

Large deviation in a two-servers system with dynamic routing

E.A. Pechersky, Y.M. Suhov and N.D. Vvedenskaya

Institute for Information Transmission Problems
19, Bolshoj Karetnyj, Moscow 127994, Russia

and

Isaac Newton Institute for Mathematical Sciences
20 Clarkson Road, Cambridge, CB3 0EH, UK

E-mail for correspondence: ndv@iitp.ru

Abstract. We consider a system with two infinite-buffer FCFS servers (of speed one). The arrival is formed by three independent Poisson flows Ξ_i , of rates λ_i , $i = 0, 1, 2$, each with IID task service times. The tasks from Ξ_1 are directed to server 1 and from Ξ_2 to server 2 (dedicated traffic). The tasks from Ξ_0 are directed to the server that has the shorter workload in the buffer at the time of arrival (opportunistic traffic). We analyse the large deviation (LD) probabilities for the virtual waiting time in flow Ξ_0 in the stationary regime.

1. Introduction. The description of the results

1.1. This paper focuses on queueing systems with dynamic routing, in particular on large deviation (LD) probabilities of a long delay in the stationary regime. In the past, various problems of heavy load in systems with dynamic routing were investigated in [5], [6], [7], [8]; see also the literature there. In particular, in [5], [6] a system with several servers was studied, with different speeds and a single discretionary flow where all tasks are directed to the least busy server (a $GI/GI/s/\infty$ load-balanced queue). In [7], [8] a system with two servers was considered, with different speeds and three Poisson flows, two dedicated and one discretionary, with exponentially distributed service times (a $M/M/2/\infty$ load-balanced queue).

Here, for the first time in the literature, LD probabilities are analyzed in a system with non-exponential service times. We consider a model with two servers and three Poisson flows with a general distribution of service times (a $M/GI/2/\infty$ load-balanced queue). We only discuss systems with equal server speed and do not consider LD probabilities for queue lengths (where the answers can be different).

There is also novelty in methods used: we introduce an auxiliary system where the discretionary flow is partitioned between servers in some proportion, regardless of the delays. We then determine a ‘genuine’ proportion which occurs in the original system in the stationary LD regime.

It is well-known that for a standard $M/GI/1/\infty$ FCFS single-server system (with server speed one), under the non-overload condition $\lambda \mathbf{E}S < 1$, the LD asymptotics of the probability of a large workload in the stationary regime is expressed via the rate function

$$I(d) = d\theta^*, \quad (1.1)$$

where

$$I(d) = \lim_{n \rightarrow \infty} \frac{-1}{n} \log \mathbf{P}(W > dn), \quad d > 0. \quad (1.2)$$

Here W stands for the workload (virtual waiting time) at a fixed time, $\theta^* > 0$ is the positive solution to the equation

$$\theta^* = \lambda(\varphi(\theta^*) - 1), \quad (1.3)$$

where φ is the Laplace transform of a random variable S that is distributed as the service time of the tasks,

$$\varphi(\theta) = \mathbf{E}e^{\theta S}, \quad (1.4)$$

assuming that φ is defined on some positive interval and takes all values in $[1, \infty)$ (although it may not be defined on the whole positive half-axis $\mathbf{R}_+ = [0, \infty)$). (See, e.g. [1], [2] and the references there.)

REMARK 1. For a single-server system the optimal LD trajectory can be presented on (t, W) plane (where t is the time and W is the workload) as a segment of a line through the origin, of slope $\lambda\varphi'(\theta^*)$, up to its intersection at the point $(T, T + d)$ with another line, through the point $(d, 0)$, along the angle $v = \pi/4$, where T is defined by $\lambda\varphi'(\theta^*) = (T + d)/T$.

1.2. As was said above, we focus on a system with two infinite-buffer FCFS servers of speed one. The arrival in such a system is formed by three independent Poisson flows Ξ_0, Ξ_1 and Ξ_2 , of rates $\lambda_i, i = 0, 1, 2$, each with IID service times. Flows Ξ_1 and Ξ_2 are dedicated: flow Ξ_1 is directed to server 1 and flow Ξ_2 to server 2. Flow Ξ_0 is discretionary: its tasks join the queue with the shorter workload. We denote by $S^{(i)}$ the random variable that is distributed as the service time in flow Ξ_i , the Laplace transform of $S^{(i)}$ is

$$\varphi_i(\theta) = \mathbf{E}e^{\theta S^{(i)}}. \quad (1.5)$$

We assume that functions $\varphi_i, i = 0, 1, 2$, are defined on some positive intervals and take all values in $[1, \infty)$.

