HP-INTERPOLATION OF NON-SMOOTH FUNCTIONS

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Abstract. The quasi-interpolation operators of Clément and Scott-Zhang type are generalized to the hp-context. New polynomial lifting and inverse estimates are presented as well.

Key words. Clément interpolant, quasi-interpolation, hp-FEM, polynomial inverse estimates

AMS subject classifications. 65N30, 65N35, 65N50

1. Introduction. Quasi-interpolation operators, that is, operators achieving optimal rates of convergence also for classes of functions of low regularity have a long history, for example in splines theory (see, e.g., [25] for an overview). In connection with the finite element method (FEM) such an operator was constructed by Ph. Clément in [23] where he showed how H^1-functions can be approximated by piecewise linear functions. Subsequent refinements and variations include [11, 15, 21, 22, 27, 35, 37] to account for higher order polynomials of fixed degree p, preservation of piecewise polynomial boundary conditions, curvilinear elements, and Hermite elements. Several of these refinements were done with a view to an application in residual based finite element error estimation as discussed in the monographs [3, 5, 39].
While quasi-interpolation in the context of the h-version FEM is well documented in the literature, the situation is less favorable for the p- and particularly the hp-version of the FEM, where the approximation properties of spaces of piecewise polynomials are quantified in terms of both the local mesh size and the local polynomial degree. The one-dimensional situation of polynomial approximation on an interval has been thoroughly studied, and we refer the reader to [25] for an excellent exposition of pertinent results. In higher dimensions, the situation is less developed: Approximation results suitable for the application to the p-version FEM/spectral method in higher dimensions can be found in the survey article [16] (for L^2-based weighted and unweighted Sobolev spaces) and [2] (for Sobolev spaces $W^{k,q}$); the approximation results given there are explicit in the approximation order but restricted to a fixed mesh. Approximation results (for L^2-based Sobolev spaces) that reflect both the local mesh size and the approximation order p can be found in [6, 36]. However, the constructions given there assume extra regularity, namely, the function to be approximation has to be in the Sobolev spaces H^s for some $s > d/2$, where $d \in \mathbb{N}$ is the spatial dimension.
In the present paper, we develop optimal quasi-interpolation operators suitable for an application in the framework of the hp-version of the FEM. We exhibit two kinds of closely related operators: Clément type operators (see Theorem 2.1) defined on the space L^1 and Scott/Zhang type operators (see Theorems 2.3, 2.4) defined on $W^{1,q}$ (so that traces on the boundary are defined) that preserve piecewise polynomial boundary conditions. Both operators achieve optimal rates of convergence. A particular application of the operators developed in the present paper is that they permit the extension of the h-FEM residual-based error estimation to the hp-FEM, [33].

This paper is organized as follows: In Section 1.1, we introduce the necessary notation, in particular γ-shape regular triangulations of two-dimensional domains and the hp-FEM spaces of piecewise mapped polynomials. We emphasize that the element

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maps need not be affine, which is an important aspect in *hp*-FEM, and that variable approximation order is considered. In Section 2, we present our quasi-interpolation operators; Section 2.2 is devoted to the proof of their approximation properties. A series of appendices concludes the paper: In Section A, we present optimal polynomial approximation results for $W^{k,q}$-functions on hyper cubes. Section B deals with several polynomial lifting results in two dimensions. Section C presents two polynomial extension operators in one dimension. In Section D, finally, we prove some polynomial inverse estimates in two dimensions.

1.1. Notation and Assumptions.

1.1.1. Triangulations. We start with the standard definitions of meshes and triangulations for two-dimensional domains.

A triangulation $\mathcal{T}$ of a set $\Omega \subset \mathbb{R}^2$ is a collection of elements $K \in \mathcal{T}$; associated with each element K is an element map $F_K : \widehat{K} \to K$, where the reference element $\widehat{K}$ corresponding to K is either the reference square $S = (0,1)^2$ or the reference triangle $T = \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1,\, 0 < y < \min(x, 1-x)\}$. We consider triangulations that satisfy the following standard conditions:

(M1) The element maps $F_K : \widehat{K} \to K = F_K(\widehat{K})$ are C^1-diffeomorphisms between $\overline{\widehat{K}}$ and $\overline{K}$, i.e., there exist domains $\widehat{K}'$ and K' with $\overline{\widehat{K}} \subset \widehat{K}'$, $\overline{K} \subset K'$ such that F_K is in fact a C^1-diffeomorphism between $\widehat{K}'$ and K'.

(M2) For two elements K, K' the intersection $\Gamma := \overline{K} \cap \overline{K'}$ falls into exactly one of the following categories: Γ is empty, or a vertex, or a whole edge, or K and K' coincide. (i.e., $F_K^{-1}(\Gamma)$ and $F_{K'}^{-1}(\Gamma)$ are edges, or vertices of the corresponding reference elements $\widehat{K}$, $\widehat{K}'$). Additionally, we require the map

$$Q : F_K^{-1}(\Gamma) \to F_{K'}^{-1}(\Gamma) : \quad x \mapsto (F_{K'}^{-1} \circ F_K)(x)$$

to be an affine homeomorphism.

(M3) $\Omega \setminus \cup_{K \in \mathcal{T}}$ is a set of Lebesgue measure zero.

A triangulation $\mathcal{T}$ is called γ-shape regular if additionally

$$h_K^{-1} \|F_K'\|_{L^\infty(\widehat{K})} + h_K \| \left(F_K'\right)^{-1} \|_{L^\infty(\widehat{K})} \leq \gamma, \tag{1.1}$$

where $h_K = \operatorname{diam} K$. We say that the triangulation is *affine* if all element maps F_K are affine maps. The restriction $\mathcal{T}|_\omega$ denotes the subset of $\mathcal{T}$ that represents the triangulation of $\omega \subset \Omega$ satisfying (M1)–(M3).

For each element $K \in \mathcal{T}$ we denote by $\mathcal{E}(K)$ the set of edges of K and by $\mathcal{N}(K)$ the set of vertices of K. Similarly, $\mathcal{N}(\mathcal{T})$ denotes the set of all vertices of $\mathcal{T}$ and $\mathcal{E}(\mathcal{T})$ the set of all edges. Setting

$$\hat{I} = (0,1)$$

the assumption (M2) implies that we can define *edge maps* $F_e : \hat{I} \to e$ for each $e \in \mathcal{E}(\mathcal{T})$ by taking an element K such that e is an edge of K, then identifying the edge $F_K^{-1}(e)$ of $\hat{K}$ with $\hat{I}$ via an affine map and finally taking F_e as the restriction of F_K to $F_K^{-1}(e)$; the assumption (M2) guarantees that the map F_e obtained in this way is independent of the choice of K. Additionally, we introduce the notion of the *patch* ω_V associated with a node $V \in \mathcal{N}(\mathcal{T})$ by

$$\omega_V := \{x \in \Omega \,|\, x \in \overline{K} \quad \text{for some } K \text{ with } V \in \overline{K}\}^\circ, \tag{1.2}$$

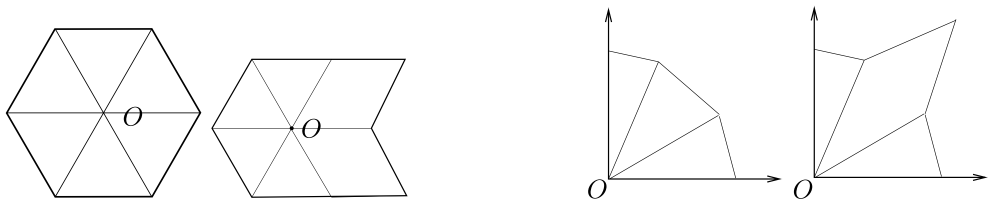

FIG. 1.1. *Left: interior reference patches $\omega_{int,6,1}$ and $\omega_{int,6,j}$ (for some $j \in \{2,\ldots,2^6\}$. Right: boundary reference patch $\omega_{bdy,3,1}$ and $\omega_{bdy,3,j}$ for some $j \in \{2,\ldots,2^3\}$.*

where A° denotes the interior of the set A. We note that the patches ω_V are open subsets of Ω. Of importance will be the connectivity of the patches. Our tool for classifying patches according to their connectivity will be the notion of *reference patches* that we make precise in the following definition:

DEFINITION 1.1 (Reference patch). *Reference patches are Lipschitz domains that are either labeled* interior *or* boundary *patches. They are characterized as follows:*

1. Interior patches: *For each $M \in \mathbb{N}$, $M \geq 3$, we define 2^M* interior reference patches *$\omega_{int,M,j}$, $j = 1,\ldots,2^M$, as follows: $\omega_{int,M,1}$ is defined to be the regular polygon with M edges of length 1 that is centered at the origin $0 \in \mathbb{R}^2$ and is triangulated with M triangles all sharing the vertex 0. The remaining $2^M - 1$ reference patches are obtained from this one by replacing one or several of these isoceles triangles by parallelograms (see Fig. 1.1).*
2. Boundary patches: *For each $M \in \mathbb{N}$ we define 2^M* boundary reference patches *$\omega_{bdy,M,j}$, $j = 1,\ldots,2^M$, in the following way: $\omega_{bdy,M,1} \subset \{(x,y)\,|\,x > 0, y > 0\}$ is the polygon that consists of M isoceles triangles all sharing the vertex $0 \in \mathbb{R}^2$ and having angle $\pi/(2M)$ at 0. The remaining $2^M - 1$ patches are obtained from this one by replacing one or several of these isoceles triangles by parallelograms (see Fig. 1.1).*

We will only consider triangulations whose patches can be related to these reference patches:

(M4) For each vertex V of the triangulation there exists a reference patch $\widehat{\omega}_V$ of the form given in Definition 1.1 together with a homeomorphism $F_V : \widehat{\omega}_V \to \omega_V$ with $F_V(0) = V$ that has the form

$$F_V^{-1}|_K = A_{K,V} \circ F_K^{-1} \qquad \forall K \in \mathcal{T}|_{\omega_V},$$

where the maps $A_{K,V} : \mathbb{R}^2 \to \mathbb{R}^2$ are *affine*.

REMARK 1.2. It is worth pointing out that slit domains are not excluded by (M4). However, other kinds of domains that fail to be Lipschitz domain are not covered by the present results: For example, domains such as the one depicted in Fig. 1.2 are not admitted since the vertex cannot be mapped to a boundary reference patch in the way condition (M4) requires. ■

We finish this subsection by noting that γ-shape regularity of the element maps implies that only a finite number of elements can meet at a vertex:

LEMMA 1.3. *Let $\mathcal{T}$ be a γ-shape regular triangulation satisfying (M1)–(M3). Then there exists a constant $M \in \mathbb{N}$, which depends only on γ, such that*

1. *no more than M elements share a common vertex;*
2. *for any two elements K, K' with $\overline{K} \cap \overline{K'} \neq \emptyset$ there holds $M^{-1} h_K \leq h_{K'} \leq M h_K$.*

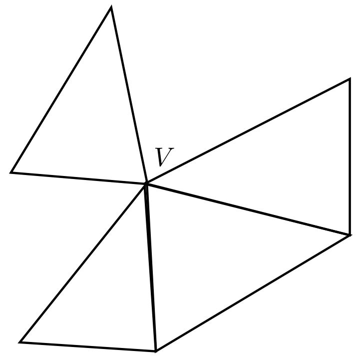

FIG. 1.2. *Example of a mesh excluded by (M4).*

If the triangulation satisfies additionally (M4), then the maps $A_{K,V}$ appearing in condition (M4) satisfy

$$\|A'_{K,V}\|_{L^\infty(\hat{K})} + \|{A'}^{-1}_{K,V}\|_{L^\infty(\hat{K})} \le C$$

for some $C > 0$ that depends only on γ. Additionally, $F_V \in W^{1,\infty}(\widehat{\omega}_V)$ and $F_V^{-1} \in W^{1,\infty}(\omega)$, and we have the bound

$$h_V^{-1}\|F'_V\|_{L^\infty(\widehat{\omega}_V)} + h_V\| \left(F'_V\right)^{-1} \|_{L^\infty(\widehat{\omega}_V)} \le C, \qquad h_V = \min_{K:V\in\mathcal{N}(K)} h_K,$$

for some $C > 0$ depending solely on γ.

Proof. 1. step: The element maps F_K are C^1 up to the boundary of the reference elements. The fact that the interior angles of the reference elements are non-degenerate and the γ-shape regularity assumption (1.1) then imply that the interior angles of elements $K \in \mathcal{T}$ are within $(\varepsilon, \pi - \varepsilon)$ for an $\varepsilon > 0$ that depends solely on γ. The first claim of the lemma then follows if we choose $M \in \mathbb{N}$ such that $M \ge 2\pi/\varepsilon$.

2. step: The γ-shape regularity assumption (1.1) also implies the existence of $C > 0$ depending solely on γ such

$$C^{-1}h_K \le |e| \le Ch_K \qquad \forall e \in \mathcal{E}(K) \qquad \forall K \in \mathcal{T}.$$

This fact together with the observation of the first step easily implies the second claim after appropriately adjusting the constant M.

3. step: We will only show that $F_V^{-1} \in W^{1\infty}(\omega)$ with the corresponding bound for the derivative. By assumption $F_V^{-1}|_K \in C^1(\overline{K})$ for each element $K \in \mathcal{T}|_{\omega_V}$. From this an elementwise integration by parts together with the observation $F_V^{-1} \in C(\omega)$ implies that the weak derivative is elementwise given by $(F_V^{-1})'|_K = A_{K,V} \cdot (F_K^{-1})'$. From this representation, we readily infer $F_V^{-1} \in W^{1,\infty}(\omega)$ and the desired bound. □

1.2. Polynomial spaces. The finite element spaces that we consider are the variable order piecewise mapped polynomials, an early implementation of which is discussed in [24]: For each element $K \in \mathcal{T}$, we choose a polynomial degree $p_K \in \mathbb{N}$ and collect these numbers in the polynomial degree vector $\mathbf{p} = (p_K)_{K\in\mathcal{T}}$. We then define the space $S^{\mathbf{p}}(\mathcal{T}) \subset W^{1,q}(\Omega)$ by

$$S^{\mathbf{p}}(\mathcal{T}) = \{u \in C(\Omega) \,|\, u|_K \circ F_K \in \Pi_{p_K}(\widehat{K})\}, \tag{1.3}$$

where we set

$$\Pi_p(\widehat{K}) = \begin{cases} \mathcal{P}_p := \operatorname{span}\{x^i y^j \,|\, 0 \le i+j \le p\} & \text{if } \widehat{K} = T \\ \mathcal{Q}_p := \operatorname{span}\{x^i y^j \,|\, 0 \le i,j \le p\} & \text{if } \widehat{K} = S. \end{cases} \tag{1.4}$$

We will write $S^p(\mathcal{T})$ if the degree vector $\mathbf{p}$ satisfies $p_K = p$ for all $K \in \mathcal{T}$. In this case, we will also permit the choice $p = 0$, where $S^0(\mathcal{T})$ reduces to a one-dimensional space.

