POINCARÉ–TYPE INEQUALITIES FOR BROKEN SOBOLEV SPACES

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Abstract. We present two versions of general Poincaré–type inequalities for functions in broken Sobolev spaces, providing bounds for the L^q–norm of a function in terms of its broken H^1–norm.

Key words. Poincaré–type inequalities, nonconforming finite elements, broken Sobolev spaces

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1. Introduction and Notation. In recent years there has been considerable interest in the development and mathematical analysis of discontinuous Galerkin finite element methods for the numerical approximation of second–order elliptic partial differential equations. Unlike classical, conforming, finite element methods which seek a continuous piecewise polynomial approximation to a weak solution u in the Sobolev space $H^1(\Omega)$, in discontinuous Galerkin finite element methods the numerical solution is sought as a discontinuous piecewise polynomial approximation to u on a suitable finite subdivision $\mathcal{T} = \{\kappa\}$, into open disjoint Lipschitz subdomains κ, of the computational domain Ω, $\overline{\Omega} = \cup_{\kappa\in\mathcal{T}}\overline{\kappa}$, where Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary.

A useful technical tool in the error analysis of classical conforming finite element methods is the Poincaré inequality which, given that Ψ is a bounded linear functional on $H^1(\Omega)$ with $\Psi(1) = 1$, asserts the existence of a positive constant $C = C(d)$, where $d = \mathrm{diam}(\Omega)$, such that

$$\|\xi\|_{L^2(\Omega)} \le C\left\{\|\nabla\xi\|_{L^2(\Omega)} + |\Psi(\xi)|\right\} \quad \text{for all} \quad \xi \in H^1(\Omega).$$

By writing $\xi - \Psi(\xi)$ in place of ξ in this inequality, we deduce that

$$\|\xi - \Psi(\xi)\|_{L^2(\Omega)} \le C\,\|\nabla\xi\|_{L^2(\Omega)} \quad \text{for all} \quad \xi \in H^1(\Omega),$$

and therefore

$$\inf_{c\in\mathbb{R}} \|\xi - c\|_{L^2(\Omega)} \le \|\nabla\xi\|_{L^2(\Omega)} \quad \text{for all} \quad \xi \in H^1(\Omega).$$

In the context of discontinuous Galerkin finite element methods it is natural to enquire what the analogue of these inequalities are when the function ξ only belongs to the *broken* Sobolev space $H^1(\Omega, \mathcal{T})$, consisting of functions ξ such that $\xi|_\kappa \in H^1(\kappa)$ for each κ in $\mathcal{T}$.

We shall present two versions of a Poincaré–type inequality for broken Sobolev spaces. A variant of a Poincaré–type inequality for functions in broken Sobolev spaces in the case of $\Omega \subset \mathbb{R}^2$ was derived in Arnold [2], Lemmas 2.1 and 2.2, where the proof relies on elliptic regularity in non–smooth domains. More general results of this kind for $\Omega \subset \mathbb{R}^n$ involving various seminorms were obtained by Brenner in [4]. The proofs in [4] heavily rely on the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$.

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Here we generalise both approaches, providing bounds on the $L^q(\Omega)$–norm, in terms of a broken H^1–norm, for $1 \le q \le 2n/(n-2)$, $n \ge 3$, and for $1 \le q < \infty$, $n = 2$, $\Omega \subset \mathbb{R}^n$. The first version of a Poincaré–type inequality for functions in broken Sobolev spaces is proved by using the Sobolev embedding theorem and elliptic regularity results, and thus relies on the regularity of the domain. For the second version, we chose to extend the approach used by Brenner in [4]. Both versions include the critical value of $q = 2n/(n-2)$ for which the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ is continuous but not compact.

Let us introduce the notation we shall be using throughout. Let $\mathcal{T}$ be a subdivision of the domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, into disjoint open elements κ such that $\overline{\Omega} = \cup_{\kappa \in \mathcal{T}} \overline{\kappa}$. We assume that the family of subdivisions $\mathcal{T}$ is shape–regular (see [3]), and that each $\kappa \in \mathcal{T}$ is an affine image of a fixed master (reference) element $\hat{\kappa}$, i.e., $\kappa = F_\kappa(\hat{\kappa})$, where $\hat{\kappa}$ is either the open unit simplex or the open unit hypercube in $\mathbb{R}^n$. For a nonnegative integer k we denote by $\mathcal{P}_k(\hat{\kappa})$ the set of polynomials of total degree k on $\hat{\kappa}$. To each $\kappa \in \mathcal{T}$ we assign a local polynomial degree $p_\kappa \ge 0$ and a local Sobolev index $s_\kappa \ge 0$. With this, we designate to the subdivision $\mathcal{T}$ the broken Sobolev space of composite order $\mathbf{s} = \{s_\kappa : \kappa \in \mathcal{T}\}$,

$$H^{\mathbf{s}}(\Omega, \mathcal{T}) = \left\{ u \in L^2(\Omega) : u|_\kappa \in H^{s_\kappa}(\kappa) \quad \forall \kappa \in \mathcal{T} \right\},$$

with the broken Sobolev norm and seminorm, respectively,

$$\|u\|_{H^{\mathbf{s}}(\Omega,\mathcal{T})} := \left(\sum_{\kappa \in \mathcal{T}} \|u\|^2_{H^{s_\kappa}(\kappa)} \right)^{\frac{1}{2}}, \quad |u|_{H^{\mathbf{s}}(\Omega,\mathcal{T})} := \left(\sum_{\kappa \in \mathcal{T}} |u|^2_{H^{s_\kappa}(\kappa)} \right)^{\frac{1}{2}}.$$

By $\mathcal{E}$ we denote the set of all open $(n-1)$–dimensional faces of the subdivision $\mathcal{T}$, containing the smallest common $(n-1)$–dimensional interfaces e of neighbouring elements. We define

$$\mathcal{E}_{\text{int}} := \bigcup_{e \in \mathcal{E} \setminus \partial\Omega} e \quad \text{and} \quad \mathcal{E}_\partial := \bigcup_{e \in \mathcal{E} \cap \partial\Omega} e.$$

Numbering the elements of the subdivision $\mathcal{T}$, and choosing any internal interface $e \in \mathcal{E}_{\text{int}}$, there exist positive integers i, j such that $i > j$ and elements $\kappa \equiv \kappa_i$ and $\kappa' \equiv \kappa_j$ which share this interface e. We define the *jump of a function* $u \in H^{\mathbf{s}}(\Omega, \mathcal{T})$ *across the face* e and the *mean value of* u *on* e by

$$[u]_e := u|_{\partial\kappa \cap e} - u|_{\partial\kappa' \cap e}, \quad \langle u \rangle_e := \frac{1}{2} \left(u|_{\partial\kappa \cap e} + u|_{\partial\kappa' \cap e} \right)$$

respectively, $\partial\kappa$ denoting the union of all open faces of element κ.

2. Sobolev–Poincaré Inequality for Broken Sobolev Spaces: I. In this section we derive broken Sobolev–Poincaré inequalities stated in Theorem 2.1 and Corollary 2.2. In the next section we shall then improve these results in various ways by using a completely different technique.

Suppose that $\hat{\kappa} \subset \mathbb{R}^n$ is a bounded open set with Lipschitz continuous boundary, and $\text{diam}_n(\hat{\kappa}) = 1$, $n \ge 2$.

By the trace inequality (see [1]), for $1 \le q \le \infty$ and $1/q + 1/q' = 1$,

$$W^1_{q'}(\hat{\kappa}) \hookrightarrow W^{1-1/q'}_{q'}(\partial\hat{\kappa}) = W^{1/q}_{q'}(\partial\hat{\kappa}).$$

Applying this with $q' = 2n/(n+2)$, it follows that

$$W^1_{q'}(\hat{\kappa}) = W^1_{\frac{2n}{n+2}}(\hat{\kappa}) \hookrightarrow W^{\frac{n-2}{2n}}_{\frac{2n}{n+2}}(\partial\hat{\kappa}), \quad n \geq 2. \tag{2.1}$$

We note that by the Sobolev embedding theorem (see [1])

$$W^s_p(\Omega) \hookrightarrow W^t_q(\Omega), \quad s - \frac{n}{p} \geq t - \frac{n}{q}, \quad \begin{array}{l} 0 \leq t \leq s < \infty \\ 1 < p \leq q < \infty, \end{array} \tag{2.2}$$

with $\Omega \subset \mathbb{R}^n$ an open set in $\mathbb{R}^n$. Applying this result on $\Omega = \partial\hat{\kappa}$ (splitting $\partial\hat{\kappa}$ into finitely many open subdomains, if necessary) with n replaced by $n-1$ in (2.2) and

$$s = \frac{n-2}{2n}, \quad p = \frac{2n}{n+2}, \quad t = 0, \quad q = 2\left(1 - \frac{1}{n}\right), \quad n \geq 2,$$

we see that

$$s - \frac{n-1}{p} = \frac{n-2}{2n} - \frac{(n-1)(n+2)}{2n} = -\frac{n}{2},$$

while

$$t - \frac{n-1}{q} = -\frac{n(n-1)}{2(n-1)} = -\frac{n}{2}.$$

The condition on the Sobolev indices, required for the embedding, holds, and hence

$$W^{\frac{n-2}{2n}}_{\frac{2n}{n+2}}(\partial\hat{\kappa}) \hookrightarrow W^0_{2\left(1-\frac{1}{n}\right)}(\partial\hat{\kappa}) = L^{2\left(1-\frac{1}{n}\right)}(\partial\hat{\kappa}). \tag{2.3}$$

