EXACT SOLUTIONS OF THE SAVAGE-HUTTER EQUATIONS FOR ONE-DIMENSIONAL GRANULAR FLOWS

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Abstract

We consider the spatially one-dimensional Savage-Hutter equations for the motion of a finite granular mass moving down an inclined planar chute, [1]. In a coordinate system moving with the centre of gravity these equations are given by the system of hyperbolic partial differential equations (1.1). They admit the Lie algebra; it consists of the direct sum of the five-dimensional Lie algebra plus an infinite Lie algebra, [4]. We construct the partially invariant solutions. Then (1.1) is transformed to a linear system of PDEs by interchanging the role of the dependent and independent variables. Exact solutions to the Cauchy and Gorse problems can be found via a transformation of the two linear equations to a single hyperbolic PDE whose Riemann function is expressible in terms of a hypergeometric function. The theoretical findings are illustrated by determining the various stages of the motion of a collapsing granular rectangle on an incline, either free or confined from above by a stationary wall. For large time, also an approximate solution is given, which serves as a basis for solutions of free avalanches starting from an arbitrary smooth or non-smooth initial profile.

1. Introduction

In 1989 Savage and Hutter [1] derived a mathematical model for the flow of a finite mass of granular materials down inclined planes. The model equations involve a height and a depth averaged velocity as functions of the time and the downslope spatial coordinate. Later, the model was also generalized to the flow down slightly curved chutes [2]. A brief summary of the equations with applications to one-dimensional flows down chutes and in rotating cylinders is provided by Gray [3]. For the flow down inclined planes the equations have the form

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial \eta}=0;\quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial \eta}+\beta\,\frac{\partial h}{\partial \eta}=0 \tag{1.1}$$

in which h, u, t, η are, respectively, the thickness, thickness averaged streamwise velocity, the time and the spatial coordinate. To be precise, η is the spatial coordinate of an observer moving with the centre of gravity of the spreading mass of the granular material and beta is a coefficient measuring the earth pressure and takes different values under dilating and contracting flows. Thus,

$$\beta = \varepsilon K \cos\zeta, \qquad K = K_{act} \quad if \quad \frac{\partial u}{\partial \eta} > 0, \qquad K = K_{pas} \quad if \quad \frac{\partial u}{\partial \eta} < 0,$$

$$K_{act/pas} = 2\sec^2\phi(1 \mp \sqrt{1-\cos^2\phi\sec^2\delta}\,) - 1,$$

$$\eta = x - s_0 t^2/2, \qquad u = Q/h - s_0 t,$$

$$s_0 = \sin\zeta - \frac{Q}{|Q|}\tan\delta\cos\zeta \quad if \quad \zeta = const,$$

in which x, ζ, ϕ, δ, in this order, are the physical spatial coordinate, the slope inclination angle, internal angle of friction and the bed friction angle, respectively, and ε is the aspect ratio "typical depth divided by typical length" of the avalanche. In the ensuing analysis, the quantities β, s_o are known and pre-assigned positive constants, which requires Q and the derivative of u with respect to η do not change the sign, and $\phi \geq \delta$.

The system of hyperbolic partial differential equations (1.1) admits the Lie algebra which has been shown by Chugunov et al. [4] to consist of the direct sum of the five-dimensional Lie algebra plus an infinite Lie algebra. With its aid, all similarity solutions of equations (1.1) were constructed in [4]. However, all the families of similarity solutions do not exhaust all possible exact solutions.

In the present work, our intention is the finding of some exact solutions different from the similarity solutions, but with physical relevance and with a potential to be reproducible by experiment.

The first example is the spreading of a pile, initially covering a finite interval and having constant height. Its early time response is a pile with the same height but decreasing width and with shoulders that spread more and more until the pile is eroded from the left and right to zero width, i. e., until the left and right edge points meet, see Fig.1. For this early phase an exact solution can be found. The construction of the exact solution for the second phase is possible because the system (1.1) admits the infinite Lie algebra and can be linearised .

The second problem is the flow down an inclined plane from an initial rectangular pile of which the upper edge is held constant by a wall. In this case, the lower edge of the pile erodes as in the previous example, but the erosion of the upper flank is now different, and the solution must be constructed to satisfy the boundary condition at the wall, see Fig. 2. This solution is valid so long as neither the two edge points at the free pile surface have met nor the wall point of the upper eroding surface has reached the slide plane. The subsequent motion then depends on which case occurs first.

