof. For any entropy pair (η, q) with convex η and for any nonnegative function $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$, we have

(3.8)
$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \left(\eta(\mathbf{Q}^\ell)\partial_t\varphi + q(\mathbf{Q}^\ell)\partial_x\varphi\right) dxdt + \int_{\mathbb{R}} \eta(\mathbf{Q}^\ell(x,0))\varphi(x,0)\, dx \\
&= S^\ell(\varphi) + \sum_{i,n} \Bigg(\varphi_i^n \int_{x_{i-\frac12}}^{x_{i+\frac12}} \left(\eta(\mathbf{Q}_-^n) - \eta(\mathbf{Q}_i^n)\right)\, dx \\
&\qquad\qquad + \varphi_{i+\frac12}^{n+\frac12} \int_{x_i}^{x_{i+1}} \left(\eta(\mathbf{Q}_-^{n+\frac12}) - \eta(\mathbf{Q}_{i+\frac12}^{n+\frac12})\right)\, dx \Bigg) \\
&\quad + \sum_{i,n} \Bigg(\int_{x_{i-\frac12}}^{x_{i+\frac12}} \left(\varphi(x,t_n) - \varphi_i^n\right)\left(\eta(\mathbf{Q}_-^n) - \eta(\mathbf{Q}_i^n)\right) dx \\
&\qquad + \int_{x_i}^{x_{i+1}} \left(\varphi(x,t_{n+\frac12}) - \varphi_{i+\frac12}^{n+\frac12}\right) \left(\eta(\mathbf{Q}_-^{n+\frac12}) - \eta(\mathbf{Q}_{i+\frac12}^{n+\frac12})\right)\, dx \Bigg).
\end{aligned}$$

Since η is a convex function, then the entropy inequality for shocks in the approximate solutions $\mathbf{Q}^\ell(x,t)$ yields

$$S^\ell(\varphi) = \int_0^T \sum_{\text{shocks } x(t)} \left(x'(t)\ [\eta](t) - [q](t)\right) dt \geq 0, \tag{3.9}$$

$$\sum_{i,n} \varphi_i^n \int_{x_{i-\frac12}}^{x_{i+\frac12}} \int_0^1 (\mathbf{Q}_-^n - \mathbf{Q}_i^n)\nabla^2\eta_*(\mathbf{Q}_i^n + \tau(\mathbf{Q}_-^n - \mathbf{Q}_i^n))(\mathbf{Q}_-^n - \mathbf{Q}_i^n)^\top\,(1-\tau)\, d\tau dx \geq 0,$$

and

$$\begin{aligned}
\sum_{i,n} \varphi_{i+\frac12}^{n+\frac12} \int_{x_i}^{x_{i+1}} \int_0^1 &(\mathbf{Q}_-^{n+\frac12} - \mathbf{Q}_{i+\frac12}^{n+\frac12})\nabla^2\eta_*(\mathbf{Q}_{i+\frac12}^{n+\frac12} + \tau(\mathbf{Q}_-^{n+\frac12} - \mathbf{Q}_{i+\frac12}^{n+\frac12})) \\
&\times(\mathbf{Q}_-^{n+\frac12} - \mathbf{Q}_{i+\frac12}^{n+\frac12})^\top\,(1-\tau)\, d\tau dx \geq 0.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\Big|\sum_{i,n} \int_{x_{i-\frac12}}^{x_{i+\frac12}} \left(\varphi(x,t_n) - \varphi_i^n\right)\, \left(\eta(\mathbf{Q}_-^n) - \eta(\mathbf{Q}_i^n)\right)\, dx\Big| \\
&\leq C\ell^{1/2}\|\varphi\|_{C_0^1} \left(\sum_{i,n} \int_{x_{i-\frac12}}^{x_{i+\frac12}} |\mathbf{Q}_-^n - \mathbf{Q}_i^n|^2\, dx \right)^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\sum_{i,n} \int_{x_i}^{x_{i+1}} \left(\varphi(x,t_{n+\frac12}) - \varphi_{i+\frac12}^{n+\frac12}\right)\, \left(\eta(\mathbf{Q}_-^{n+\frac12}) - \eta(\mathbf{Q}_{i+\frac12}^{n+\frac12})\right)\, dx\Big| \\
&\leq C\ell^{1/2}\|\varphi\|_{C_0^1} \left(\sum_{i,n} \int_{x_i}^{x_{i+1}} |\mathbf{Q}_-^n - \mathbf{Q}_{i+\frac12}^{n+\frac12}|^2\, dx \right)^{1/2}.
\end{aligned}$$

Thus, when $\ell \to 0$,

$$\begin{aligned}
&|\sum_{i,n}\int_{x_{i-\frac12}}^{x_{i+\frac12}} (\varphi(x,t_n)-\varphi_i^n)\ \big(\eta(\mathbf{Q}_-^n)-\eta(\mathbf{Q}_i^n)\big)\ dx| \\
&+|\sum_{i,n}\int_{x_i}^{x_{i+1}} \Big(\varphi(x,t_{n+\frac12})-\varphi_{i+\frac12}^{n+\frac12}\Big)\ \Big(\eta(\mathbf{Q}_-^{n+\frac12})-\eta(\mathbf{Q}_{i+\frac12}^{n+\frac12})\Big)\ dx| \\
&\le C\,\ell^{1/2} \to 0.
\end{aligned} \tag{3.10}$$

Since $\left|\mathbf{Q}^\ell(x,t)\right| \le C$ and $\mathbf{Q}^\ell(x,t) \longrightarrow \mathbf{Q}(x,t)$ a.e. (x,t), we have $|\mathbf{Q}(x,t)| \le C$ almost everywhere. We also conclude from (3.8)–(3.10) that $\mathbf{Q}(x,t)$ satisfies the entropy inequality

$$\int_0^\infty\int_{\mathbb{R}}(\eta(\mathbf{Q})\partial_t\phi+q(\mathbf{Q})\partial_x\phi)\,dxdt+\int_{\mathbb{R}}\eta(\mathbf{Q}_0(x))\,\phi(x,0)\,dx\ \ge\ 0,$$

for any nonnegative function $\phi \in C_0^\infty(\mathbb{R}\times[0,\infty))$. This completes the proof of Theorem 3.1.

