Rapid motion of free-surface avalanches over natural terrains and their simulations through curved and twisted channels

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This paper presents a new theory and discussions about the motion of avalanches from initiation to run-out over arbitrarily curved and twisted complicated mountain topographies and its numerical simulations down curved and twisted channels incorporating variable cross sectional widths. To this end, a well established and widely used depth-averaged avalanche model of *Savage & Hutter* has been generalised by *Pudasaini & Hutter* to arbitrary channelised topographies, the intention being that the new model would be able not only to describe the flow of a finite mass of snow, gravel, debris or mud, down a corrie of arbitrary curvature and twist and arbitrary cross sectional profile, but equally also the transportation of grains or pills in the agricultural and pharmaceutical industry, respectively. The emerging equations for the distribution of the avalanche thickness and the topography-parallel depth-averaged velocity components are a set of hyperbolic partial differential equations. The model equations are solved for different topographic configurations, from simple to complex, by applying a high resolution Non-Oscillatory Central (NOC) differencing scheme with Total Variation Diminishing (TVD) limiters. The new theory and its numerical simulations disclose unknown physics of flow of avalanching debris or snow through strongly curved and twisted channels of general types and open an enormous spectrum of applications.

Keywords: Rapid granular avalanches, free-surface motion, natural terrains, curved and twisted channels, hyperbolic equations, Mohr-Coulomb behaviour

1. Introduction

Natural hazards such as avalanches, debris- and mud-flows as well as landslides are common natural phenomena to the inhabitants of high-mountain areas. People and municipal authorities in these areas who have learned to accept their occasional occurrence and to avoid the damage that accompanies them are always seeking to minimise such unpleasant, and sometimes unavoidable happenings, causing the death and damage of the life and property of the people. In the second half of the last century significant efforts have been made to understand the mechanisms of formation of avalanches at high elevations, dynamics of the motion along the complicated and non-trivial mountain tracks and settlements of such huge and catastrophic events in the flat valleys. Special attentions have been paid to the

mechanical, dynamical and the geometrical parts of the problem separately or together. To this end, several theories - ranging from statistical and mass point to hydraulic and molecular dynamics or the kinetic theories - have been proposed (see, Jenkins & Savage 1983; Eglit 1983; Jenkins & Richman 1985; Savage 1989; Savage & Hutter 1989, 1891; Hutter *et al.* 1993; Greve *et al.* 1994; Hwang & Hutter 1995; Iverson 1997; Gray *et al.* 1999; Iverson & Denlinger 2001; Pudasaini & Mohring 2002; Pudasaini *et al.* 2003a, 2003b), different numerical techniques have been developed and well implemented, and a number of experiments both in the laboratory and the field have been performed (Hutter & Koch 1991; Greve & Hutter 1993; Gray *et al.* 1999; Wieland *et al* 1999; Denlinger & Iverson 2001; Tai *et al.* 2002; Koschdon, K. & Schäfer 2003; Issler 2003; Zwinger *et al.* 2003; Pudasaini 2003; Denlinger & Iverson 2003; Iverson *et al.* 2003; Pitman *et al.* 2003; Patra *et al.* 2003). The main aim behind all these scientific and technical activities is to forecast the occurrence of avalanches and debris flows and to predict zones of effects either on their tracks or down in the valley as they come to settlements. This essentially means the construction of hazard maps into the regions of dangerous, less dangerous and danger-free zones. Nevertheless, accidents causing damage of property and loss of life have regularly occurred in the past and continue to occur today (Hutter & Pudasaini 2003). This apparently signifies the need of study and research of avalanches and debris flows from even higher and intensified levels, and makes it a topic of permanent public concern in mountainous regions.

The physics of the release or failure of a large mass of soil, gravel or snow and the dynamics of its motion must be understood if the concomitant danger should be avoided or the impact of a moving mass on the avalanche track or on obstructing buildings be estimated. One hopes that understanding their physical basis will enable the appropriate defensive measures to be taken. Natural avalanches and debris flows are always associated with the complicated mountain topographies which make the prediction and defensive measurements very difficult. In this regard, an exact analysis of avalanching debris flow is a very difficult and challenging task. Nevertheless, the last few years have witnessed increased efforts devoted to the physical understanding of avalanche formation and modelling its dynamical motion in complex topography compatible to the mountain surfaces. To some extent, success has been achieved in this direction, but it still needs serious and unified research to acquire the desired goal.

In this paper we present a new theory and discussions about the motion of avalanches from initiation to run-out over a generally curved and twisted complicated mountain topography and its numerical simulations for flows down curved and twisted channels of different kind. The model computations allow inferences as to the distribution of the mass of different granulates, like gravel or snow, in the deposition zone as well as to the forces exerted on structures that are affected by the motion of the avalanche through general tracks. To achieve this aim, a well established and widely used hydraulic avalanche model of Savage and Hutter (SH, 1989) has been recently generalised by Pudasaini and Hutter (PH, 2003) to describe the motion of flowing geo-materials over arbitrary channelised topographies, the intention being that the model would be able not only to predict the flow of a finite mass of snow, gravel, debris or mud, down a corrie of arbitrary curvature and twist and arbitrary cross sectional profile, but equally also the transportation of grains or pills in the agricultural and pharmaceutical industry, respectively. The

emerging model equations for the distribution of the avalanche thickness and the topography-parallel depth-averaged velocity components, to be presented here, are a set of nonlinear hyperbolic partial differential equations with possible discontinuities in the variables and the coefficients. Once they are derived, a new significant question concerns the solution of the model equations for physically significant and mathematically interesting configurations in order to judge their adequacy and applicability: How are these equations solved numerically, in order to simulate the dynamics of flow avalanches through such complex and non-trivial topographies?

The model equations are solved by implementing the NOC scheme with TVD limiters (see, Nessyahu & Tadmor 1990; Jiang & Tadmor 1998). These are high-resolution numerical techniques that are able to resolve the steep height and velocity gradients and moving sharp fronts often observed in experiments and field events but not captured by traditional finite difference schemes (Wang *et al.* 2003). We performed several numerical tests for avalanching masses down curved and twisted bed topographies (Pudasaini 2003). Uniformly and non-uniformly curved and twisted channels as well as channels which incorporate continuous transition zones merging into the horizontal run-out zones are considered. Both, confined and unconfined transition zones with constant and variable inclination angle of the topography are taken into account. These computations reveal fantastic and fascinating results that we were imaging while developing the theory. They demonstrate the combined effects of curvature, torsion and the radial acceleration (the radial effects due to the curvilinear coordinates) associated with the bed topography. Thus, we are able to quantify the intrinsic effects of the topography on the dynamics of flow avalanches. Such sophisticated studies have not been carried out before, and it was possible here only with the new model equations of PH.

