FREE BOUNDARY PROBLEMS AND TRANSONIC SHOCKS FOR THE EULER EQUATIONS IN UNBOUNDED DOMAINS

GUI-QIANG CHEN AND MIKHAIL FELDMAN

ABSTRACT. We establish the existence and stability of multidimensional transonic shocks (elliptic-hyperbolic shocks), which are not nearly orthogonal to the flow direction, for the Euler equations for steady compressible potential fluids in unbounded domains in $\mathbf{R}^n, n \geq 3$. The Euler equations can be written as a second order nonlinear equation of mixed elliptic-hyperbolic type for the velocity potential. The transonic shock problem can be formulated into the following free boundary problem: The free boundary is the location of the multidimensional transonic shock which divides two regions of $C^{2,\alpha}$ flow, and the equation is hyperbolic in the upstream region where the $C^{2,\alpha}$ perturbed flow is supersonic. In this paper, we develop a new approach to deal with such free boundary problems and establish the existence and stability of multidimensional transonic shocks. We first reformulate the free boundary problem into a fixed conormal boundary value problem for a nonlinear elliptic equation of second order in unbounded domains and then develop nonlinear techniques to solve this nonlinear elliptic problem. Our results indicate that there exists a solution of the free boundary problem such that the equation is always elliptic in the unbounded downstream region, the uniform velocity state at infinity in the downstream direction is uniquely determined by the given hyperbolic phase, and the free boundary is $C^{2,\alpha}$, provided that the hyperbolic phase is close in $C^{2,\alpha}$ to a uniform flow. We further prove that the free boundary is stable under the $C^{2,\alpha}$ steady perturbation of the hyperbolic phase. Moreover, we extend our existence results to the case that the regularity of the steady perturbation is only $C^{1,1}$, and we also introduce another simpler nonlinear approach to deal with the existence and stability problem when the regularity of the steady perturbation is $C^{3,\alpha}$ or higher.

1. INTRODUCTION

We are concerned with the existence and stability of multidimensional steady transonic shocks, which are not nearly orthogonal to the flow direction, in inviscid compressible potential flows. The Euler equations for such fluid flows consist of the conservation law of mass and the Bernoulli law for velocity, and can be formulated into the following nonlinear second-order equations of mixed elliptic-hyperbolic type for the velocity potential $\varphi : \Omega \subset \mathbf{R}^n \to \mathbf{R}$:

$$\operatorname{div}(\rho(|D\varphi|^2)D\varphi) = 0, \tag{1.1}$$

where the density $\rho(q^2)$ is

$$\rho(q^2) = (1 - \theta q^2)^{\frac{1}{2\theta}}, \tag{1.2}$$

with $\theta = \frac{\gamma-1}{2} > 0$ for the adiabatic exponent $\gamma > 1$.

The second-order nonlinear equation (1.1) is elliptic at $D\varphi$ with $|D\varphi| = q$ if

$$\rho(q^2) + 2q^2\rho'(q^2) > 0; \tag{1.3}$$

Date: August 7, 2003.

1991 *Mathematics Subject Classification.* 35M10,35J65,35R35,76H05,76L05,35B45,35B35.

Key words and phrases. Elliptic-hyperbolic, nonlinear equations, second-order, mixed type, transonic shocks, free boundary problems, unbounded domains, existence, uniqueness, stability, Euler equations, compressible flow.

and is hyperbolic if

$$\rho(q^2) + 2q^2\rho'(q^2) < 0. \tag{1.4}$$

Some efforts were made in solving the nonlinear equation (1.1) of mixed elliptic-hyperbolic type in [4, 13, 16, 31, 34], [9, 10, 19, 22, 27, 33, 35], and the references cited therein. A similar problem was considered in [5] for the two-dimensional transonic small-disturbance (TSD) model.

In [6], we developed a nonlinear approach by combining an iteration scheme with a fixed point technique to establish the existence and stability of multidimensional transonic shocks that are nearly orthogonal to the flow directions.

In Sections 3–4, we develop a new, different approach to deal with other difficulties for more general multidimensional transonic shock problems, especially including the essential non-orthogonality of transonic shocks to the flow direction; such situations arise in several important physical problems.

In this paper, we focus on multidimensional transonic shocks near planar transonic shocks in $\mathbf{R}^n, n \geq 3$; multidimensional transonic shocks near spherical transonic shocks will be handled in [8]. Such a transonic shock problem can be formulated into the corresponding free boundary problem: The free boundary is the location of the multidimensional transonic shock which divides two regions of $C^{2,\alpha}$ flow in $\mathbf{R}^n$, and the equation is hyperbolic in the upstream region where the $C^{2,\alpha}$ perturbed flow is supersonic.

One of the main ingredients in our new approach is to employ a partial hodograph transform to reduce the free boundary problem into a conormal boundary value problem for the corresponding nonlinear elliptic equation of divergence form in the half space. In order to solve the conormal boundary value problem in the unbounded domain, our strategy is to first construct solutions in a series of half balls with radius R, then make uniform estimates in R, and finally send $R \to \infty$. To achieve this requires delicate apriori estimates. We first obtain a uniform bound in a weighted L^∞-norm by employing a comparison principle and identifying a global function with the same decay rate as the fundamental solution of the elliptic equation with constant coefficients which controls the solutions. Of course, this decay rate is the same as for the fundamental solution of the Laplace equation. Then, by scaling arguments, we obtain the uniform estimates in a weighted Hölder norm for the solutions. Thus we obtain the existence of a solution in the half space and the algebraic rate of decay of this solution at infinity. For such decaying solutions in the half space, a comparison principle holds, which implies the uniqueness for the conormal problem. Finally, by a gradient estimate, we show that the limit function is a solution of the multidimensional transonic shock problem. We further prove that the multidimensional transonic shock solution is stable with respect to the $C^{2,\alpha}$ supersonic perturbation in Section 5.

In Section 6, we extend the existence results to the case that the regularity of the steady perturbation is only $C^{1,1}$, and we also introduce another simpler nonlinear approach to deal with the existence and stability problem when the regularity of the steady perturbation is $C^{3,\alpha}$ or higher.

We remark that the case $n = 2$ exhibits special features, different from the case $n \geq 3$, and requires a different approach, which will be a part of the content of our forthcoming paper [8].

2. Multidimensional Transonic Shocks in the Whole Space

In this section, we first set up the multidimensional transonic shock problem near planar transonic shocks in $\mathbf{R}^n$ and present the main theorems of this paper.

A function $\varphi \in W^{1,\infty}(\Omega)$ is a weak solution of (1.1) in an unbounded domain Ω if

(i) $|D\varphi(x)| \leq 1/\sqrt{\theta}$ a.e.

(ii) For any $w \in C_0^\infty(\Omega)$,

$$\int_\Omega \rho(|D\varphi|^2) D\varphi \cdot Dw \, dx = 0. \tag{2.1}$$

We are interested in weak solutions with shocks. Let Ω^+ and Ω^- be open subsets of Ω such that

$$\Omega^+ \cap \Omega^- = \emptyset, \qquad \overline{\Omega^+} \cup \overline{\Omega^-} = \overline{\Omega},$$

and $S = \partial\Omega^+ \cap \Omega$. Let $\varphi \in W^{1,\infty}(\Omega)$ be a weak solution of (1.1) and be in $\varphi \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ so that $D\varphi$ experiences a jump across S that is an $(n-1)$-dimensional smooth surface. The requirement $\varphi \in W^{1,\infty}(\Omega)$ yields $curl(D\varphi) = 0$ in the sense of distributions, which implies

$$D_\tau \varphi^+ = D_\tau \varphi^- \qquad \text{on} \quad S,$$

where $D_\tau \varphi^\pm := D\varphi^\pm - (D\varphi^\pm \cdot \nu)\nu$ are the tangential gradients of φ in the $(n-1)$-dimensional tangential space on the $\Omega^\pm$ sides of S, respectively, and ν is the unit normal to S from Ω^- to Ω^+. Then we simply write $D_\tau \varphi := D_\tau \varphi^\pm$ on S and assume

$$\varphi^+ = \varphi^- \qquad \text{on } S. \tag{2.2}$$

Thus, we use (1.1), (2.1), and (2.2) to conclude the Rankine-Hugoniot jump condition on S:

$$\Big[\rho(|D\varphi|^2) D\varphi \cdot \nu\Big]_S = 0, \tag{2.3}$$

where the bracket denotes the difference between the values of the function along S on the $\Omega^\pm$ sides of S, respectively. We can also write (2.3) as

$$\rho(|D\varphi^+|^2)\varphi_\nu^+ = \rho(|D\varphi^-|^2)\varphi_\nu^- \qquad \text{on} \quad S, \tag{2.4}$$

where $\varphi_\nu^\pm = D\varphi^\pm \cdot \nu$ are the normal derivatives on the $\Omega^\pm$ sides, respectively.

The function

$$\Phi(p) := \left(1 - \theta p^2\right)^{\frac{1}{2\theta}} p, \tag{2.5}$$

defined for $p \in \left[0, \sqrt{1/\theta}\right]$, satisfies

$$\lim_{p \to 0+} \Phi(p) = \lim_{p \to \sqrt{1/\theta}-} \Phi(p) = 0, \qquad \Phi(p) > 0 \quad \text{for } p \in \left(0, \sqrt{1/\theta}\right), \tag{2.6}$$

$$0 < \Phi'(p) < 1 \text{ on } (0, p_{sonic}), \text{ and } \Phi'(p) < 0 \quad \text{on } \left(p_{sonic}, \sqrt{1/\theta}\right), \tag{2.7}$$

$$\Phi''(p) < 0 \text{ on } (0, p_{sonic}], \tag{2.8}$$

where

$$p_{sonic} = \sqrt{1/(\theta+1)} \tag{2.9}$$

is the sonic speed.

Suppose that $\varphi(x)$ is a solution satisfying

$$|D\varphi(x)| < p_{sonic} = \frac{1}{\sqrt{\theta+1}} \quad \text{in } \Omega^+, \qquad |D\varphi(x)| > p_{sonic} \quad \text{in } \Omega^-, \tag{2.10}$$

and

$$D\varphi^\pm \cdot \nu > 0 \qquad \text{on } S, \tag{2.11}$$

besides (2.2) and (2.3). Then $\varphi(x)$ is a *transonic shock solution* with *transonic shock* S dividing *subsonic region* Ω^+ and *supersonic region* Ω^- and satisfying the physical entropy condition (see Courant-Friedrichs [11]; also see Dafermos [12] and Lax [21]):

$$\rho(|D\varphi^-|^2) < \rho(|D\varphi^+|^2) \qquad \text{along } S, \tag{2.12}$$

which implies, by (2.11), that the density increases in the flow direction. Note that equation (1.1) is elliptic in the subsonic region and is hyperbolic in the supersonic region.

Let (x', x_n) be the coordinates in $\mathbf{R}^n$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. Fix $V_0 \in \mathbf{R}^n$, and let

$$\varphi_0(x) := V_0 \cdot x, \quad x \in \mathbf{R}^n.$$

If $|V_0| \in (0, p_{sonic})$ (resp. $|V_0| \in (p_{sonic}, 1/\sqrt{\theta})$), then $\varphi_0(x)$ is a subsonic (resp. supersonic) solution in $\mathbf{R}^n$, and $V_0 = D\varphi_0$ is its velocity.

Let $V_0' \in \mathbf{R}^{n-1}$ and $q_0^+ > 0$ be such that the vector $V_0^+ := (V_0', q_0^+)$ satisfies $|V_0^+| < p_{sonic}$. Then, using the properties of function (2.5), we conclude from (2.6)–(2.9) that there exists a unique $q_0^- > 0$ such that

$$\big(1 - \theta(|V_0'|^2 + |q_0^+|^2)\big)^{\frac{1}{2\theta}} q_0^+ = \big(1 - \theta(|V_0'|^2 + |q_0^-|^2)\big)^{\frac{1}{2\theta}} q_0^-. \tag{2.13}$$

The entropy condition (2.12) implies $q_0^- > q_0^+$. By denoting $V_0^- := (V_0', q_0^-)$ and defining functions $\varphi_0^\pm(x) := V_0^\pm \cdot x$ on $\mathbf{R}^n$, then φ_0^+ (resp. φ_0^-) is a subsonic (resp. supersonic) solution. Furthermore, from (2.4) and (2.13), the function

$$\varphi_0(x) := \min(\varphi_0^+(x), \varphi_0^-(x)) = \begin{cases} V_0^- \cdot x, & x \in \Omega_0^- := \{x \in \mathbf{R}^n \,:\, x_n < 0\}, \\ V_0^+ \cdot x, & x \in \Omega_0^+ := \{x \in \mathbf{R}^n \,:\, x_n > 0\} \end{cases} \tag{2.14}$$

is a flat transonic shock solution in $\mathbf{R}^n$, Ω_0^+ and Ω_0^- are respectively its subsonic and supersonic regions, and $S = \{x_n = 0\}$ is a transonic shock. Note that, if $V_0' = 0$, then the velocities $V_0^\pm$ are orthogonal to the shock S and, if $V_0' \neq 0$, then the velocities are not orthogonal to S.

The multidimensional transonic shock problem near $\varphi_0(x)$ with $V_0' = 0$ has been handled in Chen-Feldman [6, 7]. In this paper, we develop a new, nonlinear approach to handle with the case $V_0' \neq 0$ in the whole space $\mathbf{R}^n$. We first study perturbations of the uniform transonic shock solution (2.14) in the whole space $\mathbf{R}^n, n \geq 3$, in Sections 3–5. In order to state our problem, we first introduce weighted Hölder seminorms and norms on unbounded domains. Note that later we consider our fixed boundary value problems on the subsonic region Ω^+, which is expected to be close to the half-space $\Omega_0^+ = \{x_n > 0\}$.

Let $\mathcal{D} = \{x_n > f(x')\}$, where $f(x')$ is a Lipschitz function. For $x = (x', x_n) \in \mathcal{D}$, let $\hat{\delta}_x = 1 + |x|$ and, for $x, y \in \mathcal{D}$, let $\hat{\delta}_{x,y} = 1 + \min(\delta_x, \delta_y)$. Let $\theta \in \mathbf{R}$, $\alpha \in (0,1)$, and k a nonnegative integer. We define

$$\begin{aligned}
[[u]]^{(\theta)}_{k;0;\mathcal{D}} &= \sup_{x \in \mathcal{D}} \left(\hat{\delta}_x^{k+\theta} |D^k u(x)| \right), \\
[[u]]^{(\theta)}_{k;\alpha;\mathcal{D}} &= \sup_{x,y \in \mathcal{D}, x \neq y} \left(\hat{\delta}_{x,y}^{k+\alpha+\theta} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha} \right), \\
\|u\|^{(\theta)}_{k;0;\mathcal{D}} &= \sum_{j-0}^{k} [[u]]^{(\theta)}_{j;0;\mathcal{D}}, \\
\|u\|^{(\theta)}_{k;\alpha;\mathcal{D}} &= \|u\|^{(\theta)}_{k;0;\mathcal{D}} + [[u]]^{(\theta)}_{k;\alpha;\mathcal{D}}.
\end{aligned} \tag{2.15}$$

We study the existence and stability of multidimensional transonic shocks near the plane transonic shock (2.14) under small perturbations of the supersonic flow. It suffices to prescribe the perturbed supersonic flow only near the unperturbed shock surface $S_0 = \{x_n = 0\}$. Thus, we introduce a domain $\Omega_1 := \mathbf{R}^{n-1} \times (-1, 1)$ and focus our discussion on the domain $\Omega := \mathbf{R}^{n-1} \times (-1, \infty)$.