3.4. **Convergence of the FORCE Scheme for 2×2 Strictly Hyperbolic Systems.**

Theorem 3.2. *Let system* (3.1) *be 2×2 strictly hyperbolic and genuinely nonlinear. Let $\mathbf{Q}^\ell(x,t)$ be the FORCE approximate solutions satisfying*

$$|\mathbf{Q}^\ell(x,t)| \le C,$$

where $C>0$ is independent of ℓ. Then there exists a subsequence $\mathbf{Q}^{\ell_n}(x,t)$ converging to an L^∞ entropy solution $\mathbf{Q}(x,t)$:

$$\mathbf{Q}^{\ell_n}(x,t) \to \mathbf{Q}(x,t), \qquad n\to\infty,$$

with

$$|\mathbf{Q}(x,t)| \le C.$$

For any bounded set $\Omega \subset \mathbb{R}\times[0,T]$ and for any entropy pair (η,q), we deduce as for (3.8) that, for any $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}}\big(\eta(\mathbf{Q}^\ell)\partial_t\varphi+q(\mathbf{Q}^\ell)\partial_x\varphi\big)\,dxdt+\int_{\mathbb{R}}\eta(\mathbf{Q}^\ell(x,0))\varphi(x,0)\,dx \\
&= S^\ell(\varphi)+L_1^\ell(\varphi)+L_2^\ell(\varphi),
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
L_1^\ell(\varphi) = \sum_{i,n}\Bigg(&\varphi_i^n\int_{x_{i-\frac12}}^{x_{i+\frac12}}(\eta(\mathbf{Q}_-^n)-\eta(\mathbf{Q}_i^n))\,dx \\
&+\varphi_{i+\frac12}^{n+\frac12}\int_{x_i}^{x_{i+1}}(\eta(\mathbf{Q}_-^{n+\frac12})-\eta(\mathbf{Q}_{i+\frac12}^{n+\frac12}))\,dx\Bigg)
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
L_2^\ell(\varphi) = \sum_{i,n}\Bigg(&\int_{x_{i-\frac12}}^{x_{i+\frac12}}(\varphi(x,t_n)-\varphi_i^n)\,(\eta(\mathbf{Q}_-^n)-\eta(\mathbf{Q}_i^n))\,dx \\
&+\int_{x_i}^{x_{i+1}}(\varphi(x,t_{n+\frac12})-\varphi_{i+\frac12}^{n+\frac12})\,(\eta(\mathbf{Q}_-^{n+\frac12})-\eta(\mathbf{Q}_{i+\frac12}^{n+\frac12}))\,dx\Bigg).
\end{aligned} \tag{3.13}$$

Then

$$|S^\ell(\varphi)| \le C\,\|\varphi\|_{\mathcal{C}_0}\int_0^T \sum (x'(t)[\eta_*]-[q_*])\,dt \le C\,\|\varphi\|_{\mathcal{C}_0(\Omega)},$$

$$\begin{aligned}
&|L_1^\ell(\varphi)| \\
&\le C\,\|\varphi\|_{\mathcal{C}_0}\sum_{i,n}\Bigg(\int_{x_{i-\frac12}}^{x_{i+\frac12}} dx\int_0^1 (\mathbf{Q}_-^n-\mathbf{Q}_i^n)\nabla^2\eta_*(\mathbf{Q}_i^n+\tau(\mathbf{Q}_-^n-\mathbf{Q}_i^n))(\mathbf{Q}_-^n-\mathbf{Q}_i^n)^\top(1-\tau)\,d\tau \\
&\qquad +\int_{x_i}^{x_{i+1}} dx\int_0^1 (\mathbf{Q}_-^{n+\frac12}-\mathbf{Q}_{i+\frac12}^{n+\frac12})\nabla^2\eta_*(\mathbf{Q}_-^{n+\frac12}+\tau(\mathbf{Q}_-^{n+\frac12}-\mathbf{Q}_{i+\frac12}^{n+\frac12})) \\
&\qquad\qquad \times(\mathbf{Q}_-^{n+\frac12}-\mathbf{Q}_{i+\frac12}^{n+\frac12})^\top(1-\tau)\,d\tau\Big)\le C.
\end{aligned}$$

Hence, $|(M^\ell+S^\ell+L_1^\ell)(\varphi)| \le C\,\|\varphi\|_{\mathcal{C}_0}$, which yields a uniform bound in the space $\mathcal{M}(\Omega)$ of bounded measures for $M^\ell+S^\ell+L_1^\ell$, considered as a functional on the space of continuous functions:

$$\|M^\ell+S^\ell+L_1^\ell\|_{\mathcal{M}(\Omega)} \le C.$$

The embedding theorem $\mathcal{M}(\Omega)\overset{\text{compact}}{\hookrightarrow} W^{-1,q_0}(\Omega)$, $1<q_0<2$, yields that

(3.14) $\qquad M^\ell+S^\ell+L_1^\ell \quad$ is a compact sequence in $W^{-1,q_0}(\Omega)$.

It remains to treat $L_2^\ell(\varphi)$. Let $\varphi\in\mathcal{C}_0^\alpha(\Omega)$. Then

(3.15)

$$\begin{aligned}
|L_2^\ell(\varphi)| &\le \ell^\alpha\,\|\varphi\|_{\mathcal{C}_0^\alpha}\sum_{i,n}\Bigg(\int_{x_i}^{x_{i+1}}|\eta(\mathbf{Q}_-^{n+\frac12})-\eta(\mathbf{Q}_{i+\frac12}^{n+\frac12})|^2dx \\
&\qquad +\int_{x_{i-\frac12}}^{x_{i+\frac12}}|\eta(\mathbf{Q}_-^n)-\eta(\mathbf{Q}_i^n)|^2\,dx\Bigg) \\
&\le \ell^{\alpha-1/2}\|\nabla\eta\|_{L^\infty}\|\varphi\|_{\mathcal{C}_0^\alpha}\Bigg(\sum_{i,n}\Bigg(\int_{x_i}^{x_{i+1}}|\mathbf{Q}_-^{n+\frac12}-\mathbf{Q}_{i+\frac12}^{n+\frac12}|^2dx \\
&\qquad +\int_{x_{i-\frac12}}^{x_{i+\frac12}}|\mathbf{Q}_-^n-\mathbf{Q}_i^n|^2\,dx\Bigg)\Bigg)^{1/2} \\
&\le C\,\ell^{\alpha-1/2}\|\varphi\|_{W_0^{1,p}(\Omega)},\quad \text{for all } p>\frac{2}{1-\alpha}.
\end{aligned}$$

Estimate (3.15) implies

(3.16) $\|L_2^\ell\|_{W^{-1,q_0}(\Omega)}\le C\ell^{\alpha-1/2}\longrightarrow 0$, when $\ell\to 0$, for $1<q_0<\dfrac{2}{1+\alpha}<2$.

Finally, we combine (3.14) with (3.16) to obtain that

(3.17) $\qquad M^\ell+S^\ell+L_1^\ell+L_2^\ell \quad$ is compact in $\quad W^{-1,q_0}(\Omega)$.

Since $|\mathbf{Q}^\ell(x,t)|\le C$, we have that

(3.18) $\qquad M^\ell+S^\ell+L_1^\ell+L_2^\ell \quad$ is bounded in $\quad W^{-1,r}(\Omega), r>2$.

The interpolation lemma in [14], (3.17), and (3.18) imply that

$$M^\ell + S^\ell + L_1^\ell + L_2^\ell \quad \text{is compact in} \quad W^{-1,2}(\Omega),$$

which implies that

$$\partial_t \eta(\mathbf{Q}^\ell) + \partial_x q(\mathbf{Q}^\ell) \quad \text{is compact in} \quad W^{-1,2}(\Omega). \tag{3.19}$$

In view of the uniform boundedness of $\mathbf{Q}^\ell(x,t)$, (3.19), Theorem 3.1, and the compactness theorem in DiPerna [16], we conclude that there exists a subsequence $\mathbf{Q}^{\ell_n}(x,t)$ converging for a.e. (x,t) to an entropy solution $\mathbf{Q} \in L^\infty$.