From (2.1) and (2.3) we deduce that

$$W^1_{\frac{2n}{n+2}}(\hat{\kappa}) \hookrightarrow L^{2\left(1-\frac{1}{n}\right)}(\partial\hat{\kappa}), \quad n \geq 2.$$

By the continuity of the embedding operator, there exists $C = C(\hat{\kappa})$ such that

$$\|\hat{u}\|_{L^{2\left(1-\frac{1}{n}\right)}(\partial\hat{\kappa})} \leq C\, \|\hat{u}\|_{W^1_{\frac{2n}{n+2}}(\hat{\kappa})}, \quad n \geq 2,$$

for all $\hat{u} \in W^1_{\frac{2n}{n+2}}(\hat{\kappa})$, i.e.,

$$\|\hat{u}\|_{L^{2\left(1-\frac{1}{n}\right)}(\partial\hat{\kappa})} \leq C\left(\|\hat{u}\|_{L^{\frac{2n}{n+2}}(\hat{\kappa})} + \left\|\hat{\nabla}\hat{u}\right\|_{L^{\frac{2n}{n+2}}(\hat{\kappa})}\right), \quad n \geq 2.$$

Applying (2.2) with $\Omega = \hat{\kappa}$ and

$$s = 1, \quad p = \frac{2n}{n+2}, \quad t = 0, \quad q = 2, \quad n \geq 2,$$

we deduce that

$$W^1_{\frac{2n}{n+2}}(\hat{\kappa}) \hookrightarrow L^2(\hat{\kappa}), \quad n \geq 2;$$

hence $\hat{u} \in W^1_{\frac{2n}{n+2}}(\hat{\kappa})$ has $\|\hat{u}\|_{L^2(\hat{\kappa})} < \infty$.

Therefore, noting that $1 \le 2n/(n+2) < 2$, Hölder's inequality gives

$$\|\hat{u}\|_{L^{2\left(1-\frac{1}{n}\right)}(\partial\hat{\kappa})} \le C\left(\|\hat{u}\|_{L^2(\hat{\kappa})} + \left\|\hat{\nabla}\hat{u}\right\|_{L^{\frac{2n}{n+2}}(\hat{\kappa})}\right), \quad n \ge 2. \tag{2.4}$$

Let κ be defined as $h \cdot \hat{\kappa}$, by simply scaling $\hat{\kappa}$. Hence $h = \mathrm{diam}(\kappa)$, and we shall write $h_\kappa = h$. By rescaling (2.4), we see that

$$h_\kappa^{0-\frac{n-1}{2(1-1/n)}} \|u\|_{L^{2\left(1-\frac{1}{n}\right)}(\partial\kappa)} \le C\left(h_\kappa^{-\frac{n}{2}} \|u\|_{L^2(\kappa)} + h_\kappa^{1-\frac{n}{2n/(n+2)}} \|\nabla u\|_{L^{\frac{2n}{n+2}}(\kappa)}\right),$$

where $u(x) = \hat{u}(\hat{x})$, $\hat{x} \in \hat{\kappa}$. Thus,

$$\|u\|_{L^{2\left(1-\frac{1}{n}\right)}(\partial\kappa)} \le C\left(\|u\|_{L^2(\kappa)} + \|\nabla u\|_{L^{\frac{2n}{n+2}}(\kappa)}\right) \tag{2.5}$$

for each $u \in W^1_{\frac{2n}{n+2}}(\kappa)$, $n \ge 2$.

Let $\xi \in H^1(\Omega, \mathcal{T})$. Then $\xi \in L^q(\Omega, \mathcal{T}) = L^q(\Omega)$, ($1 \le q \le 2n/(n-2)$ for $n \ge 3$ and $1 \le q < \infty$ for $n = 2$). Now, for $n \ge 3$, with $q = 2n/(n-2)$, $q' = 2n/(n+2)$,

$$\|\xi\|_{L^q(\Omega)} = \sup_{\chi \ne 0} \frac{(\xi, \chi)}{\|\chi\|_{L^{q'}(\Omega)}}. \tag{2.6}$$

Let $\psi \in H^1_0(\Omega)$ denote the weak solution to the following elliptic boundary value problem, with $\chi \in L^{q'}(\Omega)$,

$$\left.\begin{aligned} -\Delta\psi &= \chi \quad \text{in} \quad \Omega \\ \psi &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}\right\}. \tag{2.7}$$

Since $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ by the embedding theorem, it follows by the duality theorem that

$$\chi \in L^{q'}(\Omega) = (L^q(\Omega))' \hookrightarrow \left(H^1_0(\Omega)\right)' = H^{-1}(\Omega),$$

and therefore the existence and uniqueness of a weak solution to (2.7) in $H^1_0(\Omega)$ is guaranteed (for example, by the Lax–Milgram theorem, see [5]).

We shall suppose from now on that Ω is a $W^2_{q'}$–regular domain in $\mathbb{R}^n$ (see [8]). By elliptic regularity theory it then follows that

$$\|\psi\|_{W^2_{q'}(\Omega)} \le C\|\chi\|_{L^{q'}(\Omega)}. \tag{2.8}$$

Furthermore,

$$\|\nabla\psi\|^2_{L^2(\Omega)} = (\nabla\psi, \nabla\psi) = (-\Delta\psi, \psi) = (\chi, \psi) \le \|\chi\|_{L^{q'}(\Omega)} \|\psi\|_{L^q(\Omega)} \le C\|\chi\|_{L^{q'}(\Omega)} \|\nabla\psi\|_{L^2(\Omega)},$$

and therefore

$$\|\nabla\psi\|_{L^2(\Omega)} \le C\|\chi\|_{L^{q'}(\Omega)}. \tag{2.9}$$

Now, returning to (2.6) with $\psi = \psi_\chi$,

$$\|\xi\|_{L^q(\Omega)} = \sup_{\chi \ne 0} \frac{(\xi, -\Delta\psi)}{\|\chi\|_{L^{q'}(\Omega)}} = \sup_{\chi \ne 0} \frac{\sum_\kappa (\nabla\xi, \nabla\psi)_\kappa - \sum_e \int_e [\xi]\,(\nabla\psi \cdot \mathbf{n})\,\mathrm{d}s}{\|\chi\|_{L^{q'}(\Omega)}}. \tag{2.10}$$

Here we used the notational convention that, for $e \subset \partial\Omega$, $[\xi]|_e = \xi$. Clearly,

$$\sum_\kappa (\nabla\xi, \nabla\psi)_\kappa \le \sum_\kappa \|\nabla\xi\|_{L^2(\kappa)} \|\nabla\psi\|_{L^2(\kappa)} \le \left(\sum_\kappa \|\nabla\xi\|^2_{L^2(\kappa)}\right)^{\frac{1}{2}} \left(\sum_\kappa \|\nabla\psi\|^2_{L^2(\kappa)}\right)^{\frac{1}{2}}$$
$$= \left(\sum_\kappa \|\nabla\xi\|^2_{L^2(\kappa)}\right)^{\frac{1}{2}} \|\nabla\psi\|_{L^2(\Omega)} \le C \left(\sum_\kappa \|\nabla\xi\|^2_{L^2(\kappa)}\right)^{\frac{1}{2}} \|\chi\|_{L^{q'}(\Omega)}, \quad (2.11)$$

where in the transition to the last inequality we used (2.9).

On the other hand,

$$-\sum_e \int_e [\xi]\,(\nabla\psi\cdot\mathbf{n})\,\mathrm{d}s \le \sum_e \int_e |[\xi]|\,|\nabla\psi\cdot\mathbf{n}|\,\mathrm{d}s \le \sum_e \|[\xi]\|_{L^{\frac{2n-2}{n-2}}(e)} \|\nabla\psi\cdot\mathbf{n}\|_{L^{2\left(1-\frac{1}{n}\right)}(e)}, \quad (2.12)$$

where we used Hölder's inequality on e with $s = (2n-2)/(n-2) > 2$ and $s' = 2(1-1/n) < 2$, $1/s + 1/s' = 1$.

However,

$$\|\nabla\psi\cdot\mathbf{n}\|_{L^{2\left(1-\frac{1}{n}\right)}(e)} \le \|\nabla\psi\|_{L^{2\left(1-\frac{1}{n}\right)}(e)} \le \|\nabla\psi\|_{L^{2\left(1-\frac{1}{n}\right)}(\partial\kappa)}$$
$$\le C\left(\|\nabla\psi\|_{L^2(\kappa)} + |\psi|_{W^2_{\frac{2n}{n+2}}(\kappa)}\right) \quad (2.13)$$

for each element κ that contains the face e, and in the transition to the last inequality we used (2.5).