Problem A. Given a supersonic solution $\varphi^-(x)$ of (1.1) in Ω_1 satisfying that, for some $\alpha > 0$,

$$\|\varphi^- - \varphi_0^-\|^{(n-1)}_{2,\alpha,\Omega_1} \leq \sigma, \tag{2.16}$$

with $\sigma > 0$ small, find a transonic shock solution $\varphi(x)$ in Ω such that $\Omega^- \subset \Omega_1$ and $\varphi(x) = \varphi^-(x)$ in Ω^-, where $\Omega^- := \Omega \setminus \Omega^+$ and $\Omega^+ := \{x \in \Omega \quad : \quad |D\varphi(x)| < p_{sonic}\}$.

Remark 2.1. *Note that, since $n \geq 3$, our assumptions imply that the perturbation is not only small in $C^{2,\alpha}$, but also "localized", i.e., has an appropriate algebraic decay at infinity.*

One of our main results of this paper is the following.

Theorem 2.1. *Let $|(V_0', q_0^+)| \in (0, p_{sonic})$ and $q_0^- \in \left(p_{sonic}, 1/\sqrt{\theta}\right)$ satisfy (2.13), and let $\varphi_0(x)$ be the transonic shock solution (2.14). Then there exist positive constants σ_0, C_1, and C_2 depending only on n, γ, α, $|V_0'|$, and q_0^+ such that, for every $\sigma \leq \sigma_0$ and any supersonic solution $\varphi^-(x)$ of (1.1) satisfying the conditions stated in Problem A, there exists a unique solution $\varphi(x)$ of Problem A satisfying*

$$\|\varphi - \varphi_0^+\|_{2,\alpha,\Omega^+}^{(n-2)} \leq C_1\sigma, \tag{2.17}$$

with Ω^+ defined in Problem A. In addition,

$$\Omega^+ = \{x_n > f(x')\} \tag{2.18}$$

where $f : \mathbf{R}^{n-1} \to \mathbf{R}$ satisfies

$$\|f\|_{2,\alpha,\mathbf{R}^{n-1}}^{(n-2)} \leq C_2\sigma, \tag{2.19}$$

that is, the shock surface $S = \{(x', x_n) \ : \ x_n = f(x'), x' \in \mathbf{R}^{n-1}\}$ is in $C^{2,\alpha}$ and converges at infinity, with an appropriate algebraic rate, to the hyperplane $S_0 = \{x_n = 0\}$.

Furthermore, we have the following stability theorem.

Theorem 2.2. *There exist a nonnegative nondecreasing function $\Psi \in C([0,\infty))$ satisfying $\Psi(0) = 0$ and a constant σ_0 depending only on n, γ, α, $|V_0'|$, and q_0^+ such that, if $\sigma < \sigma_0$ and smooth supersonic solutions $\varphi^-(x)$ and $\hat{\varphi}^-(x)$ of (1.1) satisfy (2.16), then the unique solutions $\varphi(x)$ and $\hat{\varphi}(x)$ of Problem A for $\varphi^-(x)$ and $\hat{\varphi}^-(x)$, respectively, satisfy*

$$\|f_\varphi - f_{\hat{\varphi}}\|_{2,\alpha,\mathbf{R}^{n-1}}^{(n-2)} \leq \Psi\left(\|\varphi^- - \hat{\varphi}^-\|_{2,\alpha,\Omega_1}^{(n-1)}\right), \tag{2.20}$$

where $f_\varphi(x')$ and $f_{\hat{\varphi}}(x')$ are the free boundary functions (2.18) of $\varphi(x)$ and $\hat{\varphi}(x)$, respectively.

The proof of these theorems is obtained first by reducing Problem A to a free boundary problem for a nonlinear, uniformly elliptic equation and then by develop partial hodograph transform techniques to solve the free boundary problem.

3. Free boundary problems and A Partial hodograph transform

In this section, we first extend φ^- to the whole space $\mathbf{R}^n$, then formulate the transonic shock problems into free boundary problems, and finally reformulate the free boundary problems into fixed conormal boundary value problems for a nonlinear elliptic equation.

3.1. Extension of φ^- to the Whole Space $\mathbf{R}^n$.

Since φ^- satisfies (2.16) in the domain $\Omega_1 := \mathbf{R}^{n-1} \times (-1, 1)$, then we use a standard extension procedure to extend φ^- to $\mathbf{R}^n$ so that the extension (still denoted) φ^- is in $C^{2,\alpha}(\mathbf{R}^n)$ and satisfies

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\mathbf{R}^n}^{(n-1)} \leq C(n,\alpha)\sigma, \tag{3.1}$$

$$\operatorname{supp}(\varphi^- - \varphi_0^-) \subset \mathbf{R}^{n-1} \times (-2, 2). \tag{3.2}$$

We fix this extension operator with the properties described above.

Consider the function

$$g = \operatorname{div}\left(\rho(|D\varphi^-|^2)D\varphi^-\right) \qquad \text{in } \mathbf{R}^n. \tag{3.3}$$

Since $\varphi^-(x)$ satisfies (1.1) in $\mathbf{R}^{n-1}\times(-1,1)$, then, from (3.1)–(3.2), we have that g satisfies

$$\begin{aligned}&g\in C^{\alpha}(\mathbf{R}^n),\\ &\|g\|^{(n+1)}_{0,\alpha,\mathbf{R}^n}\le C\sigma,\\ &\operatorname{supp}(g)\subset\big(\mathbf{R}^{n-1}\times(1,2)\big)\cup\big(\mathbf{R}^{n-1}\times(-2,-1)\big).\end{aligned}\tag{3.4}$$

Define

$$F(x',x_n)=\int_0^{x_n} g(x',s)ds\qquad\text{in }\mathbf{R}^n.\tag{3.5}$$

Then, from (3.4) and (3.5), we have

$$\begin{aligned}&F,F_{x_n}\in C^{\alpha}(\mathbf{R}^n),\\ &\|F\|^{(n+1)}_{0,\alpha,\mathbf{R}^{n-1}\times(-1,1)}\le C\sigma,\\ &\sup_{(x',x_n)\in\mathbf{R}^n}\big((1+|x'|)^{n+1}|F(x',x_n)|\big)\le C\sigma,\\ &\|F_{x_n}\|^{(n+1)}_{0,\alpha,\mathbf{R}^n}\le C\sigma,\\ &F\equiv 0\qquad\text{in }\mathbf{R}^{n-1}\times(-1,1).\end{aligned}\tag{3.6}$$

From now on, we use the extended function $\varphi^-=\varphi^-(x)$, and C may denote a different constant at each occurrence, depending only on the data, i.e. on n, γ, α, $|V_0'|$, and q_0^+, unless otherwise is specified.

3.2. **Free Boundary Problems.** Similarly to [6, 7], we first reformulate Problem A into a free boundary problem.

Problem B. Find $\varphi\in C(\mathbf{R}^n)$ such that

(i) In $\mathbf{R}^n$,
$$\varphi\le\varphi^-;\tag{3.7}$$
(ii) $\varphi\in C^{2,\alpha}(\overline{\Omega^+})$ with $\Omega^+=\{\varphi<\varphi^-\}$, the noncoincidence set;
(iii) φ is a solution of (1.1) in Ω^+;
(iv) The free boundary $S=\partial\Omega^+$ is given by the equation $x_n=f(x')$ for $x'\in\mathbf{R}^{n-1}$ so that $\Omega^+=\{x_n>f(x')\}$ with $f\in C^{2,\alpha}(\mathbf{T}^{n-1})$;
(v) The free boundary condition (2.3) holds on S.

For the motivation of this reduction, see [6, 7]. Note that Problem B is not equivalent to Problem A in general, but a solution of Problem B satisfying (2.17), (2.18), and (2.19) is a solution of Problem A, provided that σ is sufficiently small.

3.3. **Partial Hodograph Transform.** We attempt to find a solution φ of Problem B, which satisfies (2.17)–(2.19). Let $\varphi(x)$ be such a solution. Define a function u in Ω^+ by

$$u(x)=\varphi^-(x)-\varphi(x).$$

Then (2.17) and (3.1) imply

$$\|u-(q_0^--q_0^+)x_n\|^{(n-2)}_{2,\alpha,\Omega^+}\le C\sigma.\tag{3.8}$$

In particular, if σ is sufficiently small,

$$0<\frac{q_0^--q_0^+}{2}\le u_{x_n}(x)\le 2(q_0^--q_0^+)\quad\text{ for any }\ x\in\Omega^+.\tag{3.9}$$

Now we show that $u(x)$ is a solution of a boundary value problem for a uniformly elliptic equation. From (2.17) with sufficiently small σ, $\varphi(x)$ satisfies (1.1) in Ω^+ and (2.4) on S. Then, using (3.3), we see that $u(x)$ is a solution of the following problem:

$$\begin{aligned}&\operatorname{div}\big(A(x,Du)\big)=-g\qquad\text{in }\ \Omega^+,\\ &A(x,Du)\cdot\nu=0\qquad\text{on }\ S,\end{aligned}$$

where

$$A(x,P)=\tilde{\rho}(|D\varphi^-(x)-P|^2)(D\varphi^-(x)-P)-\rho(|D\varphi^-(x)|^2)D\varphi^-(x), \tag{3.10}$$

for $x\in\Omega^+$ and $P\in\mathbf{R}^n$. Note that, from (2.18) and (2.19), for sufficiently small σ, the free boundary S lies within the domain $\Omega_1=\mathbf{R}^{n-1}\times(-1,1)$. Then, by (3.6), the function F defined by (3.5) vanishes on S, and thus we can rewrite the boundary value problem as the following conormal boundary value problem:

$$\operatorname{div}\left(A(x,Du)+F(x)e_n\right)=0 \qquad \text{in} \quad \Omega^+, \tag{3.11}$$

$$\left(A(x,Du)+F(x)e_n\right)\cdot\nu=0 \qquad \text{on} \quad S, \tag{3.12}$$

where $e_n=(0,\dots,0,1)$. Equation (3.11) is uniformly elliptic on u if σ is sufficiently small, which follows from (2.16) and (3.8) since

$$0<c_0\le\Phi'(q)=\tilde{\rho}(q^2)+2q^2\rho'(q^2)\le C \qquad \text{for } q \text{ near } q_0^+,$$

for some constants c_0 and $C>0$. Note that the weak form of problem (3.11)–(3.12) is

$$\int_{\Omega^+}\left(A(x,Du)+F(x)e_n\right)\cdot D\eta\,dx=0 \qquad \text{for any } \eta\in C_0^1(\mathbf{R}^n). \tag{3.13}$$

Since $\varphi=\varphi^-$ on S, it follows that

$$u=0 \qquad \text{on S.} \tag{3.14}$$

Now we make the change of variables. Define a mapping $\Phi:\overline{\Omega^+}\to\mathbf{R}^n$ by

$$(x',x_n)\to(y',y_n)=(x',u(x',x_n)).$$

The nondegeneracy property (3.9) implies that the map Φ is one-to-one on $\overline{\Omega^+}$ and, from (3.9) and (3.14),

$$\Phi(\Omega^+)=\mathbf{R}^n_+, \qquad \Phi(S)=\partial\mathbf{R}^n_+,$$

i.e. the free boundary S is mapped to the fixed boundary $\partial\mathbf{R}^n_+$. Also, by (3.9), there exists a function $v\in C^{2,\alpha}(\overline{\mathbf{R}^n_+})$ such that, for $(x',x_n)\in\overline{\Omega^+}$ and $y_n\ge0$,

$$u(x',x_n)=y_n \qquad \text{if and only if} \qquad v(x',y_n)=x_n. \tag{3.15}$$

Thus

$$\Phi^{-1}(y',y_n)=(y',v(y',y_n)).$$

Differentiating the identity $u(x',v(x',y_n))=y_n$, which holds for any $(x',y_n)\in\overline{\mathbf{R}^n_+}$, we find

$$v_{y_n}>0 \qquad \text{in } \mathbf{R}^n_+,$$

and

$$D_{x'}u=-\frac{1}{v_{y_n}}D_{y'}v, \qquad u_{y_n}=\frac{1}{v_{y_n}}, \tag{3.16}$$

where the left-hand and right-hand sides are taken at the points (x',x_n) and $\Phi(x',x_n)$, respectively. In particular, (3.9) implies

$$0<\frac{1}{2(q_0^--q_0^+)}\le v_{y_n}\le\frac{2}{q_0^--q_0^+} \quad \text{for any} \quad x\in\Omega^+. \tag{3.17}$$

From this and (2.17), we get

$$\|v-v_0\|^{(n-2)}_{2,\alpha,\Omega^+}\le C\sigma, \tag{3.18}$$

where

$$v_0(y)=\frac{y_n}{q_0^--q_0^+}. \tag{3.19}$$

Now, since $u(x)$ is a solution of the conormal boundary value problem (3.11)–(3.12) in Ω^+, then $v(x)$ is a solution of the corresponding problem in $\mathbf{R}^n_+$. In order to show that this problem has also a conormal structure, we make the change of variables $x\to y=\Phi(x)$ in

the weak form (3.13) of problem (3.11)–(3.12). In order to do that, we in particular need to change variables in the test function η. For that, we note that the function $\psi(y) := \eta \circ \Phi^{-1}(y) = \eta(y', v(y', y_n))$ satisfies $\psi \in C^1(\overline{\mathbf{R}^n_+})$ and, if $\eta \equiv 0$ on $\mathbf{R}^n \setminus B_R$, then $\psi \equiv 0$ on $\mathbf{R}^n_+ \setminus B_{R_1}$ for some R_1, i.e., $\psi = \tilde{\psi}|_{\overline{\mathbf{R}^n_+}}$ for some $\tilde{\psi} \in C^1_0(\mathbf{R}^n)$. Similarly, for any $\psi \in C^1_0(\mathbf{R}^n)$, there exists $\eta \in C^1_0(\mathbf{R}^n)$ such that $\psi|_{\overline{\mathbf{R}^n_+}} = \eta \circ \Phi^{-1}$ and $\eta(x) = \psi \circ \Phi(x)$ for $x \in \overline{\Omega^+}$. We differentiate the identity $\eta(x) = \psi(x', u(x', x_n))$ and use (3.16) to obtain

$$D_{x'}\eta = D_{y'}\psi - \frac{\psi_{y_n}}{v_{y_n}} D_{y'}v, \qquad \eta_{y_n} = \frac{\psi_{y_n}}{v_{y_n}}. \tag{3.20}$$