As a direct application of the analysis above and a compactness theorem recently estalished in Chen-Li-Li [10], we conclude the convergence of approximate solutions $\mathbf{Q}^\ell = (v^\ell, u^\ell)$ of the Euler equations for one-dimensional media with unit reference density and zero body force in Lagrangian coordinates:

$$\begin{aligned} &\partial_t v - \partial_x u = 0, \\ &\partial_t u - \partial_x \sigma = 0, \end{aligned} \tag{3.20}$$

where u denotes the velocity, σ the stress, and v the strain of the medium. When the medium is elastic, the stress at the material point x and time t is determined solely by the value of the strain at $(x,t) \in \mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$ via a constitutive relation:

$$\sigma(x,t) = \sigma(v(x,t)). \tag{3.21}$$

Under the standard assumption $\sigma'(v) > 0$, system (3.20)–(3.21) is strictly hyperbolic. In elastodynamics, genuine nonlinearity is typically precluded by the fact that the medium in question can sustain discontinuities in both the compressive and expansive phases of the motion. In the simplest model for common rubber, one postulates that the stress σ as a function of the strain v switches from concave in the compressive mode $v < \hat{v}$ to convex in the expansive mode $v > \hat{v}$, i.e.,

$$\operatorname{sign}((v - \hat{v})\sigma''(v)) \geq 0. \tag{3.22}$$

Moreover, we assume that σ satisfies that

$$\text{there is no interval on which } \sigma \text{ is affine,} \tag{3.23}$$

and there exists an integer $m \in [1, \infty]$ such that, on an interval $(\hat{v}, \hat{v}+\delta)$ or $(\hat{v}-\delta, \hat{v})$ for some $\delta > 0$,

$$\sum_{k=1}^{m} |\sigma^{(2k)}(v)| \neq 0. \tag{3.24}$$

Conditions (3.22)-(3.24) are very general, which especially include the stress-strain relation

$$\sigma''(v) = sign(v) e^{-\frac{1}{|v|}} \left(sin \frac{1}{v} \right)^{2n}, \qquad n = 1, 2, \cdots. \tag{3.25}$$

System (3.20)–(3.21) is strictly hyperbolic, whose genuinely nonlinearity fails on the set $\{v : \sigma''(v) = 0\}$. Nevertheless, this system is endowed bounded convex invariant regions for Riemann solutions and thus for the FORCE approximate solutions, which implies the uniform boundedness of $\mathbf{Q}^\ell(x,t)$. Then the compactness theorem in [10] combining the H^{-1} compactness (3.19) and the entropy-consistency theorem (Theorem 3.1) implies the convergence of the FORCE scheme for this physical system.

4. Convergence and Entropy-Consistency of the FORCE Scheme for the Euler Equations for Isentropic Fluids with Vacuum

Now we follow the same line in Chen [2] and Ding-Chen-Luo [14], [15] to show the convergence and entropy-consistency of the force scheme for the isentropic Euler equations.

4.1. **Isentropic Euler Equations.**

The system of isentropic Euler equations reads:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p \right) = 0, \end{cases} \tag{4.1}$$

where ρ is the density, v the fluid velocity, with $\rho v = m$ the momentum, and p the scalar pressure.

As usual, assume that the pressure function $p(\rho)$ satisfies that, when $\rho > 0$,

$$\begin{aligned} & p(\rho) > 0, \;\; p'(\rho) > 0 \, (\text{hyperbolicity}), \\ & \rho p''(\rho) + 2p'(\rho) > 0 \, (\text{genuine nonlinearity}), \end{aligned} \tag{4.2}$$

and, when ρ tends to zero,

$$p(\rho), \;\; p'(\rho) \to 0, \tag{4.3}$$

which is different from the isothermal case. In addition, we assume that there exists an exponent $\gamma \in (1, \infty)$, a smooth function $P = P(\rho)$, and some real $\epsilon > 0$ such that

$$\begin{aligned} & p(\rho) = \kappa \, \rho^\gamma \left(1 + \rho^{\theta(1+\epsilon)} \, P(\rho)\right), \\ & P(\rho) \;\; \text{and} \, \rho^3 \, P'''(\rho) \;\; \text{are bounded as } \rho \to 0, \end{aligned} \tag{4.4}$$

where $\kappa := (\gamma - 1)^2/(4\gamma)$ after normalisation. Of course, the function $P(\rho)$ may exhibit some singularities at $\rho = 0$. In fact, (4.4) implies solely that $\rho \, P'(\rho)$ and $\rho^2 \, P''(\rho)$ remain bounded.

For a polytropic gas,

$$p(\rho) = \kappa_0 \rho^\gamma, \qquad \gamma > 1, \tag{4.5}$$

where $\kappa_0 > 0$ is any constant under scaling. The pressure-density laws above especially include the example

$$p(\rho) = \kappa_1 \, \rho^{\gamma_1} + \kappa_2 \, \rho^{\gamma_2}, \qquad \gamma_1, \gamma_2 > 1, \;\; \kappa_1, \kappa_2 > 0. \tag{4.6}$$

The eigenvalues of system (4.1) are

$$\lambda_j = m/\rho + (-1)^i \sqrt{p'(\rho)}, \quad j = 1, 2, \tag{4.7}$$

and the corresponding right-eigenvectors are

$$\mathbf{r}_j = \alpha_j(\rho)(1, \lambda_j)^\top, \quad \alpha_j(\rho) = (-1)^j \frac{2\rho\sqrt{p'(\rho)}}{\rho p''(\rho) + 2p'(\rho)}, \tag{4.8}$$

so that $\nabla \lambda_j \cdot \mathbf{r}_j = 1, j = 1, 2$. The Riemann invariants are

$$w_j = \frac{m}{\rho} + (-1)^{j-1} \int_0^\rho \frac{\sqrt{p'(s)}}{s} ds, \quad j = 1, 2. \tag{4.9}$$

From (4.3) and (4.7),

$$\lambda_2 - \lambda_1 = 2\sqrt{p'(\rho)} \to 0, \qquad \rho \to 0.$$

Therefore, system (4.1) is strictly hyperbolic in the nonvacuum states $\{(\rho, v) : \rho > 0,\ |v| \le C_0\}$. However, strict hyperbolicity fails near the vacuum states $\{(\rho, m/\rho) : \rho = 0,\ |m/\rho| \le C_0\}$.

A pair of mappings $(\eta, q) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is called an entropy-entropy flux pair (or entropy pair for short) of system (4.1) if it satisfies the hyperbolic system:

$$\nabla q(\rho, m) = \nabla \eta(\rho, m) \nabla \mathbf{F}(\rho, m). \tag{4.10}$$

Furthermore, $\eta(\rho, m)$ is called a weak entropy if

$$\eta\Big|_{\substack{\rho=0\\ v=m/\rho \text{ fixed}}} = 0. \tag{4.11}$$

For example, the mechanical energy (a sum of the kinetic and internal energy) and the mechanical energy flux

$$\eta_*(\rho, m) = \frac{m^2}{2\rho} + \rho \int_0^\rho \frac{p(s)}{s^2} ds, \quad q_*(\rho, m) = \frac{m^3}{2\rho^2} + m \int_0^\rho \frac{p'(s)}{s} ds \tag{4.12}$$

form a special entropy pair; $\eta_*(\rho, m)$ is convex for any $\gamma > 1$ and strictly convex (even at the vacuum states) if $\gamma \le 2$, in any bounded region in $\rho \ge 0$.

