# $HP$-VERSION DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS FOR SEMILINEAR PARABOLIC PROBLEMS

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**Abstract.** We consider the $hp$–version interior penalty discontinuous Galerkin finite element method ($hp$–DGFEM) for semilinear parabolic equations with mixed Dirichlet and Neumann boundary conditions. Our main concern is the error analysis of the $hp$–DGFEM on shape–regular spatial meshes. We derive error bounds under various hypotheses on the regularity of the solution, for both the symmetric and non–symmetric versions of DGFEM.

**Key words.** $hp$–finite element methods, discontinuous Galerkin methods, semilinear parabolic PDEs

**AMS subject classifications.** 65N12, 65N15, 65N30

**1. Introduction.** Discontinuous Galerkin finite element methods (DGFEMs) were introduced in the early 1970's for the numerical solution of first–order hyperbolic problems (see [30, 26, 24, 23, 16, 17, 18, 19, 31, 32]). Simultaneously, but independently, they were proposed as non–standard schemes for the numerical approximation of second–order elliptic equations [29, 36, 4]. In recent years there has been renewed interest in discontinuous Galerkin methods due to their favourable properties, such as a high degree of locality, stability in the absence of streamline–diffusion stabilisation for convection–dominated diffusion problems [21], and the flexibility of locally varying the polynomial degree in $hp$–version approximations, since no pointwise continuity requirements are imposed at the element interfaces. Much attention has been paid to the analysis of DG methods applied to non–linear hyperbolic equations and hyperbolic systems [20, 13, 14], several other types of non–linear equations (including the Hamilton–Jacobi equation [22], the non–linear Schrödinger equation [25], and other non–linear problems [15]). The analysis of the spatial discretisation of non–linear parabolic problems by the Interior Penalty type of the DGFEM (see [4]) has been pursued by Rivière & Wheeler in [33], where the non–linearities were assumed to be uniformly Lipschitz continuous with respect to the unknown solution. The resulting error bounds were based on the projection operator described in [34], and were not $p$–optimal in the $H^1$–norm.

In this work we shall be concerned with the error analysis of the $hp$–version interior penalty discontinuous Galerkin finite element method ($hp$–DGFEM), for an initial–boundary value problem for a semilinear PDE of parabolic type in $n \geq 2$ spatial dimensions on shape–regular quadrilateral meshes (see (2.1) below). Here, we consider only the spatial discretisation of the problem, leaving the choice of time–stepping techniques and their analysis for a future work. We shall suppose that the non–linearity satisfies the local Lipschitz condition (2.2).

The paper is structured as follows. In Section 2 we state the model problem, followed by the definition of function spaces used throughout our work (Section 3). Next, we state the broken weak formulation (Section 4). After selecting the finite element space that will be used for the discretisation of the model problem in space

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(Section 5), we state the *hp*–DGFEM (Section 6). Section 7 contains the approximation theory, required in the subsequent error analysis. The error analysis of the *hp*–DGFEM for semilinear parabolic equations is discussed in Section 8. We begin by establishing the local Lipschitz continuity of the mapping $f : L^q(\Omega) \to L^2(\Omega)$. Section 8.1 contains the error analysis of the non–symmetric version of the interior penalty *hp*–DGFEM: we prove an $h$–optimal and $p$–suboptimal (by half an order of $p$) *a priori* error bound. The bound indicates that the presence of the non–linearity, obeying condition (2.2), does not degrade the accuracy of the *hp*–DGFEM in the $H^1$–norm. Section 8.2 is concerned with the derivation of the $L^2$–norm error bounds in the case of the symmetric version of the interior penalty *hp*–DGFEM. For this purpose, we first derive error bounds on the broken elliptic projector (Section 8.2.1) defined by the symmetric version of the *hp*–DGFEM. Section 8.2.2 is concerned with the error analysis and derivation of the *a priori* error bound for the $L^2$–norm, and is largely based on the techniques developed in the analysis of the non–symmetric version of the *hp*–DGFEM. Section 9 contains some final comments on the results in this work.

**2. Model Problem.** Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$, $n \geq 2$, with a sufficiently smooth boundary $\partial\Omega$. We consider the semilinear partial differential equation of parabolic type

$$\dot{u} - \Delta u = f(u) \quad \text{in} \quad \Omega \times (0, T], \tag{2.1}$$

where $\dot{u} \equiv \partial u/\partial t$, $T > 0$, and $f \in C^1(\mathbb{R})$.

We also assume the following growth–condition on the function $f$:

$$|f(v) - f(w)| \leq C_{\mathrm{g}}(1 + |v| + |w|)^{\alpha}\, |v - w| \quad \text{for all} \quad v,\, w \in \mathbb{R}, \tag{2.2}$$

where $C_{\mathrm{g}} > 0$ and $\alpha > 0$.

Upon decomposing the boundary $\partial\Omega$ into two parts, $\Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{N}}$, so that $\overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}} = \partial\Omega$, we impose Dirichlet and Neumann boundary conditions respectively:

$$\begin{aligned} u &= g_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}} \times [0, T], \\ \nabla u \cdot \mathbf{n} &= g_{\mathrm{N}} \quad \text{on} \quad \Gamma_{\mathrm{N}} \times [0, T], \end{aligned} \tag{2.3}$$

where $\mathbf{n} = \mathbf{n}(x)$ denotes the unit outward normal vector to $\partial\Omega$ at $x \in \partial\Omega$.

Finally, we impose the initial condition

$$u = u_0 \quad \text{on} \quad \overline{\Omega} \times \{0\}\,, \tag{2.4}$$

where $u_0 = u_0(x)$.

As the solution of this problem may exhibit blow–up in finite time, we shall assume that, for the potential blow–up time $T^\star \in (0, \infty]$, the time interval $[0, T]$ on which the problem is defined is bounded by the blow–up time, i.e., $T < T^\star$.

**3. Function Spaces.** Since *hp*-DGFEM is a non–conforming method, it is necessary to introduce Sobolev spaces defined on a subdivision $\mathcal{T}$ of the domain $\Omega$; we call such 'piecewise Sobolev spaces' *broken Sobolev spaces*.

A *subdivision* $\mathcal{T}$ of the domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a family of disjoint open sets (elements) $\kappa$ such that $\overline{\Omega} = \cup_{\kappa\in\mathcal{T}}\overline{\kappa}$. Before we define broken Sobolev spaces, we shall introduce the basic principles of constructing a subdivision $\mathcal{T}$.

Let $\mathcal{T}$ be a subdivision of the polyhedral domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, into disjoint open polyhedra (elements) $\kappa$ such that $\overline{\Omega} = \cup_{\kappa\in\mathcal{T}}\overline{\kappa}$, where $\mathcal{T}$ is regular or 1–irregular, i.e., each face of $\kappa$ has at most one hanging node. We assume that the family of subdivisions $\mathcal{T}$ is shape–regular (see pages 61, 113, and Remark 2.2 on page 114 in [10]), and require each $\kappa \in \mathcal{T}$ to be an affine image of a fixed master element $\hat{\kappa}$, i.e., $\kappa = F_\kappa(\hat{\kappa})$ for all $\kappa \in \mathcal{T}$, where $\hat{\kappa}$ is either the open unit simplex or the open unit hypercube in $\mathbb{R}^n$, $n \geq 2$.

**Definition 3.1** *The broken Sobolev space of composite order* $\mathbf{s} = \{s_\kappa : \kappa \in \mathcal{T}\}$ *on a subdivision* $\mathcal{T}$ *of* $\Omega$ *is defined as*

$$W_p^{\mathbf{s}}(\Omega, \mathcal{T}) := \left\{ u \in L^p(\Omega) : u|_\kappa \in W_p^{s_\kappa}(\kappa) \text{ for all } \kappa \in \mathcal{T} \right\},$$

$s_\kappa$ *being the local Sobolev index on the element* $\kappa$.

*The associated broken norm and seminorm are defined as*

$$\|u\|_{W_p^{\mathbf{s}}(\Omega,\mathcal{T})} := \left( \sum_{\kappa \in \mathcal{T}} \|u\|^p_{W_p^{s_\kappa}(\kappa)} \right)^{1/p}, \quad |u|_{W_p^{\mathbf{s}}(\Omega,\mathcal{T})} := \left( \sum_{\kappa \in \mathcal{T}} |u|^p_{W_p^{s_\kappa}(\kappa)} \right)^{1/p}.$$

*When* $s_\kappa = s$, *we write* $W_p^s(\Omega, \mathcal{T})$, *and for* $p = 2$ *we denote* $H^{\mathbf{s}} \equiv W_2^{\mathbf{s}}$.

As our main concern are time–dependent problems, we need to introduce Sobolev spaces comprising functions that map a closed bounded subinterval of $\mathbb{R}$, with the interval in question thought of as a time interval, into Banach spaces.

For further reference, let $X$ denote a Banach space, with the norm $\|\cdot\|$, and let the time interval of interest be $[0, T]$ with $T > 0$.

**Definition 3.2** *The space*

$$L^p(0, T; X)$$

*consists of all strongly measurable functions* $\mathbf{u} : [0, T] \to X$ *with the norm*

$$\|\mathbf{u}\|_{L^p(0,T;X)} := \left( \int_0^T \|\mathbf{u}(t)\|^p \, \mathrm{d}t \right)^{1/p} < \infty \quad \text{for} \quad 1 \leq p < \infty,$$

*and*

$$\|\mathbf{u}\|_{L^\infty(0,T;X)} := \operatorname*{ess.sup}_{0 \leq t \leq T} \|\mathbf{u}(t)\| < \infty.$$

In order to move to Banach–space–valued Sobolev spaces, we shall define the weak derivative of a function belonging to $L^1(0, T; X)$

**Definition 3.3** *The function* $\mathbf{v} \in L^1(0, T; X)$ *is the weak derivative of* $\mathbf{u} \in L^1(0, T; X)$, *written*

$$\dot{\mathbf{u}} = \mathbf{v},$$

*provided that, for all scalar test functions* $\varphi \in C_0^\infty(0, T)$, *we have*

$$\int_0^T \dot{\varphi}(t)\mathbf{u}(t) \, \mathrm{d}t = -\int_0^T \varphi(t)\mathbf{v}(t) \, \mathrm{d}t.$$

**Definition 3.4** *The Sobolev space*

$$W_p^1(0,T;X)$$

*consists of all functions* $\mathbf{u} \in L^p(0,T;X)$ *such that* $\dot{\mathbf{u}}$ *exists in the weak sense and belongs to* $L^p(0,T;X)$*, with the associated norm*

$$\|\mathbf{u}\|_{W_p^1(0,T;X)} := \left(\int_0^T \{\|\mathbf{u}(t)\|^p + \|\dot{\mathbf{u}}(t)\|^p\} \,\mathrm{d}t\right)^{1/p} < \infty \quad \textit{for} \quad 1 \le p < \infty,$$

*and*

$$\|\mathbf{u}\|_{W_\infty^1(0,T;X)} := \operatorname*{ess.sup}_{0\le t\le T}(\|\mathbf{u}(t)\| + \|\dot{\mathbf{u}}(t)\|).$$

Further, for simplicity, we shall write $H^1(0,T;X) \equiv W_2^1(0,T;X)$.

**4. Broken Weak Formulation.** Before presenting the broken weak formulation of the problem described in Section 2, we shall introduce some notation. Let $\mathcal{T}$ be a subdivision of $\Omega \subset \mathbb{R}^n$, $n \ge 2$, into disjoint open polyhedra $\kappa$ as in Section 3. By $\mathcal{E}$ we denote the set of all open $(n-1)$-dimensional faces of the subdivision $\mathcal{T}$, containing the smallest common $(n-1)$-dimensional interfaces $e$ of neighbouring elements. We define

$$\mathcal{E}_{\mathrm{int}} := \bigcup_{e\in\mathcal{E}\setminus\partial\Omega} e \quad \text{and} \quad \mathcal{E}_\partial := \bigcup_{e\in\mathcal{E}\cap\partial\Omega} e.$$

Numbering the elements of the subdivision $\mathcal{T}$, and choosing any internal interface $e \in \mathcal{E}_{\mathrm{int}}$, there exist positive integers $i$, $j$ such that $i > j$ and elements $\kappa \equiv \kappa_i$ and $\kappa' \equiv \kappa_j$ which share this interface $e$. We define the *jump of a function* $u \in H^{\mathbf{s}}(\Omega, \mathcal{T})$ *across the face* $e$ and the *mean value of* $u$ *on* $e$ by

$$[u]_e := u|_{\partial\kappa\cap e} - u|_{\partial\kappa'\cap e} \quad \text{and} \quad \langle u\rangle_e := \frac{1}{2}\left(u|_{\partial\kappa\cap e} + u|_{\partial\kappa'\cap e}\right),$$

respectively, with $\partial\kappa$ denoting the union of all open faces of the element $\kappa$. With each face $e$ we associate the unit normal vector $\nu$ pointing from the element $\kappa_i$ to $\kappa_j$ when $i > j$; when the face belongs to $\mathcal{E}_\partial$, we choose $\nu$ to be the unit outward normal vector $\mathbf{n}$. Finally, we decompose the set of all faces on the boundary $\mathcal{E}_\partial$ into two sets, $\mathcal{E}_{\mathrm{D}}$ and $\mathcal{E}_{\mathrm{N}}$, such that $\Gamma_{\mathrm{D}} = \cup_{e\in\mathcal{E}_{\mathrm{D}}} e$ and $\Gamma_{\mathrm{N}} = \cup_{e\in\mathcal{E}_{\mathrm{N}}} e$.