REMARK 1.4. By writing integrals over Ω as a sum of integrals over elements, it can indeed be checked $S^{\mathbf{p}}(\mathcal{T}) \subset W^{1,\infty}(\Omega)$. ∎

A key property of the spaces $S^{\mathbf{p}}(\mathcal{T})$ is that we can identify "nodal shape functions" that form a partition of unity, i.e., for each vertex $V \in \mathcal{N}(\mathcal{T})$, we can find a function $\varphi_V \in S^1(\mathcal{T})$ such that

$$\varphi_V|_{\Omega\setminus\omega_V} \equiv 0 \qquad \text{and} \qquad \sum_{V\in\mathcal{N}(\mathcal{T})} \varphi_V \equiv 1 \quad \text{on } \Omega. \tag{1.5}$$

A well-known consequence of the γ-shape regularity of the triangulation is that these nodal shape functions satisfy for some constant $C > 0$ that depends solely on γ

$$\|\varphi_V\|_{L^\infty(\Omega)} \le 1, \qquad \|\nabla\varphi_V\|_{L^\infty(\Omega)} \le C h_K^{-1} \qquad \forall K \in \mathcal{T}|_{\omega_V}. \tag{1.6}$$

In the present paper we consider only γ-shape regular triangulations. Such triangulations have the property that neighboring elements are comparable in size (cf. Lemma 1.3). We will impose a similar condition on the polynomial degree distribution:

$$\gamma^{-1} p_K \le p_{K'} \le \gamma p_K \qquad \forall K, K' \in \mathcal{T} \quad \text{s.t. } \overline{K} \cap \overline{K'} \ne \emptyset. \tag{1.7}$$

We will also employ the notation

$$p_V := \min\{p_K \,|\; V \in \mathcal{N}(K)\}, \qquad p_e := \min\{p_K \,|\; e \in \mathcal{E}(K)\}. \tag{1.8}$$

1.3. Notation for Sobolev spaces. For domains $\Omega \subset \mathbb{R}^2$ and $k \in \mathbb{N}_0$, $q \in [1,\infty]$ we employ standard Sobolev spaces $W^{k,q}(\Omega)$ as described in, e.g., [1]. For the reference interval $\hat{I} = (0,1)$, $\kappa \in (0,1)$ and $q \in [1,\infty)$, we equip the space $W^{\kappa,q}(\hat{I})$ with the Slobodeckij norm

$$\|u\|^q_{W^{\kappa,q}(\hat{I})} = \|u\|^q_{L^q(\hat{I})} + \int_{\hat{I}}\int_{\hat{I}} \frac{|u(x)-y(y)|^q}{|x-y|^{1+q\kappa}}\,dx\,dy. \tag{1.9}$$

We will also require the spaces $\widetilde{W}^{\kappa,p}(\hat{I})$ which consist of the functions $u \in W^{\kappa,p}(\hat{I})$ such that their trivial extension (i.e., by zero) to $\mathbb{R}$ is an element of $W^{\kappa,p}(\mathbb{R})$. This space is equipped with the norm

$$\|u\|^q_{\widetilde{W}^{\kappa,p}(\hat{I})} = \|u\|^q_{W^{\kappa,p}(\hat{I})} + \int_0^1 \frac{|u(x)|^q}{x^{q-1}}\,dx + \int_0^1 \frac{|u(x)|^q}{(1-x)^{q-1}}\,dx. \tag{1.10}$$

In analogy to the spaces $\widetilde{W}^{\kappa,p}(\hat{I})$ we can define the spaces $\widetilde{W}^{\kappa,p}_l(\hat{I})$ if the trivial extension to $I' = \{x \in \mathbb{R} \,|\, x < 1\}$ is in $W^{\kappa,q}(I')$. This space is equipped with the norm

$$\|u\|^q_{\widetilde{W}^{\kappa,p}_l(\hat{I})} = \|u\|^q_{W^{\kappa,p}(\hat{I})} + \int_0^1 \frac{|u(x)|^q}{x^{q-1}}\,dx. \tag{1.11}$$

We finally record how functions transform under concatenation with the patch maps F_V:

LEMMA 1.5. *Let $\mathcal{T}$ be a γ-shape regular triangulation satisfying (M1)–(M4). Let $q \in [1, \infty]$. Then for every patch ω_V, $V \in \mathcal{N}(\mathcal{T})$, and every $u \in W^{1,q}(\omega_V)$ we have that $\widehat{u} := u \circ F_V \in W^{1,q}(\widehat{\omega}_V)$ and*

$$\|\widehat{u}\|_{L^q(\widehat{\omega}_V)} \sim h_V^{2/q} \|u\|_{L^q(\omega_V)}, \qquad \|\nabla \widehat{u}\|_{L^q(\widehat{\omega}_V)} \sim h_V^{1-2/q} \|\nabla u\|_{L^q(\omega_V)}, \tag{1.12}$$

where $h_V = \min_{K: K \subset \omega_V} h_K$. The constants hidden in the $\sim$-notation depend solely on γ and q.

Proof. We claim that the pull-back $\widehat{u}$ is in $W^{1,q}(\widehat{\omega}_V)$. To see this, we first consider the case $q < \infty$. For each element K of the patch ω_V and its corresponding element $K' := F_V^{-1}(K) \subset \widehat{\omega}_V$, the assumption (M1) guarantees that $F_V|_{K'} \in C^1(\overline{K'})$ and likewise $F_V^{-1} \in C^1(\overline{K})$. Hence by standard properties of Sobolev space (see, e.g., [1, Chap. III, Thm. 3.35]) we have for each element K that $u \circ F_V|_{K'} \in W^{1,q}(K')$ and the derivative satisfies $(\nabla (u \circ F_V))|_{K'} = (\nabla u \circ F_V) F_V'$. In order to see that $u \circ F_V$ is in $W^{1,q}(\widehat{\omega}_V)$ we have to check that the traces on the edges shared by two elements K_1, K_2 of $\widehat{\omega}_V$ coincide. This follows easily from the assumption (M3). The case $q = \infty$ is obtained by inspection: Since the weak derivative has been identified as $(\nabla u \circ F_V) F_V'$, one merely has to check that it is in $L^\infty(\widehat{\omega}_V)$, which is indeed the case. The bounds (1.12) now follow from (1.1). ∎

2. Quasi-interpolation of non-smooth functions.

2.1. Approximation results.

We present two types of quasi-interpolation results for $W^{1,q}$-functions: In Theorem 2.1 we exhibit a quasi-interpolation operator of Clément type; in Theorem 2.3 we present an operator that additionally preserves homogeneous boundary conditions that may be imposed on parts of the boundary. This latter operator is generalized in Theorem 2.4 to an operator that preserves arbitrary piecewise polynomial Dirichlet boundary conditions.

In order to formulate these results, we introduce the following additional notation: For $e \in \mathcal{E}(\mathcal{T})$ we denote by $\mathcal{N}(e)$ the two endpoints of e, i.e., $\mathcal{N}(e) = \{V \in \mathcal{N}(\mathcal{T}) \mid V \in \overline{e}\}$. Patches of order $j \in \mathbb{N}$ associated with an element $K \in \mathcal{T}$ or an edge $e \in \mathcal{E}(\mathcal{T})$ are defined thus:

$$\omega_e^1 := \bigcup_{V \in \mathcal{N}(e)} \omega_V, \qquad \omega_e^{j+1} := \bigcup_{V \in \mathcal{N}(\mathcal{T}) : V \in \overline{\omega_e^j}} \omega_V, \quad j = 1, 2, \ldots, \tag{2.1}$$

$$\omega_K^1 := \bigcup_{V \in \mathcal{N}(K)} \omega_V, \qquad \omega_K^{j+1} := \bigcup_{V \in \mathcal{N}(\mathcal{T}) : V \in \overline{\omega_K^j}} \omega_V, \quad j = 1, 2, \ldots, \tag{2.2}$$

2.1.1. Clément type approximation.

Quasi-interpolation of Clément type takes the following form:

THEOREM 2.1 (Clément type quasi-interpolation). *Let $\mathcal{T}$ be a γ-shape regular triangulation of a domain $\Omega \subset \mathbb{R}^2$ satisfying (M1)–(M4) and let $\mathbf{p}$ be a polynomial degree distribution satisfying (1.7). Then there exists a bounded linear operator $I^{hp} : L^1(\Omega) \to S^{\mathbf{p}}(\mathcal{T}) \subset L^1(\Omega)$, and there exists a constant $C > 0$ that depends solely on $q \in [1, \infty]$ and γ such that for every $u \in W^{1,q}(\Omega)$ and all elements $K \in \mathcal{T}$ and all* *edges* $e \in \mathcal{E}(\mathcal{T})$

$$\|u - I^{hp}u\|_{L^q(K)} + \frac{h_K}{p_K}\|\nabla(u - I^{hp}u)\|_{L^q(K)} \leq C\frac{h_K}{p_K}\|\nabla u\|_{L^q(\omega_K^1)}, \tag{2.3}$$

$$\|u - I^{hp}u\|_{L^q(e)} \leq C\left(\frac{h_e}{p_e}\right)^{1-1/q}\|\nabla u\|_{L^q(\omega_e^1)}. \tag{2.4}$$

2.1.2. Scott-Zhang type approximation. The operator I^{hp} of Theorem 2.1 does not preserve boundary conditions if applied to functions of $W^{1,q}(\Omega)$. The operators of Theorem 2.1 can, however, be modified to accommodate this.

Let a set $\mathcal{B} \subset \mathcal{E}(\mathcal{T})$ of boundary edges of the triangulation $\mathcal{T}$ be given, i.e.,

$$\mathcal{B} \subset \mathcal{E}(\mathcal{T}) \quad \text{and} \quad b \subset \partial\Omega \qquad \forall b \in \mathcal{B}. \tag{2.5}$$

Next, we define for $q \in (1, \infty)$ the spaces

$$W^{1,q}_{\mathcal{B},0} := \{u \in W^{1,q}(\Omega) \,|\, u|_b = 0 \quad \text{for all } b \in \mathcal{B}\}, \tag{2.6}$$

$$W^{1,q}_{\mathcal{B},\mathbf{p}} := \{u \in W^{1,q}(\Omega) \,|\, u|_b \circ F_b \in \mathcal{P}_{p_b} \quad \text{for all } b \in \mathcal{B} \text{ and (2.8) holds}\}, \tag{2.7}$$

where the continuity condition (2.8) is:

$$\text{for all } b,\, b' \in \mathcal{B} \text{ and } V \in \mathcal{N}(b) \cap \mathcal{N}(b') \text{ there holds } \lim_{\substack{x \to V \\ x \in b}} u(x) = \lim_{\substack{x \to V \\ x \in b'}} u(x). \tag{2.8}$$

Remark 2.2. Since the edges of $\mathcal{B}$ are part of the boundary of $\partial\Omega$, the function values are understood in the sense of traces. In the case of slit domains appropriate limits have to be taken. ∎

We then have the following approximation results:

Theorem 2.3 (homogeneous boundary conditions). *Let $\mathcal{T}$ be a γ-shape regular triangulation of a domain $\Omega \subset \mathbb{R}^2$ satisfying (M1)–(M4). Let $\mathbf{p}$ be a polynomial degree distribution satisfying (1.7). Let $q \in (1, \infty)$ and a set $\mathcal{B} \subset \mathcal{E}(\mathcal{T})$ of boundary edges be given. Then there exists a linear operator $I^{hp}_{hom} : W^{1,q}_{\mathcal{B},0}(\Omega) \to S^{\mathbf{p}}(\mathcal{T}) \cap W^{1,q}_{\mathcal{B},0}(\Omega)$, and there exists a constant $C > 0$ depending solely on γ and q such that*

$$\|u - I^{hp}_{hom}u\|_{L^q(K)} + \frac{h_K}{p_K}\|\nabla(u - I^{hp}_{hom}u)\|_{L^q(K)} \leq C\frac{h_K}{p_K}\|\nabla u\|_{L^q(\omega_K^1)}, \tag{2.9}$$

$$\|u - I^{hp}_{hom}u\|_{L^q(e)} \leq C\left(\frac{h_e}{p_e}\right)^{1-1/q}\|\nabla u\|_{L^q(\omega_e^1)}. \tag{2.10}$$

A slightly different situation arises if non-homogeneous piecewise polynomial boundary conditions are to be preserved: The domain of influence in the local bounds is enlarged, and we impose a restriction on the variation in polynomial degree distribution for elements near the Dirichlet part of the boundary:

Theorem 2.4 (Scott-Zhang type quasi-interpolation). *Let $q \in (1, \infty)$, $\mathcal{T}$ be a γ-shape regular triangulation of a domain $\Omega \subset \mathbb{R}^2$ satisfying (M1)–(M4). Let $\mathbf{p}$ be a polynomial degree distribution satisfying (1.7). Let $\mathcal{B} \subset \mathcal{E}(\mathcal{T})$ be a collection of boundary edges. Assume additionally that*

$$|p_K - p_{K'}| \leq \gamma \qquad \forall K, K' \text{ s.t. } \overline{K} \cap \overline{K'} \cap \overline{b} \neq \emptyset \text{ for some } b \in \mathcal{B}. \tag{2.11}$$

Then there exists a linear operator $I^{hp}_{inhom} : W^{1,q}_{\mathcal{B},\mathbf{p}}(\Omega) \to S^{\mathbf{p}}(\mathcal{T})$ *such that*

$$(I^{hp}_{inhom} u)|_b = u|_b \qquad \forall b \in \mathcal{B}.$$

Furthermore, there exists a constant $C > 0$ *depending only on* γ *and* q *such that for all elements* $K \in \mathcal{T}$ *and all edges* $e \in \mathcal{E}(\mathcal{T})$

$$\|u - I^{hp}_{inhom} u\|_{L^q(K)} + \frac{h_K}{p_K}\|\nabla(u - I^{hp}_{inhom} u)\|_{L^q(K)} \le C\frac{h_K}{p_K}\|\nabla u\|_{L^q(\omega^4_K)},$$

$$\|u - I^{hp}_{inhom} u\|_{L^q(e)} \le C\left(\frac{h_K}{p_K}\right)^{1-1/q}\|\nabla u\|_{L^q(\omega^4_e)}.$$

Remark 2.5. The dependence on the domains ω^4_K, ω^4_e is not optimal. A careful inspection of the proof allows slightly sharper bounds. For example, for elements K such that $\omega^4_K \subset\subset \Omega$ we can replace ω^4_K with ω^1_K. ▪

2.2. Proofs.

2.2.1. Proof of Theorem 2.1. Theorem 2.1 is proved using the ideas of the partition of unity method, [32], which is based on the following result:

Lemma 2.6. *Let* $\mathcal{T}$ *be a* γ*-shape regular triangulation triangulation of a domain* $\Omega \subset \mathbb{R}^2$ *satisfying (M1)–(M3). Let* $q \in [1,\infty]$*, and let* $\mathbf{p}$ *be an arbitrary polynomial degree distribution. Assume that for a given* $u \in W^{1,q}(\Omega)$ *a function* $u_V \in S^{p_V-1}(\mathcal{T}|_{\omega_V})$ *is given for each* $V \in \mathcal{N}(\mathcal{T})$*, where* p_V *is defined in (1.8). Then there exists* $C > 0$ *depending solely on* γ *such that the function* $\widetilde{u} := \sum_{V\in\mathcal{N}(\mathcal{T})}\varphi_V u_V \in S^{\mathbf{p}}(\mathcal{T})$ *and*

$$\|u - \tilde{u}\|_{L^q(K)} \le C\sum_{V\in\mathcal{N}(K)}\|u - u_V\|_{L^q(K)},$$

$$\|\nabla(u - \tilde{u})\|_{L^q(K)} \le C\sum_{V\in\mathcal{N}(K)}\left[\|\nabla(u - u_V)\|_{L^q(K)} + \frac{1}{h_K}\|u - u_V\|_{L^q(K)}\right],$$

$$\|u - \tilde{u}\|_{L^q(e)} \le C\sum_{V\in\mathcal{N}(e)}\|u - u_V\|_{L^q(e)}.$$

Proof. We start by ascertaining $\tilde{u} \in C(\Omega)$. This follows easily from the support properties of the functions $\varphi_V \in C(\Omega)$, namely, $\varphi|_{\Omega\setminus\omega_V} \equiv 0$, together with $u_V \in C(\omega_V)\cap L^\infty(\omega_V)$. In order to see $\tilde{u} \in S^{\mathbf{p}}(\mathcal{T})$ we have to make sure that $(\varphi_V u_V)\circ F_K \in \Pi_{p_K}(\hat{K})$ for all $K \in \mathcal{T}|_{\omega_V}$ for all $V \in \mathcal{N}(\mathcal{T})$. This follows easily from $\varphi_V \in S^1(\mathcal{T})$ and $u_V \in S^{p_V-1}(\mathcal{T}|_{\omega_V})$. The essential ingredient for proving the estimates is the observation that $\sum_{V\in\mathcal{N}(K)}\varphi_V \equiv 1$ on K for every $K \in \mathcal{T}$ and $\sum_{V\in\mathcal{N}(e)}\varphi_V \equiv 1$ on e for every $e \in \mathcal{E}(\mathcal{T})$. The bounds on $(u - \tilde{u})|_K$ then follow from the observation that $(u-\tilde{u})|_K = \sum_{V\in\mathcal{N}(K)}\varphi_V(u - u_V)$, where the sum extend over at most 4 terms, and from the bounds (1.5) on the functions φ_V. □

Lemma 2.7. *Let* $\mathcal{T}$ *a* γ*-shape regular triangulation of a domain* $\Omega \subset \mathbb{R}^2$ *satisfying (M1)–(M4). Assume that the polynomial degree distribution* $\mathbf{p}$ *satisfies (1.7). Then for each vertex* V *there exists a bounded linear operator* $I_V : L^1(\omega_V) \to S^{p_V-1}(\mathcal{T}|_{\omega_V})$*, and there exists a constant* $C > 0$ *that depends solely on* γ *such that for each* $u \in$

$W^{1,q}(\omega_V)$, *each* $K \in \mathcal{T}|_{\omega_V}$, *and each edge* $e \in \mathcal{E}(\mathcal{T}|_{\omega_V})$

$$\|u - I_V u\|_{L^q(K)} + \frac{h_K}{p_K} \|\nabla(u - I_V u)\|_{L^q(K)} \leq C \frac{h_V}{p_V} \|\nabla u\|_{L^q(\omega_V)},$$
$$\|u - I_V u\|_{L^q(e)} \leq C \left(\frac{h_V}{p_V}\right)^{1-1/q} \|\nabla u\|_{L^q(\omega_V)}.$$

Proof. Consider a patch ω_V. Condition (M4) provides the patch map $F_V : \widehat{\omega}_V \to \omega_V$ and Lemma 1.5 gives $\widehat{u}_V = u|_K \circ F_V \in W^{1,q}(\widehat{\omega}_V)$ together with

$$\|\widehat{u}_V\|_{L^q(\widehat{\omega}_V)} \leq C h_V^{2/q} \|u\|_{L^q(\omega_V)}, \qquad \|\nabla \widehat{u}_V\|_{L^q(\widehat{\omega}_V)} \leq C h_V^{1-2/q} \|\nabla u\|_{L^q(\omega_V)}.$$