4.2. **Estimates of L^∞ Bound and Numerical Dissipation.** As in Section 4.1, we can construct the FORCE approximate solutions $(\rho^\ell, m^\ell)(x, t)$, using the solvability of the Riemann problems for any Riemann data with nonnegative density. We also set $v^\ell = m^\ell/\rho^\ell$ when $\rho^\ell > 0$ and $v^\ell = 0$ otherwise.

First we estimate the L^∞ bound and numerical dissipation.

Proposition 4.1. *For any $w_1^0 > w_2^0$, the region*

$$\sum(w_1^0, w_2^0) = \{(\rho, m) : w_1 \le w_1^0,\ w_2 \ge w_2^0,\ w_1 - w_2 \ge 0\}$$

is also invariant for the FORCE approximate solutions $(\rho^\ell, m^\ell)(x, t)$. In particular, there exists $C > 0$ such that

$$0 \le \rho^\ell(x, t) \le C, \qquad |m^\ell(x, t)/\rho^\ell(x, t)| \le C. \tag{4.13}$$

First it is easy to check that $\sum(w_1^0, w_2^0)$ is an invariant region for the Riemann solutions. Since the set $\sum(w_1^0, w_2^0)$ is convex in the (ρ, m)-plane, it follows from Jensen's inequality that, for any function satisfying $\{(\rho, m)(x) : a \le x \le b\} \subset \sum(w_1^0, w_2^0)$ for some (w_1^0, w_2^0),

$$(\bar\rho, \bar m) := \frac{1}{b-a} \int_a^b (\rho, m)(x) dx \in \sum(w_1^0, w_2^0).$$

Therefore, $\sum(w_1^0, w_2^0)$ is also an invariant region for the FORCE scheme, which implies estimate (4.13).

In particular, Proposition 4.1 shows that the approximate density function $\rho^\ell(x, t)$ remains nonnegative, and both $\rho^\ell(x, t)$ and $m^\ell(x, t)/\rho^\ell(x, t)$ are uniformly bounded so it is indeed possible to construct the approximate solutions globally, as described earlier.

Consider the weak entropy dissipation measures $\partial_t \eta(\mathbf{Q}^\ell) + \partial_x q(\mathbf{Q}^\ell)$ associated with a weak entropy pair (η, q). Using the Gauss-Green formula, for any test-function $\varphi(x, t)$ compactly supported in $\mathbb{R} \times [0, T]$ with $T \equiv K\Delta t$ for some integer

K, one has

$$\int_{\mathbb{R}}\int_0^T (\eta(\mathbf{Q}^\ell)\partial_t\varphi + q(\mathbf{Q}^\ell)\partial_x\varphi)\,dxdt = M^\ell(\varphi) + S^\ell(\varphi) + L_1^\ell(\varphi) + L_2^\ell(\varphi), \tag{4.14}$$

where

$$M^\ell(\varphi) := \int_{\mathbb{R}} \eta(\mathbf{Q}^\ell(x,T))\,\varphi(x,T)\,dx - \int_{\mathbb{R}} \eta(\mathbf{Q}^\ell(x,0))\,\varphi(x,0)\,dx, \tag{4.15}$$

and $S^\ell(\varphi), L_1^\ell(\varphi)$, and $L_2^\ell(\varphi)$ have the same formulas as in (3.9), (3.12), and (3.13) for $\mathbf{Q}^\ell = (\rho^\ell, m^\ell)$.

Since each $\mathbf{Q}^\ell(x,t)$ has compact support, we may substitute $(\eta, q) = (\eta_*, q_*)$ and $\varphi \equiv 1$ in the formulas (4.14) to obtain

$$\int_0^T \sum_{\text{shocks } x(t)} \Big(x'(t)[\eta_*](t) - [q_*](t)\Big)\,dt \,\le\, C, \tag{4.16}$$

and

(4.17)

$$\begin{aligned}
&\sum_{i,n}\int_{x_{i-\frac12}}^{x_{i+\frac12}} dx \int_0^1 (\mathbf{Q}_-^n - \mathbf{Q}_i^n)\nabla^2\eta_*(\mathbf{Q}_i^n + \tau(\mathbf{Q}_-^n - \mathbf{Q}_i^n))(\mathbf{Q}_-^n - \mathbf{Q}_i^n)^\top\,(1-\tau)\,d\tau\\
&+\sum_{i,n}\int_{x_i}^{x_{i+1}} dx \int_0^1 (\mathbf{Q}_-^{n+\frac12} - \mathbf{Q}_{i+\frac12}^{n+\frac12})\nabla^2\eta_*(\mathbf{Q}_-^{n+\frac12} + \tau(\mathbf{Q}_-^{n+\frac12} - \mathbf{Q}_{i+\frac12}^{n+\frac12}))\\
&\qquad\qquad \times(\mathbf{Q}_-^{n+\frac12} - \mathbf{Q}_{i+\frac12}^{n+\frac12})^\top\,(1-\tau)\,d\tau \le C.
\end{aligned}$$

by using the entropy inequality and the convexity of $\eta_*(\mathbf{Q})$. Then we observe the following:

(i) For $1 < \gamma \le 2$, entropy η_* is uniformly convex so that the Hessian matrix $\nabla^2\eta_*$ is bounded below by a positive constant, which implies

$$\sum_{i,n}\left(\int_{x_{i-\frac12}}^{x_{i+\frac12}} |\mathbf{Q}_-^n - \mathbf{Q}_i^n|^2 dx + \int_{x_i}^{x_{i+1}} |\mathbf{Q}_-^{n+\frac12} - \mathbf{Q}_{i+\frac12}^{n+\frac12}|^2) dx\right) \le C. \tag{4.18}$$

(ii) For $\gamma > 2$, the estimate (4.17) implies

$$\begin{aligned}
\sum_{i,n}\int_{x_{i-\frac12}}^{x_{i+\frac12}} \Bigg(&\frac{\rho_-^n}{2}(\frac{m_-^n}{\rho_-^n} - \frac{m_i^n}{\rho_i^n})^2\\
&+\int_0^1 \frac{p'(\rho_i^n + \tau(\rho_-^n - \rho_i^n))}{\rho_i^n + \tau(\rho_-^n - \rho_i^n)}\,(1-\tau)\,d\tau(\rho_-^n - \rho_i^n)^2\Bigg)\,dx \,\le\, C,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i,n}\int_{x_i}^{x_{i+1}} \Bigg(&\frac{\rho_-^{n+\frac12}}{2}(\frac{m_-^{n+\frac12}}{\rho_-^{n+\frac12}} - \frac{m_{i+\frac12}^{n+\frac12}}{\rho_{i+\frac12}^{n+\frac12}})^2\\
&+\int_0^1 \frac{p'(\rho_{i+\frac12}^{n+\frac12} + \tau(\rho_-^{n+\frac12} - \rho_{i+\frac12}^{n+\frac12}))}{\rho_{i+\frac12}^{n+\frac12} + \tau(\rho_-^{n+\frac12} - \rho_{i+\frac12}^{n+\frac12})}\,(1-\tau)\,d\tau(\rho_-^{n+\frac12} - \rho_{i+\frac12}^{n+\frac12})^2\Bigg)\,dx \,\le\, C.
\end{aligned}$$