From (2.12) and (2.13), for $n \ge 3$,

$$-\sum_e \int_e [\xi]\,(\nabla\psi\cdot\mathbf{n})\,\mathrm{d}s \le C\left(\sum_e \|[\xi]\|^2_{L^{\frac{2n-2}{n-2}}(e)}\right)^{\frac{1}{2}} \left(\sum_\kappa \left(\|\nabla\psi\|^2_{L^2(\kappa)} + |\psi|^2_{W^2_{\frac{2n}{n+2}}(\kappa)}\right)\right)^{\frac{1}{2}}$$
$$\le C\left(\sum_e \|[\xi]\|^2_{L^{\frac{2n-2}{n-2}}(e)}\right)^{\frac{1}{2}} \left(\|\nabla\psi\|^2_{L^2(\Omega)} + \left(\sum_\kappa |\psi|^2_{W^2_{\frac{2n}{n+2}}(\kappa)}\right)^{\frac{1}{2}\cdot 2}\right)^{\frac{1}{2}}. \quad (2.14)$$

Next, with $a_\kappa \equiv |\psi|_{W^2_{\frac{2n}{n+2}}(\kappa)}$, $q' = 2n/(n+2) < 2$,

$$\left(\sum_\kappa |\psi|^2_{W^2_{\frac{2n}{n+2}}(\kappa)}\right)^{\frac{1}{2}} \equiv \left(\sum_\kappa a_\kappa^2\right)^{\frac{1}{2}} \le \left(\sum_\kappa |a_\kappa|^{q'}\,|a_\kappa|^{2-q'}\right)^{\frac{1}{2}} \le \left(\max_\kappa |a_\kappa|^{2-q'}\right)^{\frac{1}{2}} \left(\sum_\kappa |a_\kappa|^{q'}\right)^{\frac{1}{2}}$$
$$= \left(\max_\kappa |a_\kappa|\right)^{1-\frac{q'}{2}} \left(\sum_\kappa |a_\kappa|^{q'}\right)^{\frac{1}{2}} \le \left[\left(\sum_\kappa |a_\kappa|^{q'}\right)^{\frac{1}{q'}}\right]^{1-\frac{q'}{2}} \left(\sum_\kappa |a_\kappa|^{q'}\right)^{\frac{1}{2}},$$

where we used that

$$\max_\kappa |a_\kappa| \le \left(\sum_\kappa |a_\kappa|^{q'}\right)^{\frac{1}{q'}}.$$

Therefore,

$$\left(\sum_\kappa |\psi|^2_{W^2_{\frac{2n}{n+2}}(\kappa)}\right)^{\frac12} \le \left(\sum_\kappa |a_\kappa|^{q'}\right)^{\frac{1}{q'}-\frac{1}{q'}\cdot\frac{q'}{2}+\frac12} = \left(\sum_\kappa |a_\kappa|^{q'}\right)^{\frac{1}{q'}} = \left(\sum_\kappa |\psi|^{q'}_{W^2_{\frac{2n}{n+2}}(\kappa)}\right)^{\frac{1}{q'}}$$
$$= \left(\sum_\kappa |\psi|^{\frac{2n}{n+2}}_{W^2_{\frac{2n}{n+2}}(\kappa)}\right)^{\frac{n+2}{2n}} = |\psi|_{W^2_{\frac{2n}{n+2}}(\Omega)} . \qquad (2.15)$$

Substituting (2.15) into (2.14), we have that, for $n \ge 3$,

$$-\sum_e \int_e [\xi]\,(\nabla\psi\cdot\mathbf{n})\,\mathrm{d}s \le C\left(\sum_e \|[\xi]\|^2_{L^{\frac{2n-2}{n-2}}(e)}\right)^{\frac12}\left(\|\nabla\psi\|^2_{L^2(\Omega)} + |\psi|^2_{W^2_{\frac{2n}{n+2}}(\Omega)}\right)^{\frac12}$$
$$\le C\left(\sum_e \|[\xi]\|^2_{L^{\frac{2n-2}{n-2}}(e)}\right)^{\frac12}\|\chi\|_{L^{q'}(\Omega)}, \qquad (2.16)$$

where in the transition to the last inequality we used (2.9) and (2.8).

Substituting (2.11) and (2.16) into (2.10) gives

$$\|\xi\|_{L^q(\Omega)} \le C\left\{\left(\sum_\kappa \|\nabla\xi\|^2_{L^2(\kappa)}\right)^{\frac12} + \left(\sum_e \|[\xi]\|^2_{L^{\frac{2n-2}{n-2}}(e)}\right)^{\frac12}\right\}.$$

Thus we have proved the following theorem.

Theorem 2.1 *Let $n \ge 3$, $q = 2n/(n-2)$, and suppose that $\Omega \subset \mathbb{R}^n$ is a $W^2_{q'}$–regular domain, $q' = 2n/(n+2)$. There exists $C = C(\Omega)$ such that*

$$\|\xi\|_{L^q(\Omega)} \le C\left\{\left(\sum_\kappa \|\nabla\xi\|^2_{L^2(\kappa)}\right)^{\frac12} + \left(\sum_e \|[\xi]\|^2_{L^{\frac{2n-2}{n-2}}(e)}\right)^{\frac12}\right\} \qquad (2.17)$$

for all $\xi \in H^1(\Omega, \mathcal{T})$.

In particular, if ξ is a (discontinuous) piecewise polynomial function on $\mathcal{T}$ with $p_\kappa = \deg \xi|_\kappa \ge 1$, then $[\xi]$ is a polynomial on e whose degree is $p_e = \max(p_\kappa, p_{\kappa'})$, where $e \subset \partial\kappa \cap \partial\kappa'$.

Let us scale e to $\hat e$ so that $\mathrm{diam}_{n-1}(\hat e) = 1$. By Bernstein's inequality,

$$\|\hat v\|_{L^r(\hat e)} \le C p_{\hat e}^{1-\frac{2}{r}}\,\|\hat v\|_{L^2(\hat e)} \quad \forall \hat v \in \mathbb{P}^{p_{\hat e}}(\hat e).$$

By rescaling to e,

$$h_e^{-\frac{n-1}{r}}\,\|v\|_{L^r(e)} \le C p_e^{1-\frac{2}{r}}\,h_e^{-\frac{n-1}{2}}\,\|v\|_{L^2(e)},$$

and thus

$$\|v\|_{L^r(e)} \le C p_e^{1-\frac{2}{r}}\,h_e^{(n-1)\left(\frac1r-\frac12\right)}\,\|v\|_{L^2(e)} \quad \forall v \in \mathbb{P}^{p_e}(e).$$

In particular, with

$$r = \frac{2n-2}{n-2} = \frac{2(n-1)}{n-2} \quad \text{we have} \quad 1 - \frac{2}{r} = \frac{1}{n-1} \quad \text{and} \quad (n-1)\left(\frac{1}{r} - \frac{1}{2}\right) = -\frac{1}{2}.$$

Hence, with $n \geq 3$,

$$\|v\|_{L^{\frac{2n-2}{n-2}}(e)} \leq C p_e^{\frac{1}{n-1}} h_e^{-\frac{1}{2}} \|v\|_{L^2(e)} \quad \forall v \in \mathbb{P}^{p_e}(e).$$

Taking $v = [\xi]$ yields

$$\|[\xi]\|^2_{L^{\frac{2n-2}{n-2}}(e)} \leq C p_e^{\frac{2}{n-1}} h_e^{-1} \|[\xi]\|^2_{L^2(e)},$$

where $p_e = \max(p_\kappa, p_{\kappa'})$, $p_\kappa, p_{\kappa'} \geq 1$.

Now, let us suppose that the polynomial degree $\mathbf{p} = \{p_\kappa : \kappa \in \mathcal{T}\}$ has locally bounded variation in the sense that there exists $\rho \geq 0$ such that

$$\max_{\kappa} \max_{\substack{\kappa' \\ \text{meas}_{n-1}(\bar{\kappa} \cap \bar{\kappa}') > 0}} |p_\kappa - p_{\kappa'}| \leq \rho.$$

Then if $p_\kappa \geq p_{\kappa'}$, we have $p_e = p_\kappa$, whereas if $p_\kappa \leq p_{\kappa'}$, we have

$$p_e = p_{\kappa'} = p_\kappa + p_{\kappa'} - p_\kappa = p_\kappa + |p_{\kappa'} - p_\kappa| \leq p_\kappa + \rho = p_\kappa \left(1 + \frac{\rho}{p_\kappa}\right) \leq (1+\rho) p_\kappa,$$

and hence for any face e and any element κ that contains the face e, we have

$$p_e \leq (1+\rho) p_\kappa.$$

Thus we have proved the following Sobolev–Poincaré inequality.

Corollary 2.2 *Let $n \geq 3$, $q = 2n/(n-2)$, and suppose that $\Omega \subset \mathbb{R}^n$ is a $W^2_{q'}$–regular domain, $q' = 2n/(n+2)$. There exists $C = C(\Omega)$ such that*

$$\|\xi\|_{L^q(\Omega)} \leq C \left\{ \left(\sum_\kappa \|\nabla \xi\|^2_{L^2(\kappa)} \right)^{\frac{1}{2}} + \left(\sum_e p_e^{\frac{2}{n-1}} h_e^{-1} \|[\xi]\|^2_{L^2(e)} \right)^{\frac{1}{2}} \right\}$$

for all $\xi \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$, where $p_e = \max(p_\kappa, p_{\kappa'})$, $e \subset \partial\kappa \cap \partial\kappa'$.

Further, if the polynomial degree vector has bounded local variation, then

$$\|\xi\|_{L^q(\Omega)} \leq C \left\{ \left(\sum_\kappa \|\nabla \xi\|^2_{L^2(\kappa)} \right)^{\frac{1}{2}} + \left(\sum_\kappa p_\kappa^{\frac{2}{n-1}} h_\kappa^{-1} \|[\xi]\|^2_{L^2(\partial\kappa)} \right)^{\frac{1}{2}} \right\}. \tag{2.18}$$

Remark 2.3 For $n = 2$ an identical result holds for any $1 \leq q < \infty$. For $n \geq 3$ the results of Theorem 2.1 and Corollary 2.2 hold for all q with $1 \leq q \leq 2n/(n-2)$. This follows from

$$\|\xi\|_{L^q(\Omega)} \leq C \|\xi\|_{L^{\frac{2n}{n-2}}(\Omega)}, \quad 1 \leq q \leq \frac{2n}{n-2}.$$

□

3. Sobolev–Poincaré Inequality for Broken Sobolev Spaces: II. In the previous section we obtained a broken Sobolev–Poincaré inequality for functions in the broken Sobolev space $H^1(\Omega,\mathcal{T})$ under certain restrictive conditions, i.e., $W^2_{q'}$–regularity of the domain Ω and Corollary 2.2 being shown only for functions in the finite–dimensional subspace $S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F})$ of $H^1(\Omega,\mathcal{T})$.