The tasks from Ξ_1 are directed to server 1 and from Ξ_2 to server 2 (dedicated traffic). The tasks from Ξ_0 are directed to the server that has the

shorter workload in the buffer at the time of arrival (opportunistic traffic). The non-overload domain (where \exists a unique stationary regime) is $\lambda_i \varphi'_i < 1, i = 1, 2, \sum_{i=0,1,2} \lambda_i \varphi'_i < 2$.

We analyse the LD probabilities for the delay of a virtual task (of zero length) put into the flow Ξ_0 at some not random time. In our problem it is

$$I_0(d) = \lim_{n \rightarrow \infty} \frac{-1}{n} \log \mathbf{P}(W^{\min} \geq nd), \quad d > 0, \quad (1.6)$$

where W^{\min} stands for the $\min [W^{(1)}, W^{(2)}]$. Here $W^{(i)}$ is the workload in the buffer of server i (at the fixed time), $i = 1, 2$.

Our aim is to give an explicit expression for function $d \rightarrow I_0(d)$. It is identified in terms of solution ϑ to (1.7) and the solutions $\theta_i, \theta_{0,j}, i = 1, 2, j = 3 - i$, to (1.8),(1.9),

$$\vartheta = \frac{1}{2} \left(\sum_{i=0,1,2} \lambda_i (\varphi_i(\vartheta) - 1) \right), \quad (1.7)$$

$$\theta_i + \theta_{0,j} = \lambda_i (\varphi_i(\theta_i) - 1) + \lambda_j (\varphi_j(\theta_{0,j}) - 1) + \lambda_0 (\varphi_0(\theta_{0,j}) - 1), \quad (1.8)$$

and

$$\lambda_i \varphi'_i(\theta_i) = \lambda_j \varphi'_j(\theta_{0,j}) + \lambda_0 \varphi'_0(\theta_{0,j}), \quad (1.9)$$

where

$$i = 1, 2, \quad j = 3 - i.$$

THEOREM 1.

Let ϑ be a solution to (1.7)

A. In the case

$$\lambda_0 \varphi'_0(\vartheta) \geq |\lambda_1 \varphi'_1(\vartheta) - \lambda_2 \varphi'_2(\vartheta)|, \quad (1.10)$$

$I_0(d)$ has the form

$$I_0(d) = 2d\vartheta, \quad (1.11)$$

B. In the case

$$\lambda_2 \varphi'_2(\vartheta) > \lambda_1 \varphi'_1(\vartheta) + \lambda_0 \varphi'_0(\vartheta), \quad (1.12)$$

$I_0(d)$ has the form

$$I_0(d) = d(\theta_2 + \theta_{0,1}). \quad (1.13)$$

where $\theta_2, \theta_{0,1}$ are the solution to (1.8), (1.9) with $i = 2, j = 1$.

C. In the case

$$\lambda_1 \varphi'_1(\vartheta) > \lambda_2 \varphi'_2(\vartheta) + \lambda_0 \varphi'_0(\vartheta). \quad (1.14)$$

$I(d)$ has the form

$$I_0(d) = d(\theta_1 + \theta_{0,2}), \quad (1.15)$$

where $\theta_1, \theta_{0,2}$ are the solution to (1.8), (1.9) with $i = 1, j = 2$

In general case on the set where the equality in (1.10) is attained the derivatives $\frac{\partial I_0}{\partial \lambda_i}$ are discontinues.

Observe that the expressions (1.7) and (1.11) are similar to (1.1),(1.3)

2. The large deviation calculus

For the large delay problem in a one-server system with a Poisson flow the optimal LD rate function $I(d)$ was presented by (1.1). In our problem the input flows to the servers are neither independent nor Poisson. To find I_0 we consider the auxiliary systems where there are two servers and two independent Poisson flows to these servers. Using the large deviation principal for the Poisson processes of the auxiliary system (see [2]) we find the optimal LD trajectories for achieving the large workloads in both servers. It can be shown that the probability of large workloads in the initial and auxiliary systems coincide.

Next we describe the auxiliary system. Consider a system with two servers and two Poisson flows Ξ_1^P and Ξ_2^P , that are directed to the first and the second server correspondingly. Let the rate of the flows be

$$\lambda_1^P = \lambda_1 + \alpha\lambda_0, \quad \text{and} \quad \lambda_2^P = \lambda_2 + (1 - \alpha)\lambda_0, \quad 0 \leq \alpha \leq 1,$$

then the Laplace transforms of the service times are

$$\frac{1}{\lambda_1 + \alpha\lambda_0}(\lambda_1\varphi_1 + \alpha\lambda_0\varphi_0) \quad \text{and} \quad \frac{1}{\lambda_2 + (1 - \alpha)\lambda_0}(\lambda_2\varphi_2 + (1 - \alpha)\lambda_0\varphi_0).$$

Here all φ_i , and $\lambda_i, i = 0, 1, 2$, are the same as in the initial problem, the value of α is presented below.