Now we are ready to introduce the broken weak formulation of the problem (2.1)–(2.4). We define the bilinear form $B(\cdot,\cdot)$ by

$$\begin{aligned} B(u,v) := &\sum_{\kappa\in\mathcal{T}} \int_\kappa \nabla u \cdot \nabla v \,\mathrm{d}x \\ &+ \int_{\Gamma_{\mathrm{int}}} \{\theta \langle\nabla v\cdot\nu\rangle [u] - \langle\nabla u\cdot\nu\rangle [v]\} \,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \sigma [u][v] \,\mathrm{d}s \\ &+ \int_{\Gamma_{\mathrm{D}}} \{\theta(\nabla v\cdot\mathbf{n})u - (\nabla u\cdot\mathbf{n})v\} \,\mathrm{d}s + \int_{\Gamma_{\mathrm{D}}} \sigma uv \,\mathrm{d}s, \end{aligned} \tag{4.1}$$

and the linear functional $l(\cdot)$ by

$$l(v) := \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}} v \,\mathrm{d}s + \theta \int_{\Gamma_{\mathrm{D}}} (\nabla v\cdot\mathbf{n}) g_{\mathrm{D}} \,\mathrm{d}s + \int_{\Gamma_{\mathrm{D}}} \sigma g_{\mathrm{D}} v \,\mathrm{d}s. \tag{4.2}$$

Here $\sigma$ is called the *discontinuity–penalisation parameter* and is defined by

$$\sigma|_e = \sigma_e \quad \text{for} \quad e \in \mathcal{E}_{\text{int}} \cup \mathcal{E}_\partial,$$

where $\sigma_e$ is a non–negative constant on the face $e$. The precise choice of $\sigma_e$ will be discussed in Section 8. The subscript $e$ in these definitions will be suppressed when no confusion is likely to occur. The parameter $\theta$ here takes the values $\pm 1$. The choice of $\theta = -1$ leads to a symmetric bilinear form $B(\cdot,\cdot)$; we call this method a *Symmetric Interior Penalty*, or SIP, method. On the other hand, the choice of $\theta = 1$ leads to a non–symmetric, but coercive bilinear form $B(\cdot,\cdot)$; we call such method a *Nonsymmetric Interior Penalty*, or NSIP, method. Further we shall label the bilinear form (4.1) and the linear functional (4.2) with indices S and NS in the symmetric and non–symmetric cases respectively.

Then, the broken weak formulation of the problem (2.1)–(2.4) reads:

$$\begin{gathered}\text{find } u \in H^1(0,T;\mathfrak{A}) \text{ such that}\\ \sum_{\kappa\in\mathcal{T}} \int_\kappa \dot{u} v \,\mathrm{d}x + B(u,v) - \sum_{\kappa\in\mathcal{T}} \int_\kappa f(u) v \,\mathrm{d}x = l(v), \quad \text{for all} \quad v \in H^2(\Omega,\mathcal{T}),\\ u(0) = u_0,\end{gathered} \tag{4.3}$$

where by $\mathfrak{A}$ we denote the function space

$$\mathfrak{A} = \left\{ w \in H^2(\Omega,\mathcal{T}) : w,\, \nabla w \cdot \nu \text{ are continuous across each } e \in \mathcal{E}_{\text{int}} \right\}.$$

**5. Finite Element Space.** Here we define the finite–dimensional subspace of $H^1(\Omega,\mathcal{T})$ on which the finite element method will be posed.

It makes sense to construct this space in such a way that the degree of piecewise polynomials contained in the space can be different on every element $\kappa$ of the subdivision $\mathcal{T}$. This will allow us to vary the approximation order according to the local regularity of the solution on the element by changing the degree of the polynomial on elements. As we are concerned with the discontinuous Galerkin method here, we do not need to make any additional assumptions to ensure continuity of the approximation across element interfaces. Henceforth, this method will be referred to as *hp*–DGFEM (see [35] for a description of *hp*–FEM).

For a non–negative integer $p$, we denote by $\mathcal{P}_p(\hat{\kappa})$ the set of polynomials of total degree $p$ on a bounded open set $\hat{\kappa}$. When $\hat{\kappa}$ is the unit hypercube, we also consider $\mathcal{Q}_p(\hat{\kappa})$, the set of all tensor–product polynomials on $\hat{\kappa}$ of degree $p$ in each coordinate direction. To each $\kappa \in \mathcal{T}$ we assign a non–negative integer $p_\kappa$ (the local polynomial degree) and a non–negative integer $s_\kappa$ (the local Sobolev index).

Recalling the construction of the subdivision $\mathcal{T}$ (see Section 3), we collect the $p_\kappa$ and the $F_\kappa$ into vectors $\mathbf{p} = \{p_\kappa : \kappa \in \mathcal{T}\}$ and $\mathbf{F} = \{F_\kappa : \kappa \in \mathcal{T}\}$, and consider the finite element space

$$S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F}) := \left\{ u \in L^2(\Omega) : u|_\kappa \circ F_\kappa \in \mathcal{R}_{p_\kappa}(\hat{\kappa}),\, \kappa \in \mathcal{T} \right\}, \tag{5.1}$$

where $\mathcal{R}$ is either $\mathcal{P}$ or $\mathcal{Q}$.

**6. Discontinuous Galerkin Finite Element Method.** Using the finite element space $S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F})$, defined in the previous section, and the broken weak formulation of the problem (4.3), the approximation $u_{\text{DG}}$ to the solution $u$ of the problem

(2.1)–(2.4), discretised by the discontinuous Galerkin finite element method in space, is defined as follows:

$$
\begin{aligned}
&\text{find } u_{\mathrm{DG}} \in H^1(0,T;S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F})) \text{ such that}\\
&\sum_{\kappa\in\mathcal{T}}\int_\kappa \dot{u}_{\mathrm{DG}} v \,\mathrm{d}x + B(u_{\mathrm{DG}},v) - \sum_{\kappa\in\mathcal{T}}\int_\kappa f(u_{\mathrm{DG}}) v\,\mathrm{d}x = l(v), \text{ for all } v \in S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F}),\\
&\qquad\qquad u_{\mathrm{DG}}(0) = u_0^{\mathrm{DG}},
\end{aligned}
\tag{6.1}
$$

where $u_0^{\mathrm{DG}}$ denotes the approximation of the function $u_0$ from the finite element space $S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F})$, and the parameter $\sigma$ in (4.1) and (4.2) is to be defined in the error analysis.

The equation (6.1) can be interpreted as a system of ordinary differential equations in $t$ for the coefficients in the expansion of $u_{\mathrm{DG}}(t)$ in terms of basis functions of the finite–dimensional space $S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F})$. Thus, (6.1) defines an autonomous system of ordinary differential equations with $C^1$ (and, therefore, locally Lipschitz continuous) right–hand side, given that $f \in C^1(\mathbb{R})$ and the other terms are linear. By the Cauchy–Picard theorem this, in turn, implies the existence of a unique local solution to (6.1).

Since no pointwise continuity requirement is imposed on the elements of the finite element space, the approximation $u_{\mathrm{DG}}$ in $S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F})$ to the solution $u$ will be, in general, discontinuous.

**Remark 6.1** If the continuity assumptions made in the construction of the space $\mathfrak{A}$ are violated, i.e., $u$ and $\nabla u\cdot\nu$ are discontinuous across the element interfaces, we have to modify the DGFEM accordingly. This could be done, for example, by performing a DGFEM discretisation on every subdomain of $\Omega$ where the continuity requirements are satisfied, and incorporating into the definition of the method transmission conditions on interfaces where discontinuities in the solutions occur. Such situations include, for example, heat transfer problems in heterogeneous or layered media or problems that contain different phases of material. There the solution $u$ and/or the diffusive fluxes $\nabla u\cdot\mathbf{n}$ can be discontinuous across element interfaces. This information has to be incorporated into the definition of the method and, in particular, into the choice of the discontinuity–penalisation parameter $\sigma$, to avoid penalising physical discontinuities. □

**7.** ***hp*–Error Estimates.** The first analysis of the $p$–version of FEM for Poisson's equation was given by Babuška et al. [9], and was subsequently refined by Babuška & Suri in [7] and [8]. The analysis relied on the use of the Babuška–Suri projection operator. For the special case of $n = 2$, the analysis in $W^s_q$–norms was carried out by Ainsworth & Kay in [2] and [3], where the approximation bounds were used for deriving a priori error bounds for $p$– and $hp$–version FEMs for the $r$–Laplacian, using approximation by continuous piecewise polynomials on both quadrilateral and triangular elements. The error bounds obtained in these works contain logarithmic terms in $p$, and thus are only optimal in $p$ up to a logarithmic factor.

We shall proceed with the derivation of local approximation error bounds, avoiding such suboptimal logarithmic terms by using some very recent results due to Melenk [28].

From Proposition A.2 and Theorem A.3 in [28], we conclude the following result concerning polynomial approximation of functions defined on hypercubes.

**Lemma 7.1** *Let* $Q := (-1,1)^n$, $n \geq 1$, *and let* $u \in W_q^k(Q)$, *where* $q \in [1,\infty]$; *then there exists a sequence of algebraic polynomials* $z_p(u) \in \mathcal{R}_p(Q)$, $p \in \mathbb{N}$, *such that, for any* $0 \leq l \leq k$,

$$\|u - z_p(u)\|_{W_q^l(Q)} \leq Cp^{-(k-l)} \|u\|_{W_q^k(Q)}, \quad 1 \leq q \leq \infty, \tag{7.1}$$

*where* $C > 0$ *is a constant, independent of* $u$ *and* $p$, *but dependent on* $q$ *and* $k$.

To derive the general $hp$–estimates for the projection operator $u \mapsto z_p(u)$, recall from Sections 3 and 5 the construction of the subdivision $\mathcal{T}$ of the computational domain $\Omega$. Let $\hat{\kappa}$ be the $n$–dimensional open unit hypercube, which we shall call the reference element. We construct each element $\kappa \in \mathcal{T}$ via an affine mapping from the reference element $\kappa = F_\kappa(\hat{\kappa})$, based on scaling each coordinate of the reference element by the factor $h_\kappa$.

We shall also need the following result.

**Lemma 7.2** *Suppose that* $\kappa \in \mathcal{T}$ *is an* $n$*–dimensional parallelepiped of diameter* $h_\kappa$, *and that* $u|_\kappa \in W_q^{k_\kappa}(\kappa)$ *for some* $k_\kappa \geq 0$ *and* $\kappa \in \mathcal{T}$. *Define* $\hat{u} \in W_q^{k_\kappa}(\hat{\kappa})$ *by the rule* $\hat{u}(\hat{x})|_{\hat{\kappa}} = u(F_\kappa(\hat{x}))|_\kappa$; *Then*

$$\inf_{\hat{v} \in \mathcal{R}_{p_\kappa}(\hat{\kappa})} \|\hat{u} - \hat{v}\|_{W_q^{k_\kappa}(\hat{\kappa})} \leq Ch_\kappa^{s_\kappa - n/q} \|u\|_{W_q^{k_\kappa}(\kappa)},$$

*where* $s_\kappa = \min(p_\kappa + 1, k_\kappa)$.

**Proof.** (See [7], Lemma 4.4, and [3], Lemma 1). Assume that $k_\kappa$ is an integer. If $k_\kappa = 0$, then the result follows by bounding the left–hand side of the inequality by $\|\hat{u}\|_{L^q(\hat{\kappa})}$ and scaling to $\|u\|_{L^q(\kappa)}$. Suppose, therefore, that $k_\kappa \geq 1$. For any $\hat{v} \in \mathcal{R}_{p_\kappa}(\hat{\kappa})$, we have

$$\|\hat{u} - \hat{v}\|_{W_q^{k_\kappa}(\hat{\kappa})} \leq \|\hat{u} - \hat{v}\|_{W_q^{s_\kappa}(\hat{\kappa})} + \sum_{l_\kappa = s_\kappa + 1}^{k_\kappa} |\hat{u}|_{W_q^{l_\kappa}(\hat{\kappa})},$$

with the convention that if $s_\kappa = k_\kappa$ then the summation is over an empty index set of $l_\kappa$.

Using Theorem 3.1.1 in [12], we obtain

$$\inf_{\hat{v} \in \mathcal{R}_{p_\kappa}(\hat{\kappa})} \|\hat{u} - \hat{v}\|_{W_q^{k_\kappa}(\hat{\kappa})} \leq \sum_{l_\kappa = s_\kappa}^{k_\kappa} |\hat{u}|_{W_q^{l_\kappa}(\hat{\kappa})}.$$

Scaling back to the element $\kappa \in \mathcal{T}$, we obtain the result for integer $k_\kappa$. The result for general $k_\kappa$ follows by a standard function space interpolation argument. □

Now we are ready to state our main result concerning the approximation properties of the projection operator $u \mapsto z_p(u)$.

**Theorem 7.3** *Suppose that* $\kappa \in \mathcal{T}$ *is an* $n$*–dimensional parallelepiped of diameter* $h_\kappa$, *and that* $u|_\kappa \in W_q^{k_\kappa}(\kappa)$ *for some* $k_\kappa \geq 0$ *and* $\kappa \in \mathcal{T}$; *then, there exists a sequence of algebraic polynomials* $z_{p_\kappa}^{h_\kappa}(u) \in \mathcal{R}_{p_\kappa}(\kappa)$, $p_\kappa \geq 1$, *such that for any* $l$, *with* $0 \leq l \leq k_\kappa$,

$$\left\|u - z_{p_\kappa}^{h_\kappa}(u)\right\|_{W_q^l(\kappa)} \leq C\,\frac{h_\kappa^{s_\kappa - l}}{p_\kappa^{k_\kappa - l}}\, \|u\|_{W_q^{k_\kappa}(\kappa)}, \quad 1 \leq q \leq \infty, \tag{7.2}$$

*and, for* $q = 2$,

$$\left\| u - z_{p_\kappa}^{h_\kappa}(u) \right\|_{L^2(e_\kappa)} \leq C \, \frac{h_\kappa^{s_\kappa - \frac{1}{2}}}{p_\kappa^{k_\kappa - \frac{1}{2}}} \, \|u\|_{H^{k_\kappa}(\kappa)} \, , \tag{7.3}$$

$$\left\| \nabla (u - z_{p_\kappa}^{h_\kappa}(u)) \right\|_{L^2(e_\kappa)} \leq C \, \frac{h_\kappa^{s_\kappa - \frac{3}{2}}}{p_\kappa^{k_\kappa - \frac{3}{2}}} \, \|u\|_{H^{k_\kappa}(\kappa)} \, , \tag{7.4}$$

*where* $e_\kappa$ *is any face (edge)* $e_\kappa \subset \partial\kappa$, $s_\kappa = \min(p_\kappa + 1, k_\kappa)$, *and* $C$ *is a constant independent of* $u$, $h_\kappa$, *and* $p_\kappa$, *but dependent on* $k = \max_{\kappa \in \mathcal{T}} k_\kappa$ *and* $q$.