In this section we shall obtain a more general and more precise bound on the L^q–norm of functions in $H^1(\Omega,\mathcal{T})$, removing the restrictions mentioned above. We shall closely follow the argument presented by Brenner in [4], extending it from L^2–based norms to L^q–based norms, $1 \le q \le 2n/(n-2)$, $n \ge 3$.

3.1. Seminorm on $H^1(\Omega)$. Let Φ be a seminorm on $H^1(\Omega)$, such that

$$\Phi(\xi) \le C\,\|\xi\|_{H^1(\Omega)} \quad \text{for all} \quad \xi \in H^1(\Omega),\, C > 0,$$

and such that

$$\Phi(1) = 1.$$

The latter implies that $\Phi(c) = 0$ for a constant function c if, and only if, $c = 0$. With such Φ the following generalised Poincaré–Friedrichs inequality holds: there exists a positive constant $C = C(\Omega)$ such that

$$\|\xi\|_{L^2(\Omega)} \le C\left\{|\xi|_{H^1(\Omega)} + \Phi(\xi)\right\} \quad \text{for all} \quad \xi \in H^1(\Omega);$$

this follows from the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ (see [9]).

Let $n \ge 3$ and $1 \le q \le 2n/(n-2)$. Then $L^q(\Omega)$ is continuously embedded into $H^1(\Omega)$ (note that for $q = 2n/(n-2)$ the embedding is not compact, — but this will not affect our argument).

In the following argument we shall show that the general Poincaré–Friedrichs inequality holds for the L^q–norm, with q defined above.

1. Let Ψ be a bounded linear functional on $H^1(\Omega)$, with the property $\Psi(1) = 1$ and let $\Phi(\xi) = |\Psi(\xi)|$. Then

$$\begin{aligned}\|\xi - \Psi(\xi)\|_{L^2(\Omega)} &\le C\left\{|\xi - \Psi(\xi)|_{H^1(\Omega)} + \Phi(\xi - \Psi(\xi))\right\}\\ &= C\left\{|\xi|_{H^1(\Omega)} + |\Psi(\xi - \Psi(\xi))|\right\} = C\,|\xi|_{H^1(\Omega)}\,,\end{aligned}$$

as $|\Psi(\xi - \Psi(\xi))| = |\Psi(\xi) - \Psi(\xi)\Psi(1)| = 0$ by the property $\Psi(1) = 1$.

2. From now on, we suppose that

$$q = \frac{2n}{n-2}.$$

Then, by the continuity of the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$, we have

$$\|\xi\|_{L^q(\Omega)} \le C\,\|\xi\|_{H^1(\Omega)} = C(\|\xi\|_{L^2(\Omega)} + |\xi|_{H^1(\Omega)}),$$

and thus by the results obtained in the previous step, we have

$$\begin{aligned}\|\xi - \Psi(\xi)\|_{L^q(\Omega)} &\le C\left\{\|\xi - \Psi(\xi)\|_{L^2(\Omega)} + |\xi|_{H^1(\Omega)}\right\}\\ &\le C\left\{|\xi|_{H^1(\Omega)} + |\xi|_{H^1(\Omega)}\right\} = C\,|\xi|_{H^1(\Omega)}\,. \quad (3.1)\end{aligned}$$

Thus we conclude that

$$\|\xi\|_{L^q(\Omega)} \le \|\xi - \Psi(\xi)\|_{L^q(\Omega)} + \|\Psi(\xi)\|_{L^q(\Omega)} \le C\,|\xi|_{H^1(\Omega)} + |\Psi(\xi)|\,|\Omega|^{1/q}\,,$$

and hence by the properties of Ψ we have

$$\|\xi\|_{L^q(\Omega)} \le C\left\{|\xi|_{H^1(\Omega)} + |\Psi(\xi)|\right\} = C\left\{|\xi|_{H^1(\Omega)} + \Phi(\xi)\right\}.$$

Thus, when $n \ge 3$, this inequality also holds for all q with $1 \le q \le 2n/(n-2)$. This is a simple consequence of Hölder's inequality.

When $n = 2$, the same result is true for all q with $1 \le q < \infty$.

3.2. Nonconforming and Conforming Finite Element Interpolants. Let us concentrate on the case when the finite element partition is a simplical subdivision $\mathcal{T}$ of Ω, consisting of triangles in the case of $n = 2$ and simplices in the case of $n \ge 3$.

Let us introduce some notation we shall be using throughout. We denote the minimum angle of the triangles or simplices in $\mathcal{T}$ by $\theta_{\mathcal{T}}$. To represent the statement $A \le \varsigma(\theta_{\mathcal{T}})B$, where the generic function $\varsigma : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and independent of $\mathcal{T}$, we shall use the notation $A \lesssim B$; the statement "$A \lesssim B$ and $B \lesssim A$" will be denoted as $A \approx B$.

The nonconforming $\mathcal{P}_1$ finite element space (see [7]) associated with the triangulation $\mathcal{T}$ is

$$V_{\mathcal{T}} := \big\{v \in L^2(\Omega) : v_\kappa = v|_\kappa \in \mathcal{P}_1(\kappa) \text{ for any } \kappa \in \mathcal{T} \text{ and } v \text{ is continuous at the centre of the common side (face) of any two neighbouring triangles (simplices)}\big\}.$$

A function in $V_{\mathcal{T}}$ is completely determined by its nodal values at the centres of the sides (faces) of the triangles (simplices) in $\mathcal{T}$.

Next, we introduce the interpolation operators we shall be using throughout our argument.

We define the interpolation operator $\mathcal{I} : H^1(\Omega, \mathcal{T}) \to V_{\mathcal{T}}$ by

$$(\mathcal{I}\xi)(c_e) := \frac{1}{|e|}\int_e \langle\xi\rangle\,\mathrm{d}s, \quad \text{for all} \quad e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_\partial, \quad \xi \in H^1(\Omega, \mathcal{T}), \tag{3.2}$$

where $e \subset \partial\kappa$ is an open face of the element $\kappa \in \mathcal{T}$, and c_e is the centre of this face. For $e \in \mathcal{E}_\partial$, we take $\langle\xi\rangle$ to be ξ.

We define the local interpolation operator $\Pi_\kappa : H^1(\kappa) \to \mathcal{P}_1(\kappa)$ by

$$(\Pi_\kappa\xi)(c_e) := \frac{1}{|e|}\int_e \xi\,\mathrm{d}s, \quad \text{for all} \quad e \subset \partial\kappa, \quad \xi \in H^1(\Omega, \mathcal{T}). \tag{3.3}$$

Thus the difference of the two interpolants $\mathcal{I}$ and Π_κ on an element $\kappa \in \mathcal{T}$ is

$$(\mathcal{I}\xi - \Pi_\kappa\xi)(c_e) = \begin{cases} \dfrac{1}{2\,|e|}\displaystyle\int_e [\xi]\,\mathrm{d}s & \text{if} \quad e \subset \partial\kappa \setminus \partial\Omega, \\ 0 & \text{if} \quad e \subset \partial\kappa \cap \partial\Omega. \end{cases} \tag{3.4}$$

From (3.4), by standard finite element estimates (see [5, 6]) we have

$$\begin{aligned} |\mathcal{I}\xi - \Pi_\kappa\xi|^2_{H^1(\kappa)} &\lesssim |\kappa|^{1-(2/n)} \sum_{e\subset\partial\kappa} |(\mathcal{I}\xi - \Pi_\kappa\xi)(c_e)|^2 \\ &\lesssim |\kappa|^{1-(2/n)} \sum_{e\subset\partial\kappa\setminus\partial\Omega} |e|^{-2}\left(\int_e [\xi]\,\mathrm{d}s\right)^2 \lesssim \sum_{e\subset\partial\kappa\setminus\partial\Omega} |e|^{\frac{n}{1-n}}\left(\int_e [\xi]\,\mathrm{d}s\right)^2, \end{aligned} \tag{3.5}$$

and for $n \geq 3$, $q = 2n/(n-2)$,

$$\begin{aligned}\|\mathcal{I}\xi - \Pi_\kappa\xi\|^2_{L^q(\kappa)} &\lesssim |\kappa|^{2/q} \sum_{e\subset\partial\kappa} |(\mathcal{I}\xi - \Pi_\kappa\xi)(c_e)|^2 \lesssim \sum_{e\subset\partial\kappa} |e|^{\frac{n}{n-1}\cdot\frac{2}{q}} |(\mathcal{I}\xi - \Pi_\kappa\xi)(c_e)|^2 \\ &= \sum_{e\subset\partial\kappa} |e|^{\frac{n-2}{n-1}} |(\mathcal{I}\xi - \Pi_\kappa\xi)(c_e)|^2 \lesssim \sum_{e\subset\partial\kappa\setminus\partial\Omega} |e|^{\frac{n-2}{n-1}} |e|^{-2} \left(\int_e [\xi]\,\mathrm{d}s\right)^2 \\ &= \sum_{e\subset\partial\kappa\setminus\partial\Omega} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\,\mathrm{d}s\right)^2, \end{aligned} \tag{3.6}$$

where $|\kappa|$ is an n–dimensional volume of κ, and for $e \subset \partial\kappa$ we have

$$|\kappa| \approx |e|^{\frac{n}{n-1}}\,. \tag{3.7}$$

Moreover, we have the following estimate for the local interpolation operator:

$$\|\xi - \Pi_\kappa\xi\|^2_{L^q(\kappa)} + |\Pi_\kappa\xi|^2_{H^1(\kappa)} \lesssim |\xi|^2_{H^1(\kappa)}\,. \tag{3.8}$$