Let $\widehat{S}$ be a square such that $\widehat{\omega}_V \subset\subset \widehat{S}$ and denote by $E : L^1(\widehat{\omega}_V) \to L^1(\widehat{S})$ the universal linear extension operator of [38]. We then have the existence of a constant $C_q > 0$, which depends solely on $q \in [1, \infty]$ and $\widehat{\omega}_V$, such that

$$\|E\widehat{u}_V\|_{L^q(\widehat{S})} \leq C_q \|\widehat{u}_V\|_{L^q(\widehat{\omega}_V)}, \qquad \|E\widehat{u}_V\|_{W^{1,q}(\widehat{S})} \leq C \|\widehat{u}_V\|_{W^{1,q}(\widehat{\omega}_V)}.$$

Choosing $N = \lfloor (p_V - 1)/2 \rfloor$ in the approximation result Theorem A.3, we obtain a bounded linear operator $J_{1,N} : L^1(\widehat{S}) \to \mathcal{Q}_N \subset \mathcal{P}_{p_V - 1}$ that reproduces constant functions and satisfies

$$(p_V + 1)\|v - J_{1,N} v\|_{L^q(\widehat{S})} + \|\nabla(v - J_{1,N} v)\|_{L^q(\widehat{S})} \leq C \|v\|_{W^{1,q}(\widehat{S})} \qquad \forall v \in W^{1,q}(\widehat{S}).$$

We next define the operator $J_{p_V} : L^1(\widehat{\omega}_V) \to \mathcal{P}_{p_V - 1}$ by

$$J_{p_V} v := \overline{v} + J_{1,N} \circ E(v - \overline{v}), \qquad \overline{v} := \frac{1}{|\widehat{\omega}_V|} \int_{\widehat{\omega}_V} v(x)\, dx.$$

J_{p_V} is a bounded linear operator on $L^1(\widehat{\omega}_V)$, and we obtain for $W^{1,q}$-functions:

$$(p_V + 1)\|v - J_{p_V} v\|_{L^q(\widehat{\omega}_V)} + \|\nabla(v - J_{p_V} v)\|_{L^q(\widehat{\omega}_V)} \leq C \|v - \overline{v}\|_{W^{1,q}(\widehat{\omega}_V)} \leq C \|\nabla v\|_{L^q(\widehat{\omega}_V)},$$

where in the last estimate we employed the second Poincaré inequality. Applying this operator to the pull-back $\widehat{u}_V$, we obtain

$$(p_V + 1)\|\widehat{u}_V - J_{p_V} \widehat{u}_V\|_{L^q(\widehat{\omega}_V)} + \|\nabla(\widehat{u}_V - J_{p_V} \widehat{u}_V)\|_{L^q(\widehat{\omega}_V)} \leq C \|\nabla \widehat{u}_V\|_{L^q(\widehat{\omega}_V)}$$
$$\leq C h_V^{1-2/q} \|u_V\|_{L^q(\omega_V)}.$$

Returning to the patch ω_V, we observe that the function u_{p_V} defined on ω_V by $u_{p_V} = (J_{p_V} \widehat{u}_V) \circ F_V^{-1}$ is an element of $S^{p_V - 1}(\mathcal{T}|_{\omega_V})$ (this is due to the fact that elementwise F_V is the composition of an *affine* map and the element map) and

$$(p_V + 1) h_V^{-2/q} \|u_V - u_{p_V}\|_{L^q(\omega_V)} + h_V^{1-2/q} \|\nabla(u_V - u_{p_V})\|_{L^q(\omega_V)} \leq C h_V^{1-2/q} \|\widehat{u}_V\|_{L^q(\omega_V)}.$$

This leads to the desired bound on elements $K \in \mathcal{T}|_{\omega_V}$. For the bound on an edge $e \in \mathcal{E}(\mathcal{T}|_{\omega_V})$, we employ a trace theorem on $\widehat{\omega}_V$ before transforming back to ω_V.
Checking the steps of the construction, we see that the map $u_V \mapsto u_{p_V}$ is linear and that it is at the same time a bound linear map $L^1(\omega_V) \to \mathcal{P}_{p_V - 1}$.
The constant in the last estimate does depend on the reference patch $\widehat{\omega}_V$. We observe, however, that for a given (upper bound on) γ, only finitely many reference patches

have to be considered since only finitely many elements can abut on a vertex (cf. Lemma 1.3). This concludes the argument. ▯

Proof of Theorem 2.1: The proof of Theorem 2.1 now follows from combining Lemmata 2.6 and 2.7. For each vertex V, we construct the local approximation $I_V u \in S^{p_V-1}(\mathcal{T}|_{\omega_V})$ with the aid of Lemma 2.7. The operator $I^{hp} : L^1(\Omega) \to S^{\mathbf{p}}(\mathcal{T})$ is then defined as

$$I^{hp} u = \sum_{V \in \mathcal{N}(\mathcal{T})} \varphi_V I_V u,$$

where the vertex shape functions $\varphi_V \in S^1(\mathcal{T})$ have the support properties of (1.5). The operator I^{hp} maps indeed into $S^{\mathbf{p}}(\mathcal{T})$ since $I_V u \in s^{p_V-1}(\mathcal{T}|_{\omega_V})$. By inspection, we observe that $I^{hp} : L^1(\Omega) \to S^{\mathbf{p}}(\mathcal{T})$ is a bounded linear operator. Its approximation properties, when applied to $W^{1,q}$-functions, follow from Lemmata 2.6 and 2.7. □

2.2.2. Proof of Theorem 2.3. We modify the approximation operator of Theorem 2.1 so as to enforce homogeneous Dirichlet boundary conditions. Since we need the trace theorem to hold, the operator is now defined on $W^{1,q}(\Omega)$ instead of $L^1(\Omega)$. The construction of this operator is again patch oriented. The difference is that we will change the definition of the linear maps I_V for $V \in \mathcal{N}(\mathcal{B})$. Here, we defined

$$\mathcal{N}(\mathcal{B}) := \bigcup_{b \in \mathcal{B}} \mathcal{N}(b). \tag{2.12}$$

We first analyze the prototypical situation on a boundary reference patch:

LEMMA 2.8. *Let $q \in (1,\infty)$. Let $\widehat{\omega} = \omega_{bdy,M,j}$ for some $M \in \mathbb{N}$ and $j \in \{1,\dots,2^M\}$ and denote by $\widehat{\mathcal{T}}$ the triangulation of $\widehat{\omega}$. Denote by Γ_0 the edge of $\widehat{\omega}$ lying on the x-axis and by Γ_M the edge lying on the y-axis (cf. Fig. 2.1). Let Γ_D be either Γ_0, Γ_M or $\Gamma_0 \cup \Gamma_M \cup \{0\}$. Denote $W^{1,q}_{\Gamma_D,0}(\widehat{\omega}) = \{u \in W^{1,q}(\widehat{\omega}) \,|\, u|_{\Gamma_D} = 0\}$. Then for every $p \in \mathbb{N}_0$ there exists a bounded linear map $I_p : W^{1,q}_{\Gamma_D,0}(\widehat{\omega}) \to S^p(\widehat{\mathcal{T}}) \cap W^{1,q}_{\Gamma_D,0}(\widehat{\omega})$ such that*

$$(p+1)\|u - I_p u\|_{L^q(\widehat{\omega})} + \|\nabla(u - I_p u)\|_{L^q(\widehat{\omega})} \le C\|\nabla u\|_{L^q(\widehat{\omega})}, \tag{2.13}$$

where the constant $C > 0$ is independent of p and $u \in W^{1,q}_{\Gamma_D,0}(\widehat{\omega})$.

Proof. We will demonstrate the result for the case $\Gamma_D = \Gamma_0 \cup \Gamma_M \cup \{0\}$, the other two cases being handled similarly. The construction of I_p is done in two steps: First, we let $J_p : L^1(\widehat{\omega}) \to \mathcal{P}_p$ be the linear operator of the proof of Lemma 2.7. It satisfies for $u \in W^{1,q}(\widehat{\omega})$

$$(p+1)\|u - J_p u\|_{L^q(\widehat{\omega})} + \|\nabla(u - J_p u)\|_{L^q(\widehat{\omega})} \le C\|\nabla u\|_{L^q(\widehat{\omega})}.$$

In particular, from the multiplicative trace inequality (see, e.g., [20, Thm. 1.6.6]) and the fact that $u|_{\Gamma_D} = 0$ we get

$$\|J_p u\|_{L^q(\Gamma_D)} = \|u - J_p u\|_{L^q(\Gamma_D)} \le C(p+1)^{-(1-1/q)}\|\nabla u\|_{L^q(\widehat{\omega})},$$
$$\|J_p u\|_{W^{1-1/q,q}(\Gamma_D)} = \|u - J_p u\|_{W^{1-1/q,q}(\Gamma_D)} \le C\|\nabla u\|_{L^q(\widehat{\omega})},$$

The function $J_p u$ does not, however, satisfy homogeneous boundary conditions on Γ_D. This is corrected in a second step by an element-by-element construction using appropriate polynomial liftings. To that end, we enumerate the edges of $\widehat{\mathcal{T}}$ emanating from the origin in a counterclockwise fashion as depicted in Fig. 2.1. Likewise, the

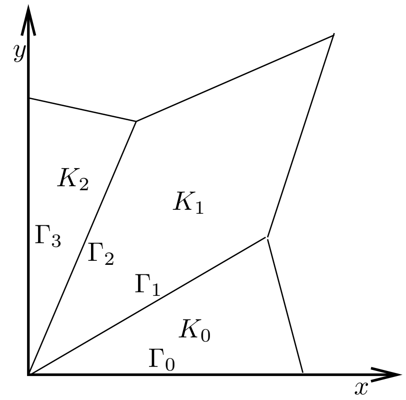

FIG. 2.1. *Numbering of elements and edges of a boundary reference patch for $M = 3$.*

elements are labeled K_i, $i = 0, \ldots, M-1$. Next, we observe that the functions $u_0 := (u - J_p u)|_{\Gamma_0} = (J_p u)|_{\Gamma_0}$ and $u_M := (u - J_p u)|_{\Gamma_M} = (J_p u)|_{\Gamma_M}$ are polynomials of degree p. Additionally, we note that each edge Γ_i, $i = 0, \ldots, M$, is homeomorphic to the reference interval $\hat{I} = (0,1)$ by means of an affine map $\gamma_i : \hat{I} \to \Gamma_i$, which we may choose to satisfy $\gamma_i(0) = 0$ for $i \in \{0, \ldots, M\}$. We then define a function $z \in C(\cup_{i=0}^M \overline{\Gamma_i})$ by

$$
\begin{aligned}
z \circ \gamma_i(x) &:= u_0 \circ \gamma_0(x), \qquad x \in \overline{\hat{I}}, \quad i = 0, \ldots, M-1, \\
z \circ \gamma_M(x) &:= u_M \circ \gamma_M(x).
\end{aligned}
$$

Clearly, for $i \in \{0, \ldots, M\}$ we have

$$
\|z\|_{L^q(\Gamma_i)} \le C \left[\|u_0\|_{L^q(\Gamma_0)} + \|u_M\|_{L^q(\Gamma_M)} \right] \le C(p+1)^{-1+1/q} \|\nabla u\|_{L^q(\widehat{\omega})}.
$$

We next define for each element K_i the set

$$
\Gamma_{i,i+1} := \Gamma_i \cup \Gamma_{i+1} \cup \{0\} \subset \partial K_i, \qquad i = 0, \ldots, M-1. \tag{2.14}
$$

We then ascertain

$$
\|z\|_{W^{1-1/q,q}(\Gamma_{i,i+1})} \le C \|J_p u\|_{W^{1-1/q,q}(\Gamma_D)} \le C \|u\|_{W^{1,q}(\widehat{\omega})}.
$$

This follows easily from the fact that for two edges Γ_i, Γ_j with $i \neq j$ we have the characterization (see, e.g., [28, Thm. 1.5.2.3])

$$
\|z\|^q_{W^{1-1/q,q}(\Gamma_i \cup \Gamma_j \cup \{0\})} \sim \|\hat{z}_i\|^q_{W^{1-1/q,q}(\hat{I})} + \|\hat{z}_j\|^q_{W^{1-1/q,q}(\hat{I})} + \int_0^1 \frac{|\hat{z}_i(x) - \hat{z}_j(x)|^q}{x^{q-1}} \, dx
$$

where we wrote $\hat{z}_i = z \circ \gamma_i$, $\hat{z}_j = z \circ \gamma_j$. Here, the constants hidden in the $\sim$-notation depend solely on Γ_i, Γ_j, and q. We finally construct with the aid of Proposition B.7 a function $Z \in S^{4p}(\widehat{\mathcal{T}})$ such that $Z|_{\Gamma_i} = z|_{\Gamma_i}$ for all $i \in \{0, \ldots, M\}$ and

$$
\begin{aligned}
(p+1)\|Z\|_{L^q(K_i)} + \|\nabla Z\|_{L^q(K_i)} &\le C \left[\|z\|_{W^{1-1/q,q}(\Gamma_{i,i+1})} + (p+1)^{1-1/q} \|z\|_{L^q(\Gamma_{i,i+1})} \right] \\
&\le C \|\nabla u\|_{L^q(\widehat{\omega})}, \qquad i = 0, \ldots, M-1.
\end{aligned}
$$

We conclude the argument by noting that the map $u \mapsto J_p u + Z$ is linear and bounded. Since $J_p u + Z \in S^{4p}(\widehat{\mathcal{T}})$, replacing p with $\lfloor p/4 \rfloor$ gives the desired result. ∎

Proof of Theorem 2.3: The proof of Theorem 2.3 now follows by the same arguments as that of Theorem 2.1. Merely for the patches ω_V with $V \in \mathcal{N}(\mathcal{B})$ we replace the local approximation $I_V u$ of Lemma 2.7 with the pushforward $(I_{p_V} \widehat{u}_V) \circ F_V^{-1}$ of $I_{p_V} \widehat{u}_V$, where $I_{p_V} \widehat{u}_V$ with $\widehat{u}_V = u|_{\omega_V} \circ F_V$ is defined in Lemma 2.8. □

2.2.3. Proof of Theorem 2.4: Lifting results. The proof of Theorem 2.4 follows along the same lines as that of Theorem 2.3. The key difference is that additionally appropriate (polynomial) liftings are required. Providing these is the purpose of the present subsection.

We start with a "vertex lifting" result on boundary reference patches that yields the correct value at a boundary vertex. Given a collection of boundary edges $\widehat{\mathcal{B}}$ of the reference patch $\widehat{\omega}$, the spaces $W^{1,q}_{\widehat{\mathcal{B}},\mathbf{p}}(\widehat{\omega})$ on $\widehat{\omega}$ are defined analogously to the way the spaces $W^{1,q}_{\mathcal{B},\mathbf{p}}(\Omega)$ are defined in (2.6). We then have:

LEMMA 2.9. *Let $\widehat{\omega} = \widehat{\omega}_{bdy,M,j}$ for some $M \in \mathbb{N}$, $j \in \{1, \dots, 2^M\}$, and denote by $\widehat{\mathcal{T}}$ the triangulation of $\widehat{\omega}$. Let $\mathbf{p}$ be a polynomial degree distribution on $\widehat{\mathcal{T}}$ and assume that*

$$|p_K - p_{K'}| \le k \qquad \forall K, K' \in \widehat{\mathcal{T}}.$$

Define $p' := \min\{p_K - 1 \,|\, K \in \widehat{\mathcal{T}}\} \in \mathbb{N}_0$.

Let $\widehat{\mathcal{B}} = \{\Gamma_0\}$ or $\widehat{\mathcal{B}} = \{\Gamma_M\}$ or $\widehat{\mathcal{B}} = \{\Gamma_0, \Gamma_M\}$ (cf. Fig. 2.1). Then there exists a constant $C > 0$ that depends solely on $\widehat{\omega}$ (i.e., on M, j) and k, q, and there exists a bounded linear operator $L : W^{1,q}_{\widehat{\mathcal{B}},\mathbf{p}}(\widehat{\omega}) \to S^{p'}(\widehat{\mathcal{T}})$ such that

$$(Lu - u)(0) = 0, \tag{2.15}$$

$$\|Lu - u\|_{W^{1,q}(\widehat{\omega})} \le C \|\nabla u\|_{L^q(\widehat{\omega})}, \tag{2.16}$$

$$\|(Lu - u) \circ \gamma_b\|_{\widetilde{W}_l^{1-1/q,q}(\hat{I})} \le C \|\nabla u\|_{L^q(\widehat{\omega})} \qquad \forall b \in \widehat{\mathcal{B}}, \tag{2.17}$$

where $\gamma_b : \hat{I} \to b$ is the affine parametrization of $b \in \widehat{\mathcal{B}}$ satisfying $\gamma_b(0) = 0$.