2. Model equations

Before presenting the new theory proposed by Pudasaini and Hutter (PH) we briefly discuss the physically justified and realistic assumptions made in the development of the model equations. Savage and Hutter (1989) developed a continuum hydraulic theory to describe the evolving geometry of a finite mass of a granular material and the associated velocity distribution as an avalanche slides down an inclined surface. In order to formulate a realistic model the following assumptions were made: (*i*) The moving dry and cohesionless granular mass is incompressible and obeys a Mohr–Coulomb yield criterion both inside the deforming mass as well as at the sliding basal surface. (*ii*) The geometries of the avalanching masses are shallow in the sense that typical avalanche thicknesses are small in comparison to the extent parallel to the sliding surface. (*iii*) To obtain a dimensionally reduced theory the field equations are integrated through the depth of the avalanche, and a nearly uniform velocity profile through the depth is assumed. (*iv*) Scaling analysis identifies the physically significant terms in the equations and isolates the terms that can be neglected. These assumptions are supported by observations of large scale snow avalanches in the fields as well as small scale laboratory avalanches of different dry granular particles sliding and deforming down different chutes and channels. These facts are well documented and can be found in a wide range of literatures (see, Savage & Hutter 1989; Hutter & Koch 1991; Keller *et al.* 1998; Dent *et al.* 1998; McElwaine & Nishimura 2001; Pudasaini 2003; Ancey & Meunier 2003;

Iverson *et al.* 2003; Denlinger & Iverson 2003). The simple spatially one-dimensional model of SH, applicable along a straight sliding surface, has been generalised to higher dimensions, to more complex geometries, and has been tested against realistic laboratory experiments and back calculations of the field events. Good to excellent agreements were obtained between the theoretical predictions and the experiments and field data (see, e.g., Savage and Hutter 1991; Hutter & Koch 1991; Greve & Hutter 1993; Hutter *et al.* 1995; Gray *et al.* 1999; Wieland *et al.* 1999; Denlinger and Iverson 2001; Zwinger *et al.* 2003; Pudasaini *et al.* 2003a,b,c; Pudasaini 2003; Denlinger & Iverson 2003; Pitman *et al.* 2003; Patra *et al.* 2003). Here we will focus on a recent three-dimensional extension of the SH–model by PH and its application to avalanche motion over a realistic three-dimensional flow paths as pointed out earlier.

(*a*) *Effects of the topography*

Curved flow path surfaces strongly influence the flow dynamics because transverse shearing and cross-stream momentum transport occur when the topography obstructs or redirects the motion due to its curvature and torsion. Local deceleration and deposition of mass may occur due to energy dissipation. Resistance due to basal friction is modified by "centrifugal forces" induced by the bed curvature as well as torsion.

Recently PH (2003) extended the SH–theory to flows of dry granular masses in a *non-uniformly curved and twisted channel.* Consider an avalanche-prone landscape and a subregion of it where the topography allows identification of the avalanche track. A space curve parallel to the talweg of the valley is singled out as a master curve C (which can be obtained, e.g., by shifting the talweg along its normal direction) from which the track topography will be modelled. The curvature and torsion of the master curve, $\kappa = \kappa(x)$, $\tau = \tau(x)$, are either assumed to be known or can be computed from digital elevation GIS (Geographic Information System) data as functions of the arc length x of the master curve. Then, an orthogonal coordinate system along the master curve is introduced and the model equations are derived in this general coordinate system. In the model equations under consideration in this paper, (x, y) form a curved reference surface, where x is the coordinate along the talweg of a mountain valley, while y is the circular arc length in a cross-sectional plane perpendicular to the talweg whose value is determined by the relation $y = \varepsilon\theta z_T$, where ε is the aspect ratio between the avalanche height and the extent, θ is the azimuthal angle which accounts for the cross-slope curvature and z_T (usually $z_T \gg 1$) is the radial distance between the master curve and the talweg and z is the coordinate perpendicular to the reference topography. Every quantity in this paper is written in non-dimensional form. The channel topography and the geometry of the avalanche in lateral and longitudinal directions are illustrated in Fig. 1. Let us discuss some terms and parameters arising in the model equations presented in the next section. g_x, g_y and g_z are the projected components of the gravitational acceleration along the down-slope, cross-slope and normal directions, for explicit computation see, Pudasaini & Hutter 2003, Pudasaini *at al.* 2003c. The aspect ratio ε, and λ, the measure of curvature relative to the typical avalanche length, are both small numbers. The basal topography (which is the deviation of

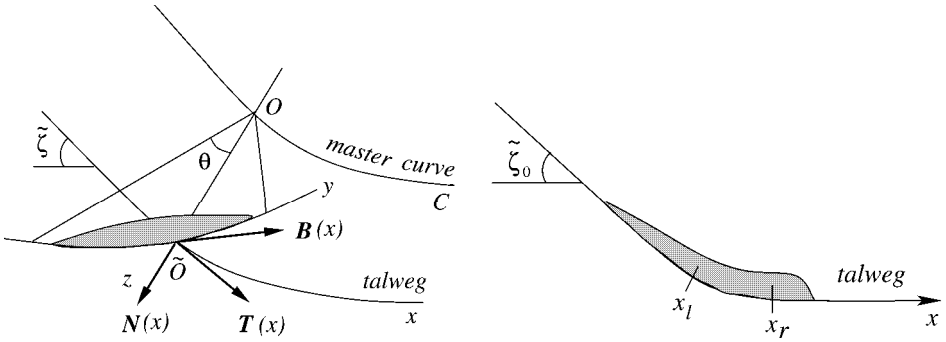

Figure 1. Left: The avalanche domain in the lateral direction occupies a region in a circular section of a plane perpendicular to the talweg of the valley and θ is the azimuthal angle in this plane. $O\tilde{O} = z_T$ is the radial distance between the master curve and the talweg. $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is the moving orthonormal unit triad following the talweg. $\tilde{\zeta}$ is the slope angle of the talweg with the horizontal. The depth of the avalanche in this section is represented by a height function $h(x, y, t)$ and is measured in the radial direction. Right: Avalanche passing through the transition into the run-out zone in a vertical plane containing the talweg of the valley. In this picture, x_l and x_r are the left and right end points of the continuous transition between the straight inclined upper part with inclination angle $\tilde{\zeta}_0$ and the horizontal run-out in the valley.

the basal topography from the reference surface $z = 0$, and includes the small-scale geometric features of the bed morphology) will be denoted by $b(x, y)$.

The extended theory is designed to model the flow of (debris) avalanches over curved and twisted channels having general curvature and torsion. Although there are other models that consider the problem of avalanche motion over curved slopes (e.g., Savage and Nohguchi 1988; Maeno and Nishimura 1987; Norem *et al.*, 1987; Zwinger *et al.* 2003; Iverson *et al.* 2003; Pitman *et al.* 2003), the model equations considered in this paper explicitly and simultaneously include the curvature and torsion effects in a systematic and rigorous manner. This makes the extended model amenable to realistic snow and debris motions down arbitrary guiding topographies. In fact, GIS data of mountainous avalanche- and debris-prone regions can be implemented to this model, which provides the geometrical basis for realistic application and tuned to practical use, and thus lays the theoretical foundation towards this end. Different from the original SH–theory and all their previous extensions (e.g., Gray *et al.* 1999; Wieland *et al.* 1999; Pudasaini *et al.* 2003a) an *arbitrary space curve* is used to define an orthogonal curvilinear coordinate system. The final governing balance laws of mass and momentum are written in these coordinates. PH (2003) are, thus, able to study the simultaneous effects of curvature and torsion on the flow avalanche in channels which have not been investigated analytically before.