In view of assumption (4.4), there exists $C_1 > 0$ depending on γ such that

$$\int_0^1 \frac{p'(\rho_i^n + \tau(\rho_-^n - \rho_i^n))}{\rho_i^n + \tau(\rho_-^n - \rho_i^n)} (1-\tau)\, d\tau \;\ge\; C_1 \min\big\{1, (\rho_-^n - \rho_i^n)^{\gamma-2}\big\},$$

which yields

$$\sum_{i,n} \int_{x_{i-\frac12}}^{x_{i+\frac12}} \Big(\rho_-^n \big(\frac{m_-^n}{\rho_-^n} - \frac{m_i^n}{\rho_i^n}\big)^2 + |\rho_-^n - \rho_i^n|^\gamma\Big)\, dx \;\le\; C.$$

The Cauchy-Schwarz inequality implies

$$\sum_{i,n} \int_{x_{i-\frac12}}^{x_{i+\frac12}} \rho_-^n \big|\frac{m_-^n}{\rho_-^n} - \frac{m_i^n}{\rho_i^n}\big|\, dx \;\le\; C\,\ell^{-1/2}, \tag{4.19}$$

and

$$\sum_{i,n} \int_{x_{i-\frac12}}^{x_{i+\frac12}} |\rho_-^n - \rho_i^n|\, dx \;\le\; C\,\ell^{1/\gamma-1}. \tag{4.20}$$

Similarly, we have

$$\sum_{i,n} \int_{x_i}^{x_{i+1}} \rho_-^{n+\frac12} \Big|\frac{m_-^{n+\frac12}}{\rho_-^{n+\frac12}} - \frac{m_{i+\frac12}^{n+\frac12}}{\rho_{i+\frac12}^{n+\frac12}}\Big|\, dx \;\le\; C\,\ell^{-1/2}, \tag{4.21}$$

and

$$\sum_{i,n} \int_{x_i}^{x_{i+1}} |\rho_-^{n+\frac12} - \rho_{i+\frac12}^{n+\frac12}|\, dx \;\le\; C\,\ell^{1/\gamma-1}. \tag{4.22}$$

4.3. **Convergence and Entropy-Consistency of the FORCE Scheme.**

Theorem 4.1. *Let $(\rho_0, m_0)(x)$ be the Cauchy data satisfying*

$$0 \le \rho_0(x) \le C_0, \quad |m_0(x)/\rho_0(x)| \le C_0,$$

for some constant $C_0 > 0$. Extracting a subsequence, if necessary, the FORCE approximate solutions $(\rho^\ell, m^\ell)(x,t)$ converge strongly almost everywhere to a limit $(\rho, m) \in L^\infty(\mathbb{R}_+^2)$ which is an entropy solution of the Cauchy problem (4.1) *with initial data $(\rho_0, m_0)(x)$.*

First, from Proposition 4.1, there exists a constant $C > 0$ depending only on C_0 and $p(\rho)$ such that

$$0 \le \rho^\ell(x,t) \le C, \quad |m^\ell(x,t)/\rho^\ell(x,t)| \le C. \tag{4.23}$$

For any bounded set $\Omega \subset \mathbb{R} \times [0,T]$ and for any weak entropy pair (η, q), we deduce from (3.9), (3.12), (3.13), (4.14), (4.15), and (4.23) that, for any $\varphi \in \mathcal{C}_0(\Omega)$,

$$\begin{aligned}
&|M^\ell(\varphi)| = 0,\\
&|S^\ell(\varphi)| \;\le\; C\,\|\varphi\|_{\mathcal{C}_0} \int_0^T \sum (x'(t)[\eta_*] - [q_*])\; dt \;\le\; C\,\|\varphi\|_{\mathcal{C}_0(\Omega)},\\
&|L_1^\ell(\varphi)| \;\le\; C\,\|\varphi\|_{\mathcal{C}_0(\Omega)}.
\end{aligned}$$

Hence $|(M^\ell + S^\ell + L_1^\ell)(\varphi)| \;\le\; C\,\|\varphi\|_{\mathcal{C}_0}$, which yields a uniform bound in $\mathcal{M}(\Omega)$ for $M^\ell + S^\ell + L_1^\ell$:

$$\|M^\ell + S^\ell + L_1^\ell\|_{\mathcal{M}(\Omega)} \;\le\; C.$$

Then

$$(4.24)\quad M^\ell + S^\ell + L_1^\ell \quad \text{is a compact sequence in } W^{-1,q_0}(\Omega), \qquad 1 < q_0 < 2.$$

It remains to treat $L_2^\ell(\varphi)$. Let $\varphi \in \mathcal{C}_0^\alpha(\Omega)$, $\frac{1}{2} < \alpha < 1$. We distinguish two cases :

(i) For $1 < \gamma \le 2$, we deduce as before from (4.18) that

(4.25)
$$\begin{aligned}
|L_2^\ell(\varphi)| &\le C\,\ell^{\alpha-1/2}\|\nabla\eta\|_{L^\infty}\|\varphi\|_{\mathcal{C}_0^\alpha} \\
&\quad\times\left(\sum_{i,n}\int_{x_{i-\frac12}}^{x_{i+\frac12}}|\mathbf{Q}_-^n-\mathbf{Q}_i^n|^2\,dx+\sum_{i,n}\int_{x_{i-1}}^{x_{i+1}}|\mathbf{Q}_-^{n+\frac12}-\mathbf{Q}_{i+\frac12}^{n+\frac12}|^2\,dx\right)^{1/2} \\
&\le C\,\ell^{\alpha-1/2}\|\varphi\|_{W_0^{1,p}(\Omega)} \qquad \text{for all } p > \frac{2}{1-\alpha}.
\end{aligned}$$

(ii) For $\gamma > 2$, estimates (4.19)–(4.22) yield

(4.26)
$$\begin{aligned}
|L_2^\ell(\varphi)| \quad &\le \quad \ell^\alpha\,\|\nabla\eta\|_{L^\infty}\|\varphi\|_{C_0^\alpha}\left(\sum_{i,n}\int_{x_{i-\frac12}}^{x_{i+\frac12}}(|\rho_-^n-\rho_i^n|+\rho_-^n|\frac{m_-}{\rho_-^n}-\frac{m_i^n}{\rho_i^n}|)\,dx\right. \\
&\qquad\left.+\sum_{i,n}\int_{x_i}^{x_{i+1}}(|\rho_-^{n+\frac12}-\rho_{i+\frac12}^{n+\frac12}|+\rho_-^{n+\frac12}|\frac{m_-^{n+\frac12}}{\rho_-^{n+\frac12}}-\frac{m_{i+\frac12}^{n+\frac12}}{\rho_{i+\frac12}^{n+\frac12}}|\right)dx) \\
&\le C\,\ell^{\alpha+1/\gamma-1}\,\|\varphi\|_{C_0^\alpha(\Omega)}.
\end{aligned}$$

Estimates (4.25) and (4.26) imply

$$(4.27)\quad \|L_2^h\|_{W^{-1,q_0}(\Omega)} \le C\ell^{\alpha_0} \longrightarrow 0, \quad \text{when } \ell\to 0,\ \text{for } 1 < q_0 < \frac{2}{1+\alpha} < 2,$$

where $\alpha_0 = \max\{\alpha - 1/2, \alpha - 1 + 1/\gamma\}$. Finally, we combine (4.24) with (4.27) to obtain that

$$(4.28)\qquad M^\ell + S^\ell + L_1^\ell + L_2^\ell \quad \text{is compact in} \quad W^{-1,q_0}(\Omega).$$