Let $W_i^P(t)$ be the workload at server i of the auxiliary system at time t .

Consider the following event \mathcal{A} :

- (i) *At some moment T both workloads $W_i^P(T) \geq nd$.*
- (ii) *The busy periods of both servers during which the workloads W_i^P where achieved coincide with each other.*

For fixed $\alpha, 0 \leq \alpha \leq 1$ we look for

$$I^P(d, \alpha) = \lim_{n \rightarrow \infty} \frac{-1}{n} \log \mathbf{P}(\mathcal{A}). \quad (2.1)$$

PROPOSITION 1. *For any α under event \mathcal{A} the conditional mean velocities v_1^P, v_2^P of the $W_i^P(t)$ are equal, $v_i^P = v^P, i = 1, 2$.*

LEMMA 1. *If the inequality (1.10) takes place (case A) there exists $\alpha = \alpha_0$ such that*

$$I^P(d, \alpha_0) = 2d\vartheta, \quad (2.2)$$

where ϑ is the solution to (1.7).

SKETCH OF THE PROOF. Let α be fixed. It is sufficient to consider the straight line trajectories (for example, see [2]). Assume that at the moment T the workload of both servers is d then $v^P = \frac{T+d}{T}$ (compare with the Remark 1). Define θ_1 and θ_2 by

$$v^P = \lambda_1 \varphi'_1(\theta_1) + \alpha \lambda_0 \varphi'_0(\theta_1) = \lambda_2 \varphi'_2(\theta_2) + (1 - \alpha) \lambda_0 \varphi'_0(\theta_2). \quad (2.3)$$

The LD rate function that corresponds to such trajectory is

$$I^P(T, \alpha) = T \left(\sup_{\theta_1, \theta_2} (v^P \theta_1 + v^P \theta_2 - \lambda_1(\varphi_1(\theta_1) - 1) - \alpha \lambda_0(\varphi_0(\theta_1) - 1) - \lambda_2(\varphi_2(\theta_2) - 1) - (1 - \alpha) \lambda_0(\varphi_0(\theta_2) - 1)) \right). \quad (2.4)$$

To find minimum of $I^P(T, \alpha)$ in T we look for the condition where $\frac{\partial I^P}{\partial T} = 0$. That gives us the equality

$$\theta_1 + \theta_2 = \lambda_1(\varphi_1(\theta_1) - 1) + \alpha \lambda_0(\varphi_0(\theta_1) - 1) + (1 - \alpha) \lambda_0(\varphi_0(\theta_2) - 1) + \lambda_2(\varphi_2(\theta_2) - 1). \quad (2.5)$$

Further, there exists $\alpha = \alpha_0$ such that

$$\varphi_0(\theta_1) = \varphi_0(\theta_2), \quad \text{thus} \quad \theta_1 = \theta_2.$$

Therefore by (2.5) we have $\theta_1 = \theta_2 = \vartheta$. After straightforward calculations we get

$$I^P(d) = I^P(d, \alpha_0) = 2\vartheta d, \quad (2.6)$$

where ϑ is the single positive solution to (1.7). The value of α_0 is found now from (2.3) where $\theta_1 = \theta_2 = \vartheta$. Under condition (1.10) $0 \leq \alpha_0 \leq 1$. \triangle

REMARK 2. In fact $\frac{\partial I^P(d, \alpha)}{\partial \alpha} |_{\alpha=\alpha_0} = 0$.

Observe that if the equality takes place in (1.10) the above calculations give $\alpha = 0$ or $\alpha = 1$.

Next we consider the cases B and C. Let (1.10) be wrong and, for example, let (1.12) take place.

Consider another auxiliary system with two Poisson flows $\Xi_1^P = \Xi_1 + \Xi_0$ and $\Xi_2^P = \Xi_2$, that are directed to the first and the second server correspondingly.

We are interested again in $I^P(d, 1)$ (see (2.1) where $\alpha = 1$).

LEMMA 2. *If the inequality (1.12) takes place (case B) the rate function has the form*

$$I^P(d, 1) = d(\theta_2^* + \theta_{0,1}^*) \quad (2.7)$$

where $\theta_2^*, \theta_{0,1}^*$ satisfy (1.8) with $i = 2, j = 1$, and on the $\theta_2, \theta_{0,1}$ plane at the solution-point $(\theta_2^*, \theta_{0,1}^*)$ the line along the angle $-\pi/4$ is tangent to the curve (1.8).