(*b*) *Description of the model equations*

As in the previous models of the SH-theory, PH (2003) formulated the balance laws of mass and momentum as well as the boundary conditions in slope-fitted curvilinear coordinates of mountain surfaces, averaged these equations over depth and then non-dimensionalised the equations. The final balance laws of mass, and

momentum in the down-slope and cross-slope directions take the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(hu^2\right) + \frac{\partial}{\partial y}(huv) = hs_x - \frac{\partial}{\partial x}\left(\frac{\beta_x h^2}{2}\right), \tag{2.2}$$

$$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}\left(hv^2\right) = hs_y - \frac{\partial}{\partial y}\left(\frac{\beta_y h^2}{2}\right), \tag{2.3}$$

where h is the depth of the avalanche measured along the normal direction of the reference surface and the factors β_x and β_y are defined, respectively, as

$$\beta_x = -\varepsilon g_z K_x, \quad \beta_y = -\varepsilon g_z K_y. \tag{2.4}$$

The terms s_x and s_y represent the net driving accelerations in the down-slope and cross-slope directions, respectively, and are given by

$$s_x = g_x - \frac{u}{|\mathbf{u}|}\tan\delta\left(-g_z + \lambda\kappa\eta u^2\right) + \varepsilon g_z \frac{\partial b}{\partial x}, \tag{2.5}$$

$$s_y = g_y - \frac{v}{|\mathbf{u}|}\tan\delta\left(-g_z + \lambda\kappa\eta u^2\right) + \varepsilon g_z \frac{\partial b}{\partial y}, \tag{2.6}$$

$|\mathbf{u}| = \sqrt{u^2 + v^2}$ is the magnitude of the velocity field tangential to the reference (basal) topography. Similarly, $\lambda\kappa$ is the local radius of curvature of the talweg, whilst η gives the accumulation of the torsion of the talweg from an initial position.

The first terms on the right-hand side of (2.5) and (2.6) are due to the gravitational accelerations in the down- and cross-slope directions, respectively. The second terms emerge from the dry Coulomb friction and the third terms are the projections of the topographic variations along the normal direction. K_x and K_y in (2.4) are called the earth pressure coefficients. Elementary geometrical arguments and Mohr's circles may be used to determine these values as functions of the internal (ϕ) and basal (δ) angles of friction, (Hutter *et al.* 1993), according as

$$\begin{aligned} K_x &= K_{x_{act/pass}} = 2\sec^2\phi\left(1 \mp \sqrt{1 - \cos^2\phi\sec^2\delta}\right) - 1, \qquad (\partial u/\partial x) \gtrless 0, \\ K_y &= K_{y_{act/pass}} = \frac{1}{2}\left(K_x + 1 \mp \sqrt{(K_x - 1)^2 + 4\tan^2\delta}\right), \qquad (\partial v/\partial y) \gtrless 0, \end{aligned} \tag{2.7}$$

where K_x and K_y are active during dilatational motion (upper sign) and passive during compressional motion (lower sign).

Given the master curve, C, the material parameters δ and ϕ and the elevation of the basal topography, b, above the curved reference surface, equations (2.1)–(2.3) allow h, u and v to be computed as functions of space and time once appropriate initial and boundary conditions are prescribed, where h is the avalanche depth, and (u, v) are the depth-averaged velocity components parallel to the flow surface.

(c) *Comparison to the previous models*

Equations (2.1)–(2.3) constitute a *two-dimensional conservative system of equations.* There are several advantages of the model equations considered in this paper.

They are as follows: (i) They simultaneously include the curvature and torsion of the basal topography. Therefore, the model equations can be utilised to describe the flow of avalanches along non-uniformly curved and twisted channels. (ii) There is a non-zero gravity term g_y in the cross-slope direction which takes into account the global effect of topographic variation in the lateral direction. This might be very crucial in designing the defence structures and when dealing with the motion of avalanches that hit obstructions or deflecting structures on their ways. The torsion effect η of the topography is included in the net driving force components s_x and s_y in the two flow directions. The components of the gravitational acceleration also depend on both the curvature and the torsion of the basal topography, see (Pudasaini & Hutter 2003). The y coordinate is curved in the cross-slope direction, including the cross-slope curvature, which was just a straight line before. For a torsion-free master curve, which lies in a vertical plane, these model equations exactly reproduce all previous extensions of the SH equations as special cases (see, e.g., Gray *et al.* 1999; Pudasaini & Hutter 2003; Pudasaini 2003). In this sense, there is an enormous application of these equations. (iii) We can form a three-dimensionally curved and twisted channel using down-slope and cross-slope coordinates x and y, and we do not necessarily need to superimpose basal topography on top of the reference topography. In principal, it is thus possible to model a given channel or avalanche gully by considering its talweg and by choosing θ appropriately as a function of the down- and cross-slope coordinates. These are considerably new contributions in the model equations which are crucial to describe the complete motion of avalanching debris flows in curved and twisted channels and mountain terrains in a more realistic manner.

3. Numerical techniques

The avalanche equations (2.1)-(2.3) comprise a *nonlinear hyperbolic* system. Shock formation is an essential mechanism in granular flows on an inclined surface merging into a horizontal run-out zone or encountering an obstacle when the velocity becomes subcritical from its supercritical state. To produce more accurate and physically reliable solutions of strongly convective nonlinear hyperbolic equations, such as ours, it is therefore natural to apply conservative high-resolution numerical techniques that are able to resolve the steep gradients of the unknown variables and moving fronts often observed in experiments and field events of avalanches. The NOC scheme proposed first by Nessyahu & Tadmor (1990) and extended to higher dimensions by Jiang & Tadmor (1998) is implemented to solve the model equations. This is a high resolution shock capturing scheme. Necessary backgrounds and full details of this method can be found in diverse literature (see, e.g., Harten 1983; Harten *et al.* 1986; Yee 1987; Nessyahu & Tadmor 1990; LeVeque 1990; Kröner 1997; Jiang & Tadmor 1998; Tai 2000; Toro 2001; Tai *et al.* 2002; Pudasaini 2003).