Now, in (3.13), we make change of variables $x \to y = \Phi(x)$, use (3.16) and (3.20), note that the Jacobian of Φ^{-1} is $J\Phi^{-1}(y) = v_{y_n}(y)$, and write $A(x', x_n, p', p_n)$ for $A(x, p)$ and $F(x', x_n)$ for $F(x)$ to obtain

$$\int_{\mathbf{R}^n_+} \left(A(y',\, v,\, -\frac{1}{v_{y_n}} D_{y'}v,\, \frac{1}{v_{y_n}}) + F(y',\, v)e_n \right) \cdot \begin{pmatrix} v_{y_n} D_{y'}\psi - \psi_{y_n} D_{y'}v \\ \psi_{y_n} \end{pmatrix} dy = 0$$

for any $\psi \in C^1_0(\mathbf{R}^n)$. This can be written as

$$\int_{\mathbf{R}^n_+} (B(y', v, Dv) + F(y',\, v)e_n) \cdot D\psi dy = 0 \qquad \text{for any } \psi \in C^1_0(\mathbf{R}^n), \tag{3.21}$$

where, for $y' \in \mathbf{R}^{n-1}$, $z \in \mathbf{R}$, $P = (p', p_n) \in \mathbf{R}^{n-1} \times \mathbf{R}_+$,

$$\begin{aligned} B^i(y', z, P) &= A^i(y', z, -\frac{p'}{p_n}, \frac{1}{p_n})p_n \qquad \text{for } \ i = 1, \dots, n-1, \\ B^n(y', z, P) &= A^n(y', z, -\frac{p'}{p_n}, \frac{1}{p_n}) - \sum_1^{n-1} A^i(y', z, -\frac{p'}{p_n}, \frac{1}{p_n})p_i. \end{aligned} \tag{3.22}$$

Thus, $v(x)$ satisfies the conormal boundary value problem:

$$\operatorname{div}(B(y', v, Dv) + F(y',\, v)e_n) = 0 \qquad \text{in } \ \mathbf{R}^n_+, \tag{3.23}$$

$$B^n(y', v, Dv) + F(y',\, v) = 0 \qquad \text{on } \ \partial\mathbf{R}^n_+. \tag{3.24}$$

Conversely, let $v(y)$ is a solution of (3.23)–(3.24) satisfying (3.18) with $C\sigma$ sufficiently small depending only on the data so that (3.17) holds. Then a function $u(x)$ can be defined on

$$\Omega^+ := \{(x', v(x', y_n)) \ : \ x' \in \mathbf{R}^{n-1},\ y_n > 0\} \tag{3.25}$$

such that (3.15) holds. Clearly, $u(x)$ satisfies (3.8) and (3.9). Making the change of variables $x = \Phi^{-1}y$ in (3.21), we see that $u(x)$ satisfies (3.13) and thus (3.11)–(3.12). Then

$$\varphi(x) := \begin{cases} \varphi_0(x) - u(x) & \text{for } \ x \in \Omega^+, \\ \varphi_0(x) & \text{otherwise} \end{cases}$$

is continuous in $\mathbf{R}^n$ and satisfies (2.17). Thus, $\varphi(x)$ is a solution of Problem A if σ is so small that (2.17) implies that, in Ω^+,

(a) φ is a solution of the non-truncated equation,

(b) φ is subsonic.

Moreover, from (3.25) and (3.17), Ω^+ satisfies (2.18) with $f(x') = v(x', 0)$. Thus, from (3.18), it follows that (2.19) holds.

Furthermore, if σ is small, depending only on the data, and φ_1 and φ_2 are two solutions of Problem A satisfying (2.17). Then both functions $u_k = \varphi^- - \varphi_k$ for $k = 1, 2$ satisfy $\partial_{x_n} u_k \geq (q_0^- - q_0^+)/2 > 0$, and thus functions $v_k \in C^{2,\alpha}(\overline{\mathbf{R}^n_+})$, $k = 1, 2$, are defined by (3.15) and satisfy (3.18) with C depending only on the data. Note that v_1 is not identically equal to v_2 if φ_1 is not identically equal to φ_2.

Therefore, we have

Proposition 3.1. *Assume that σ is small, depending only on the data. Let $\varphi^-(x)$ be a supersonic solution of* (1.1) *satisfying the conditions stated in Problem A. Assume that problem* (3.23)–(3.24), *defined by* (3.10) *and* (3.22), *has a unique solution $v \in C^{2,\alpha}(\overline{\mathbf{R}^n_+})$ satisfying* (3.18) *with $C\sigma$ sufficiently small, depending only on the data. Then there exists a unique solution φ of Problem A satisfying* (2.17). *Moreover,* (2.18) *and* (2.19) *hold for φ, and the function $u := \varphi^- - \varphi$ is related with v by* (3.15).

4. Solutions of the Conormal Boundary Value Problems

By Proposition 3.1, in order to solve Problem A, it suffices to establish the existence and uniqueness for the conormal boundary value problem (3.23)–(3.24) satisfying (3.18). First, note that (3.23) is elliptic in a neighborhood of function $v_0(y)$ defined by (3.19), that is, there exist $\Lambda > \lambda > 0$ such that

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} B^i_{p_j}(y', z, P)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{4.1}$$

for any $\xi \in \mathbf{R}^n$. From (3.22), we compute

$$\sum_{i,j-1}^{n} B^i_{p_j}(y', z, P)\xi_i\xi_j = \sum_{i,j=1}^{n} A^i_{p_j}(y', z, -\frac{p'}{p_n}, \frac{1}{p_n})\zeta_i\zeta_j,$$

where

$$\zeta_i = \xi_i - \frac{p_i}{p_n}\xi_n, \quad i = 1, \dots, n-1; \qquad \zeta_n = \frac{1}{p_n}\xi_n.$$

Since (3.11) is a uniformly elliptic equation for u satisfying (3.8) with small σ, it follows that, if P is sufficiently close to $Dv_0 = \dfrac{1}{q_0^- - q_0^+}e_n$, then (4.1) holds with constants depending only on the data.

We will modify $B(y', z, P)$ away from a neighborhood of $(y, v_0(y), Dv_0)$ to obtain a uniformly elliptic equation globally.

Note that $v_0(y)$ is a solution of the problem of form (3.23)–(3.24) with $B_0(P)$ which corresponds to the supersonic solution $\varphi_0^-(y)$, i.e. $B_0(P)$ is defined by (3.22) with $A_0(P)$, defined by (3.10) with φ_0^- instead of φ^-. Then $F_0 \equiv 0$, and A_0 and B_0 depend only on P since φ_0^- is a linear function.

Since we are interested in estimate (3.18), we introduce the function

$$w(y) = v(y) - v_0(y)$$

and rewrite (3.23)–(3.24) in terms of w. Using the fact that $v_0(y)$ is a solution of the conormal boundary value problem defined by $B_0(P)$, we find that $w(y)$ satisfies

$$\begin{aligned} &\operatorname{div}(\hat{\mathcal{N}}(y, w, Dw)) = 0 \qquad \text{in} \quad \mathbf{R}^n_+, \\ &\hat{\mathcal{N}}^n(y, w, Dw) = 0 \qquad \text{on} \quad \partial\mathbf{R}^n_+, \end{aligned}$$

where

$$\hat{\mathcal{N}}(y, z, P) = B(y', v_0(y) + z, Dv_0 + P) - B_0(Dv_0) + F(y', z).$$

From the ellipticity of $B(y', z, P)$, it follows that (4.1) holds for $\hat{\mathcal{N}}(y, z, P)$ with the same ellipticity constants, if $|P|$ is sufficiently small.

Now we define a function $\mathcal{N}(y, z, P)$ as a modification of $\hat{\mathcal{N}}(y, z, P)$. Let $\eta \in C_0^\infty(\overline{\mathbf{R}_+})$ be such that

$$\zeta(t) = \begin{cases} 1 & \text{for} \quad t < \varepsilon, \\ 0 & \text{for} \quad t > 2\varepsilon, \end{cases} \qquad \eta(t) = \zeta(\frac{t}{\varepsilon}), \tag{4.2}$$

where the small constant $\varepsilon > 0$ will be chosen below. Introduce the following notations

$$X(y,z,P) := (y', v_0(y) + z, Dv_0 + P),$$
$$L_0(P) := B_0(Dv_0) + D_P B_0(Dv_0) \cdot P.$$

Now we define the modification of $\hat{\mathcal{N}}(y,z,P)$:

$$\mathcal{N}(y,z,P) = D_P B_0(Dv_0)\cdot P + \eta\left(|P|\right)\left(B(X(y,z,P)) - L_0(P)\right) + F(y', v_0(y)+z)e_n. \quad (4.3)$$

Note that

$$\mathcal{N}(y,z,P) = \hat{\mathcal{N}}(y,z,P) \qquad \text{if } |P| < \varepsilon. \quad (4.4)$$

We will also use the function

$$\mathcal{M}(y,z,P) = D_P B_0(Dv_0)\cdot P + \eta\left(|P|\right)\left(B(X(y,z,P)) - L_0(P)\right). \quad (4.5)$$

Obviously, $\mathcal{N}(y,z,P) = \mathcal{M}(y,z,P) + F(y', v_0(y)+z)e_n$. We note the following properties of $\mathcal{N}(y,z,P)$ and $\mathcal{M}(y,z,P)$.

Proposition 4.1. *There exist ε_0, σ_0, and $\Lambda \geq \lambda > 0$ depending only on the data such that, if $\varepsilon = \varepsilon_0$ in (4.4) and $\sigma \leq \sigma_0$, then*

(i) *$\mathcal{N}$ is uniformly elliptic:*

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} \mathcal{N}^i_{p_j}(y,z,P)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \text{for every } y \in \overline{\mathbf{R}^n_+},\ z \in \mathbf{R},\ P,\xi \in \mathbf{R}^n; \quad (4.6)$$

(ii) *The following estimates hold:*

$$|\mathcal{N}((y',y_n),z,P)| + |\mathcal{M}((y',y_n),z,P)| \leq C\left(\frac{\sigma}{1+|y'|^n} + |P|\right), \quad (4.7)$$

$$\sum_{i,j=1}^{n} |\mathcal{N}^i_{p_j}(y,z,P) - B^i_{0p_j}(Dv_0)| \leq C\left(\frac{\sigma\varepsilon_0^{-1}}{1+|y+ze_n|^n} + |P|\right)\chi_{[0,2\varepsilon_0]}(|P|), \quad (4.8)$$

$$|\mathcal{N}_z(y,z,P)| + \sum_{i=1}^{n-1}\sum_{j=1}^{n} |\mathcal{N}^i_{y_j}(y,z,P)| + |\mathcal{N}^n_{y_n}(y,z,P)|$$
$$\leq \frac{C\sigma}{1+|y+ze_n|^{n+1}}\chi_{[0,2\varepsilon_0]}(|P|), \quad (4.9)$$

$$\mathcal{N}_z \in C(\overline{\mathbf{R}^n_+}\times\mathbf{R}\times\mathbf{R}^n), \quad (4.10)$$

$$|\mathcal{M}_z(y,z,P)| + \sum_{i,j=1}^{n} |\mathcal{M}^i_{y_j}(y,z,P)| \leq \frac{C\sigma}{1+|y+ze_n|^{n+1}}\chi_{[0,2\varepsilon_0]}(|P|); \quad (4.11)$$

$$|\mathcal{M}_z(y,z,P) - \mathcal{M}_z(\tilde{y},\tilde{z},P)| + \sum_{i,j=1}^{n} |\mathcal{M}^i_{y_j}(y,z,P) - \mathcal{M}^i_{y_j}(\tilde{y},\tilde{z},P)|$$
$$\leq \frac{C\sigma}{1+(\min(|y+ze_n|,|\tilde{y}+\tilde{z}e_n|))^{n+1+\alpha}}\left(|y-\tilde{y}|^2 + |z-\tilde{z}|^2\right)^{\frac{\alpha}{2}}, \quad (4.12)$$

for every $y \in \overline{\mathbf{R}^n_+}$, $z \in \mathbf{R}$, and $P \in \mathbf{R}^n$, where $\chi_{[0,2\varepsilon_0]}(\cdot)$ is the characteristic function of the interval $[0,2\varepsilon_0]$, $Dv_0 = \frac{1}{q_0^- - q_0^+}e_n$, and the constant C depends only on the data and is independent of ε_0.

Moreover, the following estimates hold:

$$|D_P^2\mathcal{N}(y,z,P)| + |D_P^3\mathcal{N}(y,z,P)| = |D_P^2\mathcal{M}(y,z,P)| + |D_P^3\mathcal{M}(y,z,P)|$$
$$\leq C\left(\frac{\sigma}{1+|y+ze_n|^n} + |P|\right)\chi_{[0,2\varepsilon_0]}(|P|), \tag{4.13}$$

$$|D_{yP}^2\mathcal{M}(y,z,P)| + |D_{zP}^2\mathcal{M}(y,z,P)| \leq \frac{C\sigma}{1+|y+ze_n|^{n+1}}\chi_{[0,2\varepsilon_0]}(|P|), \tag{4.14}$$

$$|D_{yP}^2\mathcal{M}(y,z,P) - D_{yP}^2\mathcal{M}(\tilde{y},\tilde{z},P)| + |D_{zP}^2\mathcal{M}(y,z,P) - D_{zP}^2\mathcal{M}(\tilde{y},\tilde{z},P)|$$
$$\leq \frac{C\sigma}{1+(\min(|y+ze_n|,|\tilde{y}+\tilde{z}e_n|))^{n+1+\alpha}}\left(|y-\tilde{y}|^2+|z-\tilde{z}|^2\right)^{\frac{\alpha}{2}} \tag{4.15}$$

for every $y \in \overline{\mathbf{R}^n_+}$, $z \in \mathbf{R}$, and $P \in \mathbf{R}^n$, where the constant C depends only on the data and ε_0.

Proof. We first prove (4.7). Denote

$$\begin{aligned} M &= \sup\left\{\left|D(\tilde{\rho}(|Q|^2)Q)\right| \,:\, Q\in\mathbf{R}^n,\ |Q|<|D\varphi_0^-|+2\varepsilon_0+\sigma\right\} \\ &\quad + \sup\left\{\left|D(\rho(|Q|^2)Q)\right| \,:\, Q\in\mathbf{R}^n,\ |Q|<|D\varphi_0^-|+2\varepsilon_0+\sigma\right\}. \end{aligned}$$

Clearly, M depends only on the data. Now, for $y=(y',y_n)$,

$$\begin{aligned} |\mathcal{N}(y,z,P)| &\leq |B(X(y,z,P)) - B_0(Dv_0)|\,\chi_{[0,2\varepsilon_0]}(|P|) + (1-\eta(|P|))\,|D_PB_0(Dv_0)\cdot P| \\ &\quad + |F(y',v_0(y)+z)| \\ &\leq M\left|D\varphi^-(y',z) - D\varphi_0^-(y',z)\right|\chi_{[0,2\varepsilon_0]}(|P|) + C|P| + |F(y',v_0(y)+z)| \\ &\leq \frac{C\sigma}{1+|y+ze_n|^n}\chi_{[0,2\varepsilon_0]}(|P|) + C|P| + \frac{\sigma}{1+|y'|^{n+1}} \\ &\leq \frac{C\sigma}{1+|y'|^n} + C|P|. \end{aligned}$$

The estimate of $|\mathcal{M}(y,z,P)|$ involves only the terms that do not contain F in the above inequalities, and thus we obtain the same estimate. Now (4.7) is proved.