Observe that in general case as $\alpha = 1$ or $\alpha = 0$ in (2.4) the values of $\frac{\partial I_0^P}{\partial \lambda_i}$, $i = 0, 1, 2$, along the diagonal $\theta_{0,1} = \theta_2$ and along the curve (2.5) are different.

SKETCH OF THE PROOF Consider equality (2.4) with $\alpha = 1$. The condition $\partial I^P / \partial T = 0$ gives equation (1.8), $i = 2, j = 1$. Further, the condition $\partial I^P / \partial T = 0$ uniquely defines $\theta_2^*, \theta_{0,1}^*$ on the convex curve (1.8) because by (2.3) the line along the angle $-\pi/4$ is tangent to (1.8) at this point, i.e. $\frac{d\theta_{01}}{d\theta_2} = -1$. \triangle

The case C is considered similar to case B.

PROOF OF THEOREM 1, THE SKETCH. On (W_1, W_2) plane in all cases the projections onto the diagonal $W_1 = W_2$ of trajectories of flows $\Xi^P = \Xi_1^P + \Xi_2^P$ and $\Xi = \Xi_0 + \Xi_1 + \Xi_2$ coincide. Note, that the projections of these flows onto diagonal $W_1 = W_2$ are Poisson flows by itself of intensity $\sum_{0,1,2} \lambda_i$.

The LD probability for these projected flows to have the delay $\geq \sqrt{2}dn$ is equal to the probability for each of flows $\Xi_1^P + \Xi_2^P$ and $\Xi_0 + \Xi_1 + \Xi_2$ to have the delays $\geq 2dn$.

Further, consider on (W_1, W_2) plane the projections of trajectories of Ξ_j^P , $j = 1, 2$, and Ξ_i , $i = 0, 1, 2$ flows onto the line \mathcal{C} that is orthogonal to diagonal $W_1 = W_2$. We need to show that the projections on \mathcal{C} of trajectories of flows Ξ^P and Ξ bring no contribution to the LD probabilities. Really, as $n \rightarrow \infty$, and under the condition that the trajectories reach the line $W_1 + W_2 > 2dn$, with conditional probability tending to 1 the projections of trajectories stay within the region $\mathcal{S} : |s| < n^b$, $0 < b \leq 1/2$, where s is the distance to the diagonal.

Therefore the limit LD trajectories of the auxiliary and of initial flows coincide and the LD probabilities in both problems are equal. That proves the Theorem. \triangle

REMARK 3. We have so far discussed LD probabilities for a long delay in flow Ξ_0 . A different problem arises when one considers LD probabilities for the total (summatory) overload in the system, which may have a different asymptotics.

Acknowledgement. The work was partly done at CNRI, Dublin. The authors thank John Lewis and his group for hospitality and fruitful discussions.

The work of E.A.P. and N.D.V. was partly supported by Russian RFFI grant 02-01-00068

References

- [1] Shwartz, A., and Weiss, A., *Large deviations for performance analysis: queues, communications and computing*, Chaman & Hall, 1995.
- [2] Dobrushin, R.L. and Pechersky, E.A. *Large deviations for random processes with independent increments on infinite intervals*, Probl. Inform. Transm., **34**,4, pp 354-382, 1998.
- [3] Dupuis, P., Ellis, R.S. and Weiss, A., *Large deviations for Markov processes with discontinuous statistics, I: general upper bound*. Ann. Probab., **19** (1991), 1280-1297.
- [4] Dupuis, P. and Ellis, R.S., *Large deviations for Markov processes with discontinuous statistics, II: random walks*, Probab. Theory Related Fields, **91** (1992), 153-194.
- [5] Sadowski, J.S. and Szpankowski, W., *The probability of large queue lengths and waiting times in a heterogeneous multiserver queue I: tight limits*, Adv. Appl. Prob., **27**, 532-566, 1995.
- [6] Sadowski, J.S. *The probability of large queue lengths and waiting times in a heterogeneous multiserver queue II: positive recurrence and logarithmic limits*, Adv. Appl. Prob., **27**, 567-583, 1995.
- [7] Turner, S.R.E., *Large deviations for Join the Shortest Queue* Fields Inst. Communications, **28**, 95-106, 2000.
- [8] McDonald, D.R. and Turner, S.R.E., *Resource Pooling in Distributed Queueing Networks*, Fields Inst. Communications, **28**, 107-131, 2000.