Essentially, this scheme requires the system to be written in terms of conservative variables, which are the avalanche thickness h and the depth integrated down- and cross-slope momenta, $m_x = hu$ and $m_y = hv$, respectively. With the vector of conservative variables, $\mathbf{w} = (h, m_x, m_y)^T$, the model equations (2.1)-(2.3) can be rewritten in conservative form as

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{w})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{w})}{\partial y} = \mathbf{s}(\mathbf{w}). \tag{3.1}$$

The downslope and cross-slope momentum flux vectors $\mathbf{f}$ and $\mathbf{g}$ and the vector of the source terms $\mathbf{s}$ are given by

$$\mathbf{f} = \begin{pmatrix} m_x \\ m_x^2/h + \beta_x h^2/2 \\ m_x m_y/h \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} m_y \\ m_x m_y/h \\ m_y^2/h + \beta_y h^2/2 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 0 \\ hs_x \\ hs_y \end{pmatrix}. \quad (3.2)$$

The terms β_x and β_y, defined in (2.4) incorporate the extending and contracting states of the avalanching mass through the active and passive earth pressures. Similarly, the source terms s_x and s_y, described in (2.5) and (2.6), are of crucial importance as they include the total driving force generated by gravity, friction, curvature, torsion and local details of the basal topography through its gradient terms. They jointly determine the dynamics of the flow.

4. Avalanche motion down curved and twisted channels

Our main intention while developing the avalanche theory of section 2 was to be able to include the simultaneous effects of the curvature and torsion in the dynamics of an avalanching mass over generally curved and twisted mountain topography. One might expect that there must be not only the effect of curvature but equally also that of torsion on the entire dynamics and the deposit of an avalanche when it slides down over a curved and twisted natural terrain. The model equations (2.1)-(2.3) should be able to predict the flow of an avalanche over a non-uniformly curved and twisted channel in which the cross-slope curvature (or the channel width) may equally be varying. This section is devoted to the numerical simulations of such flows, their physical explanations and analysis over such topographic configurations. The main target is the analysis of the joint effects of curvature, torsion, cross-slope curvature (i.e., the channel width) and the "centrifugal" force in the dynamics of the avalanching body down more general channels and topographies. This is an *entirely new aspect* in the field of avalanche research. On the one hand, the simulations, which we are going to present in the sequel, will *disclose the unknown physics* and will discover some fundamental insights of the avalanches, and thus allow us to judge about the applicability of the new-model equations that we have presented in section 2. On the other hand, they will open a wide spectrum of possibilities for the practitioners involved in the hazard mapping, risk management and public safety. This, then leads to the implementation of our theory to realistic mountain topography together with GIS elevation data of some specific mountain sub-regions.

Several numerical simulations of avalanching flows from simple to complex topographies incorporating curvature as well as torsion of the topography demonstrate fundamental and physically interesting and practically applicable results. The new theory and its numerical simulations makes it possible to answer the questions related to the flow of avalanching debris through strongly curved and twisted channels and opens enormous possibilities of applications. In principle, the theory can be applied to any kind of topography - from a simply inclined plane to very complicated arbitrarily curved and twisted channels in industrial as well as geophysical flows down mountain valleys from initiation to the deposits in the run-out zones.

(*a*) *Flows through uniformly curved and twisted channels*

As an example, we consider a helically curved and twisted channel. This is an academic *test example*, but there are many industrial applications of granular flows in process engineering scenarios where such flow configurations are practically used. For this reason, we consider a *helix* as a master curve so as to form a helically curved and twisted channel. Let us consider a circular helix described by

$$\mathbf{R}(\vartheta) = (A\cos\vartheta, A\sin\vartheta, -B\vartheta)\,, \tag{4.1}$$

where ϑ is the azimuthal angle. The arc length, curvature, torsion and pitch of the helix are given by

$$x = \left(A^2+B^2\right)^{1/2}\vartheta,\ \kappa = A/\left(A^2+B^2\right),\ \tau = -B/\left(A^2+B^2\right),\ \mathcal{P} = 2\pi B, \tag{4.2}$$

respectively. Based on the master curve (4.1) a helically curved and twisted channel is formed. The lateral section of the topography is the intersection of a plane perpendicular to the talweg of the channel and the channel itself. Therefore, this section is a circular arc, but note that in the following considerations the curvature of this arc changes with respect to the width of the channel while dealing with variable channel widths.

One may expect that the flowing granular mass will deviate continuously outward from the central line (i.e., the talweg) of the channel due to the radial acceleration induced by the slope-fitted curvilinear coordinates that rotates and moves according to the curvature and the torsion of the bed topography. Figure 2 displays thickness contours of an avalanche sliding down through a helically curved and twisted channel with uniform curvature and torsion given by (4.2) and a constant cross-slope channel width†. The parameter values are: $A = 13$, $B = 13$, so that the channel is inclined with the horizontal at 45°; the radius of curvature and the radius of torsion are each 26, and the internal and bed friction angles are $\phi = 33°$ and $\delta = 27°$, respectively. The radius of curvature in the cross-slope direction is $z_T = 96$. The mass held by a hemi-spherical cap centered at $(23, 0)$ with radius 6.5 is suddenly released with zero initial velocity. These contours are plotted at the time steps $12, 18, 24, 29, 31, 33$, respectively. All quantities are non-dimensional. As time increases, the avalanching mass is getting less spread laterally, but, it is rapidly moving outwards from the center line of the channel. The speed of the front is much larger than the speed of the tail. This means that the body is accelerating rapidly down- and out-ward of the talweg of the channel. Such behaviour of the deforming mass is the joint effect of the curvature, torsion, and the radial acceleration that is modelled in the theory (equations (2.1)-(2.3)) through the gravitational acceleration components g_x, g_y, g_z and the net driving force components s_x, s_y, which include the curvature and torsion of the talweg, bed topography and the cross-slope curvature of the channel. The mass is always extending and accelerating in the down-slope direction, because the channel does not merge into transition- and run-out-zones. In the sequel, we will deal with the cases in which the transition and run-out zones are included in the geometrical part of the model.

† All figures shown for helical chutes are geometrically distorted. The graphs are vertical projections of the chute and granular heaps whose circular-annular geometry are stretched to become straight. Thus, a segment of the annular ring becomes a rectangle of which the top edge is the chute outside and the bottom edge the chute inside boundary. This graphical representation is chosen because it is relatively easy to program.