Proof. We employ ideas similar to those of the proof of Theorem 2.3. For simplicity of notation, we consider the case $\widehat{\mathcal{B}} = \{\Gamma_0, \Gamma_M\}$; the other two cases are treated in a similar fashion. We denote by $\gamma_i : \hat{I} \to \Gamma_i$, $i = 0, \dots, M$, the affine parametrizations of the edges Γ_i, which are assumed without loss of generality to satisfy $\gamma_i(0) = 0$. We will construct Lu first on the edges Γ_i and in a second step define Lu on the elements via appropriate liftings.

We write $p = \max\{p_K \,|\, K \in \widehat{\mathcal{T}}\} \in \mathbb{N}$. Choose $b \in \widehat{\mathcal{B}}$. Without loss of generality, we assume that $b = \Gamma_0$. By assumption $u \circ \gamma_0 \in \mathcal{P}_p$, so that we may define $l_0 := Z_{p,p'}(u \circ \gamma_0)$, where the linear operator $Z_{p,p'} : \mathcal{P}_p \to \mathcal{P}_{p'}$ is the polynomial extension operator of Lemma C.2. We then have $l_0(0) = u(0)$ and additionally by properties of $Z_{p,p'}$ and the trace theorem

$$\|l_0 - u|_{\Gamma_0} \circ \gamma_0\|_{\widetilde{W}_l^{1-1/q,q}(\hat{I})} \le C \|u\|_{W^{1-1/q,q}(\Gamma_0)} \le C \|u\|_{W^{1,q}(\widehat{\omega})}, \tag{2.18}$$

$$\|l_0 - u|_{\Gamma_M} \circ \gamma_M\|_{\widetilde{W}_l^{1-1/q,q}(\hat{I})} \le C \|u\|_{W^{1,q}(\widehat{\omega})}. \tag{2.19}$$

Next, we define

$$(Lu)|_{\Gamma_i} = l_0 \circ \gamma_i^{-1}, \qquad i = 0, \dots, M.$$

This gives $(Lu)(0) = u(0)$. Furthermore, this definition of $(Lu)|_{\Gamma_i}$ in conjunction with the bounds (2.18), (2.19) implies

$$\|Lu\|_{W^{1-1/q,q}(\Gamma_{i,i+1})} \leq C\|u\|_{W^{1,q}(\widehat{\omega})}, \qquad i = 0, \dots, M-1,$$

where we abbreviate $\Gamma_{i,i+1} = \Gamma_i \cup \Gamma_{i+1} \cup \{0\}$ as in (2.14). From the lifting result Theorem B.4, there exists then a function $Lu \in S^{p'}(\widehat{\mathcal{T}})$ with

$$\|Lu\|_{W^{1,q}(\widehat{\omega})} \leq C\|u\|_{W^{1,q}(\widehat{\omega})}. \tag{2.20}$$

Furthermore, inspection of the construction of Lu reveals that $u \mapsto Lu$ is linear. Since the operator $Z_{p,p'}$ of Lemma C.2 satisfies $Z_{p,p'}1 = 1$ and the lifting of Theorem B.4 likewise ensures that constant functions are reproduced (cf. Remark B.5), we conclude $L1 = 1$. By a standard argument, we can therefore strengthen (2.20) to yield (2.16). The estimates (2.17) are ensured by the way we defined $(Lu)|_{\Gamma_i}$ for $i \in \{0, \dots, M\}$. ☐

Lemma 2.7 allows us to construct a lifting operator as follows:

Proposition 2.10. *Let $\mathcal{T}$ be a γ-shape regular triangulation of a domain $\Omega \subset \mathbb{R}^2$ satisfying (M1)–(M4). Let $\mathcal{B} \subset \mathcal{E}(\mathcal{T})$ be a collection of boundary edges. Let $q \in (1, \infty)$ be given. Assume that the polynomial degree distribution $\mathbf{p}$ satisfies (1.7) and additionally (2.11). Then there exists a constant $C > 0$ that depends solely on γ and q, and there exists a bounded linear operator $I^{hp}_{lift} : W^{1,q}_{\mathcal{B},\mathbf{p}}(\Omega) \to S^{\mathbf{p}}(\mathcal{T})$ such that*

$$\begin{aligned}
(I^{hp}_{lift}u)|_b &= u|_b \qquad \forall b \in \mathcal{B},\\
(I^{hp}_{lift}u)|_K &= 0 \qquad \text{if } \omega_{K,\mathcal{B}} = \emptyset,\\
\|I^{hp}_{lift}u\|_{L^q(K)} &\leq C\left[\|u\|_{L^q(\omega_{K,\mathcal{B}})} + h_K\|\nabla u\|_{L^q(\omega_{K,\mathcal{B}})}\right] \qquad \text{if } \omega_{K,\mathcal{B}} \neq \emptyset,\\
\|\nabla I^{hp}_{lift}u\|_{L^q(K)} &\leq C\left[\frac{1}{h_K}\|u\|_{L^q(\omega_{K,\mathcal{B}})} + \|\nabla u\|_{L^q(\omega_{K,\mathcal{B}})}\right] \qquad \text{if } \omega_{K,\mathcal{B}} \neq \emptyset,
\end{aligned}$$

where, for an element $K \in \mathcal{T}$, we define

$$\omega_{K,\mathcal{B}} := \bigcup_{V \in \mathcal{N}(K) \cap \mathcal{N}(\mathcal{B})} \omega_V. \tag{2.21}$$

Proof. The lifting $I^{hp}_{lift}u$ is constructed as the sum of u_1 and u_2. The term u_1 is constructed such that the correct behavior at the vertices of the triangulation is ensured. In this way, the construction of the lifting is then reduced to an edgewise construction, which defines u_2.

Given $u \in W^{1,q}_{\mathcal{B},\mathbf{p}}(\Omega)$, we construct $u_1 \in S^{\mathbf{p}}(\mathcal{T})$ patchwise as

$$u_1 = \sum_{V \in \mathcal{N}(\mathcal{T})} \varphi_V L_V u,$$

where the patch operators L_V are defined with the aid of Lemma 2.9 according to the following rules:

(a) if $V \notin \mathcal{N}(\mathcal{B})$, then $L_V u = 0$;

(b) if $V \in \mathcal{N}(\mathcal{B})$, then $L_V u$ is defined on the corresponding reference patch $\widehat{\omega}_V$ as $(L_V u) \circ F_V = L\widehat{u}$, where L is the operator of Lemma 2.9. Here, $\widehat{u} = u \circ F_V$ and the polynomial degrees p and p' are defined as $p = \max\{p_K \mid K \in \mathcal{T}|_{\omega_V}\}$ and $p' = \min\{p_K \mid K \in \mathcal{T}|_{\omega_V}\} - 1$.

By the choice of the polynomial degrees p', we get $u_1 \in S^{\mathbf{p}}(\mathcal{T})$. Additionally, the function $L_V u$ satisfies for $V \in \mathcal{N}(\mathcal{B})$

$$\begin{aligned}(L_V u)(V) &= u(V),\\ \|L_V u\|_{L^q(\omega_V)} &\le C\left[\|u\|_{L^q(\omega_V)} + h_V\|\nabla u\|_{L^q(\omega_V)}\right],\\ \|\nabla L_V u\|_{L^q(\omega_V)} &\le C\|\nabla u\|_{L^q(\omega_V)}.\end{aligned}$$

Moreover, for edges $b \in \mathcal{B}$ and vertices $V \in \mathcal{N}(b)$ we have upon denoting by $\gamma_{b,V}$ the map $\gamma_{b,V} : \hat{I} \to b$ that is determined by the element maps and the condition $\gamma_{b,V}(0) = V$ the following bound:

$$\|(u - L_V u)\circ\gamma_{b,V}\|_{\widetilde{W}_l^{1-1/q,q}(\hat{I})} \le C h_V^{1-2/q}\|\nabla u\|_{L^q(\omega_V)} \qquad \forall b \in \mathcal{B}, \quad V \in \mathcal{N}(b).$$

For elements K with $\omega_{K,\mathcal{B}} = \emptyset$, our construction implies $(u_1)|_K = 0$. For elements K with $\omega_{K,\mathcal{B}} \neq \emptyset$ we get

$$\begin{aligned}u_1(V) &= u(V) \qquad \forall V \in \mathcal{N}(\mathcal{B}),\\ \|u_1\|_{L^q(K)} &\le C\left[\|u\|_{L^q(\omega_{K,\mathcal{B}})} + h_K\|\nabla u\|_{L^q(\omega_{K,\mathcal{B}})}\right],\\ \|\nabla u_1\|_{L^q(K)} &\le C\left[\frac{1}{h_K}\|u\|_{L^q(\omega_{K,\mathcal{B}})} + \|\nabla u\|_{L^q(\omega_{K,\mathcal{B}})}\right],\\ \|(u-u_1)\circ\gamma_b\|_{\widetilde{W}^{1-1/q,q}(\hat{I})} &\le C h_b^{1-2/q}\|\nabla u\|_{L^q(\omega_b^1)} \qquad \forall b \in \mathcal{B}.\end{aligned}$$

For the last estimate, we employed additionally Lemma C.3.
We now turn to the construction of u_2. Since u_1 and u coincide in the vertices that lie on the Dirichlet boundary, we can proceed in an element-by-element fashion. For elements K with $\mathcal{E}(K) \cap \mathcal{B} = \emptyset$, we set $u_2|_K = 0$. For elements K with $\mathcal{E}(K) \cap \mathcal{B} \neq \emptyset$, we construct $u_2|_K$ using the following considerations: We set $\mathcal{B}_K = \mathcal{E}(K) \cap \mathcal{B}$, denote by $\widehat{b} := F_K^{-1}(b)$ the pull-back of an edge $b \in \mathcal{B}_K$, and construct with the aid of the lifting result Theorem B.4 on the reference element $\hat{K}$ the polynomial $\widehat{u}_{2,K} \in \mathcal{P}_{p_K}$ such that

$$\begin{aligned}\widehat{u}_{2,K}|_{\widehat{b}} &= ((u-u_1)\circ F_K)|_{\widehat{b}} \qquad \forall b \in \mathcal{B}_K,\\ \widehat{u}_{2,K}|_{F_K^{-1}(e)} &= 0 \qquad \forall e \in \mathcal{E}(K)\setminus\mathcal{B}_K,\\ \|\widehat{u}_2\|_{W^{1,q}(\widehat{K})} &\le C\sum_{b\in\mathcal{B}_K}\|(u-u_1)\circ F_b\|_{\widetilde{W}^{1-1/q,q}(\hat{I})} \le C h_K^{1-2/q}\|\nabla u\|_{L^q(\omega_{K,\mathcal{B}})},\end{aligned}$$

where $F_b : \hat{I} \to b$ denotes the parametrization of b determined by the element maps. Pushing forward these estimates to the element K, the function $u_2|_K := \widehat{u}_{2,K} \circ F_K^{-1}$ then satisfies

$$\begin{aligned}u_2|_b &= (u-u_1)|_b, \qquad \forall b \in \mathcal{B}_K,\\ u_2|_e &= 0 \qquad \forall e \in \mathcal{E}(K)\setminus\mathcal{B}_K,\\ \|u_2\|_{L^q(K)} &\le C h_K^{2/q}\|\widehat{u}_2\|_{L^q(\widehat{K})} \le C h_K\|\nabla u\|_{L^q(\omega_{K,\mathcal{B}})},\\ \|\nabla u_2\|_{L^q(K)} &\le C h_K^{2/q-1}\|\nabla\widehat{u}_2\|_{L^q(\widehat{K})} \le C\|\nabla u\|_{L^q(\omega_{K,\mathcal{B}})}.\end{aligned}$$

The sum $u_1 + u_2$ is an element of $S^{\mathbf{p}}(\mathcal{T})$, it satisfies $(u_1 + u_2)|_b = u|_b$ for all $b \in \mathcal{B}$, and we have the estimates

$$\begin{aligned}\|u_1 + u_2\|_{L^q(K)} &\le C\left[\|u\|_{L^q(\omega_{K,\mathcal{B}})} + h_K\|\nabla u\|_{L^q(\omega_{K,\mathcal{B}})}\right],\\ \|\nabla(u_1+u_2)\|_{L^q(K)} &\le C\left[\frac{1}{h_K}\|u\|_{L^q(\omega_{K,\mathcal{B}})} + \|\nabla u\|_{L^q(\omega_{K,\mathcal{B}})}\right];\end{aligned}$$

inspection of the construction shows that the map $u \mapsto u_1 + u_2$ is linear. ☐

2.2.4. Proof of Theorem 2.4. We are now in position to prove Theorem 2.4:
Proof of Theorem 2.4: We employ the lifting operator I^{hp}_{lift} of Proposition 2.10 and the approximation operators I^{hp}_{hom}, I^{hp} of Theorems 2.3, 2.1. We define

$$L := I^{hp}_{lift} \circ (\mathrm{Id} - I^{hp}) + I^{hp},$$
$$I^{hp}_{inhom} := L + I^{hp}_{hom} \circ (\mathrm{Id} - L).$$

I^{hp}_{inhom} is a linear operator mapping into $S^{\mathbf{p}}(\mathcal{T})$. We easily check that for $u \in W^{1,q}_{\mathcal{B},\mathbf{p}}(\Omega)$

$$(I^{hp}_{inhom} u)|_b = u|_b \qquad \forall b \in \mathcal{B}.$$

It remains to check the approximation properties. Upon writing

$$(\mathrm{Id} - I^{hp}_{inhom}) = (\mathrm{Id} - I^{hp}_{hom}) \circ (\mathrm{Id} - L),$$

we see that the desired approximation follow from the approximation properties of I^{hp}_{hom} together with stability properties of $\mathrm{Id} - L$. These stability properties can be inferred by writing

$$\mathrm{Id} - L = (\mathrm{Id} - I^{hp}_{lift}) \circ (\mathrm{Id} - I^{hp})$$

and then observing that Proposition 2.10 implies

$$\|\nabla(u - Lu)\|_{L^q(K)} \leq C \left[\frac{1}{h_K} \|u - I^{hp}u\|_{L^q(\omega^2_K)} + \|\nabla(u - I^{hp}u)\|_{L^q(\omega^2_K)} \right] \leq C \|\nabla u\|_{L^q(\omega^3_K)}.$$

Theorem 2.3 then implies

$$\|u - I^{hp}_{inhom} u\|_{L^q(K)} + \frac{h_K}{p_K} \|\nabla(u - I^{hp}_{inhom} u)\|_{L^q(K)} \leq C \frac{h_K}{p_K} \|\nabla u\|_{L^q(\omega^4_K)}.$$

From this, the desired estimate for the edges follows. ☐

Appendix A. Approximation results.

A.1. Polynomial approximation results on hyper cubes. The purpose of the present section is to establish polynomial approximation in Sobolev spaces $W^{r,q}(S)$, where S is a hyper cube. Similar results have been obtained in [2]. Our present exposition ignores effects related to the behavior of polynomials near the endpoints of an interval. While in the one-dimensional situation a characterization of the functions that can be approximated at a certain rate can be done using weighted spaces, these results do not easily extend to higher dimensions. We refer to [25] for an exposition of the results in one dimension and refer to [7, 8] where related results for two-dimensional are proved.
We start by recalling a one-dimensional result on simultaneous trigonometric approximation:
LEMMA A.1. *Let $\mathbb{T}$ be the one-dimensional torus and denote for $r \in \mathbb{N}_0$, $q \in [1, \infty]$ in the standard way by $W^{r,q}(\mathbb{T})$ the set of functions with r weak derivatives whose derivatives are in $L^q(\mathbb{T})$. Denote by T_N the set of trigonometric polynomials of degree $N \in \mathbb{N}$. Then for each $R \in \mathbb{N}$ and each N there exists a linear operator $J_{R,N}$:*

$L^1(\mathbb{T}) \to T_N$ and a constant $C_R > 0$ (which depends solely on R) such that for all $r \in \mathbb{N}_0$ with $0 \le r \le R$, all $q \in [1,\infty]$ and all $u \in W^{r,q}(\mathbb{T})$

$$\| (u - J_{R,N}\, u)^{(j)} \|_{L^q(\mathbb{T})} \le C_r (N+1)^{-(r-j)} \|u^{(r)}\|_{L^q(\mathbb{T})}, \qquad j = 0, \dots, r. \tag{A.1}$$

Proof. Jackson type results of this form are well-known in approximation theory. The linear operators $J_{R,N}$, whose existence is ascertained in Lemma A.1 can be chosen as in [25, Chap. 7, eqn. (2.8)]. The results concerning simultaneous approximation then follows from combining Thms. 2.3, 2.7, 2.8 of [25, Chap. 7] and a check that the case $q = \infty$ is included in the form stated in Lemma A.1. The details are worked out in Proposition E.1 below. ∎

As is well-known, trigonometric approximation result imply polynomial approximation results. For future reference, we formulate this in the following proposition.