From the definition,

$$\begin{aligned} \mathcal{N}_{p_j}^i(y,z,P) &= B_{0p_j}^i(Dv_0) + \eta'(|P|)\frac{p_j}{|P|}\left(B^i(X(y,z,P)) - (L_0)^i(P)\right) \\ &\quad + \eta(|P|)\left(B_{p_j}^i(X(y,z,P)) - (L_0)_{p_j}^i(P)\right) \\ &= B_{0p_j}^i(Dv_0) + A_1 + A_2. \end{aligned}$$

If $\varepsilon_0 \leq \frac{|Dv_0|}{8} = \frac{1}{8(q_0^- - q_0^+)}$, we use (2.16), (3.10), (3.19), (3.22), and (4.2) to estimate

$$\begin{aligned} |A_1| &\leq \chi_{[0,2\varepsilon_0]}(|P|)\frac{C}{\varepsilon_0}\left(\left|B^i(X(y,z,P)) - B_0^i(Dv_0+P)\right| + \left|B_0^i(Dv_0+P) - (L_0)^i(P)\right|\right) \\ &\leq \chi_{[0,2\varepsilon_0]}(|P|)\frac{C}{\varepsilon_0}\left(\left|D\varphi^-(y',v_0(y)+z) - D\varphi_0^-(y',v_0(y)+z)\right| + |P|^2\right) \\ &\leq \chi_{[0,2\varepsilon_0]}(|P|)\frac{C}{\varepsilon_0}\left(\frac{\sigma}{1+|y+ze_n|^n} + \varepsilon_0|P|\right) \leq C\chi_{[0,2\varepsilon_0]}(|P|)\left(\frac{\sigma\varepsilon_0^{-1}}{1+|y+ze_n|^n} + |P|\right). \end{aligned}$$

We now estimate $|A_2|$:

$$\begin{aligned} |A_2| &\leq C\chi_{[0,2\varepsilon_0]}(|P|)\left(\left|B_{p_j}^i(X(y,z,P)) - B_{0p_j}^i(Dv_0+P)\right| + \left|B_{0p_j}^i(Dv_0+P) - B_{0p_j}^i(Dv_0)\right|\right) \\ &\leq C\chi_{[0,2\varepsilon_0]}(|P|)\left(\left|D\varphi^-(y',v_0(y)+z) - D\varphi_0^-(y',v_0(y)+z)\right| + |P|\right) \\ &\leq C\chi_{[0,2\varepsilon_0]}(|P|)\left(\frac{\sigma}{1+|y+ze_n|^n} + |P|\right). \end{aligned}$$

This proves (4.8).

Now we prove (4.9). Note that

$$\mathcal{N}^n_z(y,z,P) \quad = \quad \eta\left(|P|\right) B^n_z(X(y,z,P)) + F_{x_n}(y', v_0(y)+z) = A_3 + A_4. \tag{4.16}$$

We estimate

$$\begin{aligned} |A_3| &\le C\chi_{[0,2\varepsilon_0]}(|P|)|B_z(X(y,z,P))| \\ &\le C\chi_{[0,2\varepsilon_0]}(|P|)\left|D^2\varphi^-(y', v_0(y)+z)\right| \\ &\le C\chi_{[0,2\varepsilon_0]}(|P|)\frac{\sigma}{1+|y+ze_n|^{n+1}}, \end{aligned}$$

and, by (3.6),

$$|A_4| \quad \le \quad \frac{C\sigma}{1+|(y', v_0(y)+z)|^{n+1}} \le \frac{C\sigma}{1+|y+ze_n|^{n+1}}.$$

Thus, estimate (4.9) for $\mathcal{N}^n_z$ is proved. The term $\mathcal{N}^n_{y_n}$ is estimated similarly, since $v_0(y) = \dfrac{1}{q_0^- - q_0^+} y_n$. For $i = 1, \dots, n-1$ and $j = 1, \dots, n$, we have

$$\begin{aligned} &\mathcal{N}^i_z(y,z,P) = \eta\left(|P|\right) B^i_z(X(y,z,P)), \\ &\mathcal{N}^i_{y_j}(y,z,P) = d_j\eta\left(|P|\right) B^i_{y_j}(X(y,z,P)), \end{aligned}$$

where $d_j = 1$ for $j = 1, \dots, n-1$ and $d_n = \frac{1}{q_0^- - q_0^+}$. Thus, $\mathcal{N}^i_z$ and $\mathcal{N}^i_{y_j}$, for $i = 1, \dots, n-1$ and $j = 1, \dots, n$, are estimated similar to the term A_3 above. Thus, (4.9) is proved.

Note that (4.11) is also proved, since (4.11) follows from the estimates of the term A_3 above, and these estimates hold for $\mathcal{M}^i_z$ and $\mathcal{M}^i_{y_j}$ for any $i, j = 1, \dots, n$.

Also, (4.16) implies (4.10) since all the terms on the right-hand side of (4.16) are continuous.

Now we prove (4.12). We estimate

$$\begin{aligned} &|\mathcal{M}_z(y,z,P) - \mathcal{M}_z(\tilde{y},\tilde{z},P)| \\ &\le \eta(|P|)\,|B_z(X(y,z,P)) - B_z(X(\tilde{y},\tilde{z},P))| \\ &\le C\left|D^2\varphi^-(y', v_0(y)+z) - D^2\varphi^-(\tilde{y}_2', v_0(\tilde{y})+\tilde{z})\right|\chi_{[0,2\varepsilon_0]}(|z|) \\ &\le \frac{C\sigma}{1+(\min(|y+ze_n|, |\tilde{y}+\tilde{z}e_n|))^{n+1+\alpha}}\left(|y-\tilde{y}|^2 + |z-\tilde{z}|^2\right)^{\frac{\alpha}{2}}. \end{aligned}$$

We estimate $|\mathcal{N}^i_{y_j}(y,z,P) - \mathcal{N}^i_{y_j}(\tilde{y},\tilde{z},P)|$ for $i, j = 1, \dots, n$, similarly. Thus, (4.12) is proved.

Estimates (4.13)–(4.15) are proved similarly to estimates (4.8)–(4.12), since the functions $\mathcal{M}$ and $D_P\mathcal{M}$ are of the same structure.

It remains to prove assertion (i). Since $B_0(P)$ satisfies the ellipticity condition (4.1) at $P = Dv_0$ with ellipticity constants λ and Λ depending only on the data, then, from (4.8), choosing sufficiently small ε_0 and $\sigma_0 := \varepsilon_0^2$ yields (4.6) with ellipticity constants $\lambda/2$ and 2Λ for $\sigma \le \sigma_0$, □

From now on, we assume that $\varepsilon = \varepsilon_0$ is chosen in the definition of $\mathcal{N}$ and that $\sigma \le \sigma_0$ so that Proposition 4.1 holds.

Thus, in order to construct a solution of problem (3.23)–(3.24), it suffices to construct a solution of the problem

$$\operatorname{div}\left(\mathcal{N}(y,w,Dw)\right) = 0 \qquad \text{in} \quad \mathbf{R}^n_+, \tag{4.17}$$

$$\mathcal{N}^n(y,w,Dw) = 0 \qquad \text{on} \quad \partial\mathbf{R}^n_+, \tag{4.18}$$

which is sufficiently small in an appropriate norm if σ is small. In order to construct such a solution, we will construct solutions in bounded domains

$$B_R^+ := \mathbf{R}_+^n \cap B_R, \qquad \text{with } B_R = \{|x| < R\},$$

and pass to the limit as $R \to \infty$. The main goal is now to obtain the estimates independent of R. More precisely, we consider following problems:

$$\begin{array}{ll} \operatorname{div}(\mathcal{N}(x, v, Dw)) = 0 & \text{in } B_R^+, \\ \mathcal{N}^n(x, w, Dw) = 0 & \text{on } S^R := \{x_n = 0\} \cap B_R, \\ w = 0 & \text{on } \partial B_R \cap \overline{\mathbf{R}_+^n}. \end{array} \tag{4.19}$$

Lemma 4.2. *Let $w \in C(\overline{B_R^+}) \cap C^2(B_R^+)$ be a solution of (4.19). Then, if σ is sufficiently small,*

$$\|w\|_{0,0,B_R^+}^{(n-2)} \le C\sigma. \tag{4.20}$$

Proof. We prove this lemma by constructing a comparison function that is derived from the fundamental solution of the linear elliptic operator:

$$\overline{\mathcal{L}}V = \sum_{i,j=1}^{n} B_{0p_j}^i (Dv_0) V_{x_i x_j}.$$

Let $D = (d_{ij})$ be the inverse matrix of $(B_{0p_j}^i(0))$. Then D is symmetric and strictly positive definite. For $x \in \mathbf{R}^n$, denote

$$|x|_D = \left(\sum_{i,j=1}^{n} d_{ij} x_i x_j \right)^{\frac{1}{2}}.$$

Then

$$\frac{1}{\sqrt{\Lambda}}|x| \le |x|_D \le \frac{1}{\sqrt{\lambda}}|x|.$$

For $x \in \mathbf{R}^n$, we denote

$$x^* = x + Me_n,$$

where $e_n = (0, \dots, 0, 1)$, and $M > 2$ will be chosen large enough below. Fix $\tau = 1/2$ and consider the function

$$V(x) = L\left(\frac{1}{|x^*|_D^{n-2}} - \frac{1}{|x^*|_D^{n-2+\tau}} \right), \tag{4.21}$$

where the constant $L \in (0, 1]$ will be chosen below.

Since, for $x \in \mathbf{R}_+^n$,

$$|x^*|_D > 1 \qquad \text{for any } x \in \overline{B_R^+}, \tag{4.22}$$

by choosing M large depending only on λ, then we have

$$0 < V < 1, \qquad \text{in } B_R^+.$$

In particular,

$$\frac{1}{1 + |x + V(x)e_n|} \le \frac{2}{1 + |x|} \qquad \text{for any } x \in \overline{B_R^+}. \tag{4.23}$$

Note that

$$\overline{\mathcal{L}}V = -L\frac{\tau(n - 2 + \tau)}{|x^*|_D^{n+\tau}}.$$

Now we use (4.8)–(4.9) and (4.23) to compute

$$
\begin{aligned}
\operatorname{div}(\mathcal{N}(x,V,DV)) &= \sum_{i,j=1}^{n}\mathcal{N}^i_{p_j}(x,V,DV)V_{x_ix_j}+\sum_{i=1}^{n}\mathcal{N}^i_z(x,V,DV)V_{x_i}\\
&\quad+\sum_{i=1}^{n}\mathcal{N}^i_{y_i}(x,V,DV)\\
&= \overline{\mathcal{L}}V+\sum_{i,j=1}^{n}\left(\mathcal{N}^i_{p_j}(x,V,DV)-B^i_{0p_j}(Dv_0)\right)V_{x_ix_j}\\
&\quad+\sum_{i=1}^{n}\left(\mathcal{N}^i_z(x,V,DV)V_{x_i}+\mathcal{N}^i_{y_i}(x,V,DV)\right)\\
&\le -L\frac{\tau(n-2+\tau)}{|x^*|_D^{n+\tau}}+\frac{L^2C}{|x^*|_D^{2n-1}}+\frac{LC\sigma}{|x^*|_D^n(1+|x|^n)}\\
&\quad+\frac{LC\sigma}{|x^*|_D^{n-1}(1+|x|^{n+1})}+\frac{C\sigma}{1+|x|^{n+1}},
\end{aligned}
\tag{4.24}
$$

where C depends only on the data. Since $|x^*|=|x+Me_n|\le|x|+M$, we have

$$
\frac{1}{1+|x|}\le\frac{M}{M+|x|}\le\frac{M}{|x^*|}\le\frac{C(\lambda)M}{|x^*|_D}.
$$

Also, since $x_n\ge 0$, we have $|x^*|_D\ge\sqrt{\lambda}M$. Thus, using $n\ge 3$ and $\tau=1/2$ yields

$$
\begin{aligned}
\operatorname{div}(\mathcal{N}(x,V,DV)) &\le -\frac{L\tau(n-2+\tau)}{|x^*|_D^{n+\tau}}+\frac{L^2C}{|x^*|_D^{2n-1}}+\frac{CL(M^n+M^{n+1})\sigma}{|x^*|_D^{2n}}+\frac{CM^{n+1}\sigma}{|x^*|_D^{n+1}}\\
&\le -\frac{L\tau(n-2+\tau)}{|x^*|_D^{n+\tau}}+\frac{L^2C}{M^{n-1-\tau}|x^*|_D^{n+\tau}}+\frac{CLM^\tau\sigma}{|x^*|_D^{n+\tau}}+\frac{CM^{n+\tau}\sigma}{|x^*|_D^{n+\tau}}.
\end{aligned}
$$

Since $n\ge 3$ and $\tau=1/2$, choosing M large depending only on n and the constant C in the last estimate and using that $L\in(0,1]$ lead to

$$
\operatorname{div}(\mathcal{N}(x,V,DV))\le -L\frac{\tau(n-2+\tau)}{2|x^*|_D^{n+\tau}}+C\frac{M^{n+\tau}\sigma}{|x^*|_D^{n+\tau}}. \tag{4.25}
$$

Next, we estimate the boundary operator on $\{x_n=0\}$:

$$
\begin{aligned}
\mathcal{N}^n(x,V,DV) &= \mathcal{N}^n(x,V,0)+\int_0^1\mathcal{N}^i_{p_j}(x,V,sDV)V_{x_j}\nu_i ds \qquad (4.26)\\
&= \sum_{j=1}^{n}\left\{B^n_{0p_j}(Dv_0)V_{x_j}+\int_0^1(\mathcal{N}^i_{p_j}(x,V,sDV)-B^n_{0p_j}(Dv_0))V_{x_j}ds\right\}\\
&\quad+\mathcal{N}^n(x,V,0).
\end{aligned}
$$