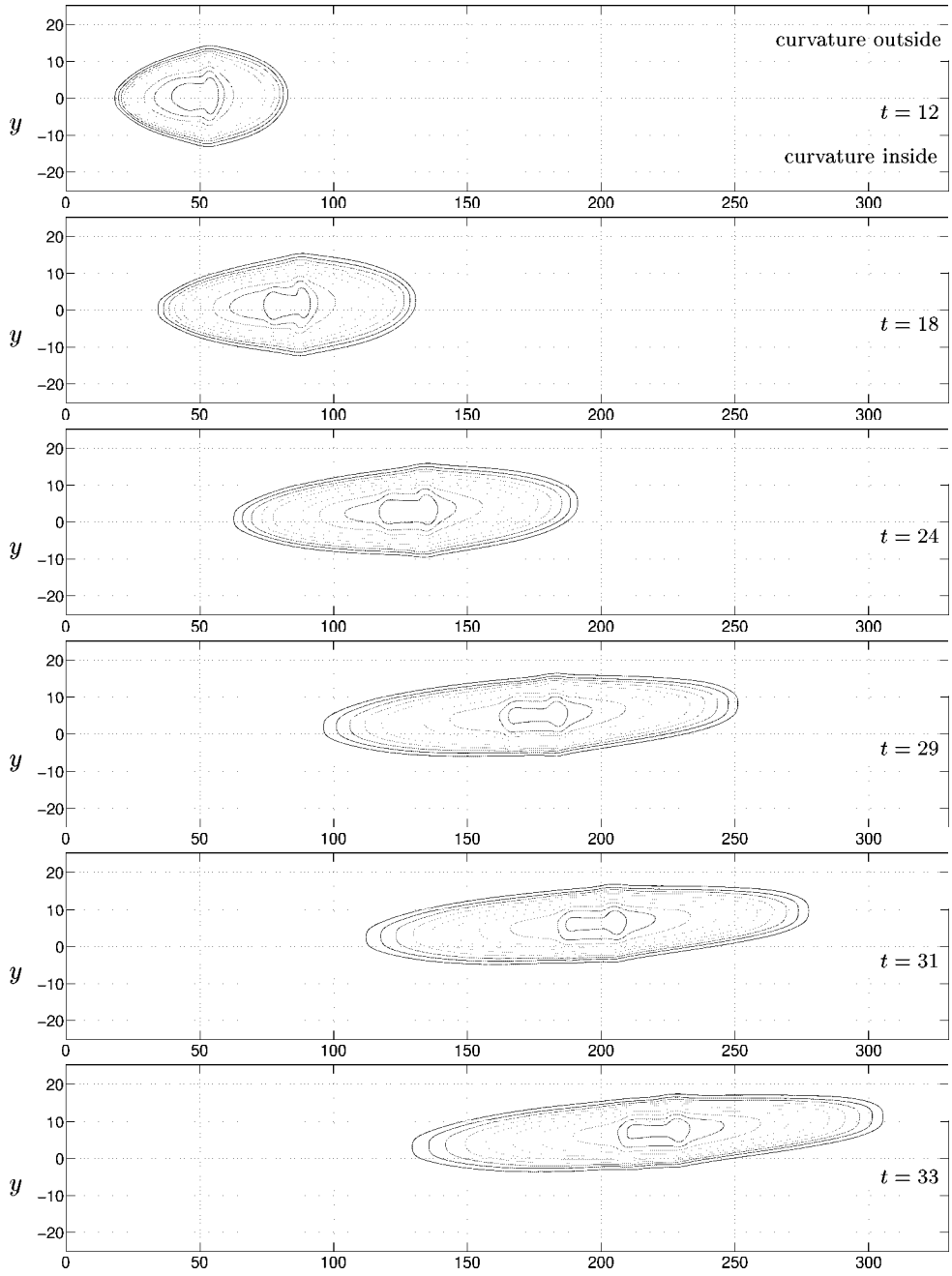

Figure 2. The height contours of an avalanching motion in a helically curved and twisted channel with uniform curvature & torsion and a constant circular cross-slope channel width. The plane rectangles are in reality helically curved and twisted in the x direction and circularly curved in the y direction.

(*b*) *Avalanching flows through non-uniformly curved and twisted channels*

In reality channels may be arbitrarily curved and twisted with variable cross-slope curvature and channel width. Another interesting aspect is the final deposit which is the ultimate and very important item of avalanche dynamics, because the masses neither can always be flowing nor be hanging on the steep slopes. The geometry must play a crucial role to make the body stand still in the valley. The settlement of the mass is directly related to the geometry of the bed topography. The concave curvature of the mountain side increases the bed friction and consequently forces the avalanche to slow down and eventually come to rest. Therefore, to achieve a deposited mass we must be able to include the run-out zone into the bed topography of the model. In this section we will present avalanche simulations through more general channels with different run-out zones.

(i) *Variable pitch*

One geometric model is such that the pitch defined in (4.2) can be modified as

$$B(x) = \begin{cases} B_0, & 0 \le x \le x_l, \\ B_0\left(\dfrac{x_r - x}{x_r - x_l}\right)^2, & x_l \le x \le x_r, \\ 0, & x \ge x_r, \end{cases} \tag{4.3}$$

so that prior to the left end point , x_l, of the continuous transition zone, the chute is exactly the same as that used in the previous subsection. However, there is a continuous decrease of the pitch from x_l to x_r. Then, for $x \ge x_r$ the pitch is always zero, and thus, the subsequent channel is forming a channelised circular run-out.

Avalanche simulations for this case are presented in Fig. 3. The chosen parameter values are as in Fig. 2, and $B_0 = 13, x_l = 300$ and $x_r = 500$. The first panel corresponds to the last panel of Fig. 2. Therefore, the deformation is presented mainly after the avalanche enters the transition zone. Since the pitch of the channel is continuously decreasing for $x > x_l$, from $t = 43$ onward, the granular mass tends to turn smoothly towards the central line of the channel. Corresponding to the decrease of the pitch, the inclination angle of the chute with the horizontal plane is also continuously decreased. Ultimately, the channel merges into a horizontal circularly curved channel, thus forming a gully-type channelised run-out zone. After $t = 33$ the sidewise pressure from the channelised bed topography exceeds the force due to radial acceleration. This leads to a continuous rotation of the body towards the center of the channel. This sidewise pressure is so strong that after $t = 63$ the mass crosses the talweg of the channel and heads towards the opposite side of the channel. Finally, the body comes to rest at time $t = 75$.

(ii) *Variable curvature and torsion*

Next, consider a channel of which curvature and torsion are redefined with the new expression for A in (4.2) as

$$A(x) = \begin{cases} A_0, & 0 \le x \le x_l, \\ A_0 \exp[(x - x_l)^a], & x_l \le x \le x_r, \\ A_0 \exp[(x_r - x_l)^a], & x \ge x_r, \end{cases} \tag{4.4}$$