**Proof.** (See also [7]). Let $u \in W_q^{k_\kappa}(\kappa)$ and define $\hat{u} \in W_q^{k_\kappa}(\hat{\kappa})$ by the rule $\hat{u}(\hat{x})|_{\hat{\kappa}} = u(F_\kappa(\hat{x}))|_\kappa$. First, we note that, by Lemma 4.1 in [7], for any $\hat{v} \in \mathcal{R}_{p_\kappa}(\hat{\kappa})$, we have the property that $\widehat{z_{p_\kappa}^{h_\kappa}}(\hat{v}) = \hat{v}$. By Lemma 7.1, (7.1), we have, for $0 \leq l \leq k_\kappa$,

$$\left\| \hat{u} - \widehat{z_{p_\kappa}^{h_\kappa}}(\hat{u}) \right\|_{W_q^l(\hat{\kappa})} \leq C p_\kappa^{-(k_\kappa - l)} \, \|\hat{u}\|_{W_q^{k_\kappa}(\hat{\kappa})} \, .$$

Noting that $\widehat{z_{p_\kappa}^{h_\kappa}}(\hat{u})(\hat{x}) = z_{p_\kappa}^{h_\kappa}(u)(F_\kappa(\hat{x}))$, and applying Lemma 7.2 with $\hat{v} \in \mathcal{R}_{p_\kappa}(\hat{\kappa})$, we obtain

$$\begin{aligned} \left\| \hat{u} - \widehat{z_{p_\kappa}^{h_\kappa}}(\hat{u}) \right\|_{W_q^l(\hat{\kappa})} &= \left\| (\hat{u} - \hat{v}) - \widehat{z_{p_\kappa}^{h_\kappa}}(\hat{u} - \hat{v}) \right\|_{W_q^l(\hat{\kappa})} \\ &\leq C p_\kappa^{-(k_\kappa - l)} \inf_{\hat{v} \in \mathcal{R}_{p_\kappa}(\hat{\kappa})} \|\hat{u} - \hat{v}\|_{W_q^{k_\kappa}(\hat{\kappa})} \leq C p_\kappa^{-(k_\kappa - l)} h_\kappa^{s_\kappa - \frac{n}{q}} \, \|u\|_{W_q^{k_\kappa}(\kappa)} \, . \end{aligned}$$

Thus, by a scaling argument, for $0 \leq m \leq l \leq k_\kappa$, we have

$$\left| u - z_{p_\kappa}^{h_\kappa}(u) \right|_{W_q^m(\kappa)} \leq C p_\kappa^{-(k_\kappa - l)} h_\kappa^{s_\kappa - m} \, \|u\|_{W_q^{k_\kappa}(\kappa)} \, ,$$

and therefore

$$\left\| u - z_{p_\kappa}^{h_\kappa}(u) \right\|_{W_q^l(\kappa)} \leq C p_\kappa^{-(k_\kappa - l)} h_\kappa^{s_\kappa - l} \, \|u\|_{W_q^{k_\kappa}(\kappa)} \, ,$$

and hence (7.2).

By setting $q = 2$ in (7.2) and using the trace inequality

$$\|u\|_{L^2(\partial\kappa)} \leq C \left( h_\kappa^{-\frac{1}{2}} \, \|u\|_{L^2(\kappa)} + \|u\|_{L^2(\kappa)}^{\frac{1}{2}} \, \|\nabla u\|_{L^2(\kappa)}^{\frac{1}{2}} \right) ,$$

we obtain (7.3) and (7.4). □

**8. Error Analysis.** This section is be concerned with the derivation of *a priori* error bounds for the initial–boundary value problem for the semilinear parabolic equation described in Section 2.

We shall assume that the polynomial degree vector $\mathbf{p}$, with $p_\kappa \geq 1$ for each $\kappa \in \mathcal{T}$, has *bounded local variation*, i.e., there exists a constant $\rho \geq 1$ such that, for any pair of elements $\kappa$ and $\kappa'$ which share an $(n-1)$–dimensional face,

$$\rho^{-1} \leq \frac{p_\kappa}{p_{\kappa'}} \leq \rho. \tag{8.1}$$

We also recall our regularity assumptions on the subdivision $\mathcal{T}$: namely, $\mathcal{T}$ is shape–regular, and regular or 1–irregular. We shall consider the error analysis of the *hp*–version of the discontinuous Galerkin finite element method on shape–regular meshes. In particular, we shall derive *a priori* error bounds for both the symmetric and the non–symmetric version of DGFEM.

Let us begin with the following lemma which establishes the local Lipschitz continuity of the non–linearity $f$, required in the *a priori* error analysis of the *hp*–version of DGFEM (6.1) for the model problem (2.1)–(2.4).

**Lemma 8.1** *Let $f \in C^1(\mathbb{R})$ satisfy the growth–condition (2.2) with $0 < \alpha < \infty$ when $n = 2$ and $0 < \alpha \leq 2/(n-2)$ when $n \geq 3$, and suppose that $2 < q < \infty$. Let*

$$\hat{q} = \max\left(q,\, \frac{2\alpha q}{q-2}\right).$$

*Then, there exists a positive constant $C = C(\alpha, C_{\rm g}, q, |\Omega|)$ such that*

$$\|f(u) - f(v)\|_{L^2(\Omega)} \leq C\, \|u - v\|_{L^q(\Omega)} \left(1 + \|u\|^{\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)} + \|v\|^{\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right) \tag{8.2}$$

*for all $u,\, v \in L^{\hat{q}}(\Omega)$.*

*Suppose that $q = 2(\alpha+1)$; then $\hat{q} = 2(\alpha+1)$. Moreover, if $n = 2$, $0 < \alpha < \infty$ then $2 < \hat{q} < \infty$, and if $n \geq 3$, $0 < \alpha \leq 2/(n-2)$ then $2 < \hat{q} \leq 2n/(n-2)$.*

**Proof.** From (2.2), by Hölder's inequality, we have

$$\begin{aligned}\|f(u) - f(v)\|^2_{L^2(\Omega)} &= \int_\Omega |f(u) - f(v)|^2 \,\mathrm{d}x \leq C_{\rm g}^2 \int_\Omega (u-v)^2 (1 + |u| + |v|)^{2\alpha}\,\mathrm{d}x \\ &\leq C_{\rm g}^2 \left(\int_\Omega |u-v|^{2\cdot\frac{q}{2}}\,\mathrm{d}x\right)^{\frac{2}{q}} \left(\int_\Omega (1+|u|+|v|)^{2\alpha\cdot\left(1-\frac{2}{q}\right)^{-1}}\,\mathrm{d}x\right)^{1-\frac{2}{q}}.\end{aligned}$$

As $1 - 2/q = (q-2)/q$ and $q > 2$, we have

$$\begin{aligned}\|f(u) - f(v)\|^2_{L^2(\Omega)} &\leq C_{\rm g}^2 \left(\int_\Omega |u-v|^q\,\mathrm{d}x\right)^{\frac{2}{q}} \left(\int_\Omega (1+|u|+|v|)^{\frac{2\alpha q}{q-2}}\,\mathrm{d}x\right)^{\frac{q-2}{q}} \\ &= C_{\rm g}^2\, \|u-v\|^2_{L^q(\Omega)} \left(\int_\Omega (1+|u|+|v|)^{\frac{2\alpha q}{q-2}}\,\mathrm{d}x\right)^{\frac{q-2}{q}} \\ &= C_{\rm g}^2\, \|u-v\|^2_{L^q(\Omega)} \left(\int_\Omega (1+|u|+|v|)^{\frac{2\alpha q}{q-2}}\,\mathrm{d}x\right)^{\frac{q-2}{2\alpha q}\cdot 2\alpha} \\ &\leq C_{\rm g}^2\, \|u-v\|^2_{L^q(\Omega)} \left(|\Omega|^{\frac{q-2}{2\alpha q}} + \|u\|_{L^{\frac{2\alpha q}{q-2}}(\Omega)} + \|v\|_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right)^{2\alpha} \\ &\leq C^2\, \|u-v\|^2_{L^q(\Omega)} \left(1 + \|u\|^{\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)} + \|v\|^{\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right)^2,\end{aligned}$$

and hence (8.2) for all $u, v \in L^{\hat{q}}(\Omega)$. The statement in the final sentence of the lemma follows from our hypothesis on the range of $\alpha$ and the fact that $q = 2(\alpha+1)$. □

**Hypothesis A.** Let $f \in C^1(\mathbb{R})$ satisfy the growth–condition (2.2) with $0 < \alpha < \infty$ when $n = 2$, and $0 < \alpha \leq 2/(n-2)$ when $n \geq 3$. We define $q = 2(\alpha+1)$.

With this hypothesis in mind, we can remove the dependence on $q$ in the constant $C$ in Lemma 8.1 in terms of $\alpha$.

For the sake of clarity of the exposition, in the rest of the paper we shall confine ourselves to the case of $n \geq 3$. Our proofs can be easily adjusted to cover the case of $n = 2$ with $0 < \alpha < \infty$.

**8.1. The Non–Symmetric Version of DGFEM.** Let the bilinear form $B$ be as in (4.1). Here we shall be concerned with the non–symmetric version of DGFEM corresponding to $\theta = 1$ in (4.1), so we write $B_{\mathrm{NS}}(\cdot,\cdot)$ in place of $B(\cdot,\cdot)$. We begin our error analysis with the following definition.

**Definition 8.2** *We define the quantity* $|||\cdot|||_{\mathrm{DG}}$ *on* $H^1(\Omega,\mathcal{T})$*, associated with the DGFEM, as follows:*

$$|||w|||_{\mathrm{DG}} := \left( \sum_{\kappa\in\mathcal{T}} \|\nabla w\|^2_{L^2(\kappa)} + \int_{\Gamma_{\mathrm{D}}} \sigma w^2 \,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \sigma\, [w]^2 \,\mathrm{d}s \right)^{\frac{1}{2}}, \tag{8.3}$$

*where* $\sigma$ *is a non–negative function on* $\Gamma$.

**Remark 8.3** Let us observe some properties of $|||\cdot|||_{\mathrm{DG}}$ defined above.

1. If $\sigma > 0$ on $\Gamma$, then $|||\cdot|||_{\mathrm{DG}}$ defines a norm in $H^1(\Omega,\mathcal{T})$.
2. If $\sigma = 0$ on $\Gamma$, then $|||\cdot|||_{\mathrm{DG}}$ defines a seminorm in $H^1(\Omega,\mathcal{T})$.
3. Clearly, $|||w|||^2_{\mathrm{DG}} = B_{\mathrm{NS}}(w,w)$, for all $w \in H^1(\Omega,\mathcal{T})$.

□

The first step in the error analysis is to decompose the error $u - u_{\mathrm{DG}}$, where $u$ denotes the analytical solution, as $u - u_{\mathrm{DG}} = \xi + \eta$, where $\xi \equiv \Pi u - u_{\mathrm{DG}}$, $\eta \equiv u - \Pi u$, with $\Pi$ defined element–wise by

$$(\Pi u)|_\kappa := \Pi(u|_\kappa),$$

and $\Pi$ denoting an appropriate projection operator on the element $\kappa$. Thus, using the triangle inequality for the $H^1$–norm, we have

$$\|u - u_{\mathrm{DG}}\|_{H^1(\Omega,\mathcal{T})} \leq \|\eta\|_{H^1(\Omega,\mathcal{T})} + \|\xi\|_{H^1(\Omega,\mathcal{T})}. \tag{8.4}$$

We assume for simplicity that the initial value is chosen as

$$u_0^{\mathrm{DG}} = \Pi u_0, \tag{8.5}$$

and thus $\xi(0) = 0$.

We shall proceed by deriving a bound on $\|\xi\|_{H^1(\Omega,\mathcal{T})}$ in terms of norms of $\eta$. Then, by choosing a suitable projection operator $\Pi$, we shall be able to use the bounds on various norms of the projection error $\eta$ derived in Section 7 to deduce an *a priori* error bound for the method.

Let us prove the continuity of the bilinear form $B_{\mathrm{NS}}$, which will also provide the necessary bound for our error analysis.

**Lemma 8.4** *Let* $\mathcal{T}$ *be a shape–regular subdivision of* $\Omega$ *and assume that the parameter* $\sigma$ *is positive on* $\Gamma_{\mathrm{int}} \cup \Gamma_{\mathrm{D}}$*; then, the following inequality holds for all* $v \in H^1(\Omega,\mathcal{T})$

*and $w \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$, with $C$ a positive constant that depends only on the dimension $n$ and the shape–regularity of $\mathcal{T}$:*

$$|B_{\mathrm{NS}}(v,w)| \leq C\,|||w|||_{\mathrm{DG}} \Bigg\{ \int_{\Gamma_{\mathrm{D}}} \sigma\,|v|^2 \,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \sigma\,[v]^2 \,\mathrm{d}s + \sum_{\kappa\in\mathcal{T}} \|\nabla v\|^2_{L^2(\kappa)} + \sum_{\kappa\in\mathcal{T}} \left( \|\sqrt{\tau} v\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{D}})} + \left\| \frac{1}{\sqrt{\sigma}} \nabla v \right\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{D}})} \right) + \sum_{\kappa\in\mathcal{T}} \left( \|\sqrt{\tau}\,[v]\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{int}})} + \left\| \frac{1}{\sqrt{\sigma}} \nabla v \right\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{int}})} \right) \Bigg\}^{\frac{1}{2}}, \tag{8.6}$$

*where $\tau_e = \langle p^2 \rangle_e / h_e$ and $h_e$ is the diameter of a face $e \subset \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$; when $e \in \mathcal{E}_{\mathrm{D}}$ the contribution from outside $\Omega$ in the definition of $\tau_e$ is set to $0$.*

**Proof.** (See also [21].) Let us decompose

$$|B_{\mathrm{NS}}(v,w)| \leq I + II + III + IV,$$

where

$$I \equiv \left| \sum_{\kappa\in\mathcal{T}} \int_\kappa \nabla v \cdot \nabla w \,\mathrm{d}x \right|, \quad II \equiv \left| \int_{\Gamma_D} \{ v(\nabla w \cdot \mathbf{n}) - (\nabla v \cdot \mathbf{n}) w \} \,\mathrm{d}s \right|,$$

$$III \equiv \left| \int_{\Gamma_{\mathrm{int}}} \{ [v] \langle \nabla w \cdot \nu \rangle - \langle \nabla v \cdot \nu \rangle [w] \} \,\mathrm{d}s \right|,$$

$$IV \equiv \left| \int_{\Gamma_{\mathrm{D}}} \sigma v w \,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \sigma\,[v]\,[w] \,\mathrm{d}s \right|.$$

For the term $I$ we have

$$I \leq |||w|||_{\mathrm{DG}} \sum_{\kappa\in\mathcal{T}} \left( \|\nabla v\|^2_{L^2(\kappa)} \right)^{\frac{1}{2}}, \tag{8.7}$$

and for the term $IV$ we have that

$$IV \leq |||w|||_{\mathrm{DG}} \left( \int_{\Gamma_{\mathrm{D}}} \sigma\,|v|^2 \,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \sigma\,[v]^2 \,\mathrm{d}s \right)^{\frac{1}{2}}. \tag{8.8}$$

To deal with the term $II$, we first note that

$$II \leq \left( \sum_{\kappa\in\mathcal{T}} \frac{1}{\gamma_\kappa} \,\|v\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{D}})} \right)^{\frac{1}{2}} \left( \sum_{\kappa\in\mathcal{T}} \gamma_\kappa \,\|\nabla w\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{D}})} \right)^{\frac{1}{2}} + \left( \sum_{\kappa\in\mathcal{T}} \left\| \frac{1}{\sqrt{\sigma}} \,\nabla v \right\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{D}})} \right)^{\frac{1}{2}} \left( \sum_{\kappa\in\mathcal{T}} \|\sqrt{\sigma}\,w\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{D}})} \right)^{\frac{1}{2}}$$

for any set of positive numbers $\{\gamma_\kappa : \kappa \in \mathcal{T}\}$. Here we can apply the inverse inequality

$$\|\nabla w\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{D}})} \leq K \,\frac{p_\kappa^2}{h_\kappa} \,\|\nabla w\|^2_{L^2(\kappa)}, \tag{8.9}$$

where $K$ depends only on the shape–regularity of $\mathcal{T}$ (see Schwab [35], Theorem 4.76, (4.6.4)). On letting $\gamma_\kappa = h_\kappa/p_\kappa^2$ and defining $\tau_e = p_\kappa^2/2h_e$ for an $(n-1)$–dimensional face $e \subset \partial\kappa \cap \Gamma_{\rm D}$, we obtain

$$II \leq C|||w|||_{\rm DG} \sum_{\kappa\in\mathcal{T}} \left( \left\|\sqrt{\tau} v\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} + \left\| \frac{1}{\sqrt{\sigma}} \nabla v \right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} \right)^{\frac{1}{2}}. \tag{8.10}$$

Similarly, we have

$$III \leq C|||w|||_{\rm DG} \sum_{\kappa\in\mathcal{T}} \left( \left\|\sqrt{\tau}\,[v]\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} + \left\| \frac{1}{\sqrt{\sigma}} \nabla v \right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} \right)^{\frac{1}{2}}. \tag{8.11}$$

By collecting the results we have the desired bound. □

Now, let us derive a bound on the $H^1$–norm of the error $u - u_{\rm DG}$.