Recalling that $D=(d_{ij})$ is the inverse matrix of $(B^i_{0p_j}(Dv_0))$, we get, on $\{x_n=0\}$,

$$
\begin{aligned}
\sum_{j=1}^{n}B^n_{0p_j}(Dv_0)V_{x_j}(x) &= L\sum_{j,k=1}^{n}B^n_{0p_j}(Dv_0)d_{jk}\left(-(n-2)\frac{x^*_k}{|x^*|_D^n}+(n-2+\tau)\frac{x^*_k}{|x^*|_D^{n+\tau}}\right)\\
&= L\left(-(n-2)\frac{M}{|x^*|_D^n}+(n-2+\tau)\frac{M}{|x^*|_D^{n+\tau}}\right)\\
&\le -\frac{L(n-2)}{2}\frac{M}{|x^*|_D^n},
\end{aligned}
$$

if M is chosen sufficiently large depending only on λ and n (recall that $\tau = 1/2$ and $|x^*|_D \geq \sqrt{\lambda}M$). Using this and estimates (4.7) (noting that, at the boundary, $x = (x', 0)$, i.e. $|x| = |x'|$), (4.8), and (4.23), we get from (4.26) that

$$\begin{aligned}
\mathcal{N}^n(x, V, DV)|_{x_n=0} &\leq -L\frac{(n-2)}{2}\frac{M}{|x^*|_D^n} + \left(\frac{C\sigma}{1+|x|^n} + |DV|\right)|DV| + \frac{C\sigma}{1+|x|^n} \\
&\leq -L\frac{(n-2)}{2}\frac{M}{|x^*|_D^n} + \frac{CL\sigma}{(1+|x|^n)|x^*|_D^{n-1}} + \frac{CL^2}{|x^*|_D^{2n-2}} + \frac{C\sigma}{1+|x|^n} \\
&\leq L\left(-\frac{(n-2)}{2}\frac{M}{|x^*|_D^n} + \frac{CL}{|x^*|_D^{2n-2}}\right) + \frac{CLM^n\sigma}{|x^*|_D^{2n-1}} + \frac{C\sigma M^n}{|x^*|_D^n} \\
&\leq L\left(-\frac{(n-2)}{2}\frac{M}{|x^*|_D^n} + \frac{CL}{M^{n-2}|x^*|_D^n}\right) + \frac{CLM\sigma}{|x^*|_D^n} + \frac{C\sigma M^n}{|x^*|_D^n}.
\end{aligned}$$

Since $n \geq 3$, then, choosing M sufficiently large depending only on n and the constant C in the last expression (i.e. on the data) and using $|L| \leq 1$, we get

$$\mathcal{N}^n(x, V, DV)|_{x_n=0} \leq -L\frac{(n-2)}{4}\frac{M}{|x^*|_D^n} + \frac{C\sigma M^n}{|x^*|_D^n}. \tag{4.27}$$

In order to show that V is a supersolution of the conormal boundary value problem, we need to choose $L \in (0, 1]$ so that the right-hand sides in (4.25) and (4.27) are negative. Since $n \geq 3$, $\tau = 1/2$, and $M > 1$, the right-hand sides in (4.25) and (4.27) are negative if we choose

$$L = CM^{n+\tau}\sigma,$$

where C depends only on n and the constant C on the right-hand sides in (4.25) and (4.27). Choosing σ_0 sufficiently small, we have $0 < L \leq 1$ if $\sigma \leq \sigma_0$.

Now, by the comparison principle, Theorem B.1 (i) (for which the ellipticity of $\mathcal{N}$, (4.9), (4.10), and (4.13) can be applied), we get

$$w \leq V \qquad \text{in } \mathbf{R}^n_+.$$

Similar argument shows that

$$w \geq -V \qquad \text{in } B_R^+.$$

Then

$$|w| \leq V.$$

Since

$$|V| \leq \frac{CL}{1+|x|^{n-2}} \leq \frac{C\sigma}{1+|x|^{n-2}},$$

where the last inequality follows from our choice of $L = C\sigma$, then the lemma is proved. □

Proposition 4.3. *If $\sigma > 0$ is sufficiently small, then, for any $R > 1$, there exists a unique solution $w \in C(\overline{B_R^+}) \cap C^{2,\alpha}(\overline{B_R^+} \setminus (\partial B_R \cap \{x_n = 0\}))$ of (4.19) such that*

$$\|w\|^{(n-2)}_{2,\alpha,B^+_{R/2}} \leq C\sigma, \tag{4.28}$$

where C depends only on the data and is independent of R.

Proof. The existence of a solution $w \in C(\Omega_R) \cap C^{2,\alpha}(\overline{B_R^+} \setminus (\partial B_R \cap \{x_n = 0\}))$ of (4.19) follows by combining the theory of mixed boundary value problems for linear elliptic equations of [25] with the estimates for Dirichlet and oblique boundary value problems for nonlinear elliptic equations of [17] and [26]. Note, in particular, that the barrier construction of [25, Lemma 2] works for the nonlinear problem (4.19). Then the proof is a direct computation, similar to Lemma 4.2.

Thus, it suffices to prove estimate (4.28). We will prove (4.28) by rescaling.

First, note that, by Lemma 4.2, for sufficiently small σ, we can rewrite (4.19) in the form:

$$\begin{aligned}
&\operatorname{div}(\mathcal{M}(x,v,Dw)) + g(x', v_0(x)+w)\Big(\frac{1}{q_0^- - q_0^+} + w_{x_n}\Big) = 0 \quad \text{in } B_R^+,\\
&\mathcal{M}^n(x,w,Dw) = 0 \quad \text{on } S^R := \{x_n = 0\} \cap B_R,\\
&w = 0 \quad \text{on } \partial B_R \cap \overline{\mathbf{R}^n_+},
\end{aligned} \tag{4.29}$$

where $\mathcal{M}$ and f are defined by (4.5) and (3.3), respectively. Indeed, since $F(x', x_n) \equiv 0$ in $\mathbf{R}^{n-1} \times [-1,1]$ and by (3.19), it follows that $F(x', v_0(x) + w(x)) \equiv 0$ in $\mathbf{R}^{n-1} \times [-\frac{q_0^- - q_0^+}{2}, \frac{q_0^- - q_0^+}{2}]$ if $|w(x)| \le \frac{1}{2}$ in $\mathbf{R}^{n-1} \times [-\frac{q_0^- - q_0^+}{2}, \frac{q_0^- - q_0^+}{2}]$, which can be achieved by (4.20) and choosing σ small.

We can assume $R > 4$. By Proposition 4.1 with the choice of $\epsilon = \epsilon_0$, the functions $\mathcal{M}(x,z,P)$ and $B(x,z,P) := g(x', v_0(x)+z)(\frac{1}{q_0^- - q_0^+} + p_n)$ satisfy the conditions of Theorem A.1 in B_1^+ with the constants λ, Λ, and M depending only on the data, and

$$\|\mathcal{M}(\cdot,\cdot,0)\|_{1,\alpha,B_1^+ \times \mathbf{R}} \le C\sigma, \qquad \|B(\cdot,\cdot,0)\|_{1,\alpha,B_1^+ \times \mathbf{R}} \le C\sigma.$$

Then, using Lemma 4.2, we can apply Theorem A.1 in B_1^+ to obtain

$$\|w\|_{2,\alpha,B_{1/2}^+} \le C(\|w\|_{0,0,B_1^+} + \sigma) \le C\sigma.$$

Thus, we get

$$\|w\|^{(n-2)}_{2,\alpha,B_{1/2}^+} \le C\|w\|_{2,\alpha,B_{1/2}^+} \le C\sigma. \tag{4.30}$$

Let $x^0 \in B_{R/2}^+ \setminus B_{1/2}^+$. Consider the following two cases:

Case 1. $|x^0| \le 16 x_n^0$. Then, denoting $\rho := |x^0|/32$, we get $B_{2\rho}(x^0) \subset B_R^+$. Rescale

$$W(y) := \frac{1}{2\rho} w(x^0 + 2\rho y) \tag{4.31}$$

for $y \in B_1(0) =: B_1$. Then $W \in C^{2,\alpha}(B_1)$. For $y \in B_1$ and $x = x^0 + 2\rho y \in B_{2\rho}(x^0)$, we have $Dw(x) = DW(y)$, and thus

$$\begin{aligned}
0 &= \operatorname{div}_x(\mathcal{M}(x, w(x), Dw(x))) + g(x', v_0(x) + w(x))\Big(\frac{1}{q_0^- - q_0^+} + w_{x_n}(x)\Big)\\
&= \frac{1}{\rho}\operatorname{div}_y(\mathcal{N}(x^0 + 2\rho y, 2\rho W(y), DW(y)))\\
&\quad + g(x^0 + 2\rho y, v_0(x^0 + 2\rho y) + 2\rho W(y))\Big(\frac{1}{q_0^- - q_0^+} + W_{y_n}(y)\Big).
\end{aligned}$$

Thus, defining in $B_1 \times \mathbf{R} \times \mathbf{R}^n$,

$$\begin{aligned}
&A(y,z,P) = \mathcal{N}(x^0 + 2\rho y,\ 2\rho z, P),\\
&B(y,z,P) = g(x^0 + 2\rho y,\ v_0(x^0 + 2\rho y) + 2\rho z)\Big(\frac{1}{q_0^- - q_0^+} + p_n\Big),
\end{aligned} \tag{4.32}$$

we see that W satisfies

$$\operatorname{div}(A(y, W, DW)) + B(y, W, DW) = 0 \qquad \text{in } B_1.$$

Note that, since $\rho = |x^0|/32 \ge 1/64$, we have

$$31\rho \le \frac{1}{4} \le |x^0 + \rho y| \le 33\rho \qquad \text{for } y \in B_1. \tag{4.33}$$

Also, by Lemma 4.2, (4.31), and (4.33),

$$\|W\|_{L^\infty(B_1)} \le \frac{C\sigma}{\rho^{n-1}}. \tag{4.34}$$

Note that, since $\rho = |x_0|/32 \geq 1/64$, then, for sufficiently small σ,

$$\frac{1}{1+|(x_0+2\rho y)+2\rho z e_n|} \leq \frac{C}{\rho} \qquad \text{for any } y \in B_1 \text{ and } |z| \leq \|W\|_{L^\infty(B_1)}. \tag{4.35}$$

Now, by Proposition 4.1 with the choice of $\epsilon = \epsilon_0$, and by (3.4), the functions in (4.32) satisfy the conditions of Theorem A.1(i) in the ball B_1 with the constants λ, Λ, M, and $M_1 = \|W\|_{0,0,L^\infty(B_1)}$ depending only on the data, and

$$\|A_x(\cdot,\cdot,0)\|_{0,\alpha,B_1\times\mathbf{R}} \leq \frac{C\sigma}{1+\rho^n}, \qquad \|(1+p^2)^{-1}B\|_{0,\alpha,B_1\times[-M_1,M_1]\times\mathbf{R}^n} \leq \frac{C\sigma}{1+\rho^{n+1}}.$$

Thus, by Theorem A.1(i) and (4.34) and using $\rho \geq \frac{1}{64}$,

$$\|W\|_{2,\alpha,B_{1/2}} \leq \frac{C\sigma}{\rho^{n-1}}.$$

Rescaling back, we get

$$\frac{1}{\rho}[w]_{0,0,B_\rho(x^0)} + [w]_{1,0,B_\rho(x^0)} + \rho[w]_{2,0,B_\rho(x^0)} + \rho^{1+\alpha}[w]_{2,\alpha,B_\rho(x^0)} \leq \frac{C\sigma}{\rho^{n-1}}.$$

Note that, for any $x \in B_{2\rho}(x^0)$, there holds $1/128 \leq \rho/\delta_x \leq 1$. Thus, multiplying the last estimate by ρ^{n-1}, we obtain

$$\|w\|^{(n-2)}_{2,\alpha,B_\rho(x^0)} \leq C\sigma. \tag{4.36}$$

Case 2. $|x^0| > 16x_n^0$. Let $z = (x',0) \in \partial B_R^+ \cap \{x_n = 0\}$. Then $|z| \geq |x^0| - x_n^0 \geq 15|x^0|/16$. Let $\rho = |x^0|/8$. Then $x^0 \in B_\rho^+(z)$. Define $W(y)$ for $y \in B_1^+$ by (4.31). Then $W \in C^{2,\alpha}(B_1^+)$ and satisfies

$$\begin{aligned} &\operatorname{div}(A(y,W,DW)) + B(y,W,DW) = 0 \qquad \text{in } B_1^+,\\ &A^n(y,W,DW) = 0 \qquad \text{on } \Gamma_0 := \partial B_1^+ \cap \{x_n = 0\}, \end{aligned}$$

where $A(y,z,P)$ and $B(y,z)$ are defined by (4.32) in B_1^+. Note also that (4.33) and (4.34) hold in B_1^+. Then, by Proposition 4.1, we use (4.35) to see that function (4.32) satisfies the conditions of Theorem A.1 in B_1 with the constants λ, Λ, M, and $M_1 = \|W\|_{L^\infty(B_1)}$ depending only on the data. Moreover, using the fact that $|x'| \geq |x|/2$ for $x \in B_\rho^+(z)$ in (4.7) and the other estimates of Proposition 4.1, we get

$$\|A(\cdot,\cdot,0)\|_{1,\alpha,B_1^\downarrow\times\mathbf{R}} \leq \frac{C\sigma}{1+\rho^n}.$$

By (3.4),

$$\|(1+|P|^2)^{-1}B\|_{0,\alpha,B_1^\downarrow\times[-M_1,M_1]\times\mathbf{R}^n} \leq \frac{C\sigma}{1+\rho^{n+1}}.$$

Thus, by Theorem A.1(ii),

$$\|W\|_{2,\alpha,B_{\frac{1}{2}}} \leq \frac{C\sigma}{\rho^{n-1}}.$$

Rescaling back, we get

$$\frac{1}{\rho}[w]_{0,0,B_\rho^\downarrow(z)} + [w]_{1,0,B_\rho^\downarrow(z)} + \rho[w]_{2,0,B_\rho^\downarrow(z)} + \rho^{1+\alpha}[w]_{2,\alpha,B_\rho^\downarrow(z)} \leq \frac{C\sigma}{\rho^{n-1}}.$$

Multiplying this estimate by ρ^{n-1} and using that $1/128 \leq \rho/\delta_y \leq 1$ for every $y \in B_{2\rho}^+(z)$, we get

$$\|w\|^{(n-2)}_{2,\alpha,B_\rho^+(z)} \leq C\sigma. \tag{4.37}$$

Estimates (4.30), (4.36), and (4.37) imply (4.28): Indeed, it only remains to estimate

$$\delta_{x,y}^{n-1+\alpha}\frac{|Dw(x)-Dw(y)|}{|x-y|^\alpha}$$

for $x, y \in B^+_{R/2}$ in the case $x > 2$, $|x| \geq |y|$, and $|x-y| > |x|/32$. Then $|x-y| > \delta_x/64$. As (4.30), (4.36), and (4.37) imply $|Dw(z)| \leq C\sigma/\delta_z^{n-1}$ for any $z \in B^+_{R/2}$, we get

$$\delta_{x,y}^{n-1+\alpha} \frac{|Dw(x) - Dw(y)|}{|x-y|^\alpha} \leq C\sigma \delta_y^{n-1+\alpha} \frac{\frac{1}{\delta_x^{n-1}} + \frac{1}{\delta_y^{n-1}}}{\delta_x^\alpha} \leq C\sigma.$$

The uniqueness follows from the comparison principle, Theorem B.1(i). □

Theorem 4.1. *There exist $\sigma > 0$ and C depending only on the data such that, if $\sigma \leq \sigma_0$, there exists a unique solution $w \in C^{2,\alpha}(\overline{\mathbf{R}^n_+})$ of the problem:*

$$\begin{aligned} &div(\mathcal{N}(x, w, Dw)) = 0 && \text{in } \mathbf{R}^n_+, \\ &\mathcal{N}^n(x, w, Dw) = 0 && \text{on } \{x_n = 0\}, \\ &\|w\|^{(n-2)}_{0,0,\mathbf{R}^n_+} < \infty, \end{aligned} \tag{4.38}$$

satisfying

$$\|w\|^{(n-2)}_{2,\alpha,\mathbf{R}^n_+} \leq C\sigma. \tag{4.39}$$

Proof. Fix a sequence $R_j \to \infty$ as $j \to \infty$. Let

$$w_{R_j} \in C(\overline{B^+_{R_j}}) \cap C^{2,\alpha}(\overline{B^+_{R_j}} \setminus (\partial B_{R_j} \cap \{x_n = 0\}))$$

be the solution of (4.19) with $R = R_j$, constructed in Proposition 4.3. By (4.28), a subsequence of u_{R_j} converges in $C^{2,\frac{\alpha}{2}}(\overline{B^+_{10}})$. A further subsequence converges in $C^{2,\frac{\alpha}{2}}(\overline{B^+_{20}})$, etc. By the diagonal procedure, we extract a sequence $w_{R_{j_k}}$ which converges in $C^{2,\frac{\alpha}{2}}$ on every compact subset of $\overline{\mathbf{R}^n_+}$. The limit w is thus a solution of (4.38). By (4.28), the limit w satisfies (4.39) with the same constant C as in (4.28).