PROPOSITION A.2. *Let $I \subset \mathbb{R}$ be a bounded interval. Let $R \in \mathbb{N}$ and $q \in [1,\infty]$. Then for each $N \in \mathbb{N}_0$ there exists a linear operator $J_{R,N} : L^1(I) \to \mathcal{P}_N$ and a constant $C > 0$ that depends only on R, q, I, such that for each $0 \le r \le R$*

$$\|u - J_{R,N} u\|_{W^{j,q}(I)} \le C(N+1)^{-(r-j)} \|u\|_{W^{r,q}(I)}, \qquad j = 0, \dots, r. \tag{A.2}$$

Furthermore, the linear operator $J_{R,N}$ may be constructed such that for $0 \le r \le R$ and $N \ge R-1$

$$J_{R,N} u = u \qquad \forall u \in \mathcal{P}_{R-1} \tag{A.3}$$

$$\|u - J_{R,N} u\|_{W^{j,q}(I)} \le C(N+1)^{-(r-j)} |u|_{W^{r,q}(I)}, \qquad j = 0, \dots, r. \tag{A.4}$$

Proof. The proof for $N = 0$ is trivial; we will therefore assume $N \in \mathbb{N}$. We will obtain the results for polynomial approximation from those for trigonometric approximation. We construct for given $u \in W^{r,q}(I)$ the approximant $J_N\, u \in \mathcal{P}_N$; tracing the steps of the construction then reveals that $u \mapsto J_N\, u$ is in fact a linear operator. Without loss of generality, we may assume that I is such that the closed interval $\overline{I}$ satisfies $\overline{I} = [-\cos\varepsilon, \cos\varepsilon]$ for some chosen $\varepsilon \in (0, \pi/2)$.

1. step: Define the interval $\Theta = (\varepsilon, \pi - \varepsilon)$. For every function v (defined on I) we define a function v_θ on Θ by $v_\theta(\theta) = v(\cos\theta)$. Then for every $j \in \mathbb{N}_0$, $q \in [1,\infty]$ there exists a constant $C > 0$ that depends only on j, q, and ε, such that

$$C^{-1} \|v\|_{W^{j,q}(I)} \le \|v_\theta\|_{W^{j,q}(\Theta)} \le C \|v\|_{W^{j,q}(I)}. \tag{A.5}$$

2. step: We construct a function $\widetilde{u}_\theta$ on the torus $\mathbb{T}$ with the properties that a) $\widetilde{u}_\theta = u_\theta$ on Θ; b) $\widetilde{u}_\theta$ is symmetric with respect to $\theta = 0$; and c) $\|\widetilde{u}_\theta\|_{W^{r,q}(\mathbb{T})} \le C\|u\|_{W^{r,q}(I)}$. To that end, we extend u_θ to a function in $W^{r,q}(\mathbb{R})$ such that the extended function (again denoted u_θ) satisfies

$$\|u_\theta\|_{W^{r,q}(\mathbb{R})} \le C \|u_\theta\|_{W^{r,q}(\Theta)};$$

such an extension is constructed, for example, in [38]. Furthermore, using smooth cut-off functions, we may assume that this extension satisfies $\operatorname{supp} u_\theta \subset [\varepsilon/2, \pi - \varepsilon/2]$. We then define on the interval $(-\pi, \pi)$ the symmetric extension of u_θ by

$$\widetilde{u}_\theta(x) := \begin{cases} u_\theta(x) & \text{if } x \in (0,\pi) \\ u_\theta(-x) & \text{if } x \in (-\pi, 0) \end{cases}$$

By the support properties of u_θ we then conclude

$$\|\widetilde{u}_\theta\|_{W^{r,q}(\mathbb{T})} \leq C\|u\|_{W^{r,q}(I)}.$$

3. step: From Lemma A.1, we get for the trigonometric polynomial $J_N := J_{R,N}\,\widetilde{u}_\theta$

$$\|\widetilde{u}_\theta - J_N\|_{W^{j,q}(\mathbb{T})} \leq CN^{-(r-j)}\|u\|_{W^{r,q}(I)}. \tag{A.6}$$

We wish to approximate $\widetilde{u}_\theta$ by a symmetric trigonometric polynomial. Since $\widetilde{u}_\theta$ is symmetric with respect to $\theta = 0$, we get that the trigonometric polynomial $\widetilde{J}_N$ defined by $\widetilde{J}_N(x) = J_N(-x)$ also satisfies

$$\|\widetilde{u}_\theta - \widetilde{J}_N\|_{W^{j,q}(\mathbb{T})} \leq CN^{-(r-j)}\|u\|_{W^{r,q}(I)}. \tag{A.7}$$

Combining (A.6), (A.7), we conclude that the symmetric trigonometric polynomial

$$\widehat{J}_N := \frac{1}{2}\left(J_N + \widetilde{J}_N\right)$$

satisfies

$$\|\widetilde{u}_\theta - \widehat{J}_N\|_{W^{j,q}(\mathbb{T})} \leq CN^{-(r-j)}\|u\|_{W^{r,q}(I)}. \tag{A.8}$$

4. step: Since the symmetric trigonometric polynomial $\widehat{J}_N$ can be written in the form $\widehat{J}_N(\theta) = P_N(\cos(\theta))$ for a polynomial $P_N \in \mathcal{P}_N$, we get the desired operator $J_{r,N}$ and the bound (A.2) from (A.8) and (A.5).
5. step: As a preparation to the final step, we ascertain for $r \geq 1$

$$\inf_{v\in\mathcal{P}_{r-1}} \|u - v\|_{W^{r,q}(I)} \leq C\|u^{(r)}\|_{L^q(I)}. \tag{A.9}$$

We see this as follows: Since $r \geq 1$, we note that $W^{r,q}(I) \subset C^{r-1}(\overline{I})$. Choosing an arbitrary point $x_0 \in I$ and setting $T_{r-1}u \in \mathcal{P}_{r-1}$ the Taylor polynomial of $u \in W^{r,q}(I)$ about x_0, we have (see, e.g., [25, Chap. 2, Prop. 5.5])

$$\|u - T_{r-1}u\|_{L^q(I)} \leq \|u^{(r)}\|_{L^q(I)}.$$

Furthermore, we have from, e.g., [25, Chap. 2, Prop. 5.6]

$$\|v\|_{W^{r,q}(I)} \leq C\left[\|v\|_{L^q(I)} + \|v^{(r)}\|_{L^q(I)}\right],$$

so that

$$\begin{aligned}\inf_{w\in\mathcal{P}_{r-1}} \|v - w\|_{W^{r,q}(I)} \leq \|v - T_{r-1}u\|_{W^{r,q}(I)} &\leq C\left[\|v - T_{r-1}v\|_{L^q(I)} + \|v^{(r)}\|_{L^q(I)}\right] \\ &\leq C\|v^{(r)}\|_{L^q(I)}.\end{aligned}$$

6. step: It remains to show that $J_{R,N}$ can be chosen such that such that the properties (A.3), (A.4) also hold.
We choose a linear operator $Q : L^1(I) \to \mathcal{P}_{R-1}$ such that $Qv = v$ for all $v \in \mathcal{P}_{R-1}$. Since the range of Q is finite dimensional, we get for any $q \in [1, \infty]$ the existence of $C > 0$ such that for every $0 \leq r \leq R$

$$\|Qv\|_{W^{r,q}(I)} \leq C\|v\|_{W^{r,q}(I)}.$$

Therefore, for $r \geq 1$ and $0 \leq j \leq r$ we get

$$\begin{aligned}\|u - (J_{R,N}(u - Qu) + Qu)\|_{W^{j,q}(I)} &= \|u - Qu - J_{r,N}(u - Qu)\|_{W^{j,q}(I)} \\ &\leq C\|u - Qu\|_{W^{r,q}(I)} \leq C\|u^{(r)}\|_{L^q(I)}.\end{aligned}$$

Hence, since $Qv = v$ for all $v \in \mathcal{P}_{r-1}$ we get from (A.9)

$$\|u - (J_{r,N}(u - Qu) + Qu)\|_{W^{j,q}(I)} \leq C \inf_{v \in \mathcal{P}_{r-1}} \|u - v\|_{W^{r,q}(I)} \leq C\|u^{(r)}\|_{L^q(I)}.$$

We conclude that the operator $u \mapsto J_{r,N}(u - Qu) + Qu$ satisfies the desired bound (A.4) for $r \geq 1$. A direct calculation shows that it also satisfies the desired bounds for $r = 0$. It maps into $\mathcal{P}_N$ provided that $N \geq R - 1$. □

The one-dimensional operator $J_{R,N}$ of Proposition A.2 can be tensorized to yield polynomial approximations of functions defined on hyper cubes.

THEOREM A.3. *Let $d \in \mathbb{N}$ and I_i, $i = 1, \ldots, d$ be bounded intervals. Set $I = I_1 \times \cdots \times I_d$. Let $R \in \mathbb{N}$. Then for each $N \in \mathbb{N}_0$ there exists a bounded linear operator $J_{R,N} : L^1(I) \to \mathcal{Q}_N(I)$ with the following properties: For each $q \in [1, \infty]$ there exists a constant $C > 0$, which depends only on R, q, and I, such that for all $N \geq R - 1$ and all $0 \leq r \leq R$*

$$J_{R,N}u = u \qquad \forall u \in \mathcal{Q}_{R-1}, \tag{A.10}$$

$$\|u - J_{R,N}u\|_{W^{l,q}(I)} \leq C(N+1)^{-(r-l)}|u|_{W^{r,q}(I)}, \qquad l = 0, \ldots, r. \tag{A.11}$$

Proof. The operator $J_{R,N}$ is taken as the tensor product of the one-dimensional ones given by Proposition A.2. To simplify the notation, we will drop the indices R, N and write $J_1, \ldots, J_d$ to denote these one-dimensional operators and J to denote the tensor product. From Proposition A.2 we obtain the following stability and approximation results:

$$|J_i u|_{W^{l,q}(I_i)} \leq C|u|_{W^{l,q}(I_i)}, \qquad l = 0, \ldots, r, \tag{A.12}$$

$$|u - J_i u|_{W^{l,q}(I_i)} \leq C(N+1)^{-(r-l)}|u|_{W^{l,q}(I_i)}, \qquad 0 \leq l \leq r \leq R, \tag{A.13}$$

where $i = 1, \ldots, d,$. These stability estimates then allow us to obtain approximation results in the standard way. We illustrate the procedure for the case $d = 2$. Let α, $\beta \geq 0$ with $\alpha + \beta = l \leq r$. Then, since the operators ∂_i and J_j commute if $i \neq j$, we get

$$\begin{aligned}\|\partial_1^\alpha \partial_2^\beta (u - J_1 \otimes J_2 u)\|_{L^q(I)} &\leq \|\partial_1^\alpha \partial_2^\beta (u - J_1 u)\|_{L^q(I)} + \|\partial_1^\alpha \partial_2^\beta J_1(u - J_2 u)\|_{L^q(I)} \\ &\leq \|\partial_1^\alpha \left((\partial_2^\beta u) - J_1(\partial_2^\beta u)\right)\|_{L^q(I)} + \|\partial_1^\alpha J_1 \partial_2^\beta (u - J_2 u)\|_{L^q(I)}.\end{aligned}$$

We consider the first term. The function $v(\cdot, x_2) = \partial_2^\beta u(\cdot, x_2)$ is defined for a.e. $x_2 \in I_2$ and $v(\cdot, x_2) \in W^{r-\beta,q}(I_1)$. Hence, we obtain from (A.13) for a.e. $x_2 \in I_2$

$$\|\partial_1^\alpha (v(\cdot, x_2) - J_1 v(\cdot, x_2))\|_{L^q(I_1)} \leq C(N+1)^{-(r-\beta-\alpha)}|v(\cdot, x_2)|_{W^{r-\beta,q}(I_1)}.$$

Substituting again the definition of v and integrating over I_2 yields

$$\|\partial_1^\alpha \left((\partial_2^\beta u) - J_1(\partial_2^\beta u)\right)\|_{L^q(I)} \leq C(N+1)^{-(r-\beta-\alpha)}|u|_{W^{r,q}(I)}.$$

By similar arguments (here, we additionally employ the stability result (A.12)) we can bound

$$\|\partial_1^\alpha J_1 \partial_2^\beta (u - J_2 u)\|_{L^q(I)} \le C(N+1)^{-(r-\alpha-\beta)} |u|_{W^{r,q}(I)}.$$

Combining these last two estimates and summing over all combinations of α, β with $\alpha + \beta = l$ yields the desired bound. The fact (A.10) follows readily from the property (A.3) of the one-dimensional operators. ◻

Appendix B. Polynomial liftings. A general trace lifting operator of the form (B.2) was studied for example in [26,30]. The subsequent observation that it also maps polynomials to polynomials (cf. Proposition B.1) was the basis for polynomial liftings from $H^{1/2}(\partial\hat{K})$ to $H^1(\hat{K})$, where $\hat{K}$ is the reference square or triangle, [4,13,31]. We generalize these results to the L^q-setting. In principle, the techniques employed here are applicable to three-dimensional problems as well, although they are technically more involved. Polynomial lifting results for hexahedra, prisms, and tetrahedra are available (in Hilbert space settings) in [9,10,14,34].

B.1. The operator $F^{[f]}$. We recall the definition of the reference triangle T as

$$T = \{(x,y) \,|\, 0 < x < 1,\, 0 < y < \min(x, 1-x)\}. \tag{B.1}$$

The bottom side of T is denoted by $\Gamma = \{(x,0) \,|\, 0 < x < 1\}$. We will view Γ as embedded in $\mathbb{R}$ in the natural way. We choose $\alpha \in (0,1)$ and define for a function $f \in L^q(\mathbb{R})$ the extension operator by

$$f \mapsto F^{[f]}(x,y) = \frac{1}{2\alpha y} \int_{x-\alpha y}^{x+\alpha y} f(t)\, dt. \tag{B.2}$$

Proposition B.1. *Let the extension operator be given by (B.2). Then $f \mapsto F^{[f]}$ is linear and $F^{[f]} \in \mathcal{P}_p$ if $f \in \mathcal{P}_p$. Furthermore, $F^{[f]}|_T$ depends only on the values of f on Γ, and for each $q \in (1,\infty)$ there exists a constant $C > 0$ such that for functions f defined on Γ the following bounds holds (provided that the right-hand side is finite):*

$$\|F^{[f]}\|_{L^q(T)} \le C \|(x(1-x))^{1/q} f\|_{L^q(\Gamma)}, \tag{B.3}$$

$$\|F^{[f]}\|_{W^{1,q}(T)} \le C \|f\|_{W^{1-1/q,q}(\Gamma)}, \tag{B.4}$$

$$\|(x-y) F^{[f/t]}\|_{L^q(T)} \le C \|f\|_{L^q(\Gamma)}, \tag{B.5}$$

$$\|(x-y)(1-x-y) F^{[f/(t(1-t))]}\|_{L^q(T)} \le C \|f\|_{L^q(\Gamma)}, \tag{B.6}$$

$$\|(x-y) F^{[f/t]}\|_{W^{1,q}(T)} \le C \left[\|f\|_{W^{1-1/q,q}(\Gamma)} + \|\frac{f(x)}{x^{1-1/q}}\|_{L^q(\Gamma)} \right], \tag{B.7}$$

$$\|(x-y)(1-x-y) F^{[f/(t(1-t))]}\|_{W^{1,q}(T)} \le C \|f\|_{\widetilde{W}^{1-1/q,q}(\Gamma)}. \tag{B.8}$$

Here, we employed the shorthand f/t to indicate the function $t \mapsto f(t)/t$ and $(x-y)F^{[f/t]}$ to denote the function $(x,y) \mapsto (x-y)F^{[f/t]}(x,y)$. Additionally, we have

$$\|F^{[f]}\|_{L^q(\partial T)} \le C \|f\|_{L^q(\Gamma)}.$$

Proof. We first show (B.3). From (B.12) of Lemma B.2 below we bound for each fixed $x \in (0,1)$

$$\int_{y=0}^{\min(x,1-x)} |F^{[f]}(x,y)|^q \, dy \le C \int_{x-\alpha\min(x,1-x)}^{x+\alpha\min(x,1-x)} |f(y)|^q \, dy.$$

Integrating over $x \in (0,1)$ we get from Lemma B.3

$$\|F^{[f]}\|_{L^q(T)}^q \leq C \int_{y=0}^{1} y(1-y)|f(y)|^q \, dy = C\|(x(1-x))^{1/q} f\|_{L^q(\Gamma)}^q.$$

We now turn to the estimate (B.5). From (B.14) of Lemma B.2 we get

$$\|(x-y)F^{[f/t]}\|_{L^q(T)}^q \leq \int_T \left| \frac{x-y}{\alpha y} \int_{x-\alpha y}^{x+\alpha y} \frac{f(t)}{t} \, dt \right|^q \, dy \, dx \leq C\|f\|_{L^q(\Gamma)}.$$