To prove this, first note that by standard finite element estimates we have

$$|\Pi_\kappa\xi|_{H^1(\kappa)} \lesssim |\xi|_{H^1(\kappa)}\,.$$

Applying (3.1) with $\Psi(\hat{\xi}) := \hat{\Pi}_{\hat{\kappa}}\hat{\xi} = |\hat{\kappa}|^{-1}\int_{\hat{\kappa}} \hat{\xi}\,\mathrm{d}\hat{x}$ yields

$$\left\|\hat{\xi} - \hat{\Pi}_{\hat{\kappa}}\hat{\xi}\right\|_{L^q(\hat{\kappa})} \lesssim \left|\hat{\xi}\right|_{H^1(\hat{\kappa})}\,.$$

Scaling back, we obtain

$$\|\xi - \Pi_\kappa\xi\|_{L^q(\kappa)} \lesssim |\xi|_{H^1(\kappa)}\,,$$

and hence (3.8).

On writing $\mathcal{I}\xi = (\mathcal{I}\xi - \Pi_\kappa\xi) + \Pi_\kappa\xi$, combining the estimates (3.5) and (3.8), and summing over $\kappa \in \mathcal{T}$, we obtain

$$|\mathcal{I}\xi|^2_{H^1(\Omega,\mathcal{T})} \lesssim |\xi|^2_{H^1(\Omega,\mathcal{T})} + \sum_{e\in\mathcal{E}_{\mathrm{int}}} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\,\mathrm{d}s\right)^2. \tag{3.9}$$

By noting that $q = 2n/(n-2) > 2$, and thus

$$\left(\sum_{\kappa\in\mathcal{T}} |a_\kappa|^q\right)^{1/q} \leq \left(\sum_{\kappa\in\mathcal{T}} |a_\kappa|^2\right)^{\frac{1}{2}}, \tag{3.10}$$

we have, with $a_\kappa = \|\xi - \mathcal{I}\xi\|_{L^q(\kappa)}$,

$$\begin{aligned}\|\xi - \mathcal{I}\xi\|^2_{L^q(\Omega)} &= \left(\sum_{\kappa\in\mathcal{T}} \|\xi - \mathcal{I}\xi\|^q_{L^q(\kappa)}\right)^{2/q} \leq \sum_{\kappa\in\mathcal{T}} \|\xi - \mathcal{I}\xi\|^2_{L^q(\kappa)} \lesssim \sum_{\kappa\in\mathcal{T}} \|\xi - \Pi_\kappa\xi\|^2_{L^q(\kappa)} \\ &+ \sum_{\kappa\in\mathcal{T}} \|\mathcal{I}\xi - \Pi_\kappa\xi\|^2_{L^q(\kappa)} \lesssim |\xi|^2_{H^1(\Omega,\mathcal{T})} + \sum_{e\in\mathcal{E}_{\mathrm{int}}} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\,\mathrm{d}s\right)^2, \end{aligned} \tag{3.11}$$

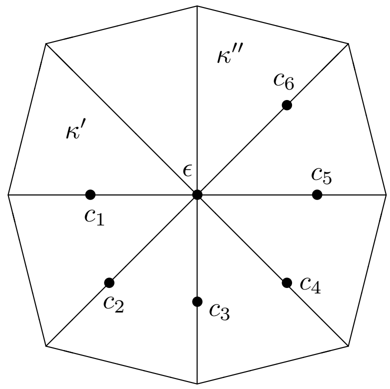

FIG. 3.1. *Sequence of c_j's*

where in the transition to the last inequality we used (3.8) and (3.6).

Now, let $W_{\mathcal{T}} \subset H^1(\Omega)$ be the Lagrange finite element space associated with $\mathcal{T}$, consisting of continuous piecewise polynomials on $\mathcal{T}$ of degree n, such that the shape functions of $V_{\mathcal{T}}$ are the shape functions of $W_{\mathcal{T}}$, and the nodal variables of $V_{\mathcal{T}}$ are the nodal variables of $W_{\mathcal{T}}$.

We denote the set of the centres of the sides of $\kappa \in \mathcal{T}$ by $\mathcal{C}(\kappa)$, the set of other nodes by $\mathcal{N}(\kappa)$, and define $\mathcal{C}(\mathcal{T}) := \bigcup_{\kappa\in\mathcal{T}} \mathcal{C}(\kappa)$ and $\mathcal{N}(\mathcal{T}) := \bigcup_{\kappa\in\mathcal{T}} \mathcal{N}(\kappa)$.

We define on the finite element spaces $V_{\mathcal{T}}$ and $W_{\mathcal{T}}$ the operators E and F by

$$(Ev)(\epsilon) = \frac{1}{|\Xi_\epsilon|} \sum_{\kappa\in\Xi_\epsilon} v_\kappa(\epsilon), \quad \text{for all} \quad \epsilon \in \mathcal{N}(\mathcal{T}) \cup \mathcal{C}(\mathcal{T}), \quad v \in V_{\mathcal{T}} \tag{3.12}$$

and

$$(Fw)(\epsilon) = w(\epsilon), \quad \text{for all} \quad \epsilon \in \mathcal{C}(\mathcal{T}), \quad w \in W_{\mathcal{T}}, \tag{3.13}$$

where $v_\kappa = v|_\kappa$, $\Xi_\epsilon = \{\kappa \in \mathcal{T} : \epsilon \in \partial\kappa\}$ is the set of elements sharing ϵ as a vertex, and $|\Xi_\epsilon|$ is the cardinal number of Ξ_ϵ, i.e., the number of elements in Ξ_ϵ. For $\epsilon \in \mathcal{C}(\mathcal{T})$ we have $(Ev)(\epsilon) = v(\epsilon)$ since v is continuous at the centres of the sides.

Let us state the following estimate for the operators E and F.

Lemma 3.1 *For the operators E and F defined by* (3.12) *and* (3.13)*, the following estimates hold:*

$$\|Ev - v\|^2_{L^q(\Omega)} \lesssim \sum_{\kappa\in\mathcal{T}} |v|^2_{H^1(\kappa)} \quad \textit{for all} \quad v \in V_{\mathcal{T}}, \tag{3.14}$$

$$\|Fw - w\|^2_{L^q(\Omega)} \lesssim \sum_{\kappa\in\mathcal{T}} |w|^2_{H^1(\kappa)} \quad \textit{for all} \quad w \in W_{\mathcal{T}}. \tag{3.15}$$

Proof. Assume that $\epsilon \in \mathcal{N}(\mathcal{T})$ and $\kappa', \kappa'' \in \Xi_\epsilon$. Then we can find a sequence $c_1, \ldots, c_m$ in $\mathcal{C}(\mathcal{T})$ so that $c_1 \in \partial\kappa'$, $c_m \in \partial\kappa''$, and $c_j, c_{j+1} \in \partial\kappa_j$, $\kappa_j \in \Xi_\epsilon$ (see Figure 3.1 for an example with triangular elements in $\mathbb{R}^2$ and $m = 6$). Note that $|\Xi_\epsilon|$ and thus m is bounded by a constant depending only on the shape–regularity of $\mathcal{T}$. Thus from the Cauchy–Schwarz inequality and the Mean Value Theorem we have, for any $\epsilon \in \mathcal{N}(\mathcal{T})$ and any $v \in V_{\mathcal{T}}$, that

$$[v_{\kappa'}(\epsilon) - v_{\kappa''}(\epsilon)]^2 \lesssim [v_{\kappa'}(\epsilon) - v_{\kappa'}(c_1)]^2 + \sum_{j=1}^{m-1} \left[v_{\kappa_j}(c_j) - v_{\kappa_j}(c_{j+1})\right]^2 + [v_{\kappa''}(c_m) - v_{\kappa''}(\epsilon)]^2$$
$$\lesssim \sum_{\kappa''' \in \Xi_\epsilon} |\kappa'''|^{(2/n)-1} |v|^2_{H^1(\kappa''')} .$$

Using this estimate together with (3.12) and the Cauchy–Schwarz inequality, for any $\kappa \in \Xi_\epsilon$ and any $\epsilon \in \mathcal{N}(\kappa)$, we have

$$[(Ev - v_\kappa)(\epsilon)]^2 \lesssim \sum_{\kappa''' \in \Xi_\epsilon} |\kappa'''|^{(2/n)-1} |v|^2_{H^1(\kappa''')} \quad \text{for all} \quad v \in V_{\mathcal{T}}. \tag{3.16}$$