Incidentally, this solution may be solved with an active or a passive earth pressure coefficient at the upper eroding surface. Different results are obtained in the two cases. Intuition would advocate for the passive case, but experiments will have to resolve the matter.

2. Partially invariant solutions

System (1.1) is invariant under rigid translations in the spatial coordinate $\eta' = \eta + \alpha$. The infinitesimal operator

$$X = \frac{\partial}{\partial \eta} \tag{2.1}$$

determines this transformation group. Therefore, the surface tangent to the class of these transformations is given by the equation

$$F(t,u,h) = 0 \quad or \quad h = f(t,u). \tag{2.2}$$

Since the union of equations (1.1) and (2.2) constitutes an overdetermined system if *f* is arbitrary, *f* must in fact be restricted. Indeed, by substituting (2.2) into (1.1), one finds

$$\frac{\partial f}{\partial t} + \left[f - \beta \left(\frac{\partial f}{\partial u} \right)^2 \right] \frac{\partial u}{\partial \eta} = 0 \tag{2.3}$$

as well as

$$\frac{\partial u}{\partial t} + \left(u + \beta \frac{\partial f}{\partial u} \right) \frac{\partial u}{\partial \eta} = 0. \tag{2.4}$$

Because *f* depends on *t* and *u*, but not on η, equation (2.4) is a quasilinear first order partial differential equation, of which the general solution is given by

$$\eta = \Phi(u) + ut + \beta \int \frac{\partial f}{\partial u} dt, \tag{2.5}$$

in which $\Phi(u)$ is an arbitrary function.

If the function *f* is known, then (2.2) and (2.5) together determine the functions *h(t, u)* and *u(t,* η*)* implicitly. Thus, with the use of (2.2), (2.3) and (2.5) various different exact solutions can be constructed. Let us examine a series of special cases.

2.1. Classical partially invariant solution

This case assumes that f depends on u only: f=f(u). In the terminology of Ovsynnikov [7] this case corresponds to the so-called partially invariant solutions. From equation (2.3), on assuming that $\partial u/\partial\eta \neq 0$, we deduce

$$\left[f - \beta\left(\frac{\partial f}{\partial u}\right)^2 \right] = 0,$$

of which the general integral is given by

$$f = \left(\frac{u}{2\sqrt{\beta}} + C\right)^2, \tag{2.6}$$

in which *C* is an arbitrary constant. Thus, knowing *f*, we infer from (2.2) and (2.5) the solution in implicit form, namely as

$$h = \left(\frac{u}{2\sqrt{\beta}} + C\right)^2, \qquad \eta = \Phi(u) + (1.5u + \sqrt{\beta}C)t. \tag{2.7}$$

By analogy with gas dynamics one may call this a "simple wave". The formulas comprise, since *C* and $\Phi(u)$ are arbitrary, an entire class of solutions.

2.2. Generalized partially invariant solution

We now return to the situation when f depends on both t and u, but restrict considerations to u-functions that are piecewise linear in η. So, we may assume u to have the form $u = \omega(t)\eta + p(t)$. It can be shown that $p(t) = C\omega(t)$, where C is a constant. It follows, because (1.1) is invariant under the operator (2.1), that it suffices to consider

$$u = \omega(t)\eta\,. \tag{2.8}$$

Thus u is affined to η. With this choice a solution of (2.3) for f will be sought by restricting f to the form

$$f = \varphi(t) + \psi(t)u^2\,. \tag{2.9}$$

If this ansatz is substituted into (2.3), an identity of the form $a+bu^2=0$ is obtained, which must hold for arbitrary u; this implies a=0 and b=0, or

$$\varphi' + \varphi\omega = 0, \; \psi' + \psi\omega - 4\beta\psi^2 = 0,$$

two differential equations which possess the solution

$$\varphi = C_1 \exp\left(-\int \omega dt\right) \text{ and } \psi = \frac{1}{4\beta}\left[1 - C_2 \exp\left(\int \omega dt\right)\right]^{-1},$$

in which C_1 and C_2 are constants of integration. It follows from (2.2), (2.8) and (2.9) that h and u can be expressed as

$$h = C_1 \exp\left(-\int \omega dt\right) + \frac{u^2}{4\beta}\left[1 - C_2 \exp\left(\int \omega dt\right)\right]^{-1}; \; u = \omega\eta \; . \tag{2.10}$$