By symmetry we conclude

$$\|(1-x-y)F^{[f/(1-t)]}\|_{L^q(T)} \leq C\|f\|_{L^q(\Gamma)}.$$

Since

$$\frac{f(t)}{t(1-t)} = \frac{f(t)}{t} + \frac{f(t)}{1-t}$$

we can easily obtain (B.6).
It remains to obtain the bounds (B.4), (B.7), (B.8). We compute

$$\begin{aligned}
\partial_x F^{[f]}(x,y) &= \frac{1}{2\alpha y} \left[f(x-\alpha y) - f(x+\alpha y) \right], \\
\partial_y F^{[f]}(x,y) &= \frac{1}{2\alpha} \left[-\frac{1}{y^2} \int_{x-\alpha y}^{x+\alpha y} f(t) \, dt + \frac{\alpha}{y} \left(f(x+\alpha y) + f(x-\alpha y) \right) \right] \\
&= -\frac{1}{\alpha y^2} \int_{x-\alpha y}^{x+\alpha y} f(t) - f(x) \, dt + \frac{f(x) - f(x-\alpha y)}{y} + \frac{f(x) - f(x+\alpha y)}{y}.
\end{aligned}$$

From the definition of the $W^{1-1/q,q}$-norm and the bound (B.11) of Lemma B.2 we get

$$\|\nabla F^{[f]}\|_{W^{1,q}(T)} \leq C\|f\|_{W^{1-1/q,q}(\Gamma)},$$

which shows (B.4). For the bound (B.7), we compute

$$\begin{aligned}
\partial_x \left((x-y)F^{[f/t]} \right) &= F^{[f/t]} + (x-y)\partial_x F^{[f]}, \\
\partial_y \left((x-y)F^{[f/t]} \right) &= -F^{[f/t]} + (x-y)\partial_y F^{[f]},
\end{aligned}$$

and

$$\begin{aligned}
\partial_x F^{[f/t]} &= \frac{1}{\alpha y} \left[\frac{f(x+\alpha y)}{x+\alpha y} - \frac{f(x-\alpha y)}{x-\alpha y} \right], \\
\partial_y F^{[f/t]} &= \frac{1}{\alpha y^2} \int_{x-\alpha y}^{x+\alpha y} \frac{f(t)}{t} \, dt + \frac{\alpha}{y} \left[\frac{f(x+\alpha y)}{x+\alpha y} + \frac{f(x-\alpha y)}{x-\alpha y} \right] \\
&= \frac{1}{\alpha y^2} \int_{x-\alpha y}^{x+\alpha y} \frac{f(t)}{t} - \frac{f(x)}{x} \, dt + \frac{\alpha}{y} \left[\frac{f(x+\alpha y)}{x+\alpha y} - \frac{f(x)}{x} + \frac{f(x-\alpha y)}{x-\alpha y} - \frac{f(x)}{x} \right].
\end{aligned}$$

With (B.10) and Lemma B.3 we get

$$\begin{aligned}
\|F^{[f/t]}\|_{L^q(T)}^q &\leq C \int_{x=0}^{1} \int_{x-\alpha \min(x,1-x)}^{x+\alpha \min(x,1-x)} \left| \frac{f(t)}{t} \right|^q \, dt \, dx \leq C \int_{x=0}^{1} x(1-x) \left| \frac{f(x)}{x} \right|^q \, dx \\
&\leq C \int_{x=0}^{1} \left| \frac{f(x)}{x^{1-1/q}} \right|^q \, dx. \qquad \text{(B.9)}
\end{aligned}$$

With (B.13) of Lemma B.2, we can bound

$$\int_T \left|\frac{1}{y^2}\int_{x-\alpha y}^{x+\alpha y} \frac{f(t)}{t} - \frac{f(x)}{x}\, dt\right|^q dx\, dy \le C \left[\|f\|^q_{W^{1-1/q,q}(\Gamma)} + \|f(x)/x\|_{L^q(\Gamma)}\right].$$

It remains to bound terms of the form

$$\left\|\frac{x-y}{y}\left(\frac{f(x\pm\alpha y)}{x\pm\alpha y} - \frac{f(x)}{x}\right)\right\|_{L^q(T)}.$$

Rearranging terms, we arrive at

$$\left\|\frac{x-y}{y}\left(\frac{f(x\pm\alpha y)}{x\pm\alpha y} - \frac{f(x)}{x}\right)\right\|_{L^q(T)} =$$
$$\left\|\frac{x-y}{x\pm\alpha y}\frac{f(x\pm\alpha y)-f(x)}{y} - \frac{\pm\alpha(x-y)}{x\pm\alpha y}\frac{f(x)}{x}\right\|_{L^q(T)} \le |f|_{W^{1-1/q,q}(\Gamma)} + \|\frac{f(x)}{x}\|_{L^q(T)},$$

where we employed the observation $|\frac{x-y}{x\pm\alpha y}| \le 1$ for $0<y<x$. The term $\|\frac{f(x)}{x}\|_{L^q(T)}$ is now controlled in the desired fashion as in (B.9).
It remains to show (B.8). By symmetry considerations we obtain analogous to (B.7)

$$\|(1-x-y)F^{[f/(1-t)]}\|_{W^{1,q}(T)} \le C\left[\|f\|_{W^{1-1/q,q}(\Gamma)} + \|\frac{f(x)}{(1-x)^{1-1/q}}\|_{L^q(\Gamma)}\right].$$

Since $\frac{f(t)}{t(1-t)} = \frac{f(t)}{t} + \frac{f(t)}{1-t}$, the desired bound (B.8) now follows easily. ∎
The following lemma contains estimates of Hardy type:
LEMMA B.2. *Let $a\le b$, $\alpha\in(0,1)$, T be defined as in (B.1). Then for $q\in(1,\infty)$*

$$\int_a^b \left|\frac{1}{x-a}\int_a^x |g(\xi)|\, d\xi\right|^q \le \left(\frac{q}{q-1}\right)^q \int_a^b |g(\xi)|^q\, d\xi, \tag{B.10}$$
$$\int_a^b \left|\frac{1}{(x-a)^2}\int_a^x g(\xi)-g(a)\, d\xi\right|^q dx \le \left(\frac{q}{q-1}\right)^q \int_a^b \left|\frac{g(\xi)-g(a)}{\xi-a}\right|^q d\xi. \tag{B.11}$$

Furthermore, for each $x\in(0,1)$ we have upon setting $m:=\min(x,1-x)$

$$\int_{y=0}^m \left|\frac{1}{y}\int_{x-\alpha y}^{x+\alpha y} g(t)\, dt\right|^q \le \left(\frac{q}{q-1}\right)^q \alpha^{q-1}\int_{x-\alpha m}^{x+\alpha m} |g(y)|^q\, dy. \tag{B.12}$$
$$\int_{y=0}^m \frac{1}{y^2}\int_{x-\alpha y}^{x+\alpha y} |g(t)-g(x)|\, dt\, dy \le \alpha^{2q-1}\left(\frac{q}{q-1}\right)^q \int_{x-\alpha m}^{x+\alpha m}\left|\frac{g(t)-g(x)}{t-x}\right|^q dt. \tag{B.13}$$

Finally, we have for some constant $C>0$ that depends only on q and α:

$$\int_T \left|(x-y)\frac{1}{y}\int_{x-\alpha y}^{x+\alpha y}\frac{g(t)}{t}\, dt\right|^q dx\, dy \le C\|g\|^q_{L^q(\Gamma)}, \tag{B.14}$$
$$\int_T \left|\frac{x-y}{y^2}\int_{x-\alpha y}^{x+\alpha y}\frac{g(t)}{t} - \frac{g(x)}{x}\, dt\right|^q dxdy \le C\left[\|g\|^q_{W^{1-1/q,q}(\Gamma)} + \left\|\frac{g(x)}{x}\right\|_{L^q(\Gamma)}\right]. \tag{B.15}$$

In all the above estimates, it is implicitly assumed that the right-hand side is finite.

Proof. The first estimate is the well-known Hardy inequality, [29, Thm. 327]. For the second estimate, we note

$$\int_a^b \left| \frac{1}{|x-a|^2} \int_a^x |g(\xi)-g(a)| \right|^q dx = \int_a^b \left| \frac{1}{|x-a|^2} \int_a^x \frac{|g(\xi)-g(a)|}{|\xi-a|} |\xi - a| \, d\xi \, dx \right|^q$$
$$\leq \int_a^b \left| \frac{1}{|x-a|} \int_a^x \frac{|g(\xi)-g(a)|}{|\xi-a|} d\xi \right|^q dx.$$

The result (B.11) now follows from (B.10). The bounds (B.12), (B.13) follow easily from (B.10) and (B.11), respectively. To proceed further, we note that for $x \in (0,1)$ we have

$$(1-\alpha)\, x \leq x - \alpha \min(x, 1-x) \leq x + \alpha \min(x, 1-x) \leq 1. \tag{B.16}$$

We are now in position to prove (B.14). From (B.12) and (B.16) we get (again with the abbreviation $m = \min(x, 1-x)$)

$$\int_T \left| \frac{(x-y)}{y} \int_{x-\alpha y}^{x+\alpha y} \frac{f(t)}{t} dt \right|^q dx \leq C \int_{x=0}^1 x^q \int_{y=0}^m \left| \frac{1}{y} \int_{x-\alpha y}^{x+\alpha y} |f(t)| \, dt \right|^q dy \, dx$$
$$\leq C \int_{x=0}^1 x^q \int_{t=x-\alpha m}^{x+\alpha m} \left| \frac{f(t)}{t} \right|^q dt \, dx \leq C \|f\|^q_{L^q(\Gamma)}.$$

We now turn to the proof of the last inequality, (B.15). We employ (B.13) and obtain

$$\int_T \left| \frac{x-y}{y^2} \int_{x-\alpha y}^{x+\alpha y} \frac{f(t)}{t} - \frac{f(x)}{x} dt \right|^q dy \, dx \leq C \int_{x=0}^1 x^q \int_{t=x-\alpha(1-x)}^{x+\alpha(1-x)} \left| \frac{\frac{f(t)}{t} - \frac{f(x)}{x}}{t-x} \right|^q dt \, dx.$$

We next rewrite the integrand as

$$\frac{f(t)}{t} - \frac{f(x)}{x} = \frac{f(t)-f(x)}{t} - \frac{t-x}{tx} f(x)$$

and arrive at

$$\int_T \left| \frac{x-y}{y^2} \int_{x-\alpha y}^{x+\alpha y} \frac{f(t)}{t} - \frac{f(x)}{x} dt \right|^q dy \, dx \leq$$
$$C \int_{x=0}^1 x^q \int_{t=x-\alpha(1-x)}^{x+\alpha(1-x)} t^{-q} \left| \frac{f(t)-f(x)}{t-x} \right|^q + C \int_{x=0}^1 x^q \int_{t=x-\alpha(1-x)}^{x+\alpha(1-x)} |f(x)|^q \frac{1}{|tx|^q} dt \, dx$$
$$\leq C \int_{\Gamma\times\Gamma} \left| \frac{f(t)-f(x)}{t-x} \right|^q dt \, dx + C \int_\Gamma \left| \frac{f(x)}{x^{1-1/q}} \right|^q dx.$$

This concludes the proof of the lemma. ☐

Lemma B.3. *Let $\alpha \in (0,1)$. Then there exists a constant $C > 0$ that depends only on α such that*

$$\int_{x=0}^1 \int_{y=x-\alpha \min(x,1-x)}^{x+\alpha\min(x,1-x)} |g(y)| \, dy \, dx \leq C \int_{y=0}^1 y(1-y)|g(y)| \, dy.$$

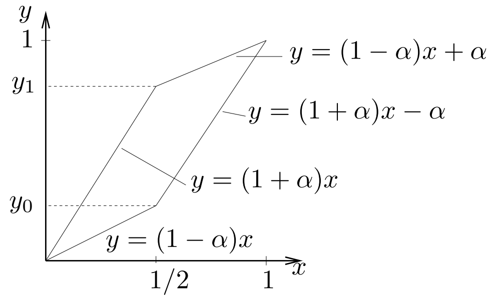

FIG. B.1. *Integration domain in Lemma B.3.*

Proof. The integration domain is sketched in Fig. B.1. Interchanging the order of integration, we get

$$\int_{x=0}^{1} \int_{y=x-\alpha\min(x,1-x)}^{x+\alpha\min(x,1-x)} |g(y)|\,dy\,dx = \int_{y=0}^{y_0} \int_{x=y/(1+\alpha)}^{y/(1-\alpha)} |g(y)|\,dx\,dy$$
$$+ \int_{y=y_0}^{y_1} \int_{x=y/(1+\alpha)}^{(y+\alpha)/(1+\alpha)} |g(y)|\,dx\,dy + \int_{y=y_1}^{1} \int_{x=(y-\alpha)/(1-\alpha)}^{(y+\alpha)/(1+\alpha)} |g(y)|\,dx\,dy,$$

where $y_0 = \frac{1-\alpha}{2}$ and $y_1 = \frac{1+\alpha}{2}$. The result follows now by elementary calculations. ∎

B.2. Polynomial lifting from the boundary.

B.2.1. $W^{1,q}$-stable liftings. The operator $f \mapsto F^{[f]}$ is the basic building block for polynomial trace liftings.

THEOREM B.4. *Let K be the reference triangle (B.1) or the reference square. Let $\Gamma = \overline{\Gamma} \subset \partial K$ be the union of (closed) edges of K. Let $q \in (1,\infty)$. Then there exists a constant $C > 0$ with the following property: for each $f \in C(\Gamma)$ such that f is a polynomial of degree p on each edge contained in Γ, there exists a polynomial $F \in \mathcal{P}_p$ if K is the triangle or $F \in \mathcal{Q}_p$ if K is the square such that $F|_\Gamma = f$ and*

$$\|F\|_{W^{1,q}(T)} \le C\|f\|_{W^{1-1/q,q}(\Gamma)},$$
$$\|F\|_{L^q(T)} \le C\|f\|_{L^q(\Gamma)}.$$

Moreover, the mapping $f \mapsto F$ is linear.

Proof. We consider the case of a triangle. Three cases may occur:

Γ *is a single edge:* We may assume $\Gamma = \{(x,0)\,|\,0 < x < 1\}$ and then choose F as $F^{[f]}$.

Γ *consists of two edges* Γ_1, Γ_2*:* The function F is defined in two steps. First, the lifting operator F of Section B.1 is employed to construct a function $F_1 \in \mathcal{P}_p$ with $F_1|_{\Gamma_1} = f|_{\Gamma_1}$ and

$$\|F_1\|_{W^{1,q}(T)} \le C\|f\|_{W^{1-1/q,q}(\Gamma_1)}, \quad \|F_1\|_{L^q(\Gamma_2)} + \|F_1\|_{L^q(T)} \le C\|f\|_{L^q(\Gamma_1)}.$$

We have thus reduced the problem to one where f vanishes on one of the sides Γ_1, Γ_2. Without loss of generality, we assume that $\Gamma_1 = \{(x,0)\,|\,0 < x < 1/2\}$ and $f|_{\Gamma_2} = 0$ with $\Gamma_2 = \{(x,x)\,|\,0 < x < 1\}$. The mapping $f \mapsto (x-y)F^{[f/t]}(x,y)$ of Section B.1 then has the desired properties.