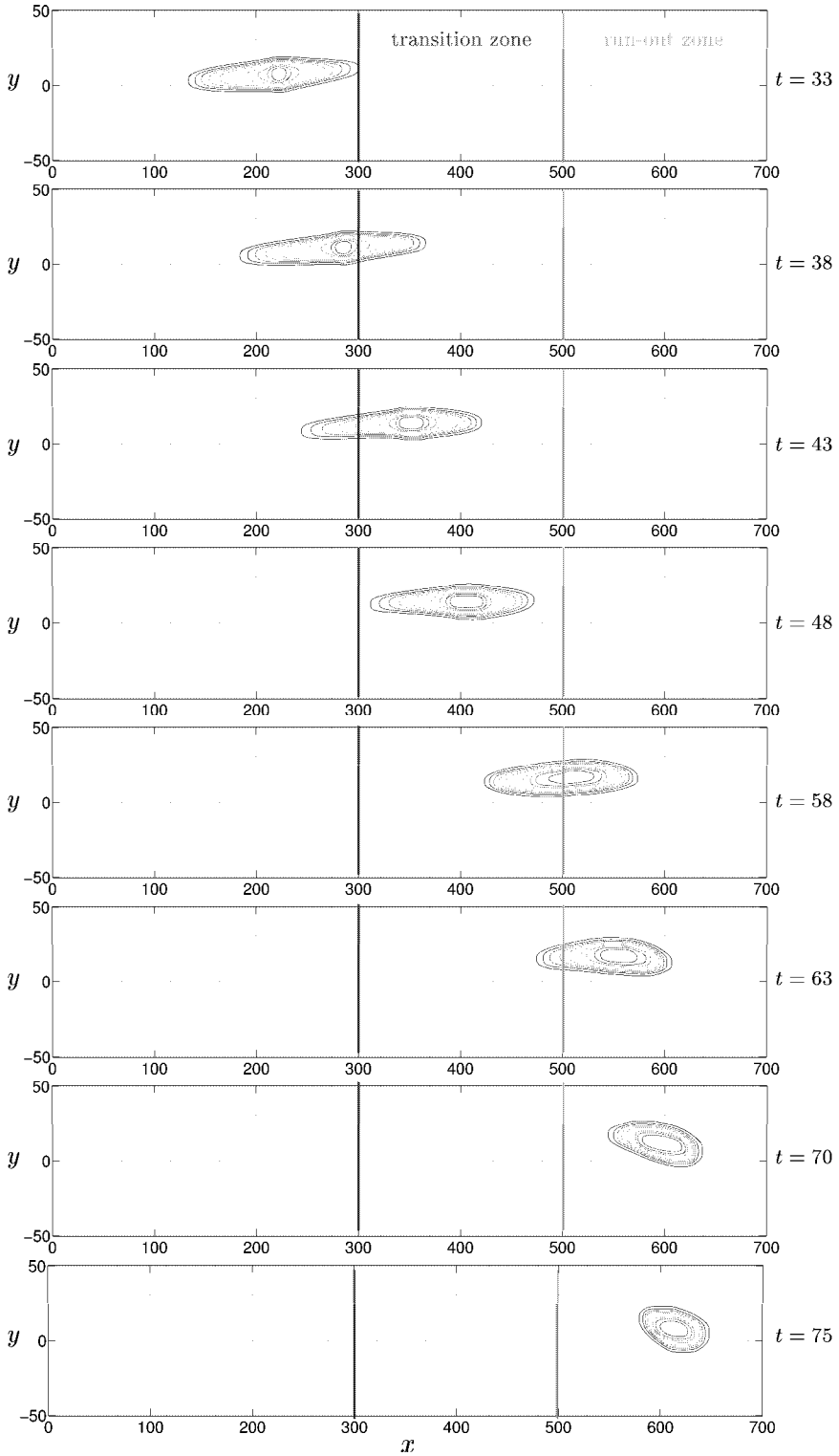

Figure 3. The height contours of an avalanching motion down a helically curved and twisted channel with variable pitch and a constant circular cross-slope channel width.

where a is an exponent that determines the intensity of decrease of the curvature and torsion. For the simulations, we have set $a = 1$. The other parameters are the same as before with $A_0 = 13$. Equation (4.4) tells us that the radius of curvature and the torsion of the channel increase rapidly as the arc-length x becomes larger than x_l. Before this transition point, the channel has uniform radius of curvature, torsion and pitch. This increase of the radius of the curvature and torsion forces the channel quickly to merge (approximately) into an increasingly less curved horizontal channel. This horizontal portion of the channel also forms the run-out zone for the avalanche.

The results of the avalanche simulation for this configuration are presented in Fig. 4. There is a great difference in the avalanche motion between Figs. 3 and 4, specially in the run-out zones. For the present case, since the radius of curvature and torsion increase rapidly from $x = x_l$, the avalanche quickly turns back to the central line of the channel and suddenly comes to rest, much earlier than in Fig. 3.

The differences manifest themselves for $t > 48$. In particular, for $t = 58$, the pile in Fig. 3 has left the transition zone by more that half of its mass, whereas it is still almost inside the transition zone in Fig. 4. This can physically be understood: The increasing radius of curvature of the channel axis in the transition zone for case (ii) reduces the local slope angle of the channel axis much faster than for case (i), so that within the transition zone of case (ii) the avalanching mass encounters deposition-prone conditions quicker than in case (i). Comparing the deposits for $t \geq 58$ in the two figures shows that the run-out distance of the avalanche mass is greatly affected.

(iii) *Decreasing pitch and variable channel width*

Real channels may be diverging or converging (with respect to their channel width or cross-slope curvature) along the down-hill direction. Therefore, the avalanche theory must be able to deal with more general channels and natural valleys or gullies with generally varying cross-slope curvature. At this point, we simulate the avalanche motion in a channel of which the pitch is defined by (4.3), as for case (i), but now we vary the channel width starting from its left boundary of the transition zone at which the pitch starts to decrease. This can be achieved by defining a channel which merges continuously into an open flat run-out zone according to

$$\theta(x,y) = \begin{cases} y/z_T, & 0 \leq x \leq x_l, \\ (y/z_T)f(x), & x_l \leq x \leq x_r, \\ 0^\circ, & x \geq x_r, \end{cases} \tag{4.5}$$

where z_T is the distance between the master curve and the talweg in the upper inclined part of the channel (hence a constant) and $f(x) = \left(1 - (x - x_l) \,/\, (x_r - x_l)\right)^2$. Thus, the continuous transition of the parametric function θ from its higher value (y/z_T) in the upper part to its zero value in the open run-out zone constitutes a required three-dimensional channel which has variable pitch and variable curvature both in the longitudinal as well as in the lateral direction. Figure 5 depicts the contours of the avalanche motion from its transition to the open run-out zone. The graphs describe the fascinating deformation of the avalanche. Although the pitch is decreasing, after reaching the transition zone the avalanching body is heading radi-