**Lemma 8.5** *Let $\mathcal{T}$ be a shape–regular subdivision of $\Omega$ and assume that $f \in C^1(\mathbb{R})$ satisfies Hypothesis A. Suppose further that the positive parameter $\sigma$ is defined on $\Gamma_{\rm int} \cup \Gamma_{\rm D}$ and*

$$\sigma_e = \sigma|_e \geq h_e^{-1}$$

*on each face $e \in \mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm D}$. In addition, suppose that*

*a) the local polynomial degree $p_\kappa \geq 2$ on each $\kappa \in \mathcal{T}$;*

*b) the local Sobolev smoothness $k_\kappa \geq 3.5$ on each $\kappa \in \mathcal{T}$;*

*c) the hp–mesh is quasi–uniform in the sense that there exists a positive constant $C_0$ such that*

$$\max_{\kappa\in\mathcal{T}} \frac{h_\kappa}{p_\kappa^2} \leq C_0 \min_{\kappa\in\mathcal{T}} \frac{h_\kappa}{p_\kappa^2}.$$

*Then, for all $t \in [0,T]$, there exists $h_0 > 0$ such that for all $h \in (0, h_0]$, $h = \max_{\kappa\in\mathcal{T}} h_\kappa$, the following inequality holds, with $C$ a positive constant that depends only on the domain $\Omega$, the quasi–uniformity of $\mathcal{T}$, on the final time $T$, the exponent $\alpha$ in the growth–condition for the function $f$, and the Lebesgue and Sobolev norms of $u$ over the time interval $[0,T]$:*

$$\begin{aligned}\int_0^t \|(u-u_{\rm DG})(s)\|^2_{H^1(\Omega,\mathcal{T})}\,{\rm d}s \leq C \sum_{\kappa\in\mathcal{T}} \int_0^t \Big\{ \|\dot\eta(s)\|^2_{L^2(\kappa)} + \|\eta(s)\|^2_{L^{2(\alpha+1)}(\kappa)} + \|\eta(s)\|^2_{H^1(\kappa)} \\ + \left\|\sqrt{\sigma}\eta(s)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} + \left\|\sqrt{\sigma}\,[\eta(s)]\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} + \left\|\sqrt{\tau}\eta(s)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} \\ + \left\|\frac{1}{\sqrt{\sigma}}\nabla\eta(s)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} + \left\|\sqrt{\tau}\,[\eta(s)]\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} + \left\|\frac{1}{\sqrt{\sigma}}\nabla\eta(s)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} \Big\}\,{\rm d}s\end{aligned} \tag{8.12}$$

*where $\tau_e = \left\langle p^2 \right\rangle_e / h_e$ and $h_e$ is the diameter of a face $e \in \mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm D}$, in which for $e \in \mathcal{E}_{\rm D}$ the contribution from outside $\Omega$ is set to 0.*

**Proof.** From the formulation of the *hp*–DGFEM (6.1), for all $v \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$, we have

$$\sum_{\kappa\in\mathcal{T}} \int_\kappa \dot u_{\rm DG} v\,{\rm d}x + B_{\rm NS}(u_{\rm DG}, v) = \sum_{\kappa\in\mathcal{T}} \int_\kappa f(u_{\rm DG}) v\,{\rm d}x + l_{\rm NS}(v). \tag{8.13}$$

On the other hand, the broken weak formulation (4.3) of the problem can be rewritten as

$$\begin{aligned}\sum_{\kappa\in\mathcal{T}}\int_\kappa(\Pi\dot{u})v\,\mathrm{d}x+B_{\mathrm{NS}}(\Pi u,v)&=\sum_{\kappa\in\mathcal{T}}\int_\kappa f(u)v\,\mathrm{d}x+l_{\mathrm{NS}}(v)\\&\quad+\sum_{\kappa\in\mathcal{T}}\int_\kappa(\Pi\dot{u}-\dot{u})v\,\mathrm{d}x+B_{\mathrm{NS}}(\Pi u-u,v).\end{aligned}\tag{8.14}$$

Upon subtracting (8.13) from (8.14) and choosing $v=\xi$, we obtain

$$\sum_{\kappa\in\mathcal{T}}\int_\kappa\dot{\xi}\xi\,\mathrm{d}x+B_{\mathrm{NS}}(\xi,\xi)=\sum_{\kappa\in\mathcal{T}}\int_\kappa\{f(u)-f(u_{\mathrm{DG}})\}\,\xi\,\mathrm{d}x-\sum_{\kappa\in\mathcal{T}}\int_\kappa\dot{\eta}\xi\,\mathrm{d}x-B_{\mathrm{NS}}(\eta,\xi).$$

By noting that

$$\sum_{\kappa\in\mathcal{T}}\int_\kappa\dot{\xi}\xi\,\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\kappa\in\mathcal{T}}\|\xi\|^2_{L^2(\kappa)}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)},$$

we can rewrite the above expression as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)}+|||\xi|||^2_{\mathrm{DG}}\le\left|\sum_{\kappa\in\mathcal{T}}\int_\kappa\{f(u)-f(\Pi u)\}\,\xi\,\mathrm{d}x\right|+\left|\sum_{\kappa\in\mathcal{T}}\int_\kappa\{f(\Pi u)-f(u_{\mathrm{DG}})\}\,\xi\,\mathrm{d}x\right|\\+\left|\sum_{\kappa\in\mathcal{T}}\int_\kappa\dot{\eta}\xi\,\mathrm{d}x\right|+|B_{\mathrm{NS}}(\eta,\xi)|.\end{aligned}\tag{8.15}$$

By the Cauchy–Schwarz and Young inequalities, with $\varepsilon_1>0$, we have

$$\left|\sum_{\kappa\in\mathcal{T}}\int_\kappa\dot{\eta}\xi\,\mathrm{d}x\right|\le\left(\sum_{\kappa\in\mathcal{T}}\|\dot{\eta}\|^2_{L^2(\kappa)}\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathcal{T}}\|\xi\|^2_{L^2(\kappa)}\right)^{\frac{1}{2}}\le\frac{\varepsilon_1}{2}\|\dot{\eta}\|^2_{L^2(\Omega)}+\frac{1}{2\varepsilon_1}\|\xi\|^2_{L^2(\Omega)},$$

and, by the same argument, with $\varepsilon_2$, $\varepsilon_3>0$,

$$\left|\sum_{\kappa\in\mathcal{T}}\int_\kappa\{f(u)-f(\Pi u)\}\,\xi\,\mathrm{d}x\right|\le\frac{\varepsilon_2}{2}\|f(u)-f(\Pi u)\|^2_{L^2(\Omega)}+\frac{1}{2\varepsilon_2}\|\xi\|^2_{L^2(\Omega)},$$

$$\left|\sum_{\kappa\in\mathcal{T}}\int_\kappa\{f(\Pi u)-f(u_{\mathrm{DG}})\}\,\xi\,\mathrm{d}x\right|\le\frac{\varepsilon_3}{2}\|f(\Pi u)-f(u_{\mathrm{DG}})\|^2_{L^2(\Omega)}+\frac{1}{2\varepsilon_3}\|\xi\|^2_{L^2(\Omega)}.$$

Further, by Lemma 8.1, upon absorbing all constants into $C$ and noting the definition of $q$ in Hypothesis A, we have

$$\begin{aligned}\|f(u)-f(\Pi u)\|^2_{L^2(\Omega)}&\le C\|\eta\|^2_{L^q(\Omega)}\left(1+\|u\|^\alpha_{L^{\frac{2\alpha q}{q-2}}(\Omega)}+\|\Pi u\|^\alpha_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right)^2\\&\le C\|\eta\|^2_{L^q(\Omega)}\left(1+\|u\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}+\|\Pi u\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right)\\&=C\|\eta\|^2_{L^q(\Omega)}\left(1+\|u\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}+\|u-\eta\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right)\\&\le C\|\eta\|^2_{L^q(\Omega)}\left(1+\|u\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}+\|\eta\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right)\\&\le C\|\eta\|^2_{L^q(\Omega)}=C\|\eta\|^2_{L^{2(\alpha+1)}(\Omega)},\end{aligned}$$

where the constant $C > 0$ depends only on the domain $\Omega$, the growth–condition for the function $f$, and on Lebesgue norms of $u$ over the time interval $[0, T]$.

By Lemma 8.4 and Young inequality, with $\varepsilon_4 > 0$, we have the bound

$$|B_{\rm NS}(\eta, \xi)| \le \frac{\varepsilon_4}{2} \,|||\xi|||^2_{\rm DG} + \frac{1}{2\varepsilon_4}\mathcal{F}_1(\eta),$$

where

$$\begin{aligned}\mathcal{F}_1(\eta) := C \sum_{\kappa\in\mathcal{T}} \Big( &\|\nabla\eta(s)\|^2_{L^2(\kappa)} + \left\|\sqrt{\sigma}\eta(s)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} + \left\|\sqrt{\sigma}\,[\eta(s)]\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} \\ &+ \left\|\sqrt{\tau}\eta(s)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} + \left\|\frac{1}{\sqrt{\sigma}}\nabla\eta(s)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} \\ &+ \left\|\sqrt{\tau}\,[\eta(s)]\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} + \left\|\frac{1}{\sqrt{\sigma}}\nabla\eta(s)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})}\Big).\end{aligned}$$

Applying these bounds on the right–hand side of (8.15) and absorbing all constants into $C_1$ and $C_2$, we obtain

$$\frac{\rm d}{{\rm d}t}\,\|\xi\|^2_{L^2(\Omega)} + (2-\varepsilon_4)|||\xi|||^2_{\rm DG} \le C_1\mathcal{F}(\eta) + C_2\,\|\xi\|^2_{L^2(\Omega)} + \varepsilon_3\,\|f(\Pi u) - f(u_{\rm DG})\|^2_{L^2(\Omega)}\,, \tag{8.16}$$

where

$$\mathcal{F}(\eta) := \mathcal{F}_1(\eta) + \|\eta\|^2_{L^{2(\alpha+1)}(\Omega)} + \|\dot\eta\|^2_{L^2(\Omega)}\,.$$

To bound $\|f(\Pi u) - f(u_{\rm DG})\|^2_{L^2(\Omega)}$, we first note that, by the same argument as above,

$$\|f(\Pi u) - f(u_{\rm DG})\|^2_{L^2(\Omega)} \le C\,\|\xi\|^2_{L^{2(\alpha+1)}(\Omega)}\left(1 + \|u_{\rm DG}\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right),$$

where the constant $C > 0$ depends only on the domain $\Omega$, the growth–condition for the function $f$, and on Lebesgue norms of $u$ over the time interval $[0, T]$.

Let us choose $u_0^{\rm DG} = \Pi u_0$, thus giving $\xi(0) = 0$, and let $0 < t_\star \le T$ be the largest time such that $u_{\rm DG}$ exists for all $t \in [0, t_\star]$ and

$$\|\xi\|^2_{H^1(\Omega,\mathcal{T})} \le 1 \quad \text{for all} \quad t \in [0, t_\star];$$

existence of such a $t_\star$ is guaranteed by the Cauchy–Picard theorem. Since, by Hypothesis A, $2\alpha q/(q-2) \le 2n/(n-2)$, this implies that

$$\|u_{\rm DG}\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)} \le Const. \quad \text{for all} \quad t \in [0, t_\star]$$

by the broken Sobolev–Poincaré inequality (see Theorem 3.7 in [27][1]); here $Const.$ is a constant that is independent of the discretisation parameters and $t$, and only

[1] Using the notation of the cited paper, we define $\Psi$ as in Example 3.6 of that paper, with

$$\psi \in L^2(\partial\Omega), \quad \psi \equiv 0 \quad \text{on} \quad \Gamma_{\rm N}$$

and

$$|\Psi(\xi)|^2 \le C \sum_{e\in\mathcal{E}_{\rm D}} h_e^{-1} \int_e \xi^2\,{\rm d}s.$$

depends on Sobolev norms of $u$ over the time interval $[0,t_\star]$.

This implies that

$$\|f(\Pi u)-f(u_{\mathrm{DG}})\|^2_{L^2(\Omega)} \le \tilde{C}|||\xi|||^2_{\mathrm{DG}},$$

where the constant $\tilde{C}>0$ depends only on the domain $\Omega$, the growth–condition for the function $f$, and on Lebesgue and Sobolev norms of $u$ over the time interval $[0,t_\star]$.