$\Gamma = \partial K$*:* After having constructed a lifting from two adjacent edges as in the above construction, we may assume that f vanishes on two sides of T. Without loss of

generality, we may therefore assume that $f|_{\Gamma_i} = 0$ for $i \in \{2,3\}$, where Γ_1 is the third side of ∂T given by $\Gamma_1 = \{(x,0) \,|\, 0 < x < 1\}$. The construction of a polynomial F with the property $F|_{\Gamma_1} = f$ and $F|_{\Gamma_i} = 0$ for $i \in \{2,3\}$ is then achieved with the operator $f \mapsto (x-y)(1-x-y)F^{[f/(t(1-t))]}(x,y)$ of Section B.1.
The case of a square is proved similarly using the ideas of [4]. Note that in the case of a square, the set Γ may be disconnected, i.e., it consists of two parallel edges of $\hat{K}$. In this event, we easily reduce the construction to the case where f vanishes on one of the two edges and construct $F \in \mathcal{Q}_p$ in the same way as the function U in the proof of Lemma C.1. □

Remark B.5. The lifting operator of Theorem B.4 is independent of p. Since for $p = 0$ the constant function is reproduced, this operator reproduces constants for any $p \in \mathbb{N}_0$. ∎

B.2.2. Polynomial liftings with improved L^q-bounds. The basis of the results of Section B.2.1 is the operator $f \mapsto F^{[f]}$ of Section B.1. We introduce a new operator $\widetilde{F}^{[f]}$ by

$$\widetilde{F}^{[f]}(x,y) = (1-y)^p F^{[f]}(x,y). \tag{B.17}$$

We note that, if $f \in \mathcal{P}_p$, then $\widetilde{F} \in \mathcal{P}_{2p}$. Furthermore, $\widetilde{F}^{[f]}|_\Gamma = f$, where $\Gamma = \{(x,0) \,|\, 0 < x < 1\}$. We also have

Lemma B.6. *Let T be the reference triangle. Then there exists a constant $C > 0$ such that for every $p \in \mathbb{N}$ the functions $\widetilde{F}^{[f]}$, $F_1 := (x-y)\widetilde{F}^{[f/t]}$, $F_2 := (x-y)(1-x-y)\widetilde{F}^{[f/(t(1-t))]}$ satisfy*

$$\begin{aligned} p\|\widetilde{F}^{[f]}\|_{L^q(T)} + \|\widetilde{F}^{[f]}\|_{W^{1,q}(T)} &\le C\|f\|_{W^{1-1/q,q}(\Gamma)} + Cp^{1-1/q}\|f\|_{L^q(\Gamma)}, \\ p\|F_1\|_{L^q(T)} + \|F_1\|_{W^{1,q}(T)} &\le C\|f\|_{W^{1-1/q,q}(\Gamma)} + C\|\frac{f(x)}{x^{1-1/q}}\|_{L^q(\Gamma)} + Cp^{1-1/q}\|f\|_{L^q(\Gamma)}, \\ p\|F_2\|_{L^q(T)} + \|F_2\|_{W^{1,q}(T)} &\le C\|f\|_{\widetilde{W}^{1-1/q,q}(\Gamma)} + Cp^{1-1/q}\|f\|_{L^q(\Gamma)}. \end{aligned}$$

Furthermore,

$$\|\widetilde{F}^{[f]}\|_{L^q(\partial T)} \le C\|f\|_{L^q(\Gamma)}.$$

Proof. The lemma follows from Lemma B.9 below and the properties of the operator $F^{[f]}$ of Section B.1. □

Proposition B.7. *Let K be the reference triangle (B.1) or the reference square. Let $\Gamma = \overline{\Gamma} \subset \partial K$ be a union of edges of K. Let $q \in (1,\infty)$. Then there exists $C > 0$ such that for every $f \in C(\Gamma)$ that is a polynomial of degree $p \in \mathbb{N}$ on each edge of K there exists a polynomial F (if K is the reference triangle then $F \in \mathcal{P}_{3p}$, otherwise $F \in \mathcal{Q}_{4p}$) such that $F|_{\partial\Gamma} = f$ and*

$$p\|F\|_{L^q(K)} + \|F\|_{W^{1,q}(K)} \le C\|f\|_{W^{1-1/q,q}(\Gamma)} + Cp^{1-1/q}\|f\|_{L^q(\Gamma)}.$$

Furthermore, the mapping $f \mapsto F$ is linear.

Proof. The proof is similar to that of Theorem B.4. The appeals to Proposition B.1 are replaced with those to Lemma B.6. □

Remark B.8. It is easy to see that in the statement of Proposition B.7 the statement $F \in \mathcal{P}_{3p}$ or $F \in \mathcal{Q}_{4p}$ can be replaced with $F \in \mathcal{P}_{\lceil \lambda p \rceil}$ or $F \in \mathcal{Q}_{\lceil \lambda p \rceil}$ for arbitrary $\lambda > 1$. The constant $C > 0$ does depends on λ, however. ∎

LEMMA B.9. *K be the reference triangle (B.1) or the reference square. Set $\Gamma = \{(x,0)\,|\,0 < x < 1\}$ and let $q \in (1,\infty)$. Then there exists $C > 0$ such that for every $p \in \mathbb{N}$ and every function $g \in W^{1,q}(K)$*

$$p\|(1-y)^p g\|_{L^q(K)} + \|(1-y)^p g\|_{W^{1,q}(K)} \leq C|g|_{W^{1,q}(K)} + p^{1-1/q}\|g\|_{L^q(\Gamma)}.$$

Proof. We express the function g for $y > 0$ as $g(x,y) = g(x,0) + \int_{t=0}^{y} g_y(x,t)\,dt$. Then

$$(1-y)^p g(x,y) = [(1-y)^p y]\,\frac{1}{y}\int_0^y g_y(x,t)\,dt + (1-y)^p g(x,0).$$

Since

$$\sup_{y\in(0,1)} (1-y)^p y \leq \frac{C}{p}, \tag{B.18}$$

we conclude with the Hardy inequality (B.10)

$$\|(1-y)^p g\|_{L^q(K)} \leq \frac{C}{p}|g|_{W^{1,q}(K)} + p^{-1/q}\|g\|_{L^q(\Gamma)}.$$

For the bound on the derivative of $(1-y)^p g$, we write

$$\nabla((1-y)^p g) = p(1-y)^{p-1} g + (1-y)^p \nabla g;$$

we treat the first term as above and for the second term we use $|1-y| \leq 1$ on K. ∎

Appendix C. One-dimensional extension operators. The liftings of Section B can be applied for the construction of one-dimensional extension operators:

LEMMA C.1. *Let $\hat{I} = (0,1)$ and $q \in (1,\infty)$. Then there exists a bounded linear operator $Z : W^{1-1/q,q}(\hat{I}) \to W^{1-1/q,q}(\hat{I})$ with the following properties:*

1. $\displaystyle\int_{x=0}^{1} \frac{|u(x) - Zu(x)|^q}{x^{q-1}}\,dx \leq C\|u\|^q_{W^{1-1/q,q}(\hat{I})}$;
2. *if $u \in \mathcal{P}_p$, then $Zu \in \mathcal{P}_p$;*
3. $Zu(1) = 0$ *and* $\displaystyle\int_{-1}^{0} \frac{|Zu(x)|^q}{(1-x)^{q-1}}\,dx \leq C\|u\|^q_{W^{1-1/q,q}(\hat{I})}$;
4. $\|Zu\|_{L^q(I)} \leq C\|u\|_{L^q(\hat{I})}$.

Proof. Let T be the reference triangle and identify $\hat{I}$ with the edge of $\hat{K}$ lying on the x-axis. Consider the trapezoid

$$\tilde{T} := \{(x,y) \in T\,|\,0 < y < 1/4\}$$

and define $\Gamma = \{(x,x)\,|\,0 < x < 1/4\}$. An elementary calculation reveals

$$\|F^{[u]}(\cdot,1/4)\|_{W^{1,\infty}((1/4,3/4))} \leq C\|u\|_{L^q(\hat{I})}$$

for some appropriate $C > 0$. Hence, the function

$$U(x,y) = F^{[u]}(x,y) - 4yF^{[u]}(x,1/4)$$

satisfies

$$\|U\|_{W^{1,q}(\tilde{T})} \leq C\|u\|_{W^{1-1/q,q}(\hat{I})}, \quad \|U\|_{L^q(\Gamma)} \leq C\|u\|_{L^q(\hat{I})}, \quad U|_{y=1/4} = 0.$$

Additionally, if $u \in \mathcal{P}_p$, then $U \in \mathcal{Q}_p$. Defining $Zu(x)$ for $x \in (0,1)$ by $(Zu)(x) = U(x/4, x/4)$ and appealing to the trace theorem concludes the proof. $\square$

Lemma C.2. *Let $\hat{I} = (0,1)$, $q \in (1,\infty)$. Let $k \in \mathbb{N}_0$. Then there exists $C > 0$ such that for every $p \in \mathbb{N}_0$ with $p \geq k$ there exists a linear operator $Z_{p,p-k} : \mathcal{P}_p \to \mathcal{P}_{p-k}$ with the following properties:*

$$\begin{aligned} Z_{p,p-k} 1 &= 1, \\ (Z_{p,p-k} u)(0) &= u(0), \\ \|Z_{p,p-k} u\|_{L^q(\hat{I})} &\leq C\|u\|_{L^q(\hat{I})}, \\ \|Z_{p,p-k} u\|_{W^{1-1/q,q}(\hat{I})} &\leq C\|u\|_{W^{1-1/q,q}(\hat{I})}, \\ \int_0^1 \frac{|(Z_{p,p-k}u)(x) - u(x)|^q}{x^{q-1}}\, dx &\leq C|u|^q_{W^{1-1/q,q}(\hat{I})}. \end{aligned}$$

In particular, therefore,

$$\|Z_{p,p-k}u - u\|_{\widetilde{W}_l^{1-1/q,q}(\hat{I})} \leq C\|u\|_{W^{1-1/q,q}(\hat{I})}.$$

Proof. The key to this result is the following approximation result of [18, Cor. 3.7]:

$$\inf_{v \in \mathcal{P}_{p-k}} \|u - v\|_{L^\infty(J)} \leq 12(4p)^{k-1}|J|^{p-k+1}\|u\|_{L^\infty(\hat{I})} \qquad \forall u \in \mathcal{P}_p, \tag{C.1}$$

where $J \subset \hat{I}$ is an arbitrary subinterval of $\hat{I}$ and $|J|$ denotes the length of J. We (arbitrarily) choose $J = (0, 1/2)$, denote by $\mathcal{I}_{p-k} : C(\overline{J}) \to \mathcal{P}_{p-k}$ the Gauß-Lobatto interpolation operator and set

$$(Z_{p,p-k}u)(x) = (\mathcal{I}_{p-k}u)(2x).$$

By construction $(Z_{p,p-k}u) \in \mathcal{P}_{p-k}$. Additionally, the fact that the endpoint 0 of J is an interpolation point implies $(Z_{p,p-k}u)(0) = u(0)$. In order to see the remaining estimates, we see that (C.1) together with standard inverse estimates (see, e.g., [25, Chap. 4, Thms. 1.4 and 2.6]) implies

$$\|u - \mathcal{I}_{p-k}u\|_{W^{1,\infty}(J)} \leq Cp^2\|u - \mathcal{I}_{p-k}u\|_{L^\infty(J)} \leq C\rho^{p-k}\|u\|_{L^\infty(\hat{I})} \leq C\tilde{\rho}^{p-k}\|u\|_{L^q(\hat{I})}$$

for some suitable $C > 0$ and $\tilde{\rho} \in (0,1)$ that are both independent of p and u. In particular, since $u(0) = (\mathcal{I}_{p-k}u)(0)$, we get

$$\max_{x \in J} \frac{|u(x) - (\mathcal{I}_{p-k}u)(x)|}{|x|} \leq \|u' - (\mathcal{I}_{p-k}u)'\|_{L^\infty(J)} \leq C\tilde{\rho}^{p-k}\|u\|_{L^q(\hat{I})}.$$

From this and the triangle inequality, we can easily infer the estimates

$$\begin{aligned} \|\mathcal{I}_{p-k}u\|_{L^q(J)} &\leq C\|u\|_{L^q(\hat{I})}, \\ \|\mathcal{I}_{p-k}u\|_{W^{1-1/q,q}(J)} &\leq C\|u\|_{W^{1-1/q,q}(\hat{I})}, \\ \int_0^{1/2} \frac{|u(x) - \mathcal{I}_{p-k}u(x)|^q}{x^{q-1}}\, dx &\leq C\|u\|^q_{L^q(\hat{I})}. \end{aligned}$$

From this and the change of variables $x \mapsto 2x$ the desired bounds follow for $Z_{p,p-k}$. $\square$

LEMMA C.3. *Let $\hat{I} = (0,1)$ and $q \in (1,\infty)$. Then $u \in W^{1-1/q,q}(\hat{I})$ implies that the function $\tilde{u} : x \mapsto xu(x)$ is in $W^{1-1/q,q}(\hat{I})$ and satisfies*

$$\int_0^1 \frac{|\tilde{u}(x)|^q}{x^{q-1}}\,dx + \|\tilde{u}\|^q_{W^{1-1/q,q}(\hat{I})} \le C\|u\|^q_{W^{1-1/q,q}(\hat{I})}$$

for some $C > 0$ that is independent of u.

Proof. The estimate $\|\tilde{u}\|_{W^{1-1/q,q}(\hat{I})} \le C\|u\|_{W^{1-1/q,q}(\hat{I})}$ follows from the smoothness of the function $x \mapsto x$. The remaining estimate follows by inspection. □

Appendix D. Polynomial inverse estimates. The companion of polynomial approximation results are inverse estimates. In the present section we generalize some well-known one-dimensional inverse estimates to the higher dimensional case. On the interval $\hat{I} = (0,1)$ we define

$$\Phi_{\hat{e}}(x) := x(1-x). \tag{D.1}$$

Then there holds the following inverse estimate:

LEMMA D.1. *Let $-1 < \alpha < \beta$, $\delta \in [0,1]$ and $\Phi_{\hat{e}}$ be defined by (D.1). Then there exist C_1, $C_2 = C(\alpha,\beta)$, $C_3 = C(\delta) > 0$ such that for all $p \in \mathbb{N}$ and all polynomials π_p of degree p*

$$\int_0^1 \Phi_{\hat{e}}(x)\left(\pi_p'(x)\right)^2\,dx \le C_1 p^2 \int_0^1 \pi_p^2(x)\,dx,$$

$$\int_0^1 \Phi_{\hat{e}}^{\alpha}\pi_p^2(x)\,dx \le C_2 p^{2(\beta-\alpha)} \int_0^1 \Phi_{\hat{e}}^{\beta}\pi_p^2(x)\,dx,$$

$$\int_0^1 \Phi_{\hat{e}}^{2\delta}\left(\pi_p'(x)\right)^2\,dx \le C_3 p^{2(2-\delta)} \int_0^1 \Phi_{\hat{e}}^{\delta}\pi_p^2(x)\,dx.$$

Furthermore,

$$\int_{-1}^1 \left(\pi_p'(x)\right)^2\,dx \le C_1 p^2 \int_{-1}^1 \Phi_{\hat{e}}^{-1}\pi_p^2(x)\,dx \qquad \textit{if additionally } \pi_p(\pm 1) = 0.$$

Proof. These one-dimensional results can be found in, e.g., [16,17]. □

The weight function $\Phi_{\hat{e}}$ in Lemma D.1 is characterized by $\Phi_{\hat{e}}(x) \sim \operatorname{dist}\left(x, \partial\hat{I}\right)$. Lemma D.1 may be generalized to higher dimensions in different ways. In spectral element methods, which are based on tensor product domains, it is natural to consider weight functions that are tensor products of the one-dimensional weight function (see, e.g., [12,17]). This approach is not suitable for hp-FEM as the case of simplices cannot be handled. We therefore base our analysis on the following weight function:

$$\Phi_{\hat{K}}(x) := \operatorname{dist}\left(x, \partial\hat{K}\right) \tag{D.2}$$

where the domain $\hat{K}$ is the reference square or the reference triangle. The two-dimensional analog of Lemma D.1 reads as follows:

THEOREM D.2. *Let $\hat{K}$ be the reference square or the reference triangle and let $\Phi_{\hat{K}}$ be given by (D.2). Let α, $\beta \in \mathbb{R}$ satisfy $-1 < \alpha < \beta$ and $\delta \in [0,1]$. Then there exist C_1,*

$C_2 = C = C(\alpha, \beta)$, and $C_3 = C_3(\delta) > 0$ such that for all polynomials $\pi_p \in \mathcal{Q}_p$

$$\int_{\hat{K}} \Phi_{\hat{K}} |\nabla \pi_p|^2 \, dx \, dy \le C_1 p^2 \int_{\hat{K}} |\pi_p|^2 \, dx \, dy, \tag{D.3}$$

$$\int_{\hat{K}} \left(\Phi_{\hat{K}}\right)^{\alpha} \pi_p^2 \, dx \, dy \le C p^{2(\beta-\alpha)} \int_{\hat{K}} \left(\Phi_{\hat{K}}\right)^{\beta} \pi_p^2 \, dx \, dy, \tag{D.4}$$

$$\int_{\hat{K}} \left(\Phi_{\hat{K}}\right)^{2\delta} |\nabla \pi_p|^2 \, dx \, dy \le C p^{2(2-\delta)} \int_{\hat{K}} \left(\Phi_{\hat{K}}\right)^{\delta} \pi_p^2 \, dx \, dy. \tag{D.5}$$

If additionally $\pi_p = 0$ on $\partial \hat{K}$, then

$$\int_{\hat{K}} |\nabla \pi_p|^2 \, dx \, dy \le C_1 p^2 \int_{\hat{K}} \left(\Phi_{\hat{K}}\right)^{-1} |\pi_p|^2 \, dx \, dy. \tag{D.6}$$

The basis of the proof of Theorem D.2 is the following quasi one-dimensional result on trapezoids:

Lemma D.3. *Let $d \in (0,1)$, a, b be given such that $-1 + ad < 1 + bd$ and define the trapezoid D by*

$$D(a,b,d) := \{(x,y) \in \mathbb{R}^2 \mid y \in (0,d) \text{ and } -1 + ay < x < 1 + by\}.$$

On D define the weight function

$$\Phi_{a,b,d}(x,y) := \min\{|x - (-1+ay)|, |x - (1+by)|\},$$

which measures the distance of the point (x,y) from the lateral edges of D. Let $-1 < \alpha < \beta$ and $\delta \in [0,1]$. Then there exist $C_1 = C_1(\alpha, \beta, a, b, d)$, $C_2 = C_2(\delta, a, b, d) > 0$ such that for all $p \in \mathbb{N}$ and all polynomials $\pi_p \in \mathcal{Q}_p$

$$\int_{D(a,b,d)} \Phi_{a,b,d}^{\alpha}(x,y) \, \pi_p^2 \, dx \, dy \le C_1 p^{2(\beta-\alpha)} \int_{D(a,b,d)} \Phi_{a,b,d}^{\beta}(x,y) \, \pi_p^2 \, dx \, dy,$$

$$\int_{D(a,b,d)} \Phi_{a,b,d}^{2\delta}(x,y) \, |\partial_x \pi_p|^2 \, dx \, dy \le C_2 p^{2(2-\delta)} \int_{D(a,b,d)} \Phi_{a,b,d}^{\beta}(x,y) \, \pi_p^2 \, dx \, dy.$$

Proof. Lemma D.1 and a scaling argument imply easily the existence of $C > 0$ independent of y such that each fixed $y \in (0,d)$ we have

$$\int_{-1+ay}^{1+by} \left(\Phi_{a,b,d}(x,y)\right)^{\alpha} \, \pi_p^2(x,y) \, dx \le C p^{2(\beta-\alpha)} \int_{-1+ay}^{1+by} \left(\Phi_{a,b,d}(x,y)\right)^{\beta} \, \pi_p^2(x,y) \, dx.$$

Integrating this last estimate over $y \in (0,d)$ completes the proof of the first estimate. The second estimate is proved similarly. ∎

Lemma D.3 is the basis for the proof of Theorem D.2.