From this estimate, for $\kappa \in \mathcal{T}$ it follows that

$$\|Ev - v\|^2_{L^2(\kappa)} \approx |\kappa| \sum_{\epsilon \in \mathcal{N}(\kappa) \cup \mathcal{C}(\kappa)} [(Ev - v_\kappa)(\epsilon)]^2 = |\kappa| \sum_{\epsilon \in \mathcal{N}(\kappa)} [(Ev - v_\kappa)(\epsilon)]^2$$
$$\lesssim \sum_{\epsilon \in \mathcal{N}(\kappa)} \sum_{\kappa''' \in \Xi_\epsilon} |\kappa'''|^{2/n} |v|^2_{H^1(\kappa''')}$$

for all $v \in V_{\mathcal{T}}$, where we used the fact that

$$|\kappa| \approx |\kappa'''| \quad \text{for} \quad \kappa''' \in \Xi_\epsilon \quad \text{and} \quad \epsilon \in \mathcal{N}(\kappa). \tag{3.17}$$

As $\|Ev - v\|_{L^q(\kappa)} \le |\kappa|^{1/q - 1/2} \|Ev - v\|_{L^2(\kappa)}$ and $1/q - 1/2 = -1/n$, from the last estimate we obtain

$$\|Ev - v\|^2_{L^q(\kappa)} \lesssim |\kappa|^{-2/n} \sum_{\epsilon \in \mathcal{N}(\kappa)} \sum_{\kappa''' \in \Xi_\epsilon} |\kappa'''|^{2/n} |v|^2_{H^1(\kappa''')} ,$$

and using (3.17) again, we obtain

$$\|Ev - v\|^2_{L^q(\kappa)} \lesssim \sum_{\epsilon \in \mathcal{N}(\kappa)} \sum_{\kappa''' \in \Xi_\epsilon} |v|^2_{H^1(\kappa''')} . \tag{3.18}$$

After summation over $\kappa \in \mathcal{T}$ and using (3.10) with $q = 2n/(n-2) > 2$, we get (3.14).

On each $\kappa \in \mathcal{T}$, Fw is a linear nodal interpolant of w with the nodes placed at the centres of the sides of κ. From standard interpolation and inverse estimates (see [5, 6]) it follows that

$$\|Fw - w\|^2_{L^2(\kappa)} \lesssim |\kappa|^{4/n} |w|^2_{H^2(\kappa)} \lesssim |\kappa|^{2/n} |w|^2_{H^1(\kappa)} \quad \text{for all} \quad w \in W_{\mathcal{T}},$$

and thus by an inverse estimate we have

$$\|Fw - w\|^2_{L^q(\kappa)} \le |\kappa|^{-2/n} \|Fw - w\|^2_{L^2(\kappa)} \lesssim |w|^2_{H^1(\kappa)} .$$

Summation over $\kappa \in \mathcal{T}$ and (3.10) with $a_\kappa = \|Fw - w\|_{L^q(\kappa)}$ give (3.15). $\square$

Corollary 3.2 *We have that*

$$\|Ev\|_{L^q(\Omega)} \approx \|v\|_{L^q(\Omega)} \quad \textit{for all} \quad v \in V_{\mathcal{T}}, \tag{3.19}$$

and

$$|Ev|_{H^1(\Omega,\mathcal{T})} \approx |v|_{H^1(\Omega,\mathcal{T})} \quad \textit{for all} \quad v \in V_{\mathcal{T}}. \tag{3.20}$$

Proof. To prove (3.19), we note that from (3.18) using Hölder's inequality for finite sums, we have

$$\begin{aligned}\|Ev\|_{L^q(\Omega)} &\le \|Ev - v\|_{L^q(\Omega)} + \|v\|_{L^q(\Omega)} = \left(\sum_{\kappa\in\mathcal{T}} \|Ev - v\|^q_{L^q(\kappa)}\right)^{1/q} + \|v\|_{L^q(\Omega)} \\ &\lesssim \left(\sum_{\kappa\in\mathcal{T}} |v|^q_{H^1(\kappa)}\right)^{1/q} + \|v\|_{L^q(\Omega)} \lesssim \left(\sum_{\kappa\in\mathcal{T}} \|v\|^q_{L^q(\kappa)}\right)^{1/q} + \|v\|_{L^q(\Omega)} \lesssim \|v\|_{L^q(\Omega)},\end{aligned} \tag{3.21}$$

where we used (3.10) and an inverse inequality with $q = 2n/(n-2)$.

Similarly, by a completely identical argument with E replaced by F and $v \in V_{\mathcal{T}}$ by $w \in W_{\mathcal{T}}$, we obtain

$$\|Fw - w\|_{L^q(\Omega)} \lesssim \|w\|_{L^q(\Omega)} \quad \text{for all} \quad w \in W_{\mathcal{T}},$$

and hence, as above,

$$\|Fw\|_{L^q(\Omega)} \lesssim \|w\|_{L^q(\Omega)} \quad \text{for all} \quad w \in W_{\mathcal{T}}. \tag{3.22}$$

From the definitions (3.12) and (3.13)

$$F(Ev) = v \quad \text{for all} \quad v \in V_{\mathcal{T}}, \tag{3.23}$$

and thus by (3.22) and (3.21)

$$\|v\|_{L^q(\Omega)} = \|F(Ev)\|_{L^q(\Omega)} \lesssim \|Ev\|_{L^q(\Omega)} \lesssim \|v\|_{L^q(\Omega)} \quad \text{for all} \quad v \in V_{\mathcal{T}}, \tag{3.24}$$

hence (3.19).

From a standard inverse estimate and (3.14), for all $v \in V_{\mathcal{T}}$ we have

$$|Ev|_{H^1(\Omega)} \le |Ev - v|_{H^1(\Omega)} + |v|_{H^1(\Omega)} \lesssim \left(\sum_{\kappa\in\mathcal{T}} \|Ev - v\|^2_{L^q(\kappa)}\right)^{\frac{1}{2}} \lesssim |v|_{H^1(\Omega,\mathcal{T})}. \tag{3.25}$$

Similarly, from (3.15),

$$|Fw|_{H^1(\Omega,\mathcal{T})} \lesssim |w|_{H^1(\Omega)} \quad \text{for all} \quad w \in W_{\mathcal{T}}. \tag{3.26}$$

The estimate (3.20) follows from (3.23), (3.26) and (3.25). □

3.3. Sobolev–Poincaré Inequalities for Nonconforming $\mathcal{P}_1$ Finite Element. We proved in Section 3.1 that, for $1 \leq q \leq 2n/(n-2)$, $n \geq 3$, there exists a positive constant C such that

$$\|\xi\|_{L^q(\Omega)} \leq C \left\{ |\xi|_{H^1(\Omega)} + |\Psi(\xi)| \right\} \quad \text{for all} \quad \xi \in H^1(\Omega), \tag{3.27}$$

where Ψ is a bounded linear functional on $H^1(\Omega)$ with the property that $\Psi(1) = 1$.

Let us now prove the Sobolev–Poincaré inequality for functions in a nonconforming $\mathcal{P}_1$ finite element space.

Theorem 3.3 *Let Ψ be a bounded linear functional on $H^1(\Omega)$, $\Psi(1) = 1$, such that*

$$|\Psi(Ev - v)| \lesssim |v|_{H^1(\Omega,\mathcal{T})} \quad \textit{for all} \quad v \in V_{\mathcal{T}}, \tag{3.28}$$

where $E : V_{\mathcal{T}} \to W_{\mathcal{T}}$ is defined by (3.12)*. Then, there exists a continuous function $\varsigma : \mathbb{R}_+ \to \mathbb{R}_+$, independent of $\mathcal{T}$, such that*

$$\|v\|_{L^q(\Omega)} \leq \varsigma(\theta_{\mathcal{T}}) \left\{ |v|_{H^1(\Omega,\mathcal{T})} + |\Psi(v)| \right\} \quad \textit{for all} \quad v \in V_{\mathcal{T}}, \tag{3.29}$$

where $q = 2n/(n-2)$ and $\theta_{\mathcal{T}}$ is the minimum angle in $\mathcal{T}$.

Proof. Combining (3.19), (3.27) and (3.20) yields, for all $v \in V_{\mathcal{T}}$,

$$\begin{aligned}\|v\|_{L^q(\Omega)} \approx \|Ev\|_{L^q(\Omega)} &\lesssim |Ev|_{H^1(\Omega)} + |\Psi(Ev)| \\ &\leq |v|_{H^1(\Omega,\mathcal{T})} + |\Psi(Ev - v)| + |\Psi(v)| \lesssim |v|_{H^1(\Omega,\mathcal{T})} + |\Psi(v)|,\end{aligned}$$

where in the transition to the last inequality we used the hypothesis (3.28). □

Let us now construct some examples of seminorms which satisfy the conditions of Theorem 3.3.