An equation determining $\omega(t)$ can now be obtained by substituting these expressions into the second equation of (1.1). This process yields

$$\omega' + \omega^2 \left\{1 + 0.5\left[1 - C_2 \exp\left(\int \omega dt\right)\right]^{-1}\right\} = 0 \, . \tag{2.11}$$

which is an integro-differential equation for $\omega(t)$. However, with

$$g(t) = \exp\left(\int \omega dt\right), \;\; g' = g(t)\omega(t) \;\text{ and}\, \omega = \frac{g'}{g}$$

(2.11) transforms into

$$\frac{g''}{g'} + \frac{g'}{2g(1 - C_2 g)} = 0 \, ,$$

which is now a second order differential equation for g. A first integral can easily be constructed and is given by

$$\ln|g'| = \frac{1}{2}\int \frac{dg}{g(C_2 g - 1)} + \widetilde{C}_3 \, ,$$

$\widetilde{C}_3$ being a constant of integration. The further integration depends upon whether $C_2 \neq 0$ or $C_2 = 0$.

a) $C_2 \neq 0$ then $g' = \pm\sqrt{\dfrac{g - C_2^{-1}}{g}}\, C_3$,

which admits the following solutions

$$a_1) \quad \sqrt{g(g - C_2^{-1})} + C_2^{-1} \ln\left[\sqrt{g} + \sqrt{g - C_2^{-1}}\right] = C_3 t + C_4 , \quad g > C_2^{-1};$$

$$a_2) \quad -\sqrt{g(C_2^{-1} - g)} + C_2^{-1} arctg \sqrt{\frac{g}{C_2^{-1} - g}} = C_3 + C_4 , \quad g < C_2^{-1}.$$

b.) If $C_2=0$, then integration of equation for $g(t)$ yields

b) $C_2 = 0$ then $g = \left(\dfrac{3}{2}C_3 t + \dfrac{3}{2}C_4\right)^{\frac{2}{3}}$,

C_1, C_2, C_3, C_4 are all constants of integration and are determined by the specific problem at hand.

Explicit expressions for h and u are obtained by substituting the relevant relations into (2.10); the following expressions are obtained:

$$\begin{cases} u = \dfrac{C_3}{g}\sqrt{\dfrac{g - C_2^{-1}}{g}}\ \eta\ ; \\ h = \dfrac{C_1}{g}\left(1 - \dfrac{C_3^2\eta^2}{4\beta C_2 C_1 g^2}\right); \end{cases} \qquad g > C_2^{-1}. \tag{2.12}$$

$$\begin{cases} u = \dfrac{C_3}{g}\sqrt{\dfrac{C_2^{-1} - g}{g}}\ \eta\ ; \\ h = \dfrac{C_1}{g}\left(1 + \dfrac{C_3^2\eta^2}{4\beta C_2 C_1 g^2}\right); \end{cases} \qquad g < C_2^{-1}. \tag{2.13}$$

$$\begin{cases} u = \dfrac{C_3}{g\sqrt{g}}\ \eta\ ; \\ h = \dfrac{C_1}{g}\left(1 + \dfrac{C_3^2\eta^2}{4\beta C_1 g^2}\right); \end{cases} \qquad C_2 = 0\ . \tag{2.14}$$

To summarize: If $C_2 > 0$, then at $|g| > C_2^{-1}$ the solution (2.12) applies and a_1) must hold; if, however, $|g| < C_2^{-1}$, then (2.13) holds together with a_2). On the other hand, if $C_2 < 0$ then at $g>0$ and $g < C_2^{-1}$, solution (2.12) with a_1) is realized, whilst for $C_2^{-1} <g<0$, solution (2.13) with a_2) applies.