Proof of Theorem D.2: Exemplarily, we will prove (D.4) and (D.5) for the case of the reference triangle $\hat{K} = T$, since the remaining cases are proved using the same ideas and techniques. For notational convenienc, we will also assume the reference triangle T to be the equilateral triangle with side lengths 1 given by

$$T = \left\{(x,y) \mid 0 < x < 1,\, 0 < y < \min\left(\sqrt{3}x, \sqrt{3}\,(1-x)\right)\right\}. \tag{D.7}$$

The basic idea of the proof is to cover $\hat{K}$ by a few (in fact, ≤ 6) trapezoids and exploiting that on each such trapezoid D, the distance to $\partial \hat{K}$ is comparable to the

distance to the lateral sides of D. Thus Lemma D.3 is applicable, and the desired result follows from a covering argument.
For any $s \in (0, \sqrt{3})$, we denote by $D(s)$ the trapezoid

$$D(s) := \Big\{(x,y) \in T \,|\, y < sx\Big\} \setminus \Big\{(x,y) \in T \,|\, y < s(x - \frac{1}{2})\Big\}. \tag{D.8}$$

For the proof of (D.4), choose $s \in (\frac{\sqrt{3}}{2}, \sqrt{3})$ and define two auxiliary trapezoids (see Fig. D.1)

$$D_1 := D(s), \qquad D_2 := \text{ mirror image of } D_1 \text{ about the line } y = \tfrac{\sqrt{3}}{2}x.$$

Note that $T \subset (D_1 \cup D_2) \cup R_{\pi/3}(D_1 \cup D_2) \cup R_{2\pi/3}(D_1 \cup D_2)$, where $R_{\pi/3}$, $R_{2\pi/3}$ denote the rotation about the barycenter of the triangle T with angles $\pi/3$, $2\pi/3$, respectively. One can easily find an affine map F and constants a, b, d such that $F(D(a,b,d)) = D_1$, where the trapezoid $D(a,b,d)$ is defined in Lemma D.3. We note that for the function $\Phi_{a,b,d}$ of Lemma D.3, we have

$$\Phi_{\hat{K}} \circ F \sim \Phi_{a,b,d} \quad \text{ uniformly on } D(a,b,d).$$

Hence, we obtain from Lemma D.3

$$\int_{D_1} \Phi_{\hat{K}}^{\alpha} |\pi_p|^2 \, dx\, dy \le C p^{2(\beta-\alpha)} \int_{D_1} \Phi_{\hat{K}}^{\beta} |\pi_p|^2 \, dx\, dy.$$

A similar estimate holds for D_2 and the sets $R_{\pi/3}D_1$, $R_{2\pi/3}D_1$, $R_{\pi/3}D_2$, $R_{2\pi/3}D_2$. Estimate (D.4) then follows by a covering argument.
The bound (D.5) is proved similarly. Choosing $\frac{\sqrt{3}}{2} < s < s' < \sqrt{3}$ we set

$$D_1 := D(s), \qquad D_1' := D(s').$$

Using the same arguments as before, we conclude with the aid of Lemma D.3 that

$$\int_{D_1} \Phi_{\hat{K}}^{2\delta} |\vec{s} \cdot \nabla \pi_p|^2 \, dx\, dy \le C p^{2(2-\delta)} \int_{D_1} \Phi_{\hat{K}}^{\delta} |\nabla \pi_p|^2 \, dx\, dy,$$
$$\int_{D_1'} \Phi_{\hat{K}}^{2\delta} |\vec{s'} \cdot \nabla \pi_p|^2 \, dx\, dy \le C p^{2(2-\delta)} \int_{D_1'} \Phi_{\hat{K}}^{\delta} |\nabla \pi_p|^2 \, dx\, dy,$$

where $\vec{s} = (s,1)$, $\vec{s'} = (s',1)$. We conclude, since $\vec{s}$, $\vec{s'}$ are linearly independent that

$$\begin{aligned} \int_{D_1 \cap D_1'} \Phi_{\hat{K}}^{2\delta} |\nabla \pi_p|^2 \, dx\, dy &\le C p^{2(2-\delta)} \int_{D_1 \cup D_1'} \Phi_{\hat{K}}^{2\delta} |\pi_p|^2 \, dx\, dy \\ &\le C p^{2(2-\delta)} \int_{T} \Phi_{\hat{K}}^{2\delta} |\pi_p|^2 \, dx\, dy. \end{aligned} \tag{D.9}$$

Using the same arguments, the bound (D.9) also holds with D_1, D_1' replaced by D_2, D_2', their mirror images with respect to the line $y = \frac{\sqrt{3}}{2}x$. Similarly, the pairs D_1, D_1', and D_2, D_2' may be replaced in (D.9) the sets obtained by rotating by $\pi/3$ and $2\pi/3$ about the barycenter of T. As all these sets together cover the triangle T, we may conclude the proof of (D.5). $\square$

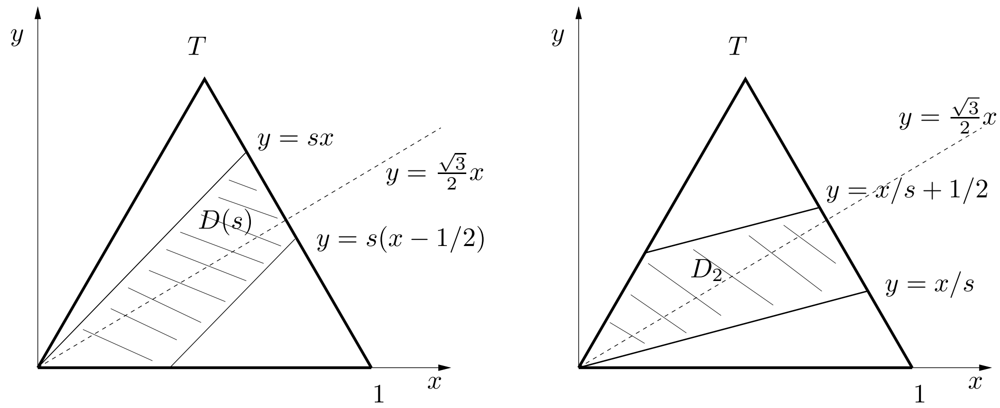

FIG. D.1. *Trapezoids $D_1 = D(s)$ and D_2 of the proof of Theorem D.2.*

REMARK D.4. The proof of Theorem D.2 is based on a) the quasi one-dimensional result Lemma D.3 and b) a covering argument. The arguments of Theorem D.2 may therefore be extended to arbitrary dimensions and to more general domains. Also other one-dimensional inverse estimates in weighted Sobolev spaces could be treated in this way.

REMARK D.5. Polynomial inverse estimates on simplices in weighted Sobolev spaces were recently obtained in [19]. While [19] provides sharp explicit bounds, the weights employed are weaker than the weight function (D.2) considered here.

Appendix E. Approximation with trigonometric polynomials. The following result can be obtained from combining Thms. 2.3, 2.7, 2.8 of [25, Chap. 7].

PROPOSITION E.1. *Let $\mathbb{T}$ be the torus and denote for $r \in \mathbb{N}_0$, $q \in [1,\infty]$ by $W^{r,q}(\mathbb{T})$ the usual (periodic) Sobolev spaces. Denote by T_N the set of trigonometric polynomials of degree N.*
Then for each $R \in \mathbb{N}$ and $N \in \mathbb{N}$ there exists a bounded linear operator $J_{R,N} : L^1(\mathbb{T}) \to T_N$ and a constant C_R (depending solely on R) with the following properties: For every $r \in \mathbb{N}_0$ with $0 \le r \le R$ and every $q \in [1,\infty]$ there holds for all $u \in W^{r,q}(\mathbb{T})$

$$\|(u - J_{R,N}u)^{(j)}\|_{L^q(\mathbb{T})} \le C_r N^{-(r-j)} \|u^{(r)}\|_{L^q(\mathbb{T})}, \qquad j = 0, \dots, r. \tag{E.1}$$

Proof. As the operator $J_{R,N}$ we take the averaging operator of Jackson type defined in [25, Chap. 7, eqn. (2.8)], i.e.,

$$J_{R,N}\, u(x) = \int_{\mathbb{T}} \left[(-1)^{R+1}\Delta_t^R(u,x) + u(x)\right] K_{N,R}(t)\, dt,$$
$$K_{N,R}(t) = \lambda_{N,R} \left(\frac{\sin(mt/2)}{\sin t/2}\right)^{2R},$$

where $\lambda_{N,R} > 0$ is such that $\int_{\mathbb{T}} K_{N,R}(t)\, dt = 1$; $m = \lfloor N/R \rfloor + 1 \in \mathbb{N}_0$, and $\Delta_t^R(u,x) = \sum_{k=0}^R \binom{R}{k}(-1)^{R-k} u(x+kt)$ is the standard higher order forward difference.
By the proof of [25, Chap. 7, Thm. 2.3] these operators have the following properties: There exists $C_R > 0$, which depends solely on R, such that for $u \in L^q(\mathbb{T})$ (if $q \in [1,\infty)$) and $u \in C(\mathbb{T})$ (if $q = \infty$)

$$\|u - J_{R,N}u\|_{L^q(\mathbb{T})} \le C_R \omega_R(u, N^{-1})_q, \tag{E.2}$$

where $\omega_R(u,\delta)_q = \sup_{|h|<\delta} \|\Delta_h^R u\|_{L^q(\mathbb{T})}$ is the standard modulus of smoothness. Furthermore, we have the following two elementary properties of the modulus of continuity (cf., e.g., [25, Chap. 2, eqns. (7.5), (7.12))]:

$$\omega_R(u,\delta)_q \le 2^{R-r}\omega_r(u,\delta)_q, \qquad 1 \le r \le R, \quad \delta > 0, \tag{E.3}$$

$$\omega_r(u,\delta)_q \le \delta^r \|u^{(r)}\|_{L^q(\mathbb{T})}, \qquad \text{if } u \in W^{r,q}(\mathbb{T}),\ r \ge 1. \tag{E.4}$$

We now prove (E.1) by induction on r. We proceed as in the the proof of [25, Chap. 7, Thm. 2.7].

1. step: The case $r = 0$. From $\int_{\mathbb{T}} K_{N,R}(t)\,dt = 1$, we get

$$\begin{aligned}\|u - J_{R,N}u\|_{L^q(\mathbb{T})} &= \| \int_{\mathbb{T}} (-1)^{R+1}\Delta_t^R(u,x)K_{N,R}(t)\,dt\|_{L^q(\mathbb{T})} \le 2^R\|u\|_{L^q(\mathbb{T})} \int_{\mathbb{T}} K_{N,R}(t)\,dt \\ &\le 2^R\|u\|_{L^q(\mathbb{T})}.\end{aligned}$$

2. step: We assume that (E.1) has been proved for some $r-1$ with $1 \le r \le R$ and show that (E.1) also holds for r. We we first note that $r \ge 1$ implies by Sobolev's embedding theorem $W^{1,q}(\mathbb{T}) \subset C(\mathbb{T})$ that any function $u \in W^{r,q}(\mathbb{T})$ is continuous. Combining (E.2), (E.3), (E.4), we get

$$\|u - J_{R,N}u\|_{L^q(\mathbb{T})} \le C_R N^{-r}\|u^{(r)}\|_{L^q(\mathbb{T})}. \tag{E.5}$$

Next, we consider the function $u' \in W^{r-1,q}(\mathbb{T})$. Then by the induction hypothesis the function $\widetilde{S} := J_{R,N}u'$ satisfies

$$\| \left(u' - \widetilde{S}\right)^{(j)} \|_{L^q(\mathbb{T})} \le C_R N^{-(r-1-j)}\|u^{(r)}\|_{L^q(\mathbb{T})}, \qquad j = 0,\dots,r-1.$$

Denoting by a_0 the constant term of the trigonometric polynomial $\widetilde{S}$, we define $S := \widetilde{S} - a_0$ and estimate with [25, Chap. 7, eqn. (2.15)]

$$\|u' - S\|_{L^q(\mathbb{T})} \le 2\|u' - \widetilde{S}\|_{L^q(\mathbb{T})}.$$

We conclude

$$\| (u' - S)^{(j)} \|_{L^q(\mathbb{T})} \le 2C_R N^{-(r-1-j)}\|u^{(r)}\|_{L^q(\mathbb{T})}, \qquad j = 0,\dots,r-1. \tag{E.6}$$

Since S has, by construction, vanishing mean, there exists a trigonometric polynomial $\widehat{S} \in T_N$ such that $\widehat{S}' = S$. We finally define the trigonometric polynomial $\widehat{R} = J_{R,N}(u - \widehat{S})$. Since $u \in W^{r,q}(\mathbb{T}) \subset C(\mathbb{T})$, we get from (E.2) and (E.6)

$$\|(u - \widehat{S}) - \widehat{R}\|_{L^q(\mathbb{T})} \le C\omega_1(u - \widehat{S}, 1/N)_q \tag{E.7}$$

$$\le CN^{-1}\|u' - \widehat{S}'\|_{L^q(\mathbb{T})} \le CN^{-r}\|u^{(r)}\|_{L^q(\mathbb{T})}. \tag{E.8}$$

Again, since $u \in W^{r,q}(\mathbb{T}) \subset C(\mathbb{T})$, we conclude from [25, Chap. 7, Lemma 2.6] (here, the estimate (E.7) is the required hypothesis) for the trigonometric polynomial $\widehat{R}$

$$\|\widehat{R}'\|_{L^q(\mathbb{T})} \le C\omega_1(u - \widehat{S}, 1/N)_q \le CN^{-1}\| \left(u - \widehat{S}\right)' \|_{L^q(\mathbb{T})} \le CN^{-(r-1)}\|u^{(r)}\|_{L^q(\mathbb{T})}. \tag{E.9}$$

Therefore, by the Bernstein inequality (see [25, Chap. 4, Thm. 2.5, 2.6])

$$\|\widehat{R}^{(j+1)}\|_{L^q(\mathbb{T})} \le N^j\|\widehat{R}'\|_{L^q(\mathbb{T})} \le CN^{-(r-1-j)}\|u^{(r)}\|_{L^q(\mathbb{T})}, \qquad j = 0,\dots,r-1. \tag{E.10}$$

We are now in position to prove the desired claim. We have to show (E.1). For $j = 0$, this follows immediately from (E.5). For derivatives, we estimate for $j = 0, \ldots, r-1$ using the Bernstein inequality

$$
\begin{aligned}
&\| (u - J_{R,N} u)^{(j+1)} \|_{L^q(\mathbb{T})} \leq \\
&\| \left(u - (\widehat{S} + \widehat{R})\right)^{(j+1)} \|_{L^q(\mathbb{T})} + \| \left(\widehat{S} + \widehat{R} - J_{R,N}\, u\right)^{(j+1)} \|_{L^q(\mathbb{T})} \leq \\
&\| (u' - S)^{(j)} \|_{L^q(\mathbb{T})} + \|\widehat{R}^{(j+1)}\|_{L^q(\mathbb{T})} + \| \left(\widehat{S} + \widehat{R} - J_{R,N}\, u\right)^{(j+1)} \|_{L^q(\mathbb{T})} \\
&\| (u' - S)^{(j)} \|_{L^q(\mathbb{T})} + \|\widehat{R}^{(j+1)}\|_{L^q(\mathbb{T})} + N^{j+1} \|\widehat{S} + \widehat{R} - J_{R,N}\, u\|_{L^q(\mathbb{T})}.
\end{aligned}
$$

We now combine the estimates (E.6), (E.10), and (E.8) to arrive at

$$
\| (u - J_{R,N} u)^{(j+1)} \|_{L^q(\mathbb{T})} \leq C N^{-(r-1-j)} \|u^{(r)}\|_{L^q(\mathbb{T})}, \qquad j = 0, \ldots, r-1,
$$

which yields (E.1). □

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