The uniqueness follows from the comparison principle, Theorem B.1(ii). □

Corollary 4.1. *Let σ_0 and C be as in Theorem 4.1, and $\sigma \leq \sigma_0$. Then there exists a unique solution of problem (3.23)–(3.24) satisfying (3.18).*

This is because v is a solution of problem (3.23)–(3.24) satisfying (3.18) if and only if $w := v - v_0(y)$ is a solution of (4.38) satisfying (4.39).

Corollary 4.1 and Lemma 3.1 imply Theorem 2.1.

5. Stability of Free Boundaries

In this section we prove Theorem 2.2.

For the supersonic perturbations φ^- and $\hat{\varphi}^-$ in Ω_1 satisfying (1.1) and (2.16), we define their extensions (still denoted) φ^- and $\hat{\varphi}^-$ to the whole space as in Section 3.1, and consider the corresponding functions g and $\hat{g}$ defined by (3.3), and F and $\hat{F}$ defined by (3.5) for φ^- and $\hat{\varphi}^-$, respectively. Furthermore, we consider the solutions φ and $\hat{\varphi}$ of Problem A for φ^- and $\hat{\varphi}^-$ whose existence is provided by Theorem 2.1, the functions

$$\begin{aligned} u(x) &= \varphi^-(x) - \varphi(x) && \text{in } \Omega^\downarrow(\varphi), \\ \hat{u}(x) &= \hat{\varphi}^-(x) - \hat{\varphi}(x) && \text{in } \Omega^\downarrow(\hat{\varphi}), \end{aligned}$$

and their hodograph-transform images $v, \hat{v} \in C^{2,\alpha}(\overline{\mathbf{R}^n_+})$ defined by (3.15). Our goal is to prove that there exists a function Ψ with the properties described in Theorem 2.2 such that, for any φ_1^- and φ_2^- as above,

$$\|v - \hat{v}\|^{(n-2)}_{2,\alpha,\mathbf{R}^n_+} \leq \Psi\left(\|\varphi^- - \hat{\varphi}^-\|^{(n-1)}_{2,\alpha,\Omega_1}\right) \tag{5.1}$$

if $\sigma > 0$ is sufficiently small. Since $f(x') = v(x',0)$ and $\hat{f}(x') = \hat{v}(x',0)$, estimate (5.1) implies (2.20), thus Theorem 2.2.

If a function Ψ described above does not exist, then there exist φ_k^- and $\hat{\varphi}_k^-$, for $k = 1, \cdots$, satisfying (1.1) and such that

$$\begin{aligned} &\varphi_k^- \text{ satisfy (2.16) with } \sigma \le \sigma_0, \\ &\|\varphi_k^- - \hat{\varphi}_k^-\|^{(n-1)}_{2,\alpha,\mathbf{R}^n_+} \le \frac{1}{k}, \\ &\|v_k - \hat{v}_k\|^{(n-2)}_{2,\alpha,\mathbf{R}^n_+} \ge \epsilon > 0. \end{aligned} \tag{5.2}$$

In order to derive a contradiction, we notice the following fact.

Lemma 5.1. *Let $\kappa > 0$. A set $\mathcal{K}_M := \{v \in C^{2,\alpha}(\overline{\mathbf{R}^n_+}) \ : \ \|v\|^{(\kappa)}_{2,\alpha,\mathbf{R}^n_+} \le M \}$ is compact in the space $C^{2,\beta}(\overline{\mathbf{R}^n_+})$ with the norm $\|\cdot\|_{2,\beta,\mathbf{R}^n_+}$ for $0 < \beta < \alpha$, where $\|\cdot\|_{2,\beta,\mathbf{R}^n_+}$ is the non-weighted Hölder norm.*

Proof. Let $v_j \in \mathcal{K}_M$ for $j = 1, 2, \ldots$. By a standard argument, we can extract a subsequence (still denoted) v_j, which converges in $C^{2,\beta}$ on every compact subset of $\overline{\mathbf{R}^n_+}$. Denote the limit by v. Then $\|v\|^{(\kappa)}_{2,\alpha,\mathbf{R}^n_+} \le M$. It remains to show that $\|v_j - v\|_{2,\beta,\mathbf{R}^n_+} \to 0$ as $j \to \infty$.

Fix $0 < \varepsilon < 1$. Then $\|v_j\|_{2,\alpha,\mathbf{R}^n_+ \setminus B_{1/\varepsilon}(0)} \le M\varepsilon^\kappa$, and the same estimate holds for v. Also, there exists j_0 such that, for $j > j_0$, $\|v_j - v\|_{2,\beta,\mathbf{R}^n_+ \cap B_{\frac{2}{\varepsilon}}(0)} \le \varepsilon^\kappa$. Then, for $j > j_0$, we have $\|v_j - v\|_{2,\beta,\mathbf{R}^n_+} \le C\varepsilon^\kappa$, and the assertion is proved. □

Denote

$$\begin{aligned} w_k(x) &= v_k(x) - v_0(x), \\ \hat{w}_k(x) &= \hat{v}_k(x) - v_0(x). \end{aligned}$$

By Theorem 4.1, both w_k and $\hat{w}_k$ satisfy (4.39). From (5.2),

$$\|w_k - \hat{w}_k\|^{(n-2)}_{2,\alpha,\mathbf{R}^n_+} \ge \epsilon > 0. \tag{5.3}$$

Denote by $A_k(x,P)$, $B_k(x',z,P)$, and $\mathcal{N}_k(x,z,P)$ the functions (3.10), (3.22), and (4.3) corresponding to φ_k^- for $k = 1, 2$. Similarly, let $\hat{A}_k(x,P)$, $\hat{B}_k(x',z,P)$, and $\hat{\mathcal{N}}_k(x,z,P)$ correspond to $\hat{\varphi}_k^-$. Each $\mathcal{N}_k$ and w_k satisfy (4.17)–(4.18). The same is true for $\hat{\mathcal{N}}_k$ and $\hat{w}_k$.

From (5.2), (4.39) (applied to w_k, $\hat{w}_k$), and Lemma 5.1, by selecting a subsequence (still kept the same notation), we have

$$\varphi_k^- \to \varphi^- \quad \text{in } C^{2,\frac{\alpha}{2}}(\overline{\mathbf{R}^n_+}), \tag{5.4}$$

$$\hat{\varphi}_k^- \to \varphi^- \quad \text{in } C^{2,\frac{\alpha}{2}}(\overline{\mathbf{R}^n_+}), \tag{5.5}$$

$$w_k \to w \quad \text{in } C^{2,\frac{\alpha}{2}}(\overline{\mathbf{R}^n_+}), \tag{5.6}$$

$$\hat{w}_k \to \hat{w} \quad \text{in } C^{2,\frac{\alpha}{2}}(\overline{\mathbf{R}^n_+}), \tag{5.7}$$

and $\varphi^- \in C^{2,\alpha}(\overline{\Omega_e})$ satisfies (2.16) with $\sigma \le \sigma_0$, and w and $\hat{w}$ satisfy (4.39). Also, both w and $\hat{w}$ satisfy (4.17)–(4.18), where $\mathcal{N}$ is defined by the limiting function φ^- through the expressions (3.10), (3.22), and (4.3). Then, by Theorem B.1(ii), $w = \hat{w}$.

On the other hand, by (5.3),

$$\|w - \hat{w}\|^{(n-2)}_{2,\alpha,\mathbf{R}^n_+} \ge \epsilon > 0. \tag{5.8}$$

This contradiction leads to (5.1), and thus Theorem 2.2.

6. Perturbations with lower and higher regularity

We now extend our results to the case that the regularity of the steady perturbation is only $C^{1,1}$, and we also introduce another nonlinear approach to deal with the existence and stability problem of multidimensional transonic shocks when the regularity of the steady perturbation is $C^{3,\alpha}$ or higher.

6.1. **Lower Regularity.** If, instead of (2.16), we assume only

$$\|\varphi^- - \varphi_0^-\|^{(n-1)}_{1,1,\Omega_1} \le \sigma, \tag{6.1}$$

then we obtain a solution of Problem A, which belongs to $C^{1,\alpha}$ for $\alpha \in (0,1)$. Precisely, if we fix any $\alpha \in (0,1)$, then the existence part of Theorem 2.1 holds with estimates (2.17) and (2.19) replaced by

$$\|\varphi - \varphi_0^+\|^{(n-2)}_{1,\alpha,\Omega^+} \le C_1\sigma, \tag{6.2}$$

and

$$\|f\|^{(n-2)}_{1,\alpha,\mathbf{R}^{n-1}} \le C_2\sigma. \tag{6.3}$$

For the proof, we first assume $\varphi^- \in C^2$ satisfying (6.1) and then follow the same scheme as above. In particular, Theorem B.1 and Lemma 4.2 are obtained without changes in the proofs. In Proposition 4.3, we do the same rescaling and use the Hölder gradient estimates of [24]. For the general $\varphi^- \in C^{1,1}$ satisfying (6.1), we approximate φ^- by $\varphi_k^- \in C^2$ with the same estimates (6.1) and then send a subsequence of the corresponding solutions φ_k and their free boundary functions f_{φ_k} to a limit, using Lemma 5.1 and estimates (6.2) for φ_k and f_{φ_k}.

6.2. **Higher Regularity.** If, instead of (2.16), we assume

$$\|\varphi^- - \varphi_0^-\|^{(n-1)}_{3,\alpha,\Omega_1} \le \sigma, \tag{6.4}$$

then we get the following stronger stability theorem.

Theorem 6.1. *There exist positive constants σ_0 and C depending only on n, γ, α, $|V_0'|$, and q_0^+ such that, if $\sigma < \sigma_0$ and smooth supersonic solutions $\varphi^-(x)$ and $\hat\varphi^-(x)$ of (1.1) satisfy (2.16), then the unique solutions $\varphi(x)$ and $\hat\varphi(x)$ of Problem A for $\varphi^-(x)$ and $\hat\varphi^-(x)$, respectively, satisfy*

$$\|f_\varphi - f_{\hat\varphi}\|^{(n-2)}_{2,\alpha,\mathbf{R}^{n-1}} \le C\|\varphi^- - \hat\varphi^-\|^{(n-1)}_{3,\alpha,\Omega_1}, \tag{6.5}$$

where $f_\varphi(x')$ and $f_{\hat\varphi}(x')$ are the free boundary functions (2.18) of $\varphi(x)$ and $\hat\varphi(x)$, respectively.

In fact, in this case, problem (4.17)–(4.18) can be solved by using the implicit function theorem as follows.

Denote by $C^{m,\alpha,(\kappa)}(\overline{\Omega})$ the set $\{u \in C^{m,\alpha}(\overline{\Omega}) \ : \ \|u\|^{(\kappa)}_{m,\alpha,\overline{\Omega}} < \infty\}$ with m a nonnegative integer, $\alpha \in (0,1)$, and $\kappa > 0$. Obviously, for the case $\Omega = \mathbf{R}^n$ or $\Omega = \mathbf{R}^n_+$, the set $C^{m,\alpha,(\kappa)}(\overline{\Omega})$ with norm $\|\cdot\|^{(\kappa)}_{m,\alpha,\Omega}$ is a Banach space.

Consider the map Φ, which assigns to $(\varphi^-, w) \in C^{3,\alpha,(n-1)}(\mathbf{R}^n) \times C^{2,\alpha,(n-2)}(\overline{\mathbf{R}^n_+})$ the left-hand sides of (4.17) and (4.18), where $\mathcal{N}$ is defined by φ^- through the expressions (3.10), (3.22), and (4.3). It is easy to see from Proposition 4.1 that Φ is a C^1 map from

$$C^{3,\alpha,(n-1)}(\mathbf{R}^n) \times C^{2,\alpha,(n-2)}(\overline{\mathbf{R}^n_+})$$

to $C^{0,\alpha,(n+1)}(\overline{\mathbf{R}^n_+}) \times C^{1,\alpha,(n)}(\mathbf{R}^{n-1})$, for which the higher regularity $\varphi^- \in C^{0,\alpha,(n+1)}(\overline{\mathbf{R}^n_+})$ is required. Also, from the definitions,

$$\Phi(\varphi_0^-, 0) = (0,0).$$

In order to apply the implicit function theorem, it suffices to show that the partial Fréchet derivative $\Phi_w(\varphi_0^-, 0)$ is invertible, that is, to show that the fixed boundary value problem for the linear elliptic equation:

$$\sum_{i,j=1}^{n} B^i_{0p_j}(Dv_0)W_{x_ix_j} = \psi \qquad \text{in} \quad \mathbf{R}^n_+,$$
$$\sum_{j=1}^{n} B^n_{0p_j}(Dv_0)W_{x_j} = h \qquad \text{on} \quad \partial\mathbf{R}^n_+$$

has a unique solution $W \in C^{2,\alpha,(n-2)}(\overline{\mathbf{R}^n_+})$ for any $(\psi, h) \in C^{0,\alpha,(n+1)}(\overline{\mathbf{R}^n_+}) \times C^{1,\alpha,(n)}(\mathbf{R}^{n-1})$, and W satisfies

$$\|W\|^{(n-2)}_{2,\alpha,\mathbf{R}^n_+} \le C(\|\psi\|^{(n+1)}_{0,\alpha,\mathbf{R}^n_+} + \|h\|^{(n)}_{1,\alpha,\mathbf{R}^{n-1}}).$$

To construct such a solution, we can repeat the argument of Section 4, that is, first construct solutions in half-balls (with zero Dirichlet data on the half-sphere) and obtain estimates independent of the radii by using the comparison function introduced in Lemma 4.2 to obtain the weighted L^∞ estimates and then rescale as in Proposition 4.3 to obtain the weighted Hölder estimates, and finally send $R \to \infty$ as in the proof of Theorem 4.1. The uniqueness follows from the obvious comparison principle for solutions with zero limit at infinity.