2.2. Partially invariant solution with respect to the operator $X = \partial / \partial t$

It is similarly possible to examine solutions which are partially invariant with respect to the operator $X = \partial / \partial t$. In this case, the tangent manifold is written as

$$h = f(\eta, u) \tag{2.15}$$

and, thus, the compatibility equation of the system (1.1) takes the form

$$\frac{\partial u}{\partial \eta}\left[f - \left(u\frac{\partial f}{\partial u} + \beta\left(\frac{\partial f}{\partial u}\right)^2 \right)\right] + u\frac{\partial f}{\partial \eta} = 0 \tag{2.16}$$

and if it is assumed that f depends only on u,

$$\left[f - \left(u\frac{df}{du} + \beta\left(\frac{df}{du}\right)^2 \right)\right] = 0\ .$$

One solution of this equation is easily seen to be $f = -(4\beta)^{-1}u^2$, but it does not satisfy the condition h>0 that must be met by physical reasons. The other solution must have the form $f = (4\beta)^{-1}(C^2 \pm 2Cu)$ and is best written as

$$f = h(u) = \frac{C^2 \pm 2Cu}{4\beta}. \tag{2.17}$$

Substituting this into the second of (1.1) yields

$$\frac{\partial u}{\partial t} + (u \pm 0.5C)\frac{\partial u}{\partial \eta} = 0,$$

which is easily integrated to give

$$\eta = (u \pm 0.5C)t + \Phi(u), \tag{2.18}$$

in which $\Phi(u)$ is an arbitrary function. The formulas (2.17) and (2.18) together define the solutions that are partially invariant with respect to $X = \partial / \partial t$.

3. Exact solutions constructed from the linear system associated with (1.1)

Because the system of differential equations (1.1) admits the infinite Lie algebra with the basis

$$X_\infty = Z(u,h)\frac{\partial}{\partial \eta} + T(u,h)\frac{\partial}{\partial t},$$

it can be transformed into a linear system by interchanging the roles of the dependent and independent variables. So, let T and Z be differentiable and locally invertible functions of h and u, such that

$$t=T(h,u),\ \eta=Z(h,u), \tag{3.1}$$

with inverse

$$h=H(t,\ \eta),\ u=U(t,\ \eta),$$

Then, the Jacobian determinants

$$J = \begin{vmatrix} \frac{\partial U}{\partial t} & \frac{\partial H}{\partial t} \\ \frac{\partial U}{\partial \eta} & \frac{\partial H}{\partial \eta} \end{vmatrix} \neq 0 \ \text{ then } D = \begin{vmatrix} \frac{\partial T}{\partial h} & \frac{\partial Z}{\partial h} \\ \frac{\partial T}{\partial u} & \frac{\partial Z}{\partial u} \end{vmatrix} \neq 0 \tag{3.2}$$

must necessarily differ from 0. It is also easy to show that

$$\frac{\partial u}{\partial t} = -\frac{1}{D}\frac{\partial Z}{\partial h}; \quad \frac{\partial h}{\partial t} = \frac{1}{D}\frac{\partial Z}{\partial u}; \quad \frac{\partial u}{\partial \eta} = \frac{1}{D}\frac{\partial T}{\partial h}; \quad \frac{\partial h}{\partial \eta} = -\frac{1}{D}\frac{\partial T}{\partial u}, \tag{3.3}$$

in which we have reverted to the notation $u=U,\ h=H$. Using transformations (3.1) and (3.3) in (1.1), we find

$$\begin{cases} \dfrac{\partial Z}{\partial u} - u\dfrac{\partial T}{\partial u} + h\dfrac{\partial T}{\partial h} = 0, \\ \dfrac{\partial Z}{\partial h} - u\dfrac{\partial T}{\partial h} + \beta\dfrac{\partial T}{\partial u} = 0. \end{cases} \tag{3.4}$$

which is linear and in which h and u are now the independent variables.

Remark 1: In the transition from system (1.1) to (3.4) solutions could be lost for which J=0. According to (3.2) these solutions must obey

$$\frac{\partial h}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial h}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)^{-1}, \quad \frac{\partial u}{\partial \eta} \neq 0. \tag{3.5}$$

When this relation is used in (1.1) and $\partial u/\partial t$ is eliminated, the following equation is obtained:

$$h\frac{\partial u}{\partial \eta} = \pm\sqrt{\beta h}\frac{\partial h}{\partial \eta}. \tag{3.6}$$

Alternatively, the second equation of (1.1) yields, if (3.6) is used

$$h\frac{\partial u}{\partial t} = \pm\sqrt{\beta h}\frac{\partial h}{\partial t}. \tag{3.7}$$