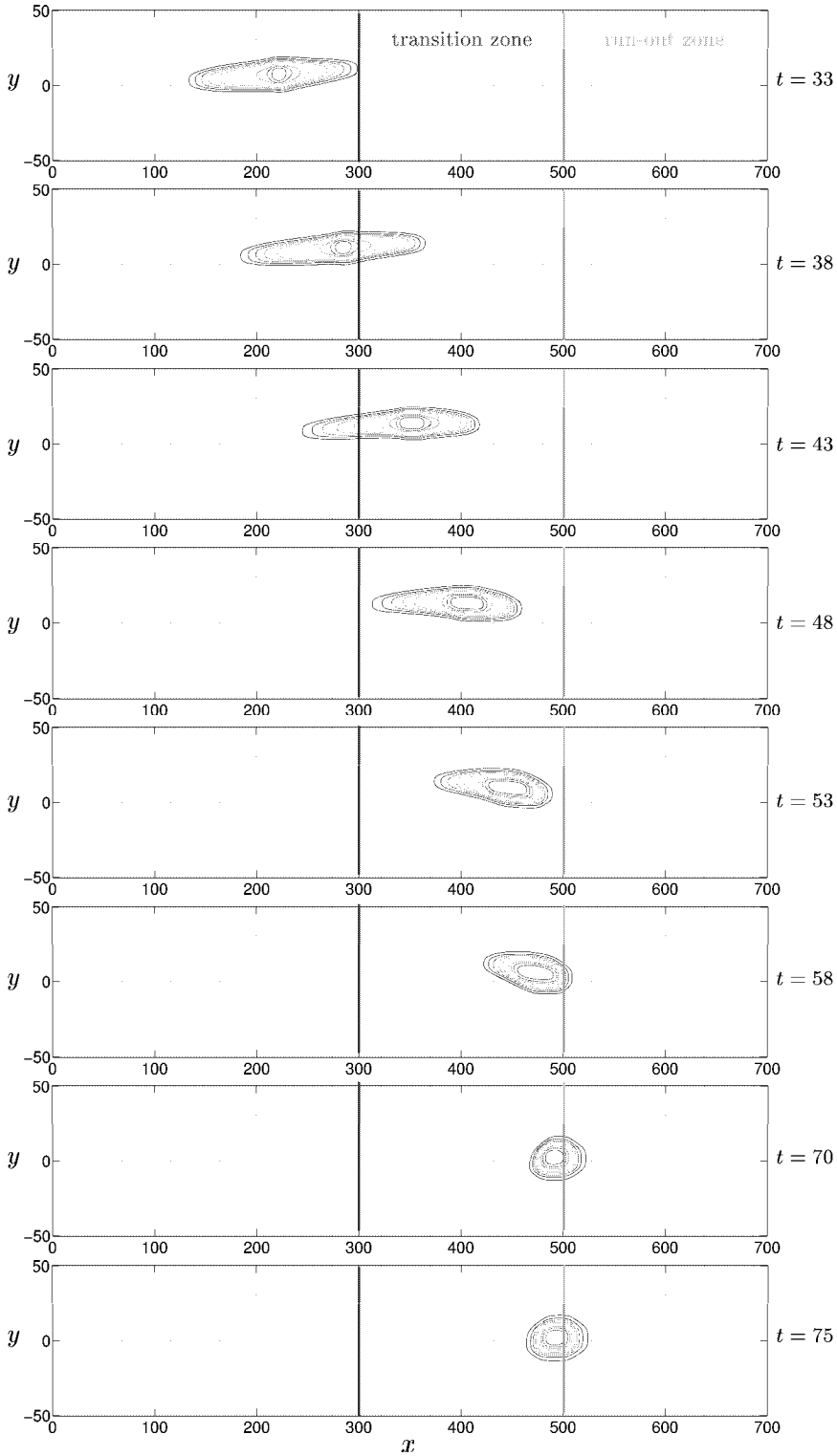

Figure 4. The height contours of an avalanching motion in a "helically" curved and twisted channel with decreasing curvature and torsion and a constant cross-slope channel width.

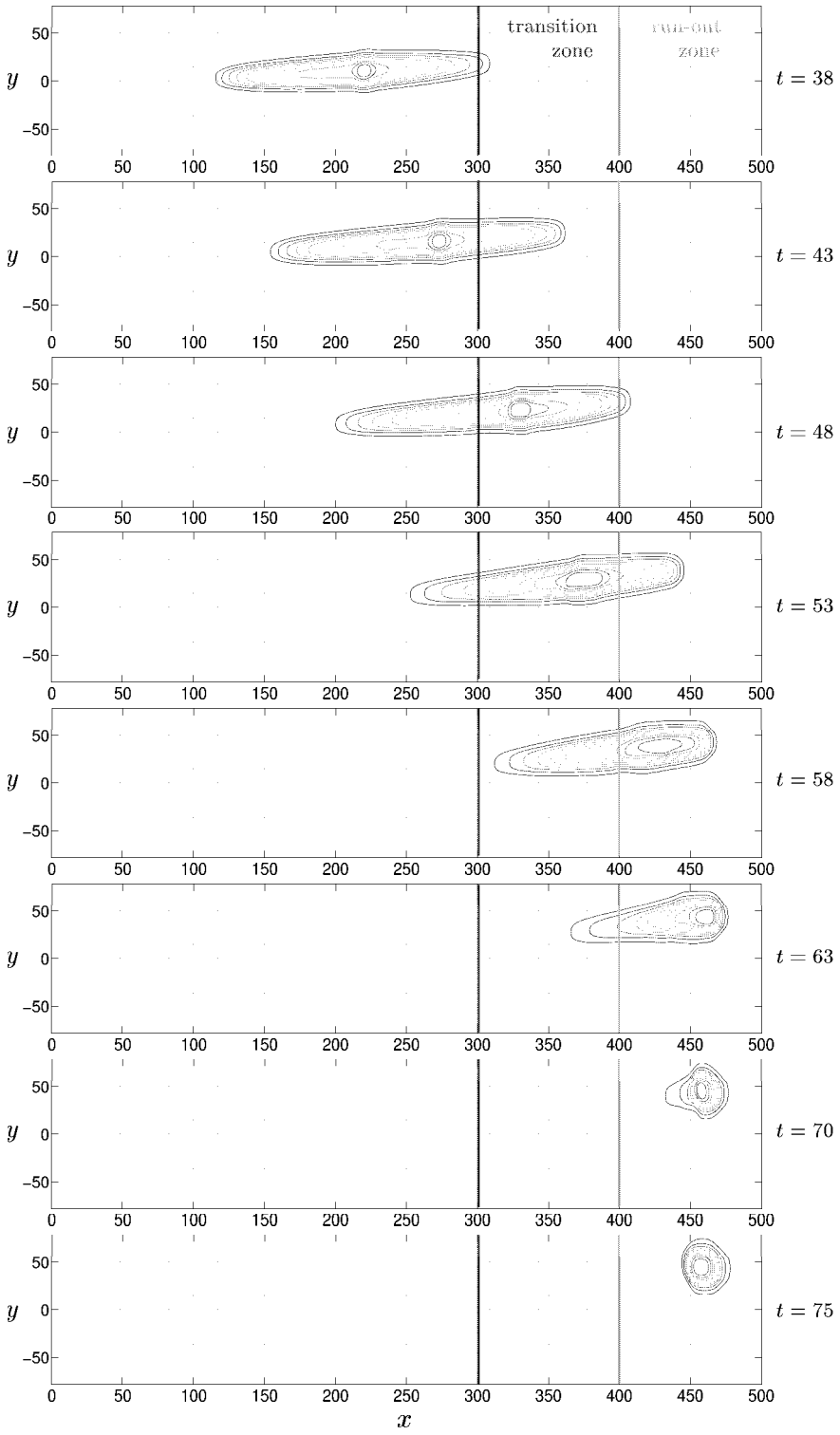

Figure 5. The height contours of an avalanching motion in a "helically" curved and twisted channel with decreasing pitch and increasing cross-slope channel width