Since $0 \le \rho^\ell(x,t) \le C, |m^\ell(x,t)/\rho^\ell(x,t)| \le C$, we have that

$$(4.29)\qquad M^\ell + S^\ell + L_1^\ell + L_2^\ell \quad \text{is bounded in} \quad W^{-1,r}(\Omega), r > 2.$$

The interpolation lemma in [14], (4.28), and (4.29) imply that

$$M^\ell + S^\ell + L_1^\ell + L_2^\ell \quad \text{is compact in} \quad W^{-1,2}(\Omega),$$

which implies that

$$(4.30)\qquad \partial_t\eta(\mathbf{Q}^\ell) + \partial_x q(\mathbf{Q}^\ell) \quad \text{is compact in} \quad W^{-1,2}(\Omega).$$

In view of Theorem in Chen-LeFloch [5], [6] (also DiPerna [17], Ding-Chen-Luo [14], Chen [2], and Lions-Perthame-Souganidis [32]) and (4.30), there exists a subsequence $\mathbf{Q}^{\ell_n}(x,t)$ converging for almost every (x,t) to a limit function $(\rho, m) \in L^\infty$.

Now we check here that $\mathbf{Q}(x,t) = (\rho, m)(x,t)$ is actually an entropy solution of the Cauchy problem (4.1) with Cauchy data $(\rho_0, m_0)(x)$.

For any weak entropy pair (η, q) with convex η and for any nonnegative function $\varphi \in C_0^\infty(\mathbb{R}\times[0,\infty))$, we obtain from (4.14) that

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}}\big(\eta(\mathbf{Q}^\ell)\partial_t\varphi+q(\mathbf{Q}^\ell)\partial_x\varphi\big)\,dxdt+\int_{\mathbb{R}}\eta(\mathbf{Q}^\ell(x,0))\varphi(x,0)\,dx\\
&= S^\ell(\varphi)+\sum_{i,n}\varphi_i^n\int_{x_{i-\frac12}}^{x_{i+\frac12}}\big(\eta(\mathbf{Q}_-^n)-\eta(\mathbf{Q}_i^n)\big)\,dx\\
&\quad+\sum_{i,n}\varphi_{i+\frac12}^{n+\frac12}\int_{x_i}^{x_{i+1}}\Big(\eta(\mathbf{Q}_-^{n+\frac12})-\eta(\mathbf{Q}_{i+\frac12}^{n+\frac12})\Big)\,dx\\
&\quad+\sum_{i,n}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\big(\varphi(x,t_n)-\varphi_i^n\big)\,\big(\eta(\mathbf{Q}_-^n)-\eta(\mathbf{Q}_i^n)\big)\,dx\\
&\quad+\sum_{i,n}\int_{x_i}^{x_{i+1}}\Big(\varphi(x,t_{n+\frac12})-\varphi_{i+\frac12}^{n+\frac12}\Big)\,\Big(\eta(\mathbf{Q}_-^{n+\frac12})-\eta(\mathbf{Q}_{i+\frac12}^{n+\frac12})\Big)\,dx.
\end{aligned}\tag{4.31}$$

Since η is a convex function, then

$$S^\ell(\varphi)\ge 0,\tag{4.32}$$

$$\begin{aligned}
&\sum_{i,n}\varphi_i^n\int_{x_{i+\frac12}}^{x_{i+\frac12}}(\eta(\mathbf{Q}_-^n)-\eta(\mathbf{Q}_i^n))\,dx\\
&=\sum_{i,n}\varphi_i^n\int_{x_i}^{x_{i+1}}\int_0^1(\mathbf{Q}_-^n-\mathbf{Q}_i^n)\nabla^2\eta(\mathbf{Q}_i^n+\tau(\mathbf{Q}_-^n-\mathbf{Q}_i^n))(\mathbf{Q}_-^n-\mathbf{Q}_i^n)^\top\,(1-\tau)\,d\tau dx\\
&\ge 0,
\end{aligned}\tag{4.33}$$

and, similarly,

$$\sum_{i,n}\varphi_{i+\frac12}^{n+\frac12}\int_{x_i}^{x_{i+1}}\Big(\eta(\mathbf{Q}_-^{n+\frac12})-\eta(\mathbf{Q}_{i+\frac12}^{n+\frac12})\Big)\,dx\ge 0.\tag{4.34}$$

Furthermore, for $1<\gamma\le 2$, one has as before that, when $\ell\to 0$,

$$\begin{aligned}
&|\sum_{i,n}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\big(\varphi(x,t_n)-\varphi_i^n\big)\big(\eta(\mathbf{Q}_-^n)-\eta(\mathbf{Q}_i^n)\big)\,dx|\\
&+|\sum_{i,n}\int_{x_i}^{x_{i+1}}\Big(\varphi(x,t_{n+\frac12})-\varphi_{i+\frac12}^{n+\frac12}\Big)\Big(\eta(\mathbf{Q}_-^{n+\frac12})-\eta(\mathbf{Q}_{i+\frac12}^{n+\frac12})\Big)\,dx|\\
&\le C\,\ell^{1/2}\to 0.
\end{aligned}\tag{4.35}$$

For $\gamma>2$, (4.19) and (4.20) imply

$$\begin{aligned}
&|\sum_{i,n}\int_{x_{i-\frac12}}^{x_{i+\frac12}}(\varphi-\varphi_i^n)\big(\eta(\mathbf{Q}_-^n)-\eta(\mathbf{Q}_i^n)\big)\,dx|\\
&\le C\,\ell\|\varphi\|_{C_0^1}\sum_{i,n}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\Big(|\rho_-^n-\rho_i^n|+\rho_-^n|\frac{m_-^n}{\rho_-^n}-\frac{m_i^n}{\rho_i^n}|\Big)\,dx\le C\,\|\varphi\|_{C_0^1}\,\ell^{1/\gamma}\to 0.
\end{aligned}\tag{4.36}$$

Similarly, we have

$$|\sum_{i,n}\int_{x_i}^{x_{i+1}} \left(\varphi-\varphi_{i+\frac12}^{n+\frac12}\right)\left(\eta(\mathbf{Q}_-^{n+\frac12})-\eta(\mathbf{Q}_{i+\frac12}^{n+\frac12})\right)dx| \le C\,\|\varphi\|_{C_0^1}\,\ell^{1/\gamma}\to 0. \tag{4.37}$$

Since $\left|\frac{m^{\ell_\ell}(x,t)}{\rho^{\ell_\ell}(x,t)}\right| \le C$ and $(\rho^{\ell_\ell}, m^{\ell_\ell})(x,t) \longrightarrow (\rho,m)(x,t)$ for almost every (x,t), we have $0 \le \rho(x,t) \le C$ and $\frac{|m(x,t)|}{\rho(x,t)} \le C$ almost everywhere. We also conclude from (4.31)–(4.36) that $\mathbf{Q}(x,t) = (\rho,m)(x,t)$ satisfies the entropy inequality

$$\int_0^\infty\int_{\mathbb{R}}(\eta(\mathbf{Q})\partial_t\phi + q(\mathbf{Q})\partial_x\phi)\,dxdt + \int_{\mathbb{R}}\eta(\mathbf{Q}_0(x))\,\phi(x,0)\,dx \,\ge\, 0,$$

for any nonnegative function $\phi \in C_0^\infty(\mathbb{R}\times[0,\infty))$. This completes the proof of Theorem 4.2.