On choosing $\varepsilon_4+\varepsilon_3\tilde{C}\le 1$, (8.16) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)} + |||\xi|||^2_{\mathrm{DG}} \le C_1\mathcal{F}(\eta) + C_2\|\xi\|^2_{L^2(\Omega)}, \tag{8.17}$$

with the constant $C_1>0$ depending only on the domain $\Omega$, the growth–condition for the function $f$, and on Lebesgue and Sobolev norms of $u$ over the time interval $[0,t_\star]$.

Upon integrating from 0 to $t\le t_\star$ and noting that $\xi(0)=0$, this yields

$$\|\xi(t)\|^2_{L^2(\Omega)} + \int_0^t |||\xi(s)|||^2_{\mathrm{DG}}\,\mathrm{d}s \le C_1\int_0^t \mathcal{F}(\eta(s))\,\mathrm{d}s + C_2\int_0^t \|\xi(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s, \tag{8.18}$$

with the constant $C_1$ as above.

According to this inequality, if $\mathcal{F}(\eta)$ were zero, we would have $\|\xi\|^2_{L^2(\Omega)}=0$ for all $t\in[0,t_\star]$. More generally, by choosing an appropriate projection operator $\Pi$, we can make $\mathcal{F}(\eta)$ as small as we like (for example, by fixing the local polynomial degree $p_\kappa$ on each element $\kappa\in\mathcal{T}$ and reducing $h=\max_{\kappa\in\mathcal{T}}h_\kappa$).

Let us choose $C_3=C_2 2^{2\alpha}$ and $h_0>0$ so small that, for all $h\le h_0$ and $t\in[0,t_\star]$, the following inequality holds:

$$C_1\int_0^t \mathcal{F}(\eta(s))\,\mathrm{d}s < \frac{1}{1+T}\,\mathrm{e}^{-C_3T}\times C_{\mathrm{inv}}^{-1}C_0^{-2}\left(\max_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa^2}\right)^2,$$

where $C_{\mathrm{inv}}$ is the constant from the inverse inequality

$$\|\xi\|^2_{H^1(\Omega,\mathcal{T})} \le C_{\mathrm{inv}}\left(\max_{\kappa\in\mathcal{T}}\frac{p_\kappa^2}{h_\kappa}\right)^2\|\xi\|^2_{L^2(\Omega)}. \tag{8.19}$$

We note in passing that in order to be able to extract the factor $(\max_{\kappa\in\mathcal{T}}(h_\kappa/p_\kappa^2))^2$ from $\mathcal{F}(\eta)$, we need hypotheses a) and b) above.

Hence (8.18) becomes

$$\|\xi(t)\|^2_{L^2(\Omega)} + \int_0^t |||\xi(s)|||^2_{\mathrm{DG}}\,\mathrm{d}s < \frac{1}{1+T}\,\mathrm{e}^{-C_3T}\times C_{\mathrm{inv}}^{-1}C_0^{-2}\left(\max_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa^2}\right)^2 + C_2\int_0^t \|\xi(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s,$$

which, by the Gronwall–Bellman inequality, implies that

$$\|\xi(t)\|^2_{L^2(\Omega)} < C_{\mathrm{inv}}^{-1}C_0^{-2}\left(\max_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa^2}\right)^2 \quad \text{for all} \quad t\in[0,t_\star].$$

By the inverse inequality (8.19) we have that,

$$\|\xi(t)\|^2_{H^1(\Omega,\mathcal{T})} < C_0^{-2}\left(\max_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa^2}\right)^2\left(\max_{\kappa\in\mathcal{T}}\frac{p_\kappa^2}{h_\kappa}\right)^2 = C_0^{-2}\left(\max_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa^2}\right)^2\left(\min_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa^2}\right)^{-2},$$

for all $t \in [0, t_\star]$, which, by the quasi–uniformity hypothesis c) above, is $\leq 1$. Hence, then, for $h \leq h_0$, we have

$$\|\xi\|^2_{H^1(\Omega,\mathcal{T})} < 1 \quad \text{for all} \quad t \in [0, t_\star].$$

By continuity of the mapping $t \mapsto \|\xi(t)\|^2_{H^1(\Omega,\mathcal{T})}$, the assumption $t_\star < T$ implies that either $\|\xi(t)\|^2_{H^1(\Omega,\mathcal{T})} \leq 1$ for all $t \in [0, T]$, or that there exists a time $t_{\star\star} \in (t_\star, T]$ such that $\|\xi(t_{\star\star})\|^2_{H^1(\Omega,\mathcal{T})} = 1$.

In either case, we arrive at a contradiction with the fact that $t_\star$ is the largest time in the interval $[0, T]$ such that, for all $t \in [0, t_\star]$, we have $\|\xi(t)\|^2_{H^1(\Omega,\mathcal{T})} \leq 1$. Thus we deduce that $t_\star = T$, for $0 < h \leq h_0$.

From (8.18) by the Gronwall–Bellman inequality we obtain

$$\|\xi(t)\|^2_{L^2(\Omega)} + \int_0^t \|\xi(s)\|^2_{H^1(\Omega,\mathcal{T})}\,\mathrm{d}s \leq C \int_0^t \mathcal{F}(\eta(s))\,\mathrm{d}s, \quad 0 \leq t \leq T,$$

and hence

$$\int_0^t \|\xi(s)\|^2_{H^1(\Omega,\mathcal{T})}\,\mathrm{d}s \leq C \int_0^t \mathcal{F}(\eta(s))\,\mathrm{d}s, \quad 0 \leq t \leq T,$$

with the constant $C > 0$ depending only on the domain $\Omega$, the quasi–uniformity of $\mathcal{T}$, on the time $T$, the growth–condition for the function $f$, and on Lebesgue and Sobolev norms of $u$ over the time interval $[0, T]$.

Employing the triangle inequality yields

$$\int_0^t \|(u - u_{\mathrm{DG}})(s)\|^2_{H^1(\Omega,\mathcal{T})}\,\mathrm{d}s \leq C \int_0^t \left\{ \|\eta\|^2_{H^1(\Omega,\mathcal{T})} + \mathcal{F}(\eta(s)) \right\} \mathrm{d}s, \quad 0 \leq t \leq T,$$

and hence (8.12). □

Our next result concerns the accuracy of the $hp$–version NSIP DGFEM (6.1).

**Theorem 8.6** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded polyhedral domain, $\mathcal{T} = \{\kappa\}$ a shape–regular and quasi–uniform subdivision of $\Omega$ into $n$–parallelepipeds, and $\mathbf{p}$ a polynomial degree vector of bounded local variation. Let each face $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$ be assigned a positive real number*

$$\sigma_e = \frac{\langle p \rangle_e}{h_e}, \tag{8.20}$$

*where $h_e$ is the diameter of $e$, with the convention that for $e \in \mathcal{E}_{\mathrm{D}}$ the contributions from outside $\Omega$ in the definition of $\sigma_e$ are set to 0. Suppose that the function $f \in C^1(\mathbb{R})$, that $f$ satisfies the growth–condition (2.2) for some positive constant $C_{\mathrm{g}}$, and that Hypothesis A holds. Then, if $u(\cdot, t)|_\kappa \in H^{k_\kappa}(\kappa)$ with $k_\kappa \geq 3.5$ on each $\kappa \in \mathcal{T}$, there exists $h_0 > 0$ such that for all $h \in (0, h_0]$, $h = \max_{\kappa \in \mathcal{T}} h_\kappa$, and all $t \in [0, T]$, the solution $u_{\mathrm{DG}}(\cdot, t) \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$ of the NSIP DGFEM (6.1) satisfies the following error bound:*

$$\|u - u_{\mathrm{DG}}\|^2_{L^2(0,T;H^1(\Omega,\mathcal{T}))} \leq C \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{\mathfrak{X}}\,; \tag{8.21}$$

*with* $1 \le s_\kappa \le \min(p_\kappa + 1, k_\kappa)$, $p_\kappa \ge 2$ *on each* $\kappa \in \mathcal{T}$, *where* $C$ *is a positive constant depending only on the domain* $\Omega$, *the shape–regularity and quasi–uniformity of* $\mathcal{T}$, *the time* $T$, *the growth–condition on the function* $f$, *the parameter* $\rho$ *in* (8.1), *on* $k = \max_{\kappa\in\mathcal{T}} k_\kappa$, *and on the Lebesgue and Sobolev norms of* $u$ *over the time interval [0,T]; the norm* $\|u\|^2_{\mathfrak{X}}$ *signifies the collection of norms* $\|u\|^2_{L^2(0,T;H^{k_\kappa}(\kappa))} + \|\dot u\|^2_{L^2(0,T;H^{k_\kappa-1}(\kappa))}$.

**Proof.** Let us choose the projector $\Pi$ to be the projection operator $u \mapsto z^{h_\kappa}_{p_\kappa}(u)$, defined in Section 7. From Theorem 7.3, inequalities (7.2)–(7.4), we have the estimates

$$\|\eta\|^2_{L^2(\partial\kappa)} \le C\frac{h_\kappa^{2s_\kappa-1}}{p_\kappa^{2k_\kappa-1}}\|u\|^2_{H^{k_\kappa}(\kappa)}, \qquad \|\nabla\eta\|^2_{L^2(\partial\kappa)} \le C\frac{h_\kappa^{2s_\kappa-3}}{p_\kappa^{2k_\kappa-3}}\|u\|^2_{H^{k_\kappa}(\kappa)},$$
$$\|\eta\|^2_{H^1(\kappa)} \le C\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-2}}\|u\|^2_{H^{k_\kappa}(\kappa)}, \qquad \|\eta\|^2_{L^2(\kappa)} \le C\frac{h_\kappa^{2s_\kappa}}{p_\kappa^{2k_\kappa}}\|u\|^2_{H^{k_\kappa}(\kappa)}.$$

Let us collect all the terms on the right–hand side of (8.12), except $\|\eta\|^2_{L^{2(\alpha+1)}(\kappa)}$:

$$\begin{aligned} I \equiv C\sum_{\kappa\in\mathcal{T}}\int_0^t \Big\{ &\|\dot\eta(s)\|^2_{L^2(\kappa)} + \|\eta(s)\|^2_{H^1(\kappa)} + \|\sqrt{\sigma}\eta(s)\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} \\ &+ \|\sqrt{\sigma}\,[\eta(s)]\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} + \|\sqrt{\tau}\eta(s)\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} + \left\|\frac{1}{\sqrt{\sigma}}\nabla\eta(s)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} \\ &+ \|\sqrt{\tau}\,[\eta(s)]\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} + \left\|\frac{1}{\sqrt{\sigma}}\nabla\eta(s)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})}\Big\}\,{\rm d}s. \end{aligned}$$

From the above approximation results, by choosing $\sigma_e$ as in (8.20), noting (8.1) and the shape–regularity of $\mathcal{T}$ to relate $h_e$ to $h_\kappa$, and taking the maximum over $t \in [0,T]$, we obtain

$$\begin{aligned} I &\le C\sum_{\kappa\in\mathcal{T}}\left\{\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-2}}\|\dot u\|^2_{L^2(0,T;H^{k_\kappa-1}(\kappa))} + \left(\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-2}} + \frac{p_\kappa^2}{h_\kappa}\frac{h_\kappa^{2s_\kappa-1}}{p_\kappa^{2k_\kappa-1}}\right)\|u\|^2_{L^2(0,T;H^{k_\kappa}(\kappa))}\right\} \\ &\le C\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}}\|u\|^2_{\mathfrak{X}}, \qquad (8.22) \end{aligned}$$

with $1 \le s_\kappa \le \min(p_\kappa+1, k_\kappa)$, $p_\kappa \ge 2$ on each $\kappa \in \mathcal{T}$, where $C$ is a positive constant depending only on the domain $\Omega$, the shape–regularity and quasi–uniformity of $\mathcal{T}$, the time $T$, the growth–condition for the function $f$, the parameter $\rho$ in (8.1), on $k = \max_{\kappa\in\mathcal{T}} k_\kappa$, and on the Lebesgue and Sobolev norms of $u$ over the time interval $[0,T]$.

Further, by the broken Sobolev–Poincaré inequality [27] we have the bound

$$\|\eta\|^2_{L^{2(\alpha+1)}(\Omega)} \le C\left(\sum_{\kappa\in\mathcal{T}}\|\nabla\eta\|^2_{L^2(\kappa)} + \sum_{e\in\mathcal{E}_{\rm int}} h_e^{-1}\int_e [\eta]^2\,{\rm d}s + \sum_{e\in\mathcal{E}_D} h_e^{-1}\int_e \eta^2\,{\rm d}s\right),$$

and thus by the above approximation bounds, by noting the shape–regularity of $\mathcal{T}$ to relate $h_e$ to $h_\kappa$, we obtain

$$\sum_{\kappa\in\mathcal{T}}\|\eta\|^2_{L^{2(\alpha+1)}(\kappa)} \le C\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-2}}\|u\|^2_{H^{k_\kappa}(\kappa)},$$

with the constant $C$ as above.

Applying this bound to the right–hand side of (8.12), noting (8.22), and taking the maximum over $0 \leq t \leq T$, we obtain the desired bound. $\square$

**Remark 8.7**

1. The estimate (8.21) is optimal in $h$ and $p$–suboptimal by $p^{\frac{1}{2}}$.
2. By the broken Sobolev–Poincaré inequality, the same bound holds for the $L^2$–norm of the error. The bound in this case is not $hp$–optimal.
3. From the error bound we conclude that the presence of the non–linearity $f(\cdot)$, satisfying the conditions stated in Section 2, does not diminish the rate of $hp$–convergence rate in the $H^1$–norm compared to the linear case. $\square$

**8.2. The Symmetric Version of DGFEM.** The symmetric version of the interior penalty discontinuous Galerkin finite element method appeared in the literature much earlier than the non–symmetric formulation, (see Wheeler [36]). It was not widely accepted as an effective numerical method until very recently, due to an additional condition on the size of the penalty parameter which is required in order to ensure the coercivity of the bilinear form of the method; this will be discussed in the next section. The renewed interest in the symmetric formulation of the IP DGFEM is due to the optimality of its convergence rate in the $L^2$–norm and for linear functionals of the solution.

The non–symmetric formulation of the IP method suffers from lack of adjoint consistency (see [6, 5]), and results in suboptimal *a priori* error bounds in the $L^2$–norm and in linear functionals of the solutions. The symmetric version, due to its adjoint consistency, does not suffer from these drawbacks.

We start our *a priori* error analysis in the $L^2$–norm by deriving the error bounds on the *broken elliptic projector* defined by the symmetric version of the interior penalty discontinuous Galerkin finite element method. This part of the $L^2$–norm error analysis is crucial, as it will allow us to remove the terms in the error bound, containing the $H^1$–seminorm, which would otherwise result in a suboptimal convergence rate in the $L^2$–norm.