Now, from the implicit function theorem, there exist σ_0 and $\rho > 0$ such that, for any $\varphi^- \in C^{3,\alpha,(n-1)}(\mathbf{R}^n)$ satisfying

$$\|\varphi^- - \varphi_0^-\|^{(n-1)}_{3,\alpha,\mathbf{R}^n} \le \sigma_0, \tag{6.6}$$

there exists a unique $w \in C^{2,\alpha,(n-2)}(\overline{\mathbf{R}^n_+})$ with $\|w\|^{(n-2)}_{2,\alpha,\mathbf{R}^n_+} \le \rho$ such that $\Phi(\varphi^-, w) = 0$ and, moreover, the function Ψ which assigns w to φ^- is a C^1 function from

$$U := \{\varphi^- \in C^{3,\alpha,(n-1)}(\mathbf{R}^n) \ : \ \|\varphi^- - \varphi_0^-\|^{(n-1)}_{3,\alpha,\mathbf{R}^n} < \sigma_0\}$$

into $C^{2,\alpha,(n-2)}(\overline{\mathbf{R}^n_+})$. This implies the existence and uniqueness of solutions in Problem A with σ_0 defined above and the stability in the form:

$$\|f_\varphi - f_{\hat\varphi}\|^{(n-2)}_{2,\alpha,\mathbf{R}^{n-1}} \le C\|\varphi^- - \hat\varphi^-\|^{(n-1)}_{3,\alpha,\mathbf{R}^n}, \tag{6.7}$$

where φ^- and $\hat\varphi^-$ are the extensions of the supersonic perturbations from Ω_1 to $\mathbf{R}^n$ defined in Section 3.1. Estimate (6.7) implies (6.5) by the linearity and continuity of the extension operator.

Appendix A. Local estimates for solutions of the conormal boundary value problems

The following facts follow by combining some standard results on elliptic equations.

Theorem A.1. (i) *Let $B_r := B_r(0) \subset \mathbf{R}^n$. Let $u \in C^2(B_1)$ be a solution of the equation*

$$div\, A(x, u, Du) + B(x, u, Du) = 0 \qquad in \ \ B_1.$$

Assume also that

$$\|u\|_{L^\infty(B_1)} \le M_1. \tag{A.1}$$

Denote $\mathcal{D} := B_1 \times [-M_1, M_1] \times \mathbf{R}^n$ and $\mathcal{D}_{x,z} := B_1 \times [-M_1, M_1]$. Assume that $A(x, z, P)$ and $B(x, z, P)$ satisfy

$$\begin{aligned} &\|A(\cdot,\cdot,P)\|_{0,\alpha,\mathcal{D}_{x,z}} \le M(1 + |P|) \quad \text{for any} \ \ P \in \mathbf{R}^n, \\ &\|(D_PA, \ A_z, \ D_xA, \ D^2_{Px}A)\|_{0,\alpha,\mathcal{D}} \le M, \\ &\|(1 + |P|^2)^{-1}B\|_{0,\alpha,\mathcal{D}} \le M \quad \text{for any} \ \ P \in \mathbf{R}^n. \end{aligned} \tag{A.2}$$

Assume that $A(x,z,P)$ is elliptic, i.e., there exist $\Lambda \geq \lambda > 0$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} A^i_{p_j}(x,z,P)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \text{for any } \xi \in \mathbf{R}^n, (x,z,P) \in \mathcal{D}. \tag{A.3}$$

Then $u \in C^{2,\alpha}(B_{1/2})$ and there exists C depending only on $n, \lambda, \Lambda, M, M_1$, and α such that

$$\|u\|_{2,\alpha,B_{1/2}} \leq C(\|u\|_{L^\infty(B_1)} + \|D_x A(\cdot,\cdot,0)\|_{0,\alpha,\mathcal{D}_{x,z}} + \|(1+|P|^2)^{-1}B\|_{0,\alpha,\mathcal{D}}).$$

(ii) *Let $B_r^+ = B_r(0) \cap \{x_n > 0\} \subset \mathbf{R}^n$. Let $u \in C^2(B_1^+)$ be a solution of the conormal boundary value problem:*

$$\begin{aligned} &div\, A(x,u,Du) + B(x,u,Du) = 0 \qquad \text{in } B_1^+, \\ &A^n(x,u,Du) = 0 \qquad \text{on } \Gamma_0 := \partial B_1^+ \cap \{x_n = 0\}. \end{aligned} \tag{A.4}$$

Let $u(x)$ and $A(x,z,P)$ satisfy all the assumptions of (i) *above in the domains B_1^+, $\mathcal{D}^+ := B_1^+ \times [-M_1, M_1] \times \mathbf{R}^n$, and $\mathcal{D}^+_{x,z} := B_1^+ \times [-M_1, M_1]$. In addition, assume that the function $(x',z) \to A((x',0),z,0)$ satisfies*

$$\|(1+|P|)^{-1}A\|_{1,\alpha,\mathcal{D}'} \leq M, \tag{A.5}$$

where $\mathcal{D}' := (B_1 \cap \{x_n = 0\}) \times [-M_1, M_1] \times \mathbf{R}^n$. Then $u \in C^{2,\alpha}(\overline{B^+_{1/2}})$ and there exists C depending only on $n, \lambda, \Lambda, M, M_1$, and α such that

$$\begin{aligned} \|u\|_{2,\alpha,B^+_{1/2}} \quad \leq \quad & C(\|u\|_{L^\infty(B_1^+)} + \|D_x A(\cdot,\cdot,0)\|_{0,\alpha,\mathcal{D}_{x,z}} + \|A(\cdot,\cdot,0)\|_{1,\alpha,\mathcal{D}'_{x,z}} \\ & + \|(1+|P|^2)^{-1}B\|_{0,\alpha,\mathcal{D}^+}), \end{aligned}$$

where $\mathcal{D}'_{x,z} := (B_1 \cap \{x_n = 0\}) \times [-M_1, M_1]$.

Proof. We sketch only the proof of assertion (ii) since the proof of (i) is similar. The constant C below depends only on $n, \lambda, \Lambda, M, M_1$, and α, and may be different at each occurrence.

Using condition (A.1) and the assumptions on $A(x,z,P)$ and $B(x,z,P)$, we can apply a local version of the estimates in [24, Section 5] to obtain

$$\|Du\|_{0,\alpha,B^+_{7/8}} \leq C. \tag{A.6}$$

Now we rewrite the equation in (A.4) in the nondivergence form:

$$\sum_{i,j=1}^{n} A^i_{p_j}(x,u,Du)u_{x_ix_j} + \sum_{i=1}^{n} \left(A^i_z(x,u,Du)u_{x_i} + A^i_{x_i}(x,u,Du)\right) + B(x,u,Du) = 0.$$

Using (A.1)–(A.3), (A.5), and (A.6), we can apply the local estimates from the proof of [24, Theorem 2] to obtain

$$\|u\|_{2,\alpha,B^+_{3/4}} \leq C. \tag{A.7}$$

Now, in $B^+_{5/8}$, we can rewrite the conormal boundary value problem as a linear problem:

$$\begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} = f(x) \qquad \text{in } B_1^+, \\ &\sum_{i=1}^{n} c_i(x)u_{x_i} = g(x) \qquad \text{on } \Gamma_0 := \partial B_1^+ \cap \{x_n = 0\}, \end{aligned}$$

where

$$
\begin{aligned}
&a_{ij}(x) = A^i_{p_j}(x, u(x), Du(x)),\\
&b_i(x) = A^i_z(x, u(x), Du(x)) + \sum_{j-1}^{n} \int_0^1 A^j_{x_j p_i}(x, u(x), tDu(x))dt,\\
&f(x) = -\sum_{i=1}^{n} A^i_{x_i}(x, u(x), 0) - B(x, u(x), Du(x)),\\
&c_i(x) = \int_0^1 A^n_{p_i}(x, u(x), tDu(x))dt,\\
&g(x) = -A^n(x, u(x), 0).
\end{aligned}
$$

From the ellipticity of A, condition (A.2), and estimate (A.7), we have

$$
\begin{aligned}
&\sum_{i,j} \|a_{ij}\|_{1,\alpha,B^+_{5/8}} + \sum_i \|b_i\|_{0,\alpha,B^+_{5/8}} + \sum_i \|c_i\|_{1,\alpha,B^+_{5/8}} \le C,\\
&c_n(x', 0) \ge \lambda,\\
&\|f\|_{0,\alpha,B^+_{5/8}} \le C\left(\|D_x A(\cdot,\cdot,0)\|_{0,\alpha,\mathcal{D}^+_{x,z}} + \|B\|_{0,\alpha,\mathcal{D}^+\cap\{|P|\le\|u\|_{C^1(B^+_{3/4})}\}}\right)\\
&\qquad\qquad \le C\left(\|D_x A(\cdot,\cdot,0)\|_{0,\alpha,\mathcal{D}^+_{x,z}} + \|(1+|P|^2)^{-1}B\|_{0,\alpha,\mathcal{D}^+}\right),\\
&\|g\|_{1,\alpha,B_{5/8}\cap\{x_n=0\}} \le C\|A(\cdot,\cdot,0)\|_{1,\alpha,\mathcal{D}'_{x,z}}.
\end{aligned}
$$

Now assertion (ii) follows from the standard linear estimates, see e.g. [17, Lemma 6.29]. □

Appendix B. Comparison principles for the conormal boundary value problems

We now show the following comparison principles for the conormal boundary value problems.

Theorem B.1. (i) *Suppose* $u_1, u_2 \in C(\overline{B^+_R}) \cap C^1(B^+_R)$, *and*

$$
\begin{aligned}
&div\, A(x, u_1, Du_1) \le div\, A(x, u_2, Du_2) \qquad \text{in } B^+_R,\\
&A^n(x, u_1, Du_1) \le A^n(x, u_2, Du_2) \qquad \text{on } \Gamma_0 := \partial B^+_R \cap \{x_n = 0\},\\
&u_1 \ge u_2 \qquad \text{on } \Gamma_1 := \partial B^+_R \cap \{x_n > 0\}
\end{aligned}
$$

in the weak sense, i.e.,

$$
\int_{B^+_R} A(x, u_1, Du_1) D\varphi(x)dx \ge \int_{B^+_R} A(x, u_2, Du_2) D\varphi(x)dx \tag{B.1}
$$

for any nonnegative $\varphi \in C^1(\overline{B^+_R})$ *satisfying* $\varphi = 0$ *on* Γ_1. *Assume that*

$$
A, A_z, D_P A \in C^1(\overline{B^+_R} \times \mathbf{R} \times \mathbf{R}^n)
$$

with $\|(A_z, D_P A)\|_{L^\infty(B^+_R\times\mathbf{R}\times\mathbf{R}^n)} \le M < \infty$ *and that the operator* A *is elliptic, i.e.,* (A.3) *holds for all* $(x, z, P) \in \overline{B^+_R} \times \mathbf{R} \times \mathbf{R}^n$. *Then*

$$
u_1 \ge u_2 \qquad \text{in } B^+_R.
$$

(ii) *Let $n \geq 3$. Suppose $u_1, u_2 \in C^{1,\alpha}(\overline{\mathbf{R}^n_+})$ with $\|u_k\|^{(n-2)}_{1,\alpha,\mathbf{R}^n_+} \leq M_1 < \infty$ for $k = 1, 2$, and*

$$\begin{aligned} &div\, A(x, u_1, Du_1) \leq div\, A(x, u_2, Du_2) \qquad \text{in } \mathbf{R}^n_+, \\ &A^n(x, u_1, Du_1) \leq A^n(x, u_2, Du_2) \qquad \text{on } \Gamma_0 := \{x_n = 0\} \end{aligned}$$

in the weak sense, i.e.,

$$\int_{\mathbf{R}^n_+} A(x, u_1, Du_1) D\varphi(x) dx \geq \int_{\mathbf{R}^n_+} A(x, u_2, Du_2) D\varphi(x) dx$$

for any nonnegative $\varphi \in C^1_c(\mathbf{R}^n)$. Assume that

$$A, A_z, D_P A \in C^1(\overline{B^+_R} \times \mathbf{R} \times \mathbf{R}^n)$$

with $|D_P A(x, z, P)| + |(1 + |x|^m) A_z(x, z, P)| \leq M < \infty$, for some $m > \frac{n}{2}$ and for any $(x, z, P) \in \overline{B^+_R} \times \mathbf{R} \times \mathbf{R}^n$ with $|z| + |P| \leq M_1$. Assume that the operator A is elliptic, i.e., (A.3) *holds for any $(x, z, P) \in \overline{B^+_R} \times \mathbf{R} \times \mathbf{R}^n$. Then*

$$u_1 \geq u_2 \qquad \text{in } \mathbf{R}^n_+.$$

Proof. (i). We follow and modify the proof of [17, Theorem10.7(ii)] to solve our conormal boundary value problems. Let

$$w = u_2 - u_1.$$

Then

$$w \leq 0 \qquad \text{on } \Gamma_0,$$

and, from (B.1),

$$\int_{B^+_R} \left(\sum_{i,j-1}^{n} a_{ij}(x) w_{x_j} \varphi_{x_i} + \sum_{i=1}^{n} b_i(x) w \varphi_{x_i} \right) dx \leq 0 \tag{B.2}$$

for any nonnegative $\varphi \in C^1(\overline{B^+_R})$ satisfying $\varphi = 0$ on Γ_1, where

$$\begin{aligned} a_{ij}(x) &= \int_0^1 A^i_{p_j}(x, (1-t)u_1(x) + tu_2(x), (1-t)Du_1(x) + tDu_2(x)) dt, \\ b_i(x) &= \int_0^1 A^i_z(x, (1-t)u_1(x) + tu_2(x), (1-t)Du_1(x) + tDu_2(x)) dt. \end{aligned} \tag{B.3}$$

Note that $a_{ij}, b_i \in C(B^+_R)$ with $\|(a_{ij}, b_i)\|_{L^\infty(B^+_R)} \leq M$, by the assumptions.