Example 3.4 By the trace inequality and the Sobolev embedding theorem, we have

$$H^1(\Omega) \hookrightarrow H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^r(\partial\Omega),$$

for

$$\frac{1}{2} - \frac{n-1}{2} = 0 - \frac{n-1}{r},$$

i.e., with

$$r = \frac{2(n-1)}{n-2} \quad \text{and} \quad r' = 2\left(1 - \frac{1}{n}\right); \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Let $\psi \in L^{2(1-\frac{1}{n})}(\partial\Omega)$, such that $\int_{\partial\Omega} \psi \,\mathrm{d}s \neq 0$ and define

$$\Psi_1(\xi) = \left(\int_{\partial\Omega} \psi \,\mathrm{d}s \right)^{-1} \int_{\partial\Omega} \psi\xi \,\mathrm{d}s, \quad \xi \in L^r(\partial\Omega). \tag{3.30}$$

Thus $\Psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded linear functional on $H^1(\Omega)$ and $\Psi_1(1) = 1$.
Let us check (3.28). With $r = 2(n-1)/(n-2) > 2$ and $r' = 2(1-1/n)$, for all $v \in V_{\mathcal{T}}$ we have

$$|\Psi_1(Ev - v)| \leq \|\psi\|_{L^{2\left(1-\frac{1}{n}\right)}(\partial\Omega)} \|Ev - v\|_{L^{\frac{2(n-1)}{n-2}}(\partial\Omega)}$$

$$= \|\psi\|_{L^{2\left(1-\frac{1}{n}\right)}(\partial\Omega)} \left(\sum_{e \in \mathcal{E}_\partial} \|Ev - v\|^r_{L^r(e)} \right)^{1/r}.$$

By subsequently using (3.10) and the inverse inequality to switch from the L^r–norm to the L^2–norm in $(n-1)$ dimensions, we obtain

$$\begin{aligned}
\left(\sum_{e \in \mathcal{E}_\partial} \|Ev - v\|^r_{L^r(e)} \right)^{1/r} &\leq \left(\sum_{e \in \mathcal{E}_\partial} \|Ev - v\|^2_{L^r(e)} \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{e \in \mathcal{E}_\partial} \|Ev - v\|^2_{L^2(e)} \left\{ (\mathrm{diam}_{n-1}(e))^{\frac{n-1}{r} - \frac{n-1}{2}} \right\}^2 \right)^{\frac{1}{2}} \\
&= C \left(\sum_{e \in \mathcal{E}_\partial} \|Ev - v\|^2_{L^2(e)} |e|^{\left(\frac{1}{r} - \frac{1}{2}\right)\cdot 2} \right)^{\frac{1}{2}} \\
&= C \left(\sum_{e \in \mathcal{E}_\partial} \|Ev - v\|^2_{L^2(e)} |e|^{\frac{2}{r} - 1} \right)^{\frac{1}{2}}.
\end{aligned}$$

Let $\mathcal{N}(e)$ denote the set of the nodes on $\overline{e}$ excluding the centre of e, and let κ_e denote the member of $\mathcal{T}$ whose boundary contains e. Then, as

$$\frac{2}{r} - 1 = \frac{n-2}{n-1} - 1 = -\frac{1}{n-1},$$

we have (by the previous estimates)

$$\begin{aligned}
\left(\sum_{e \in \mathcal{E}_\partial} \|Ev - v\|^r_{L^r(e)} \right)^{1/r} &\leq C \left(\sum_{e \in \mathcal{E}_\partial} \|Ev - v\|^2_{L^2(e)} |e|^{\frac{2}{r} - 1} \right)^{\frac{1}{2}} \\
&\leq C \left\{ \sum_{e \in \mathcal{E}_\partial} |e|^{1 - \frac{1}{n-1}} \left(\sum_{\epsilon \in \mathcal{N}(e)} |(Ev - v_{\kappa_e})(\epsilon)| \right)^2 \right\}^{\frac{1}{2}}.
\end{aligned}$$

By noting that $\mathcal{N}(e)$ is a finite set whose cardinality is uniformly bounded (which follows by the assumed shape–regularity of mesh), by the Cauchy–Schwarz inequality applied to the finite sum $\sum_{\epsilon\in\mathcal{N}(e)}\dots$ and (3.16) we have

$$\left(\sum_{e\in\mathcal{E}_\partial}\|Ev-v\|^r_{L^r(e)}\right)^{1/r} \le C\left\{\sum_{e\in\mathcal{E}_\partial}|e|^{1-\frac{1}{n-1}}\left(\sum_{\epsilon\in\mathcal{N}(e)}|(Ev-v_{\kappa_e})(\epsilon)|\right)^2\right\}^{\frac12}$$

$$\le C\left(\sum_{e\in\mathcal{E}_\partial}|e|^{1-\frac{1}{n-1}}\sum_{\epsilon\in\mathcal{N}(e)}|(Ev-v_{\kappa_e})(\epsilon)|^2\right)^{\frac12}$$

$$\le C\left(\sum_{e\in\mathcal{E}_\partial}|e|^{1-\frac{1}{n-1}}\sum_{\epsilon\in\mathcal{N}(e)}\sum_{\kappa'''\in\Xi_\epsilon}|\kappa'''|^{\frac{2}{n}-1}|v|^2_{H^1(\kappa''')}\right)^{\frac12}.$$

Finally, we note that $|e|^{1-\frac{1}{n-1}}=|e|^{\frac{n-2}{n-1}}\approx|\kappa|^{\frac{n-2}{n}}$, and that $|\kappa'''|^{\frac{2}{n}-1}=|\kappa'''|^{\frac{2-n}{n}}\approx|\kappa|^{\frac{2-n}{n}}$; thus, from the previous estimate, we obtain

$$\left(\sum_{e\in\mathcal{E}_\partial}\|Ev-v\|^r_{L^r(e)}\right)^{1/r} \le C\left(\sum_{e\in\mathcal{E}_\partial}|e|^{1-\frac{1}{n-1}}\sum_{\epsilon\in\mathcal{N}(e)}\sum_{\kappa'''\in\Xi_\epsilon}|\kappa'''|^{\frac{2}{n}-1}|v|^2_{H^1(\kappa''')}\right)^{\frac12}$$

$$\lesssim\left(\sum_{\substack{\kappa\in\mathcal{T}\\|\partial\kappa\cap\partial\Omega|>0}}|v|^2_{H^1(\kappa)}\right)^{\frac12},$$

which completes the proof that Ψ_1 satisfies (3.28). □

Example 3.5 Let $\psi\in L^{q'}(\Omega)$, $q'=2n/(n+2)$, such that $\int_\Omega\psi\,\mathrm{d}x\neq0$. Define

$$\Psi_2(\xi)=\left(\int_\Omega\psi\,\mathrm{d}x\right)^{-1}\int_\Omega\psi\xi\,\mathrm{d}x. \tag{3.31}$$

Clearly, $\Psi_2:\mathbb{R}_+\to\mathbb{R}_+$ is a bounded linear functional on $H^1(\Omega)$ and $\Psi_2(1)=1$. Moreover, by (3.14) we have, for all $v\in V_\mathcal{T}$,

$$|\Psi_2(v-Ev)|\lesssim\|v-Ev\|_{L^q(\Omega)}\|\psi\|_{L^{q'}(\Omega)}\lesssim\left(\sum_{\kappa\in\mathcal{T}}|v|^2_{H^1(\kappa)}\right)^{\frac12}=|v|_{H^1(\Omega,\mathcal{T})},$$

and thus Φ_2 satisfies the conditions of Theorem 3.3. □

Example 3.6 Let us split the boundary $\partial\Omega$ into the finite number of parts $\Gamma_1,\dots,\Gamma_m$, such that $\bigcup_{i=1,\dots,m}\overline{\Gamma}_i=\partial\Omega$.

Let $\psi\in L^{2\left(1-\frac1n\right)}(\partial\Omega)$, such that $\int_{\partial\Omega}\psi\,\mathrm{d}s\neq0$, and such that for some index set $\mathfrak{I}$ with $|\mathfrak{I}|<m$, we have $\psi\equiv0$ on Γ_i, $i\in\mathfrak{I}$. We define

$$\Psi_3(\xi)=\left(\int_{\partial\Omega}\psi\,\mathrm{d}s\right)^{-1}\int_{\partial\Omega}\psi\xi\,\mathrm{d}s. \tag{3.32}$$

Clearly, this example is a special case of Example 3.4 and therefore it also satisfies the conditions of Theorem 3.3. □

3.4. Sobolev–Poincaré Inequalities for Broken Sobolev Spaces. Suppose that

$$|\Psi(\mathcal{I}\xi-\xi)|^2 \lesssim |\xi|^2_{H^1(\Omega,\mathcal{T})} + \sum_{e\in\mathcal{E}_{\mathrm{int}}} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\,\mathrm{d}s\right)^2. \tag{3.33}$$

This condition is satisfied by Ψ_1 (and thus Ψ_3) and Ψ_2 above, if $\psi\in L^{2\left(1-\frac{1}{n}\right)}(\partial\Omega)$ and $\psi\in L^{\frac{2n}{n+2}}(\Omega)$, respectively, with $\int_{\partial\Omega}\psi\,\mathrm{d}s\neq 0$ and $\int_\Omega\psi\,\mathrm{d}s\neq 0$, respectively.

Let us start with the proof of (3.33) for Ψ_1. With $r=2(n-1)/(n-2)$, we have

$$\begin{aligned}
|\Psi_1(\mathcal{I}\xi-\xi)| \le C\,\|\mathcal{I}\xi-\xi\|_{L^r(\partial\Omega)} &= C\left(\sum_{e\subset\partial\Omega}\|\mathcal{I}\xi-\xi\|^r_{L^r(e)}\right)^{1/r}\\
&\le C\left\{\sum_{\substack{\kappa\in\mathcal{T}\\|\partial\kappa\cap\partial\Omega|>0}}\left(h_\kappa^{-1}\,\|\mathcal{I}\xi-\xi\|_{L^2(\kappa)}+\|\nabla(\mathcal{I}\xi-\xi)\|_{L^2(\kappa)}\right)^r\right\}^{1/r}\\
&\le C\left\{\sum_{\substack{\kappa\in\mathcal{T}\\|\partial\kappa\cap\partial\Omega|>0}}\left(h_\kappa^{-1}\,\|\mathcal{I}\xi-\xi\|_{L^2(\kappa)}+\|\nabla(\mathcal{I}\xi-\xi)\|_{L^2(\kappa)}\right)^2\right\}^{\frac12},
\end{aligned}$$

where we used the trace inequality

$$\|w\|_{L^r(e)} \le C\left(h_\kappa^{-1}\,\|w\|_{L^2(\kappa)}+\|\nabla w\|_{L^2(\kappa)}\right) \quad \text{for all} \quad w\in H^1(\kappa),$$

with $w=\mathcal{I}\xi-\xi$, and (3.10).