Remark 4.1. Combining the analysis above with the argument in Chen-Li [9], we can also establish the convergence and consistency of fractional-step FORCE-type schemes to an entropy solutions for the Euler equations for isothermal fluids (i.e. $\gamma = 1$) with spherical symmetry.

Remark 4.2. The convergence and consistency of the Lax-Friedrichs scheme (the bottom of the hierarchy) and the Godunov scheme to an entropy solutions for the Euler equations for isentropic fluids were first established in Chen [2], [3] and Ding-Chen-Luo [14], [15] (also see Chen-LeFloch [5], [6]).

The Lax-Wendroff scheme is a second-order scheme, which is an important part of the FORCE scheme. The convergence and consistency of the Lax-Wendroff scheme were also established for scalar conservation laws in Chen-Liu [11].

5. Convergence of Fractional Step FORCE Scheme for the Shallow Water Equations

In this section, through an example, we show the convergence of fractional-step FORCE schemes for hyperbolic systems of conservation laws with source terms.

The example we consider is the shallow water equations:

$$\begin{cases} \partial_t h + \partial_x(hv) = 0, \\ \partial_t(hv) + \partial_x(hv^2 + \frac12 gh^2) = -gb'(x)h, \end{cases} \tag{5.1}$$

where v is the velocity of water, g is the universal gravitational acceleration that is a constant, $z = b(x)$ is the function of bottom or bed of the flow of water, and $h(x,t)$ is the depth of water, the vertical distance between the bottom and the free-surface position $z = b(x) + h(x,t)$. Define $m = hu$ as the momentum.

First we construct the fractional FORCE approximate solutions $\mathbf{Q}^\ell(x,t) = (h^\ell, m^\ell)(x,t)$. As before, the fractional FORCE scheme satisfies the property of propagation with finite speed, and the convergence result applies without assumption on the decay of initial data at infinity.

We now construct $\mathbf{Q}^\ell(x,t)$, similar to these in Section 4, by taking care of the lower-order term. As before, it is assumed that the ratio $\Delta t/\ell$ is constant and satisfies the Courant-Friedrichs-Lewy stability condition:

$$\frac{\Delta t}{\ell}\|\lambda_j(\rho^\ell, m^\ell)\|_{L^\infty} \le 1,$$

that is, the CFL number is 1, and each strip $\{(x,t)\,:\,t_n \le t < t_{n+1}\}$ is divided into two substrips, $\{(x,t)\,:\,t_n \le t < t_{n+1/2}\}$ and $\{(x,t)\,:\,t_{n+1/2} \le t < t_{n+1}\}$.

In the first strip $\{(x,t)\,:\,0 \le t < t_1\}$, we can construct the approximate solutions as follows:

In the first substrip $\{(x,t)\,:\,x_i < x < x_{i+1},\ 0 \le t < t_{1/2},\ i \in \mathcal{N}\}$, we define $\mathbf{Q}_0^\ell(x,t)$ by solving a sequence of Riemann problems for (5.1) corresponding to the Riemann data:

$$\mathbf{Q}_0^\ell(x,0) = \begin{cases} \mathbf{Q}_i^0, & x < x_{i+1/2}, \\ \mathbf{Q}_{i+1}^0, & x > x_{i+1/2}, \end{cases}$$

with

$$\mathbf{Q}_i^0 = \frac{1}{\ell}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}_0(x)\,dx.$$

Then we define $\mathbf{Q}^h(x,t)$ in the strip $0 \le t < t_{1/2}$ by the fractional step scheme:

$$\mathbf{Q}^\ell(x,t) = \mathbf{Q}_0^\ell(x,t) - gb'(x)(0, h_0^\ell(x,t)).$$

We set

$$\mathbf{Q}_{i+1/2}^{1/2} = \frac{1}{\ell}\int_{x_i}^{x_{i+1}} \mathbf{Q}^\ell(x, t_{1/2}-0)\,dx.$$

In the second substrip $\{(x,t)\,:\,x_i < x < x_{i+1},\ t_{1/2} \le t < t_1,\ i \in \mathcal{N}\}$, we define $\mathbf{Q}_0^h(x,t)$ by solving a sequence of Riemann problems for (5.1) corresponding to the Riemann data:

$$\mathbf{Q}^h(x,0) = \begin{cases} \mathbf{Q}_{i-1/2}^{1/2}, & x < x_i, \\ \mathbf{Q}_{i+1/2}^{1/2}, & x > x_i. \end{cases}$$

Then we define $\mathbf{Q}^h(x,t)$ in the strip $t_{1/2} \le t < t_1$ by the fractional step scheme:

$$\mathbf{Q}^\ell(x,t) = \mathbf{Q}_0^\ell(x,t) - gb'(x)(0, h_0^\ell(x,t)).$$

Then we set

$$\mathbf{Q}_i^1 = \frac{1}{\ell}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}^\ell(x, t_1-0)\,dx.$$

If $\mathbf{Q}^\ell(x,t)$ is known for $t < t_n$, we set

$$\mathbf{Q}_i^n = \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}^\ell(x, t_n-0)\,dx.$$

In the substrip $\{(x,t)\,:\,x_i < x < x_{i+1}, t_n < t < t_{n+1/2}\}$, we define $\mathbf{Q}_0^h(x,t)$ by solving the Riemann problems with the data:

$$\mathbf{Q}^h(x,t_n) = \begin{cases} \mathbf{Q}_i^n, & x < x_i, \\ \mathbf{Q}_{i+1}^n, & x > x_i. \end{cases}$$

Then we define $\mathbf{Q}^h(x,t)$ in the strip $t_n \le t < t_{n+1/2}$ by the fractional step scheme:

$$\mathbf{Q}^\ell(x,t) = \mathbf{Q}_0^\ell(x,t) - gb'(x)(0, h_0^\ell(x,t)).$$

We set

$$\mathbf{Q}_{i+1/2}^{n+1/2} = \frac{1}{\ell}\int_{x_i}^{x_{i+1}} \mathbf{Q}^\ell(x, t_{n+1/2}-0)\,dx.$$

In the second substrip $\{(x,t) : x_i < x < x_{i+1},\ t_{n+1/2} \le t < t_{n+1},\ i \in \mathcal{N}\}$, we define $\mathbf{Q}_0^h(x,t)$ by solving a sequence of Riemann problems for (4.1) corresponding to the Riemann data:

$$\mathbf{Q}^\ell(x,t_{n+1}) = \begin{cases} \mathbf{Q}_{i-1/2}^{n+1/2}, & x < x_i, \\ \mathbf{Q}_{i+1/2}^{n+1/2}, & x > x_i. \end{cases}$$

Then we define $\mathbf{Q}^\ell(x,t)$ in the strip $t_{n+1/2} \le t < t_{n+1}$ by the fractional step scheme:

$$\mathbf{Q}^\ell(x,t) = \mathbf{Q}_0^\ell(x,t) - gb'(x)h_0^\ell(x,t).$$

This completes the construction of the fractional FORCE approximate solutions $\mathbf{Q}^\ell(x,t)$.