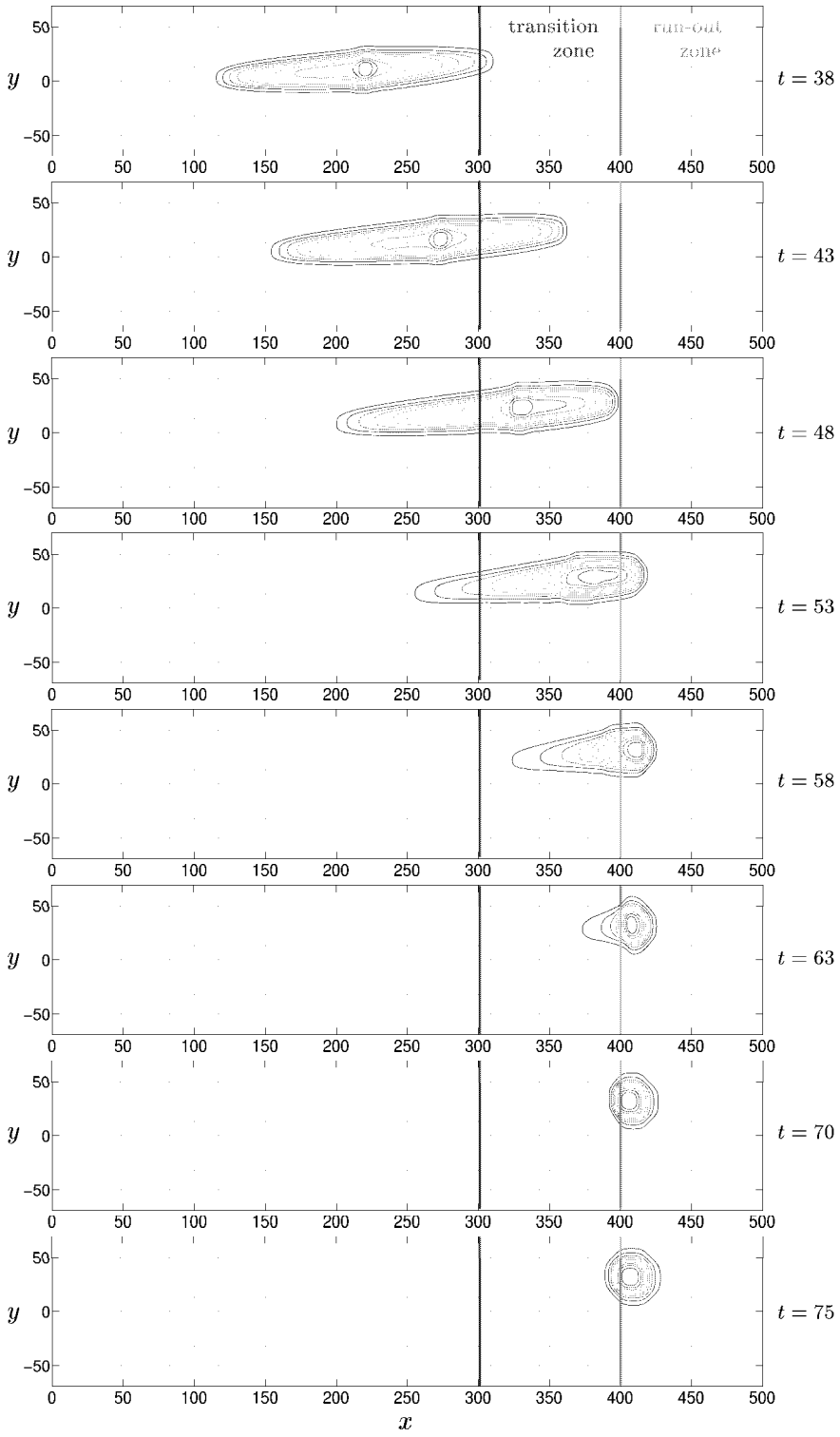

Figure 6. The height contours of an avalanching motion in a "helically" curved and twisted channel with decreasing curvature and torsion and increasing cross-slope channel width.

ally outwards of the flat run-out zone until it comes to rest close to the outside edge of the chute. The main mechanism for this is that, as soon as the mass enters the run-out zone the radial acceleration decreases rapidly, but, since the chute is flattening in the cross-slope direction, the decreasing radial acceleration must keep the mass further and further away from the center line. The direction and the process of the deposition is in conformity with our physical intuition and expectation.

(iv) *Decreasing curvature & torsion, and variable channel width*

A further interesting geometrical model is a channel whose curvature and torsion decrease from the beginning of the continuous transition zone as described by equation (4.4). The channel opens and merges continuously into the horizontal plane as described by (4.5). This case is more important in the geophysical applications because curvature and torsion generally decrease as one enters into the horizontal run-out zone of a mountain valley. The avalanching motion from the transition to the run-out zone in such a channel is presented in Fig. 6. The principal mechanism for the deformation and the deposition of the mass is analogous to case (iii) (i.e., Fig. 5), but it stops quite earlier in time and at a shorter run-out distances than before. Given the results of cases (i) and (ii), this was to be expected.

5. Concluding remarks

We presented and applied a new model describing the flow through curved and twisted channels of a cohesionless mass of granular materials. We are now able to include the simultaneous effects of curvature and torsion of the topography systematically in the avalanche motion, which could not be achieved by earlier models. The applicability of the present model equations is, therefore, much broader than in previous cases. The advantage of this formulation lies in its flexibility of application. Analysis of the motion of avalanches in channels with different cross-slope curvatures and widths is now possible. The flow down an inclined surface or within a channel with its axis in a vertical plane which may be curved can be described. The flow down complicated mountain valleys with arbitrarily curved and twisted talwegs and bed topographies can genuinely be predicted by these model equations. Thus, the theory provides an entirely new direction in the field of avalanche and debris flow research. It also opens a large spectrum of applications in different geophysical problems connected with the use of GIS and digital elevation data.

To avoid any spurious oscillations and include naturally induced shock phenomena into the solution of the nonlinear hyperbolic model equations, with possible discontinuities in the unknown variables and coefficients, we implemented two-dimensional high-resolution Non-Oscillatory Central shock-capturing numerical schemes with Total Variation Diminishing limiters. One of the most basic and fundamental questions related to the new theory is: are these model equations really able to predict flows in chutes and channels which simultaneously incorporate curvature, torsion and the cross-slope curvature effects of the bed topography? To answer this question, several numerical tests were performed for avalanching masses down curved and twisted bed topographies. Uniformly and non-uniformly curved and twisted channels as well as channels which incorporate continuous transition zones merging into the horizontal run-out zones are considered. Both, confined and

unconfined transition zones with constant and variable inclination angle of the topography are taken into account. Computational findings clearly demonstrate the combined effects of curvature, torsion and the radial acceleration associated with the bed topography. Such sophisticated studies have not been carried out before, and it is possible here only with the new model equations.

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