We need to prove that $w \leq 0$ in B^+_R. By approximation, (B.2) holds for any nonnegative $\varphi \in W^{1,2}(B^+_R)$ satisfying $\varphi = 0$ on Γ_1. Thus, for any $\varepsilon > 0$, we can substitute

$$\varphi = \frac{w^+}{w^+ + \varepsilon}$$

into (B.2) with $w^+ = \max(w, 0)$. Then, repeating the calculations in [17, page 270], we obtain

$$\int_{B^+_R} \left| D \log \left(1 + \frac{w^+}{\varepsilon} \right) \right|^2 dx \leq C(\Lambda, \lambda, M, R).$$

Since $\log \left(1 + \dfrac{w^+}{\varepsilon} \right) = 0$ on Γ_1, it follows from the Poincaré inequality that

$$\int_{B^+_R} \left| \log \left(1 + \frac{w^+}{\varepsilon} \right) \right|^2 dx \leq C(\Lambda, \lambda, M, R).$$

Letting $\varepsilon \to 0$, we conclude $w^+ \equiv 0$ in B^+_R, i.e., $w \leq 0$.

(ii) Similar to the case of the half-ball, we consider the function $w = u_2 - u_1$. It satisfies

$$\|w\|^{(n-2)}_{1,\alpha,\mathbf{R}^n_+} \le 2M_1.$$

Also,

$$\int_{\mathbf{R}^n_+} \left(\sum_{i,j=1}^n a_{ij}(x) w_{x_j} \varphi_{x_i} + \sum_{i=1}^n b_i(x) w \varphi_{x_i} \right) dx \le 0 \tag{B.4}$$

for any nonnegative $\varphi \in C^1_c(\mathbf{R}^n)$, where a_{ij} and b_i are defined by (B.3). Note that $a_{ij}, b_i \in C(\overline{\mathbf{R}^n})$ with

$$|a_{ij}(x)| + |(1+|x|^m) b_i(x)| \le M \qquad \text{for any } \ x \in \mathbf{R}^n_+, \tag{B.5}$$

by the assumptions.

By approximation, (B.4) holds for any nonnegative $\varphi \in W^{1,2}(\mathbf{R}^n_+)$ satisfying $\varphi = 0$ a.e. on $\mathbf{R}^n_+ \setminus B_R(0)$ for some $R > 0$. Thus, for any $\varepsilon > 0$ and $R > 0$, we can substitute

$$\varphi = \frac{w^+}{w^+ + \varepsilon} \eta_R^2$$

into (B.4), where $\eta_R(x) = \eta\left(\frac{x}{R}\right)$ with $\eta \in C^\infty_c(\mathbf{R}^n)$ satisfying

$$\begin{aligned} &\eta \ge 0 \quad \text{in } \ \mathbf{R}^n, \\ &\eta \equiv 1 \quad \text{in } \ B_1(0), \\ &\eta \equiv 0 \quad \text{in } \ \mathbf{R}^n \setminus B_2(0). \end{aligned}$$

Then

$$\begin{aligned} &0 \le \eta_R \le 1 \quad \text{in } \ \mathbf{R}^n, \\ &|D\eta_R| \le \frac{C}{R} \quad \text{in } \ \mathbf{R}^n, \qquad \operatorname{supp}(D\eta_R) \subset B_{2R}(0) \setminus B_R(0), \\ &\eta_R \equiv 1 \quad \text{in } \ B_R(0), \\ &\eta \equiv 0 \quad \text{in } \ \mathbf{R}^n \setminus B_{2R}(0). \end{aligned}$$

Substituting φ defined above into (B.4), we use the summation convention to get

$$\begin{aligned} \int_{\mathbf{R}^n_+} \Bigg(& a_{ij} w^+_{x_i} w^+_{x_j} \frac{\varepsilon \eta_R^2}{(w^+ + \varepsilon)^2} + b_j w^+ w^+_{x_j} \frac{\varepsilon \eta_R^2}{(w^+ + \varepsilon)^2} \\ & + 2 a_{ij} w^+_{x_i} \eta_R (\eta_R)_{x_j} \frac{w^+}{w^+ + \varepsilon} + b_j w^+ \eta_R (\eta_R)_{x_j} \frac{w^+}{w^+ + \varepsilon} \Bigg) dx \le 0. \end{aligned}$$

Thus, using the ellipticity, estimate (B.5) with $m > \frac{n}{2}$, and the properties of η_R, we have

$$\begin{aligned}
\lambda \int_{B_R^+} \left| D\log\left(1+\frac{w^+}{\varepsilon}\right)\right|^2 dx &\le \lambda \int_{\mathbf{R}^n} \eta_R^2 \left| D\log\left(1+\frac{w^+}{\varepsilon}\right)\right|^2 dx \\
&= \lambda \int_{\mathbf{R}^n} \eta_R^2 \frac{|Dw^+|^2}{(w^+ + \varepsilon)^2} dx \\
&\le \int_{\mathbf{R}^n_+} \left(a_{ij} w^+_{x_i} w^+_{x_j} \frac{\varepsilon \eta_R^2}{(w^+ + \varepsilon)^2} \right) dx \\
&\le C(n,M,\Lambda) \int_{B_{2R}} \frac{1}{1+|x|^m} \frac{w^+}{w^+ + \varepsilon} \left| D\log\left(1+\frac{w^+}{\varepsilon}\right)\right| dx \\
&\quad + \frac{C(n,M,\Lambda)}{\varepsilon R} \int_{B_{2R}\setminus B_R} \left(\frac{w^+ |Dw^+|}{w^+ + \varepsilon} + \frac{1}{1+|x|^m} \frac{|w^+|^2}{w^+ + \varepsilon} \right) dx \\
&\le C(n,M,\Lambda) \left(\int_{\mathbf{R}^n} \frac{1}{1+|x|^{2m}} dx \right)^{\frac12} \left(\int_{B_{2R}} \left| D\log\left(1+\frac{w^+}{\varepsilon}\right)\right|^2 dx \right)^{\frac12} \\
&\quad + \frac{C(n,M,\Lambda)}{\varepsilon^2 R} \int_{B_{2R}\setminus B_R} \left(w^+ |Dw^+| + \frac{|w^+|^2}{1+|x|^m} \right) dx \\
&\le C(n,M,m,\Lambda) \left(\int_{B_{2R}} \left| D\log\left(1+\frac{w^+}{\varepsilon}\right)\right|^2 dx \right)^{\frac12} \\
&\quad + \frac{C(n,M,M_1,\Lambda)}{\varepsilon^2 R} \left(\frac{R^n}{R^{n-2} R^{n-1}} + \frac{R^n}{R^m R^{2(n-2)}} \right),
\end{aligned}$$

where, in the last inequality, we used the following estimates:

$$|w(x)| \le \frac{M_1}{1+|x|^{n-2}}, \qquad |Dw(x)| \le \frac{M_1}{1+|x|^{n-1}}. \tag{B.6}$$

Thus, we obtain

$$\int_{B_R^+} \left| D\log\left(1+\frac{w^+}{\varepsilon}\right)\right|^2 dx \le C(n,m,M,M_1,\Lambda,\lambda) \left(1 + \frac{1}{\varepsilon^2 R^{n-2}}\right).$$

Since $n \ge 3$, then sending $R \to \infty$ yields

$$\int_{\mathbf{R}^n_+} \left| D\log\left(1+\frac{w^+}{\varepsilon}\right)\right|^2 dx \le C(n,m,M,M_1,\Lambda,\lambda). \tag{B.7}$$

Now we extend w to $\mathbf{R}^n$ by the reflection $w(x',-x_n) := w(x',x_n)$ for $x_n > 0$ and continue to denote the extension by w. Then $w \in C^{0,1}(\mathbf{R}^n)$, and (B.6) holds in $\mathbf{R}^n$ (the estimate of $|Du|$ holds a.e.). Also, from (B.7),

$$\int_{\mathbf{R}^n} \left| D\log\left(1+\frac{w^+}{\varepsilon}\right)\right|^2 dx \le C(n,m,M,M_1,\Lambda,\lambda). \tag{B.8}$$

Now consider the functions

$$v(x) := \log\left(1+\frac{w^+}{\varepsilon}\right), \qquad v_R(x) := \eta_R(x) \log\left(1+\frac{w^+}{\varepsilon}\right)$$

in $\mathbf{R}^n$, where $R > 0$ and η_R is defined above. Then $v \in C^{0,1}(\mathbf{R}^n)$ and $v_R \in C_c^{0,1}(\mathbf{R}^n)$. Thus, we use $n \ge 3$ to obtain

$$\left(\int_{\mathbf{R}^n} |v_R|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \le C(n) \int_{\mathbf{R}^n} |Dv_R|^2 dx. \tag{B.9}$$

Since

$$|v(x)| \le \frac{1}{\varepsilon} w^+(x) \le \frac{M_1}{\varepsilon} \frac{1}{1+|x|^{n-2}},$$

and $Dv \in L^2(\mathbf{R}^n)$ by (B.8), we use the similar properties of η_R as in the estimates above and $n \ge 3$ to see that the left-hand and right-hand sides of (B.9) converge, as $R \to \infty$, to the left-hand and right-hand sides of the inequality

$$\left(\int_{\mathbf{R}^n} |v|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \le C(n) \int_{\mathbf{R}^n} |Dv|^2 dx,$$

respectively. Now, by (B.8),

$$\int_{\mathbf{R}^n} \left| \log \left(1 + \frac{w^+}{\varepsilon} \right) \right|^{\frac{2n}{n-2}} dx \le C(n, m, M, M_1, \Lambda, \lambda).$$

Since this is true for any $\varepsilon > 0$, we conclude that $w^+ \equiv 0$ in $\mathbf{R}^n$, i.e. $w \le 0$. □

Acknowledgments. Gui-Qiang Chen's research was supported in part by the National Science Foundation under Grants DMS-0244473, DMS-0204225, DMS-0204455, and INT-9987378. Mikhail Feldman's research was supported in part by the National Science Foundation under Grants DMS-0200644 and DMS-0074037. The main part of this paper was completed when the first author visited the Isaac Newton Institute for Mathematical Sciences, University of Cambridge, UK, March–June, 2003.

References

[1] H. W. Alt and L. A. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.* **325** (1981), 105–144.
[2] H. W. Alt, L. A. Caffarelli, and A. Friedman, A free-boundary problem for quasilinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **11** (1984), 1–44.
[3] H. W. Alt, L. A. Caffarelli, and A. Friedman, Compressible flows of jets and cavities, *J. Diff. Eqs.* **56** (1985), 82–141.
[4] L. Bers, Existence and uniqueness of subsonic flows past a given profile, *Comm. Pure Appl. Math.* **7** (1954), 441-504.
[5] S. Canić, B. L. Keyfitz, and G. Lieberman, A proof of existence of perturbed steady transonic shocks via a free boundary problem, *Comm. Pure Appl. Math.* **53** (2000), 0484-0511.
[6] G.-Q. Chen and M. Feldman, Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, *J. Amer. Math. Soc.* **16** (2003), 461-494.
[7] G.-Q. Chen and M. Feldman, Steady transonic shocks and free boundary problems in infinite cylinders for the Euler equations, *Comm. Pure Appl. Math.* (2003) (to appear).
[8] G.-Q. Chen and M. Feldman, In preparation.
[9] S.-X. Chen, Existence of stationary supersonic flows past a point body, *Arch. Rational Mech. Anal.* **156** (2001), 141-181.
[10] S.-X. Chen, Asymptotic behavior of supersonic flow past a convex combined wedge, *Chinese Ann. Math.* **19B** (1998), 255-264.
[11] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Springer-Verlag: New York, 1948.
[12] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag: Berlin, 2000.
[13] G. Dong, *Nonlinear Partial Differential Equations of Second Order*, Transl. Math. Monographs, **95**, AMS: Providence, RI, 1991.
[14] L. C. Evans and W. Gangbo, Differential equations methods for the Monge-Kantorovich mass transfer problem, *Mem. Amer. Math. Soc.* **137** (1999), no. 653.
[15] R. Finn and D. Gilbarg, Asymptotic behavior and uniqueness of plane subsonic flows, *Comm. Pure Appl. Math.* **10** (1957), 23–63.
[16] R. Finn and D. Gilbarg, Three-dimensional subsonic flows, and asymptotic estimates for elliptic partial differential equations, *Acta Math.* **98** (1957), 265-296.
[17] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* 2nd Ed., Springer-Verlag: Berlin, 1983.

[18] J. Glimm and A. Majda, *Multidimensional Hyperbolic Problems and Computations*, Springer-Verlag: New York, 1991.
[19] C.-H. Gu, A method for solving the supersonic flow past a curved wedge (in Chinese), *Fudan J.* **7** (1962), 11-14.
[20] D. Kinderlehrer and L. Nirenberg, Regularity in free boundary problems, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 4 (1977), 373–391.
[21] P. D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, CBMS-RCSM, SIAM: Philiadelphia, 1973.
[22] T.-T. Li, On a free boundary problem, *Chinese Ann. Math.* **1** (1980), 351-358.
[23] G. Lieberman, Regularity of solutions of nonlinear elliptic boundary value problems, *J. Reine Angew. Math.* **369** (1986), 1–13.
[24] G. Lieberman, Hölder continuity of the gradient of solutions of uniformly parabolic equations with conormal boundary conditions, *Ann. Mat. Pura Appl.* (4) **148** (1987), 77–99.
[25] G. Lieberman, Mixed boundary value problems for elliptic and parabolic differential equations of second order. *J. Math. Anal. Appl.* **113** (1986), no. 2, 422–440.
[26] G. Lieberman and N. Trudinger, Nonlinear oblique boundary value problems for nonlinear elliptic equations, *Trans. Amer. Math. Soc.* **295** (1986), 509–546.
[27] W.-C. Lien and T.-P. Liu, Nonlinear stability of a self-similar 3-dimensional gas flow, *Commun. Math. Phys.* **204** (1999), 525–549.
[28] A. Majda, The stability of multidimensional shock fronts, *Mem. Amer. Math. Soc.* **41** (1983), no. 275, AMS: Providence.
[29] A. Majda, The existence of multidimensional shock fronts, *Mem. Amer. Math. Soc.* **43** (1983), no. 281, AMS: Providence.
[30] G. Métivier, Stability of multi-dimensional weak shocks, *Commun. Partial Diff. Eqs.* **15** (1990), 983-1028.
[31] C. S. Morawetz, On the non-existence of continuous transonic flows past profiles I–III, *Comm. Pure Appl. Math.* **9** (1956), 45–68; **10** (1957), 107–131; **11** (1958), 129–144.
[32] O. Oleinik, *Some Asymptotic Problems in the Theory of Partial Differential Equations*, Lezioni Lincee. [Lincei Lectures] Cambridge University Press: Cambridge, 1996.
[33] D. G. Schaeffer, Supersonic flow past a nearly straight wedge, *Duke Math. J.* **43** (1976), 637-670.
[34] M. Shiffman, On the existence of subsonic flows of a compressible fluid, *J. Rational Mech. Anal.* **1** (1952), 605–652.
[35] Y. Zhang, Global existence of steady supersonic potential flow past a curved wedge with a piecewise smooth boundary, *SIAM J. Math. Anal.* **31** (1999), 166-183.

Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, USA
E-mail address: gqchen@math.northwestern.edu

Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388, USA
E-mail address: feldman@math.wisc.edu