**8.2.1. The Broken Elliptic Projector.** Consider the boundary value problem for the elliptic equation in the form

$$\begin{aligned} -\Delta u &= 0 \quad \text{in} \quad \Omega, \\ u &= g_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}}, \\ \nabla u \cdot \mathbf{n} &= g_{\mathrm{N}} \quad \text{on} \quad \Gamma_{\mathrm{N}}, \end{aligned} \tag{8.23}$$

with $\overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}} = \partial\Omega$, $\Gamma_{\mathrm{D}}$ having positive measure, and $g_{\mathrm{D}} \in H^{\frac{1}{2}}(\Gamma_{\mathrm{D}})$, $g_{\mathrm{N}} \in L^2(\Gamma_{\mathrm{N}})$. We shall also assume that the solution $u$ exists, that it is unique, and that $u \in \mathfrak{A}$.

In view of Section 4, the SIP formulation of the DGFEM for this problem is

$$\text{find } u_{\mathrm{DG}} \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F}) \text{ such that } B_{\mathrm{S}}(u_{\mathrm{DG}}, v) = l_{\mathrm{S}}(v) \text{ for all } v \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F}), \tag{8.24}$$

where the symmetric bilinear form $B_{\mathrm{S}}$ is defined by (4.1), and the linear functional $l_{\mathrm{S}}$ is defined by (4.2), with $\theta = -1$.

Let us check whether and under what conditions the solution $u_{\mathrm{DG}}$ to (8.24) exists and is unique.

The proof of continuity of the symmetric bilinear form $B_{\rm S}(u,v)$ is essentially the same as in the non–symmetric case (see Lemma 8.4). The coercivity, though, requires further investigation.

For the symmetric bilinear form (4.1) (with $\theta=-1$), we have, for any $w\in S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F})$,

$$B_{\rm S}(w,w)=\sum_{\kappa\in\mathcal{T}}\|\nabla w\|^2_{L^2(\kappa)}+\int_{\Gamma_{\rm D}}\big(\sigma w^2-2w(\nabla w\cdot\mathbf{n})\big){\rm d}s+\int_{\Gamma_{\rm int}}\Big(\sigma\,[w]^2-2\,[w]\,\langle\nabla w\cdot\nu\rangle\Big){\rm d}s.$$

Clearly the integrands in the last two terms need not be positive unless $\sigma$ is chosen sufficiently large: the purpose of the analysis that now follows is to explore just how large $\sigma$ needs to be to ensure coercivity of $B_{\rm S}(\cdot,\cdot)$ over $S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F})\times S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F})$.

For any positive number $\tau_e$ we have

$$-2\int_{\Gamma_{\rm D}}w(\nabla w\cdot\mathbf{n})\,{\rm d}s\geq-\sum_{e\in\mathcal{E}_{\rm D}}\left(\int_e\tau_e w^2\,{\rm d}s+\int_e\tau_e^{-1}(\nabla w\cdot\mathbf{n})^2\,{\rm d}s\right).$$

Omitting the summations, the second term on the right–hand side can be further bounded by using the inverse inequality (8.9), the shape–regularity condition (to relate $h_\kappa$ to $h_e$, where $\kappa$ is the element whose face is $e\in\mathcal{E}_{\rm D}$) and the bounded local variation condition (to relate $p_\kappa^2$ to $\langle p^2\rangle_e$), by absorbing all constants into $C_\tau$, we obtain

$$-\int_e\tau_e^{-1}(\nabla w\cdot\mathbf{n})^2\,{\rm d}s\geq-\int_e\tau_e^{-1}\,|\nabla w|^2\,|\mathbf{n}|^2\,{\rm d}s\geq-\tau_e^{-1}C_\tau\frac{\langle p^2\rangle_e}{h_e}\,\|\nabla w\|^2_{L^2(\kappa)}\,.$$

Similarly, for the term involving interior faces, we have

$$-2\int_{\Gamma_{\rm int}}[w]\,\langle\nabla w\cdot\nu\rangle\,{\rm d}s\geq-\sum_{e\in\mathcal{E}_{\rm int}}\left(\int_e\tau_e\,[w]^2\,{\rm d}s+\tau_e^{-1}C_\tau\frac{\langle p^2\rangle_e}{h_e}\left(\|\nabla w\|_{L^2(\kappa')}\,\|\nabla w\|_{L^2(\kappa'')}\right)\right),$$

using the restriction imposed by the bounded local variation condition (8.1): here $\kappa'$ and $\kappa''$ are the two elements that have $e$ as their common face.

Now letting

$$\tau_e^{-1}:=\frac{1}{2n\cdot2^{n-1}}\left(C_\tau\frac{\langle p^2\rangle_e}{h_e}\right)^{-1}\quad\text{for}\quad e\in\mathcal{E}_{\rm D}\cup\mathcal{E}_{\rm int},$$

we get

$$B_{\rm S}(w,w)\geq\frac{1}{2}\sum_{\kappa\in\mathcal{T}}\|\nabla w\|^2_{L^2(\kappa)}+\int_{\Gamma_{\rm D}}(\sigma-\tau)w^2\,{\rm d}s+\int_{\Gamma_{\rm int}}(\sigma-\tau)\,[w]^2\,{\rm d}s.$$

Thus the symmetric bilinear form $B_{\rm S}(u,v)$ is coercive if

$$\sigma_e\geq\tau_e=2n\cdot2^{n-1}C_\tau\frac{\langle p^2\rangle_e}{h_e}.$$

The factor $2n\cdot2^{n-1}$ stems for the fact that in $n$ dimensions the summation over $e\in\mathcal{E}_{\rm int}$ may count, any one element $\kappa$, $2n\cdot2^{n-1}$ times, as we allow one hanging node per interface.

Choosing $\sigma_e$ appropriately, i.e.,

$$\sigma_e = C_\sigma \frac{\langle p^2 \rangle_e}{h_e} \tag{8.25}$$

with the constant $C_\sigma > 0$ large enough, $\sigma_e \geq \tau_e$ will be ensured and by the Lax–Milgram theorem the solution to (8.24) then exists and is unique.

In view of the above arguments, we conclude that the SIP DGFEM solution of the problem (8.23) uniquely determines the projection operator $\Pi_{\mathrm{e}}$ on $\mathfrak{A}$ onto the finite element space $S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$ with the property (for $u \in \mathfrak{A}$)

$$B_{\mathrm{S}}(u - \Pi_{\mathrm{e}} u, v) = 0 \quad \text{for all} \quad v \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F}). \tag{8.26}$$

Next, we state the approximation error bounds in the $H^1$– and $L^2$–norms for the *broken elliptic projector* $\Pi_{\mathrm{e}}$.

**Lemma 8.8** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded polyhedral domain, $\mathcal{T} = \{\kappa\}$ a shape–regular subdivision of $\Omega$ into $n$–parallelepipeds, and suppose that $u|_\kappa \in H^{k_\kappa}(\kappa)$ for some Sobolev index $k_\kappa \geq 2$ and $\kappa \in \mathcal{T}$. Let $\Pi_{\mathrm{e}} u$ be the projection of $u \in \mathfrak{A}$ onto $S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$, defined by (8.26), with $p_\kappa \geq 0$ for $\kappa \in \mathcal{T}$, and $\sigma_e$ chosen as in (8.25). Then, the following error estimate holds:*

$$\|u - \Pi_{\mathrm{e}} u\|^2_{H^1(\Omega,\mathcal{T})} \leq C \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{H^{k_\kappa}(\kappa)} . \tag{8.27}$$

*Furthermore, if $\Omega$ is convex, then*

$$\|u - \Pi_{\mathrm{e}} u\|^2_{L^2(\Omega)} \leq C \left( \max_{\kappa \in \mathcal{T}} \frac{h_\kappa^2}{p_\kappa} \right) \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{H^{k_\kappa}(\kappa)} , \tag{8.28}$$

*where $s_\kappa = \min(p_\kappa + 1, k_\kappa)$, and the constant $C$ is independent of $u$, $p_\kappa$ and $h_\kappa$, but dependent on $k = \max_{\kappa \in \mathcal{T}} k_\kappa$ and $C_\sigma$.*

**Proof.** By recalling the definition of the DG–norm (8.3), we have, from the assumption on $\sigma_e$, that

$$|||u|||^2_{\mathrm{DG}} \leq B_{\mathrm{S}}(u,u) \quad \text{for all} \quad u \in \mathfrak{A},$$

and thus by writing $u - \Pi_{\mathrm{e}} u = (u - \Pi u) - (\Pi u - \Pi_{\mathrm{e}} u) = \eta + \xi$, where the projection operator $\Pi$ will be chosen later, taking $v \equiv \xi$ in the definition of the broken elliptic projector (8.26), we deduce that

$$|||\xi|||^2_{\mathrm{DG}} \leq B_{\mathrm{S}}(\xi + \eta - \eta, \xi) \leq |B_{\mathrm{S}}(\xi + \eta, \xi)| + |B_{\mathrm{S}}(\eta, \xi)| = |B_{\mathrm{S}}(\eta, \xi)| .$$

By continuity of the bilinear form $B_{\mathrm{S}}(\eta, \xi)$ (see Lemma 8.4 and the comments above), after applying Young inequality we have

$$\begin{aligned} |||\xi|||^2_{\mathrm{DG}} \leq C \sum_{\kappa \in \mathcal{T}} \Big( & \left\|\sqrt{\sigma}\eta\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{D}})} + \left\|\sqrt{\sigma}\,[\eta]\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{int}})} + \|\nabla\eta\|^2_{L^2(\kappa)} + \left\|\sqrt{\tau}\eta\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{D}})} \\ & + \left\|\frac{1}{\sqrt{\sigma}}\nabla\eta\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{D}})} + \left\|\sqrt{\tau}\,[\eta]\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{int}})} + \left\|\frac{1}{\sqrt{\sigma}}\nabla\eta\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\mathrm{int}})} \Big), \end{aligned} \tag{8.29}$$

where $\tau_e = 2n \cdot 2^{n-1} C_\tau \left\langle p^2 \right\rangle_e / h_e$, $h_e$ is the diameter of a face $e \in \mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm D}$, and for $e \in \mathcal{E}_{\rm D}$ the contribution from outside $\Omega$ is set to 0.

Further, by noting that $\sum_{\kappa\in\mathcal{T}} \|\nabla\xi\|^2_{L^2(\kappa)} \leq |||\xi|||^2_{\rm DG}$, and employing the triangle inequality (8.4), we obtain the bound

$$\sum_{\kappa\in\mathcal{T}} \|u - \Pi_{\rm e} u\|^2_{H^1(\kappa)} \leq C \sum_{\kappa\in\mathcal{T}} \Bigg( \left\|\sqrt{\sigma}\eta\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} + \left\|\sqrt{\sigma}\,[\eta]\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} + \|\eta\|^2_{H^1(\kappa)}$$
$$+ \left\|\sqrt{\tau}\eta\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} + \left\|\frac{1}{\sqrt{\sigma}}\nabla\eta\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})}$$
$$+ \left\|\sqrt{\tau}\,[\eta]\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} + \left\|\frac{1}{\sqrt{\sigma}}\nabla\eta\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})}\Bigg), \quad (8.30)$$

Let us choose $\Pi$ to be the $u \mapsto z^{h_\kappa}_{p_\kappa}(u)$ (see Section 7). From Theorem 7.3, inequalities (7.2)–(7.4), we have the estimates

$$\|\eta\|^2_{L^2(\partial\kappa)} \leq C \frac{h_\kappa^{2s_\kappa-1}}{p_\kappa^{2k_\kappa-1}} \|u\|^2_{H^{k_\kappa}(\kappa)}, \quad \|\nabla\eta\|^2_{L^2(\partial\kappa)} \leq C \frac{h_\kappa^{2s_\kappa-3}}{p_\kappa^{2k_\kappa-3}} \|u\|^2_{H^{k_\kappa}(\kappa)},$$
$$\|\eta\|^2_{H^1(\kappa)} \leq C \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-2}} \|u\|^2_{H^{k_\kappa}(\kappa)}.$$

Applying these inequalities to the right–hand side of (8.30), choosing $\sigma_e$ as in (8.25), noting the bounded local variation condition (8.1) and the shape regularity of $\mathcal{T}$ to relate $h_e$ to $h_\kappa$, we obtain

$$\|u - \Pi_{\rm e} u\|^2_{H^1(\Omega,\mathcal{T})} \leq C \sum_{\kappa\in\mathcal{T}} \left( \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-2}} + \frac{p_\kappa^2}{h_\kappa} \frac{h_\kappa^{2s_\kappa-1}}{p_\kappa^{2k_\kappa-1}} \right) \|u\|^2_{H^{k_\kappa}(\kappa)},$$

and hence (8.27).

Let us note that the same bound (8.27) is also valid for the DG–norm $|||u - \Pi_{\rm e} u|||_{\rm DG}$; this follows from (8.29) and the fact that

$$|||u - \Pi_{\rm e} u|||^2_{\rm DG} \leq C \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|^2_{H^{k_\kappa}(\kappa)}.$$

To estimate $\|u - \Pi_{\rm e} u\|_{L^2(\Omega)}$, we shall use the Aubin–Nitsche duality argument (see [11]).

Let $(\cdot,\cdot)$ signify the $L^2$–inner product. Then, for every $g \in L^2(\Omega)$, by the Cauchy–Schwarz inequality we have

$$(u - \Pi_{\rm e} u, g) \leq \|u - \Pi_{\rm e} u\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)},$$

and therefore

$$\|u - \Pi_{\rm e} u\|_{L^2(\Omega)} = \sup_{\substack{g\in L^2(\Omega)\\ g\neq 0}} \frac{(u - \Pi_{\rm e} u, g)}{\|g\|_{L^2(\Omega)}}. \quad (8.31)$$

Further, let the function $w \in H^2(\Omega)$ be the solution of the problem

$$\begin{aligned} -\Delta w &= g \quad \text{in} \quad \Omega, \\ w &= 0 \quad \text{on} \quad \Gamma_{\rm D}, \\ \nabla w \cdot \mathbf{n} &= 0 \quad \text{on} \quad \Gamma_{\rm N}, \end{aligned} \quad (8.32)$$

with $g \in L^2(\Omega)$, and $\Gamma_{\rm D}$, $\Gamma_{\rm N}$ as in (8.23). Then the SIP DGFEM formulation of this problem is

$$\text{find } \ w \in \mathfrak{A} \ \text{ such that } \ B_{\rm S}(w,v) = l_g(v) \ \text{ for all } \ v \in H^2(\Omega,\mathcal{T}),$$

where $B_{\rm S}(w,v)$ is defined by (4.1) with $\theta = -1$, and

$$l_g(v) = (g,v) + l_{\rm S}(v),$$

with $l_{\rm S}(v)$ defined by (4.2) with $\theta = -1$ and $g_{\rm D} = 0$, $g_{\rm N} = 0$: clearly, then, $l_{\rm S}(v) = 0$ for all $v$ in $H^2(\Omega,\mathcal{T})$.