Then we set

$$\mathbf{Q}_i^{n+1} = \frac{1}{\ell}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}^\ell(x,t_{n+1}-0)\,dx.$$

It can be checked that the fractional-step FORCE scheme $\{\mathbf{Q}_i^n\}$ interpreted via Riemann solutions is consistent with the fractional-step FORCE scheme as described in §3.7 up to second order of accuracy.

Theorem 5.1 (Convergence and Existence)**.** *Let $(h_0,m_0)(x)$ be the Cauchy data satisfying*

$$0 \le h_0(x) \le C_0, \quad |m_0(x)/h_0(x)| \le C_0,$$

for some constant $C_0 > 0$. Extracting a subsequence, if necessary, the fractional-step FORCE approximate solutions $(h^\ell, m^\ell)(x,t)$ converge strongly almost everywhere to a limit $(h,m) \in L^\infty(\mathbb{R}_+^2)$ which is an entropy solution of the Cauchy problem for the shallow water equations (5.1) *with initial data $(h_0,m_0)(x)$.*

We now show how the L^∞ uniform bounds for the approximate solutions can be obtained. We first estimate the Riemann invariants

$$w^\ell(x,t) = w(\mathbf{Q}^\ell(x,t)) = \frac{m^\ell(x,t)}{\rho^\ell(x,t)} + 2\sqrt{g}\sqrt{h^\ell(x,t)},$$
$$z^\ell(x,t) = z(\mathbf{Q}^\ell(x,t)) = \frac{m^\ell(x,t)}{\rho^\ell(x,t)} - 2\sqrt{g}\sqrt{h^\ell(x,t)}.$$

For $t_i \le t < t_{i+1}$, we have

$$w^\ell(x,t) = w_0^\ell(x,t) - gb'(x)(t-t_i), \qquad z^\ell(x,t) = z_0^\ell(x,t) - gb'(x)(t-t_i), \tag{5.2}$$

where $w_0^\ell(x,t) = w(\mathbf{Q}_0^\ell(x,t))$ and $z_0^\ell(x,t) = z(\mathbf{Q}_0^\ell(x,t))$ are Riemann invariants corresponding to the Riemann solutions $\mathbf{Q}_0^\ell(x,t)$.

Notice that Proposition 5.1 still holds for the Riemann solutions $\mathbf{Q}_0^\ell(x,t)$.

For $t_i \le t < t_{i+1}$, Proposition 5.1 implies

$$w^\ell(x,t) = w_0^\ell(x,t) - gb'(x)(t-t_i) \le \sup_x w_0^\ell(x,t_i+0) + C\Delta t, \tag{5.3}$$

$$z^\ell(x,t) = z_0^\ell(x,t) - gb'(x)(t-t_i) \ge \inf_x z_0^\ell(x,t_i+0) - C\Delta t, \tag{5.4}$$

$$w^\ell(x,t) - z^\ell(x,t) \ge 0. \tag{5.5}$$

By the assumption that $0 \le h_0(x) \le C_0$ and $|m_0(x)| \le C_0 h_0(x)$, there exists $\alpha_0 > 0$ such that

$$\begin{gathered} \sup_x w(h_0(x),m_0(x)) \le \alpha_0, \quad \inf_x z(h_0(x),m_0(x)) \ge -\alpha_0, \\ w(h_0(x),m_0(x)) - z(h_0(x),m_0(x)) \ge 0. \end{gathered} \tag{5.6}$$

For $0 \le t < t_1$, the special features of Riemann invariants imply

$$\begin{aligned} w(h_0^\ell(x,t), m_0^\ell(x,t)) \le \alpha_0, \quad z(h_0^\ell(x,t), m_0^\ell(x,t)) \ge -\alpha_0, \\ w(h_0^\ell(x,t), m_0^\ell(x,t)) - z(h_0^\ell(x,t), m_0^\ell(x,t)) \ge 0. \end{aligned} \tag{5.7}$$

By (5.3)–(5.5), we obtain

$$\begin{aligned} w^\ell(x,t) = w(h^\ell, m^\ell) \le \alpha_0 + C\Delta t, \quad z^\ell(x,t) = z(h^\ell, m^\ell) \ge -\alpha_0 - C\Delta t, \\ w^\ell(x,t) - z^\ell(x,t) \ge 0. \end{aligned}$$

Performing the same procedure, we conclude that, for $0 \le t < T$, there exists $M = M(T) > 0$ such that

$$0 \le h^\ell(x,t) \le M, \quad |m^\ell(x,t)| \le M h^\ell(x,t). \tag{5.8}$$

With the uniform bounds for the approximate solutions $\mathbf{Q}^\ell(x,t)$ in $\overline{\Pi}_T = \mathbb{R} \times [0,T]$, we can follow the similar analysis in Section 5 (also see Ding-Chen-Luo [15]) to conclude that, for any weak entropy pair $(\eta, q)(\mathbf{Q})$,

$$\partial_t \eta(\mathbf{Q}^\ell) + \partial_x q(\mathbf{Q}^\ell) \quad \text{is compact in } H^{-1}_{loc}(\Pi_T).$$

In view of the compactness theorems in Lions-Perthame-Souganidis [32] and Chen-LeFloch [5], [6], there exists a subsequence $\mathbf{Q}^{\ell_n}(x,t)$ converging for almost every (x,t) to a limit function $(\rho, m) \in L^\infty$. As shown in Section 5.3 (also see [15]), the limit function $(\rho, m)(x,t)$ is an entropy solution of the Cauchy problem for the shallow water equations (5.1).

The analysis presented above can be applied to handling more general source terms for the isentropic Euler equations; see Ding-Chen-Luo [15].

6. Summary and conclusions

We have identified a family of centred (non-upwind), three-point schemes for solving nonlinear systems of hyperbolic equations. The schemes, written in conservative form on a non-staggered mesh, are monotone and have the optimal stability condition of Courant number unity. From the point of view of accuracy, as given by the numerical viscosity in the modified equation, this family of schemes forms a hierarchy of methods. At the bottom of the hierarchy is the classic Lax-Friedrichs, which is the least accurate, and at the top of the hierarchy is the FORCE scheme, being the optimal scheme in the family. The FORCE approach can also be extended to solve nonlinear systems in non-conservative form. Extensions of FORCE-based schemes to include source terms, multiple space dimensions, and higher order of accuracy are also discussed. Another extension of the FORCE approach is the construction of very simple and yet very accurate, upwind fluxes, where the FORCE flux forms the building block of a multi-stage predictor-corrector procedure. Preliminary results show that in this way one is able to attain the accuracy provided by an exact Riemann solver. The procedure is applicable to absolutely any hyperbolic system, regardless of their complexity. The FORCE scheme is shown to be consistent with the entropy inequality, that is, the limit functions of the FORCE approximate solutions are entropy solutions. The convergence of the FORCE scheme is also established for the isentropic Euler equations and the shallow water equations.

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(Gui-Qiang Chen) Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, USA, http://www.math.northwestern.edu/~gqchen
E-mail address: gqchen@math.northwestern.edu

(Eleuterio F. Toro) Laboratory of Applied Mathematics, Faculty of Engineering, University of Trento, Trento, Italy, http://www.ing.unitn.it/toro
E-mail address: toro@ing.unitn.it