Now,

$$\|\mathcal{I}\xi-\xi\|_{L^2(\kappa)} \le \|\mathcal{I}\xi-\Pi_\kappa\xi\|_{L^2(\kappa)}+\|\Pi_\kappa\xi-\xi\|_{L^2(\kappa)}\,.$$

As in (3.5),

$$\begin{aligned}
\|\mathcal{I}\xi-\Pi_\kappa\xi\|^2_{L^2(\kappa)} &\lesssim |\kappa|\sum_{e\subset\partial\kappa}|(\mathcal{I}\xi-\Pi_\kappa\xi)(c_e)|^2 \lesssim |\kappa|\sum_{e\subset\partial\kappa\setminus\partial\Omega}|e|^{-2}\left(\int_e[\xi]\,\mathrm{d}s\right)^2\\
&= |\kappa|^{\frac{2}{n}}\cdot|\kappa|^{1-\frac{2}{n}}\sum_{e\subset\partial\kappa\setminus\partial\Omega}|e|^{-2}\left(\int_e[\xi]\,\mathrm{d}s\right)^2 \lesssim h_\kappa^2\sum_{e\subset\partial\kappa\setminus\partial\Omega}|e|^{\frac{n}{1-n}}\left(\int_e[\xi]\,\mathrm{d}s\right)^2,
\end{aligned}$$

and thus

$$h_\kappa^{-1}\,\|\mathcal{I}\xi-\Pi_\kappa\xi\|_{L^2(\kappa)} \lesssim \left\{\sum_{e\subset\partial\kappa\setminus\partial\Omega}|e|^{\frac{n}{1-n}}\left(\int_e[\xi]\,\mathrm{d}s\right)^2\right\}^{\frac12}.$$

Identically to the proof of (3.8), we conclude that

$$\|\Pi_\kappa\xi-\xi\|_{L^2(\kappa)} \lesssim h_\kappa\,|\xi|_{H^1(\kappa)}\,,$$

and therefore, by combining these estimates, we have

$$h_\kappa^{-1}\,\|\mathcal{I}\xi-\xi\|_{L^2(\kappa)} \lesssim |\xi|_{H^1(\kappa)} + \left\{\sum_{e\subset\partial\kappa\setminus\partial\Omega} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\,\mathrm{d}s\right)^2\right\}^{\frac{1}{2}}.$$

Analogously, writing $\mathcal{I}\xi-\xi=(\mathcal{I}\xi-\Pi\xi)+\Pi\xi+(-\xi)$ and using (3.5), (3.8) we deduce that $|\mathcal{I}\xi-\xi|_{H^1(\kappa)}$ is also bounded by the same expression.

Hence,

$$\begin{aligned}
|\Psi_1(\mathcal{I}\xi-\xi)|^2 &\lesssim \sum_{\substack{\kappa\in\mathcal{T}\\ |\partial\kappa\cap\partial\Omega|>0}} \left(h_\kappa^{-1}\,\|\mathcal{I}\xi-\xi\|_{L^2(\kappa)} + \|\nabla\xi\|_{L^2(\kappa)}\right)^2\\
&\lesssim \sum_{\substack{\kappa\in\mathcal{T}\\ |\partial\kappa\cap\partial\Omega|>0}} \left\{|\xi|^2_{H^1(\kappa)} + \sum_{e\subset\partial\kappa\setminus\partial\Omega} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\,\mathrm{d}s\right)\right\}\\
&\lesssim |\xi|^2_{H^1(\Omega,\mathcal{T})} + \sum_{e\in\mathcal{E}_{\rm int}} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\,\mathrm{d}s\right)^2,
\end{aligned}$$

as required.

Next, we prove that Ψ_2, defined above, also satisfies the condition (3.33). With $r=2n/(n-2)$ we have

$$\begin{aligned}
|\Psi_2(\mathcal{I}\xi-\xi)| \le C\,\|\mathcal{I}\xi-\xi\|_{L^r(\Omega)} &\lesssim \|\mathcal{I}\xi-\Pi_\kappa\xi\|_{L^r(\Omega)} + \|\Pi_\kappa\xi-\xi\|_{L^r(\Omega)}\\
&\approx \left(\sum_{\kappa\in\mathcal{T}} \|\mathcal{I}\xi-\Pi_\kappa\xi\|^r_{L^r(\kappa)}\right)^{1/r} + \left(\sum_{\kappa\in\mathcal{T}} \|\Pi_\kappa\xi-\xi\|^r_{L^r(\kappa)}\right)^{1/r}\\
&\lesssim \left(\sum_{\kappa\in\mathcal{T}} \|\mathcal{I}\xi-\Pi_\kappa\xi\|^2_{L^r(\kappa)}\right)^{\frac{1}{2}} + \left(\sum_{\kappa\in\mathcal{T}} \|\Pi_\kappa\xi-\xi\|^2_{L^r(\kappa)}\right)^{\frac{1}{2}}\\
&\lesssim \left\{\sum_{\kappa\in\mathcal{T}}\sum_{e\subset\partial\kappa\setminus\partial\Omega} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\,\mathrm{d}s\right)^2\right\}^{\frac{1}{2}} + \left(\sum_{\kappa\in\mathcal{T}} |\xi|^2_{H^1(\kappa)}\right)^{\frac{1}{2}},
\end{aligned}$$

where we used (3.10), (3.6), and (3.8). Hence,

$$|\Psi_2(\mathcal{I}\xi-\xi)|^2 \lesssim \sum_{e\in\mathcal{E}_{\rm int}} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\,\mathrm{d}s\right)^2 + |\xi|^2_{H^1(\Omega,\mathcal{T})}.$$

Theorem 3.7 *Let Ψ be a bounded linear functional on $H^1(\Omega,\mathcal{T})$, $\Psi(1)=1$, which satisfies conditions* (3.28) *and* (3.33)*. Then, there exists a continuous function $\varsigma:\mathbb{R}_+\to\mathbb{R}_+$, independent of $\mathcal{T}$, such that*

$$\|\xi\|^2_{L^q(\Omega)} \le \varsigma(\theta_{\mathcal{T}})\left\{|\xi|^2_{H^1(\Omega,\mathcal{T})} + \sum_{e\in\mathcal{E}_{\rm int}} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\,\mathrm{d}s\right)^2 + |\Psi(\xi)|^2\right\}, \qquad (3.34)$$

for all $\xi\in H^1(\Omega,\mathcal{T})$, $q=2n/(n-2)$, $n\ge 3$.

Proof. By (3.27), (3.11), (3.9), and (3.33)

$$\begin{aligned}\|\xi\|_{L^q(\Omega)}^2 &\lesssim \|\xi - \mathcal{I}\xi\|_{L^q(\Omega)}^2 + \|\mathcal{I}\xi\|_{L^q(\Omega)}^2 \lesssim \|\xi - \mathcal{I}\xi\|_{L^q(\Omega)}^2 + \left(|\mathcal{I}\xi|_{H^1(\Omega)}^2 + |\Psi(\mathcal{I}\xi)|^2\right) \\ &\lesssim \|\xi - \mathcal{I}\xi\|_{L^q(\Omega)}^2 + |\mathcal{I}\xi|_{H^1(\Omega)}^2 + |\Psi(\mathcal{I}\xi - \xi)|^2 + |\Psi(\xi)|^2 \\ &\lesssim |\xi|_{H^1(\Omega,\mathcal{T})}^2 + \sum_{e\in\mathcal{E}_{\rm int}} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\, {\rm d}s\right)^2 + |\Psi(\xi)|^2,\end{aligned}$$

as required. □

Remark 3.8 We make some remarks about this theorem.

1. For $1 \le q \le 2n/(n-2)$, $n \ge 3$, the inequality follows from the one for $q = 2n/(n-2)$ via Hölder's inequality.
2. For $n = 2$, the argument is identical and (3.34) holds for all $q \in [1,\infty)$.
3. We have proved the inequality in the case of $\mathcal{T}$ being a simplical triangulation of Ω. The extension to the case of a general partition of Ω is discussed in Sections 6 and 7 of [4]. Proceeding in exactly the same fashion as in [4], we conclude that the inequality (3.34) holds for general partitions.
4. Let us use the Cauchy–Schwarz inequality on the term of (3.34) containing the integral of jumps of ξ over the face e. We obtain

$$\sum_{e\in\mathcal{E}_{\rm int}} |e|^{\frac{n}{1-n}} \left(\int_e [\xi]\, {\rm d}s\right)^2 \le \sum_{e\in\mathcal{E}_{\rm int}} |e|^{\frac{1}{1-n}} \int_e [\xi]^2\, {\rm d}s.$$

By noting that $|e|^{-1} = h_e^{-(n-1)}$, where $h_e \equiv \operatorname{diam} e$, we obtain

$$\|\xi\|_{L^q(\Omega)}^2 \le \varsigma(\theta_{\mathcal{T}}) \left\{ |\xi|_{H^1(\Omega,\mathcal{T})}^2 + \sum_{e\in\mathcal{E}_{\rm int}} h_e^{-1} \int_e [\xi]^2\, {\rm d}s + |\Psi(\xi)|^2 \right\}.$$

□

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