Consider the SIP DGFEM approximation of (8.32) in the form

$$\text{find } \ w_{\rm DG} \in S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F}) \ \text{ such that } \ B_{\rm S}(w_{\rm DG},v) = l_g(v) \ \text{ for all } \ v \in S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F}).$$

By Galerkin orthogonality, we have

$$B_{\rm S}(u - \Pi_{\rm e} u, w_{\rm DG}) = 0,$$

and thence

$$\begin{aligned}(u - \Pi_{\rm e} u, g) = (g, u - \Pi_{\rm e} u) &= l_g(u - \Pi_{\rm e} u) = B_{\rm S}(w, u - \Pi_{\rm e} u)\\ &= B_{\rm S}(u - \Pi_{\rm e} u, w) = B_{\rm S}(u - \Pi_{\rm e} u, w - \Pi w),\end{aligned}$$

where $\Pi$ is the projection operator $u \mapsto z_{p_\kappa}^{h_\kappa}(u)$.

Further, by Lemma 8.4, (8.6), and by noting that the bilinear form $B_{\rm S}(\cdot,\cdot)$ is symmetric, we have

$$\begin{aligned}(u-&\Pi_{\rm e} u, g) \le B_{\rm S}(u - \Pi_{\rm e} u, w - \Pi w) \le C\, |||u - \Pi_{\rm e} u|||_{\rm DG}\\ &\times \left\{ \int_{\Gamma_{\rm D}} \sigma\, |w - \Pi w|^2 \,{\rm d}s + \int_{\Gamma_{\rm int}} \sigma\, [w - \Pi w]^2 \,{\rm d}s + \sum_{\kappa\in\mathcal{T}} \|\nabla(w - \Pi w)\|^2_{L^2(\kappa)} \right.\\ &+ \sum_{\kappa\in\mathcal{T}} \left( \left\|\sqrt{\tau}(w - \Pi w)\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} + \left\| \frac{1}{\sqrt{\sigma}} \nabla(w - \Pi w) \right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm D})} \right)\\ &\left. + \sum_{\kappa\in\mathcal{T}} \left( \left\|\sqrt{\tau}\,[w - \Pi w]\right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} + \left\| \frac{1}{\sqrt{\sigma}} \nabla(w - \Pi w) \right\|^2_{L^2(\partial\kappa\cap\Gamma_{\rm int})} \right) \right\}^{\frac{1}{2}}\end{aligned} \tag{8.33}$$

with $\tau_e = 2n \cdot 2^{n-1} C_\tau \left\langle p^2 \right\rangle_e / h_e$.

By the previous argument, we have the estimate

$$|||u - \Pi_{\rm e} u|||^2_{\rm DG} \le C \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \, \|u\|^2_{H^{k_\kappa}(\kappa)}\,, \tag{8.34}$$

and from Theorem 7.3, inequalities (7.2)–(7.4), we have the estimates

$$\|w - \Pi w\|^2_{L^2(\partial\kappa)} \le C \frac{h_\kappa^3}{p_\kappa^3} \|w\|^2_{H^2(\kappa)}\,, \quad \|\nabla(w - \Pi w)\|^2_{L^2(\partial\kappa)} \le C \frac{h_\kappa}{p_\kappa} \|w\|^2_{H^2(\kappa)}\,,$$

$$\|\nabla(w - \Pi w)\|^2_{L^2(\kappa)} \le C \frac{h_\kappa^2}{p_\kappa^2} \|w\|^2_{H^2(\kappa)}\,.$$

Applying these inequalities and the estimate (8.34) to the right–hand side of (8.33), choosing $\sigma_e$ as in (8.25) and noting the bounded local variation condition (8.1) and the shape regularity of $\mathcal{T}$ to relate $h_e$ to $h_\kappa$, we obtain

$$(u - \Pi_{\mathrm{e}} u, g) \leq C \left( \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{H^{k_\kappa}(\kappa)} \times \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^2}{p_\kappa} \|w\|^2_{H^2(\kappa)} \right)^{\frac{1}{2}}.$$

Further, by noting that for a suitable constant $C > 0$ we have

$$\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^2}{p_\kappa} \|w\|^2_{H^2(\kappa)} \leq C \left( \max_{\kappa \in \mathcal{T}} \frac{h_\kappa^2}{p_\kappa} \right) \sum_{\kappa \in \mathcal{T}} \|w\|^2_{H^2(\kappa)} = C \left( \max_{\kappa \in \mathcal{T}} \frac{h_\kappa^2}{p_\kappa} \right) \|w\|^2_{H^2(\Omega)},$$

and, recalling that $\Omega$ is convex, on employing elliptic regularity, we obtain

$$(u - \Pi_{\mathrm{e}} u, g) \leq C \left( \left( \max_{\kappa \in \mathcal{T}} \frac{h_\kappa^2}{p_\kappa} \right) \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{H^{k_\kappa}(\kappa)} \right)^{\frac{1}{2}} \|g\|_{L^2(\Omega)},$$

and therefore

$$\frac{(u - \Pi_{\mathrm{e}} u, g)}{\|g\|_{L^2(\Omega)}} \leq C \left( \left( \max_{\kappa \in \mathcal{T}} \frac{h_\kappa^2}{p_\kappa} \right) \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{H^{k_\kappa}(\kappa)} \right)^{\frac{1}{2}}.$$

Noting (8.31), taking the supremum over $g \in L^2(\Omega)$, $g \neq 0$, and squaring the resulting expression yields (8.28). □

**8.2.2. *A priori* Error Bounds.** Having defined the broken elliptic projector and obtained the respective approximation error bounds, we are ready to state our main result about the accuracy of the symmetric version of the $hp$–DGFEM.

**Theorem 8.9** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded convex polyhedral domain, $\mathcal{T} = \{\kappa\}$ a shape–regular subdivision of $\Omega$ into $n$–parallelepipeds, and $\mathbf{p}$ a polynomial degree vector of bounded local variation. Let each face $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$ be assigned a real positive number*

$$\sigma_e = C_\sigma \frac{\langle p^2 \rangle_e}{h_e}, \tag{8.35}$$

*where $h_e$ is the diameter of $e$, with the convention that for $e \in \mathcal{E}_{\mathrm{D}}$ the contributions from outside $\Omega$ in the definition of $\sigma_e$ are set to 0, and $C_\sigma$ is sufficiently large. Suppose that the function $f \in C^1(\mathbb{R})$ and obeys the growth–condition (2.2) for some positive constant $C_{\mathrm{g}}$, and that Hypothesis A holds. Then, if $u(\cdot, t)|_\kappa \in H^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, $\kappa \in \mathcal{T}$, for $0 \leq t \leq T$ there exists $h_0 > 0$ such that for all $0 < h \leq h_0$, $h = \max_{\kappa \in \mathcal{T}} h_\kappa$, the solution $u_{\mathrm{DG}}(\cdot, t) \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$ of the SIP DGFEM (6.1) obeys the following error bounds:*

$$\operatorname*{ess.sup}_{0 \leq t \leq T} |||u - u_{\mathrm{DG}}|||^2_{\mathrm{DG}} \leq C \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{\mathfrak{X}_1} \tag{8.36}$$

*and*

$$\|u-u_{\rm DG}\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\Bigg(\max_{\kappa\in\mathcal{T}}\frac{h_\kappa^2}{p_\kappa}\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}}\|u\|^2_{\mathfrak{X}_2} + \max_{\kappa\in\mathcal{T}} h_\kappa^{2-\frac{\alpha n}{\alpha+1}}\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}}\|u\|^2_{L^2(0,T;H^{k_\kappa}(\kappa))}\Bigg), \quad (8.37)$$

*with* $1\le s_\kappa\le \min(p_\kappa+1,k_\kappa)$, $p_\kappa\ge 1$, *for* $\kappa\in\mathcal{T}$, *where* $C$ *is a positive constant depending only on the domain* $\Omega$, *the shape–regularity of* $\mathcal{T}$, *the final time* $T$, *the growth–condition for the function* $f$, *the parameter* $\rho$ *in* (8.1), *the Lebesgue and Sobolev norms of* $u$, *and* $k=\max_{\kappa\in\mathcal{T}}k_\kappa$; *the norms* $\|u\|^2_{\mathfrak{X}_{1,2}}$ *signify the collection of norms* $\|u\|^2_{L^\infty(0,T;H^{k_\kappa}(\Omega))}+\|u\|^2_{L^2(0,T;H^{k_\kappa}(\Omega))}+\|\dot u\|^2_{L^2(0,T;H^{k_\kappa}(\Omega))}$ *and* $\|u\|^2_{L^\infty(0,T;H^{k_\kappa}(\kappa))}+\|\dot u\|^2_{L^2(0,T;H^{k_\kappa}(\kappa))}$, *respectively.*

**Proof.** By the same argument as in the proof of Lemma 8.5, upon subtracting (8.13) from (8.14) and choosing $v=\dot\xi$, we obtain

$$\|\dot\xi\|^2_{L^2(\Omega)} + B_{\rm S}(\xi,\dot\xi) = \sum_{\kappa\in\mathcal{T}}\int_\kappa \{f(u)-f(u_{\rm DG})\}\,\dot\xi\,{\rm d}x - \sum_{\kappa\in\mathcal{T}}\int_\kappa \dot\eta\dot\xi\,{\rm d}x - B_{\rm S}(\eta,\dot\xi). \quad (8.38)$$

Let us choose the projection operator $\Pi$ to be the broken elliptic projector $\Pi_{\rm e}$. Then, by definition (8.25), $B_{\rm S}(\eta,\dot\xi)=0$.

With the constant $C_\sigma$ in (8.35) chosen large enough, the symmetric bilinear form $B_{\rm S}(\cdot,\cdot)$ is coercive, and therefore defines an inner product on $H^1(\Omega,\mathcal{T})$, which induces the norm $|||\cdot|||_{\rm DG}$ on this space. Hence we deduce that

$$B_{\rm S}(\xi,\dot\xi) = \frac{1}{2}\frac{\rm d}{{\rm d}t}|||\xi|||^2_{\rm DG}.$$

Thus, we can rewrite (8.38) in the form

$$\begin{aligned}\|\dot\xi\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\rm d}{{\rm d}t}|||\xi|||^2_{\rm DG} &= \sum_{\kappa\in\mathcal{T}}\int_\kappa \{f(u)-f(u_{\rm DG})\}\,\dot\xi\,{\rm d}x - \sum_{\kappa\in\mathcal{T}}\int_\kappa \dot\eta\dot\xi\,{\rm d}x\\ &\le \left|\sum_{\kappa\in\mathcal{T}}\int_\kappa \dot\eta\dot\xi\,{\rm d}x\right| + \left|\sum_{\kappa\in\mathcal{T}}\int_\kappa \{f(u)-f(\Pi_{\rm e}u)\}\,\dot\xi\,{\rm d}x\right|\\ &\quad + \left|\sum_{\kappa\in\mathcal{T}}\int_\kappa \{f(\Pi_{\rm e}u)-f(u_{\rm DG})\}\,\dot\xi\,{\rm d}x\right|. \quad (8.39)\end{aligned}$$

By the Cauchy–Schwarz and Young inequalities, we have

$$\left|\sum_{\kappa\in\mathcal{T}}\int_\kappa \dot\eta\dot\xi\,{\rm d}x\right| \le \left(\sum_{\kappa\in\mathcal{T}}\|\dot\eta\|^2_{L^2(\kappa)}\right)^{\frac12}\left(\sum_{\kappa\in\mathcal{T}}\|\dot\xi\|^2_{L^2(\kappa)}\right)^{\frac12} \le \frac{\varepsilon_1}{2}\|\dot\eta\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon_1}\|\dot\xi\|^2_{L^2(\Omega)},$$

and

$$\left|\sum_{\kappa\in\mathcal{T}}\int_\kappa \{f(u)-f(\Pi_{\rm e}u)\}\,\dot\xi\,{\rm d}x\right| \le \frac{\varepsilon_2}{2}\|f(u)-f(\Pi_{\rm e}u)\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon_2}\|\dot\xi\|^2_{L^2(\Omega)},$$

$$\left|\sum_{\kappa\in\mathcal{T}}\int_{\kappa}\{f(\Pi_{\mathrm{e}}u)-f(u_{\mathrm{DG}})\}\,\dot{\xi}\,\mathrm{d}x\right|\leq\frac{\varepsilon_3}{2}\left\|f(\Pi_{\mathrm{e}}u)-f(u_{\mathrm{DG}})\right\|^2_{L^2(\Omega)}+\frac{1}{2\varepsilon_3}\|\dot{\xi}\|^2_{L^2(\Omega)},$$

with $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$.

Next, by the result of Lemma 8.1, we have, upon absorbing the constants into $C$, and noting Hypothesis A,

$$\begin{aligned}
\|f(u)-f(\Pi_{\mathrm{e}}u)\|^2_{L^2(\Omega)} &\leq C\,\|\eta\|^2_{L^q(\Omega)}\left(1+\|u\|^{\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}+\|\Pi_{\mathrm{e}}u\|^{\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right)^2\\
&\leq C\,\|\eta\|^2_{L^q(\Omega)}\left(1+\|u\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}+\|\Pi_{\mathrm{e}}u\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right)\\
&= C\,\|\eta\|^2_{L^q(\Omega)}\left(1+\|u\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}+\|u-\eta\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right)\\
&\leq C\,\|\eta\|^2_{L^q(\Omega)}\left(1+\|u\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}+\|\eta\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right)\\
&\leq C\,\|\eta\|^2_{L^q(\Omega)}=C\,\|\eta\|^2_{L^{2(\alpha+1)}(\Omega)}\,,
\end{aligned}$$

where the constant $C>0$ depends only on the growth–condition for the function $f$, on Lebesgue norms of $u$ over the time interval $[0,T]$.

Choosing $\varepsilon_1,\varepsilon_2,\varepsilon_3$ such that $\varepsilon_1^{-1}+\varepsilon_2^{-1}+\varepsilon_3^{-1}\leq 2$, and inserting the above bounds into (8.39), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|||\xi|||^2_{\mathrm{DG}}\leq C_1\left(\|\dot{\eta}\|^2_{L^2(\Omega)}+\|\eta\|^2_{L^{2(\alpha+1)}(\Omega)}\right)+\tilde{C}_2\,\|f(\Pi_{\mathrm{e}}u)-f(u_{\mathrm{DG}})\|^2_{L^2(\Omega)}\,. \tag{8.40}$$

To bound $\|f(\Pi_{\mathrm{e}}u)-f(u_{\mathrm{DG}})\|^2_{L^2(\Omega)}$ we note that, by the same argument as above, we have

$$\|f(\Pi_{\mathrm{e}}u)-f(u_{\mathrm{DG}})\|^2_{L^2(\Omega)}\leq C\,\|\xi\|^2_{L^{2(\alpha+1)}(\Omega)}\left(1+\|\xi\|^{2\alpha}_{L^{\frac{2\alpha q}{q-2}}(\Omega)}\right),$$

where the constant $C>0$ depends only on the growth–condition for the function $f$, on Lebesgue norms of $u$ over the time interval $[0,T]$.

Let us choose $u_0^{\mathrm{DG}}=\Pi_{\mathrm{e}}u_0$, thus having $\xi(0)=0$, and let $0<t_\star\leq T$ be the largest time such that the solution $|||\xi(t)|||^2_{\mathrm{DG}}$ of (8.38) (and thus $u_{\mathrm{DG}}(t)$) exists and $|||\xi|||_{\mathrm{DG}}\leq 1$ for $t\in[0,t_\star]$; the existence of such $t_\star$ is guaranteed by the Cauchy–Picard theorem from the theory of ODEs.

By Hypothesis A, we have $2(\alpha+1)\leq 2n/(n-2)$, and hence by the broken Sobolev–Poincaré inequality,

$$\|f(\Pi_{\mathrm{e}}u)-f(u_{\mathrm{DG}})\|^2_{L^2(\Omega)}\leq C|||\xi|||^2_{\mathrm{DG}}.$$

Inserting this bound into (8.40), we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}|||\xi|||^2_{\mathrm{DG}}\leq C_1\left(\|\dot{\eta}\|^2_{L^2(\Omega)}+\|\eta\|^2_{L^{2(\alpha+1)}(\Omega)}\right)+C_2|||\xi|||^2_{\mathrm{DG}}, \tag{8.41}$$

which, upon integrating from 0 to $t\leq t_\star$ and noting that $\xi(0)=0$, yields

$$|||\xi(t)|||^2_{\mathrm{DG}}\leq C_1\int_0^t\left\{\|\dot{\eta}(s)\|^2_{L^2(\Omega)}+\|\eta(s)\|^2_{L^{2(\alpha+1)}(\Omega)}\right\}\mathrm{d}s+C_2\int_0^t|||\xi(s)|||^2_{\mathrm{DG}}\,\mathrm{d}s. \tag{8.42}$$

By Lemma 8.8, the first argument on the right–hand side can be bounded in terms of $h_\kappa$ and $p_\kappa$. Fixing the polynomial degree $p_\kappa$ for all $\kappa \in \mathcal{T}$ and denoting $0 < h = \max_{\kappa\in\mathcal{T}} h_\kappa$, let us define $C_3 = C_2 2^{2\alpha}$, and let $h_0 > 0$ be small enough so that for all $h \le h_0$ we have

$$C_1 \int_0^t \left\{ \|\dot{\eta}(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^{2(\alpha+1)}(\Omega)} \right\} \mathrm{d}s \le \frac{1}{1+T}\,\mathrm{e}^{-C_3 T}.$$

Thus, for $h \le h_0$ and $t \in [0, t_\star]$, from (8.42) we have

$$|||\xi(t)|||^2_{\mathrm{DG}} < \frac{1}{1+T}\,\mathrm{e}^{-C_3 T} + C_3 \int_0^t |||\xi(s)|||^2_{\mathrm{DG}}\,\mathrm{d}s;$$

using the Gronwall–Bellmann inequality, we deduce that $|||\xi(t)|||^2_{\mathrm{DG}} < 1$ for all $t \in [0, t_\star]$ with $h \le h_0$.

By continuity of the mapping $t \mapsto |||\xi(t)|||^2_{\mathrm{DG}}$, the assumption $t_\star < T$ implies that either $|||\xi(t)|||^2_{\mathrm{DG}} \le 1$ for all $t \in [0, T]$, or that there exists a time $t_{\star\star} \in (t_\star, T]$ such that $|||\xi(t_{\star\star})|||^2_{\mathrm{DG}} = 1$.

In either case, we have a contradiction with the fact that $t_\star$ is the largest time in the interval $[0, T]$ such that, for all $t \in [0, t_\star]$, we have $|||\xi(t)|||^2_{\mathrm{DG}} \le 1$. Thus we deduce that $t_\star = T$ for $0 < h \le h_0$.

Taking into account this fact, setting $h \le h_0$, and applying the Gronwall–Bellman inequality to (8.42) gives us the following bound:

$$|||\xi(t)|||^2_{\mathrm{DG}} \le C \int_0^t \left\{ \|\dot{\eta}(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^{2(\alpha+1)}(\Omega)} \right\} \mathrm{d}s, \quad 0 \le t \le T, \tag{8.43}$$

where the constant $C > 0$ depends only on the domain $\Omega$, the growth–condition for the function $f$, the time $T$, on Lebesgue and Sobolev norms of $u$ over the time interval $[0, T]$.

Further, by the broken Sobolev–Poincaré inequality, we have the bound

$$\|\eta\|^2_{L^{2(\alpha+1)}(\Omega)} \le C|||\eta|||^2_{\mathrm{DG}},$$

and, employing the triangle inequality, we thus obtain

$$|||(u - u_{\mathrm{DG}})(t)|||^2_{\mathrm{DG}} \le C \left( |||\eta(t)|||^2_{\mathrm{DG}} + \int_0^t \left\{ \|\dot{\eta}(s)\|^2_{L^2(\Omega)} + |||\eta(s)|||^2_{\mathrm{DG}} \right\} \mathrm{d}s \right), \quad 0 \le t \le T,$$

with the constant $C$ as above.

By the results of Lemma 8.8 we have that

$$\|\eta\|^2_{L^2(\Omega)} \le C \left( \max_{\kappa\in\mathcal{T}} \frac{h_\kappa^2}{p_\kappa} \right) \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|^2_{H^{k_\kappa}(\kappa)} \quad \text{and} \quad |||\eta|||^2_{\mathrm{DG}} \le C \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|^2_{H^{k_\kappa}(\kappa)},$$

with $1 \le s_\kappa \le \min(p_\kappa + 1, k_\kappa)$, $p_\kappa \ge 1$, for $\kappa \in \mathcal{T}$. Inserting these bounds in the above error bound, denoting $\|u\|^2_{\mathfrak{X}_1} := \|u\|^2_{L^\infty(0,T;H^{k_\kappa}(\Omega))} + \|u\|^2_{L^2(0,T;H^{k_\kappa}(\Omega))} + \|\dot{u}\|^2_{L^2(0,T;H^{k_\kappa}(\Omega))}$, and taking the maximum over $t \in [0, T]$ yields (8.36).

From (8.43), by the broken Sobolev–Poincaré inequality, we deduce that

$$\|\xi(t)\|^2_{L^2(\Omega)} \le C \int_0^t \left\{ \|\dot{\eta}(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^{2(\alpha+1)}(\Omega)} \right\} \mathrm{d}s, \quad 0 \le t \le T.$$

Employing the triangle inequality yields, for all $0 \le t \le T$,

$$\|u(t) - u_{\mathrm{DG}}(t)\|^2_{L^2(\Omega)} \le C \left( \|\eta(t)\|^2_{L^2(\Omega)} + \int_0^t \left\{ \|\dot{\eta}(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^{2(\alpha+1)}(\Omega)} \right\} \mathrm{d}s \right), \tag{8.44}$$

with the constant $C$ as above.

Further, by the Sobolev inequality (see [1]), we have, for $1 \le 2(\alpha+1) \le 2n/(n-2)$, $n \ge 3$, and $1 \le 2(\alpha+1) < \infty$, $n = 2$,

$$\|\eta\|_{L^{2(\alpha+1)}(\hat{\kappa})} \le C \|\eta\|_{H^1(\hat{\kappa})},$$

where $\hat{\kappa}$ is the unit reference element (the unit hypercube). By scaling back from the reference element, we obtain

$$\|\eta\|_{L^{2(\alpha+1)}(\kappa)} \le C \left( h_\kappa^{n\left(\frac{1}{2(\alpha+1)} - \frac{1}{2}\right)} \|\eta\|_{L^2(\kappa)} + h_\kappa^{1+n\left(\frac{1}{2(\alpha+1)} - \frac{1}{2}\right)} |\eta|_{H^1(\kappa)} \right),$$

and thus, upon squaring and summing over $\kappa \in \mathcal{T}$, taking the square root and noting that

$$\left( \sum_i |a_i|^q \right)^{\frac{1}{q}} \le \left( \sum_i |a_i|^2 \right)^{\frac{1}{2}}, \quad q \ge 2,$$

we obtain

$$\|\eta\|_{L^{2(\alpha+1)}(\Omega)} \le C \left( \max_{\kappa \in \mathcal{T}} h_\kappa^{n\left(\frac{1}{2(\alpha+1)} - \frac{1}{2}\right)} \|\eta\|_{L^2(\Omega)} + \max_{\kappa \in \mathcal{T}} h_\kappa^{1+n\left(\frac{1}{2(\alpha+1)} - \frac{1}{2}\right)} |\eta|_{H^1(\Omega,\mathcal{T})} \right).$$

Inserting this inequality into (8.44) gives us

$$\begin{aligned} \|(u - u_{\mathrm{DG}})(t)\|^2_{L^2(\Omega)} \le C \Bigg( \|\eta(t)\|^2_{L^2(\Omega)} + \int_0^t \Big\{ \|\dot{\eta}(s)\|^2_{L^2(\Omega)} \\ + \max_{\kappa \in \mathcal{T}} h_\kappa^{n\left(\frac{2}{2(\alpha+1)} - 1\right)} \|\eta(s)\|^2_{L^2(\Omega)} + \max_{\kappa \in \mathcal{T}} h_\kappa^{2+n\left(\frac{2}{2(\alpha+1)} - 1\right)} |\eta(s)|^2_{H^1(\Omega,\mathcal{T})} \Big\} \mathrm{d}s \Bigg). \end{aligned} \tag{8.45}$$

From Lemma 8.8, error bound (8.28), for $k_\kappa \ge 2$ and $s_\kappa = \min(p_\kappa + 1, k_\kappa)$, we have

$$\|\eta(t)\|^2_{L^2(\Omega)} \le C \left( \max_{\kappa \in \mathcal{T}} \frac{h_\kappa^2}{p_\kappa} \right) \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u(t)\|^2_{H^{k_\kappa}(\kappa)}$$

and

$$\max_{\kappa \in \mathcal{T}} h_\kappa^{n\left(\frac{2}{2(\alpha+1)} - 1\right)} \|\eta(t)\|^2_{L^2(\Omega)} \le C \left( \max_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2+n\left(\frac{2}{2(\alpha+1)} - 1\right)}}{p_\kappa} \right) \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u(t)\|^2_{H^{k_\kappa}(\kappa)}.$$

Similarly, from (8.27) we have

$$\max_{\kappa \in \mathcal{T}} h_\kappa^{2+n\left(\frac{2}{2(\alpha+1)} - 1\right)} \|\eta(t)\|^2_{H^1(\Omega,\mathcal{T})} \le C \left( \max_{\kappa \in \mathcal{T}} h_\kappa^{2+n\left(\frac{2}{2(\alpha+1)} - 1\right)} \right) \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u(t)\|^2_{H^{k_\kappa}(\kappa)},$$

Inserting these error bounds into (8.45) and taking the maximum over $t \in [0, T]$, we obtain

$$\|u - u_{\mathrm{DG}}\|^2_{L^\infty(0,T;L^2(\Omega))} \le C \left( \max_{\kappa\in\mathcal{T}} \frac{h_\kappa^2}{p_\kappa} \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|^2_{\mathfrak{X}_2} + \max_{\kappa\in\mathcal{T}} h_\kappa^{2-\frac{\alpha n}{\alpha+1}} \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|^2_{L^2(0,T;H^{k_\kappa}(\kappa))} \right),$$

with $1 \le s_\kappa \le \min(p_\kappa + 1, k_\kappa)$, $p_\kappa \ge 1$, for $\kappa \in \mathcal{T}$, where the constant $C > 0$ depends only on the domain $\Omega$, the shape–regularity of $\mathcal{T}$, the time $T$, the parameter $\rho$ in (8.1) the growth–condition for the function $f$, on $k = \max_{\kappa\in\mathcal{T}} k_\kappa$, and on Lebesgue norms of $u$ over the time interval $[0, T]$; here we denote $\|u\|^2_{\mathfrak{X}_2} := \|u\|^2_{L^\infty(0,T;H^{k_\kappa}(\Omega))} + \|\dot{u}\|^2_{L^2(0,T;H^{k_\kappa}(\Omega))}$, and hence (8.37). □

**9. Conclusions.** This work was concerned with the spatial discretisation of initial–boundary value problems with mixed Dirichlet and Neumann boundary conditions for second–order semilinear equations of parabolic type by the *hp*–version interior penalty discontinuous Galerkin finite element method. Our goal was to derive *hp*–version *a priori* error bounds. For this purpose, we derived *hp*–version error bounds in the $L^2$– and broken $H^1$–norms for the non–local broken elliptic projection operator. We also developed the techniques of handling the non–linearity in the error analysis of the *hp*–version interior penalty discontinuous Galerkin finite element method, which allows for the proofs to be conducted on the entire time interval of existence of the solution.

These enabled us to prove general error bounds for *hp*–version discontinuous Galerkin finite element methods (symmetric and non–symmetric variants) on shape–regular meshes. The bounds, in the $H^1$–norm at least, are optimal in $h$ and slightly suboptimal in $p$.

To the best of our knowledge, these are the first error bounds of this kind for semilinear parabolic equations with a non–linearity of such general type.

With these bounds, we have shown that the presence of the non–linearity, satisfying certain growth–conditions, does not degrade the convergence rate (in the $H^1$–norm) compared to the rates obtained in the linear case. In the case of the symmetric version of the DGFEM, an attempt of the $L^2$–analysis has been made; here, the impact of the non–linearity on the optimality of the convergence rate is clearly seen, as the presence of the non–linear term introduces a non–optimal term into the error bound.

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