The last two relations obviously imply the total differential $hdu = \pm\sqrt{\beta h}dh$, which integrates to

$$h = \left(\frac{u}{2\sqrt{\beta}} - \frac{C}{2\sqrt{\beta}}\right)^2, \tag{3.8}$$

with the aid of which the second equation of (1.1)

$$\eta = \left(\frac{3}{2}u - 0.5C\right)t + \Phi(u) \tag{3.9}$$

is obtained. This agrees with the classical partially invariant solution (2.7).•

Let us find a solution of the linear system (3.4) by choosing Z according to

$$Z = uT - \frac{\partial \psi}{\partial u}. \tag{3.10}$$

(This choice follows an analogous choice made in gas dynamics). If we substitute (3.10) into the first of (3.4), then

$$T + h\frac{\partial T}{\partial h} = \frac{\partial^2 \psi}{\partial u^2} \tag{3.11}$$

is obtained; alternatively, the second of (1.1) implies

$-\dfrac{\partial^2 \psi}{\partial u \partial h} + \beta\dfrac{\partial T}{\partial u} = 0$ provided β =*constant.*

This suggests to choose

$$T = \frac{1}{\beta}\frac{\partial \psi}{\partial h}. \tag{3.12}$$

Back substitution into (3.11) yields the single partial differential equation for ψ

$$h\frac{\partial^2 \psi}{\partial h^2} + \frac{\partial \psi}{\partial h} = \beta \frac{\partial^2 \psi}{\partial u^2}. \tag{3.13}$$

Once its solution has been found, the functions T and Z can be obtained from

$$\begin{cases} T = \dfrac{1}{\beta}\dfrac{\partial \psi}{\partial h}, \\ Z = uT - \dfrac{\partial \psi}{\partial u}. \end{cases} \tag{3.14}$$

A series of further transformations identifies (3.13) as an equation from which the Riemann function can directly be constructed. First, we introduce $y=y(h)$ in the form

$$y = 2\sqrt{\beta h} \tag{3.15}$$

and deduce

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{1}{y}\frac{\partial \psi}{\partial y} = \frac{\partial^2 \psi}{\partial u^2}. \tag{3.16}$$

Second, we choose

$$\xi = u + y, \quad \nu = u - y \tag{3.17}$$

and may then obtain (3.16) in the so-called canonical form

$$2\frac{\partial^2 \psi}{\partial \xi \partial \nu} - \frac{1}{\xi - \nu}\left(\frac{\partial \psi}{\partial \xi} - \frac{\partial \psi}{\partial \nu}\right) = 0. \tag{3.18}$$

Third, see [5], we replace ψ by ω, defined via

$$\omega = \psi\sqrt{\xi - \nu} \tag{3.19}$$

and then obtain

$$\frac{\partial^2 \omega}{\partial \xi \partial \eta} - \frac{\omega}{4(\xi - \nu)^2} = 0. \tag{3.20}$$

which is the standard final form of the original equation (3.13).

The Riemann function to equation (3.20) has been constructed, see [5], and is given by

$$R(\xi, \nu, \xi_0, \nu_0) = F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{(\xi - \xi_0)(\nu - \nu_0)}{(\xi_0 - \nu_0)(\xi - \nu)}\right), \tag{3.21}$$

where

$$F\left(\frac{1}{2},\frac{1}{2},1,z\right)=1+\sum_{k=1}^{\infty}\frac{(0.5)_k(0.5)_k}{(1)_k}\frac{z^k}{k!},\qquad (\alpha)_k=\alpha(\alpha+1)...(\alpha+k-1) \tag{3.22}$$

is the hypergeometric function, see [5]. Notice that, if the Riemann function is known, it is possible to write the explicit solutions for the Cauchy and Gorse problems by using the formula due to Riemann

$$(\omega R)_M=\frac{(\omega R)_P+(\omega R)_Q}{2}-\frac{1}{2}\int_{PQ}\left(\omega\frac{\partial R}{\partial\xi}-R\frac{\partial\omega}{\partial\xi}\right)d\xi+\left(R\frac{\partial\omega}{\partial\nu}-\omega\frac{\partial R}{\partial\eta}\right)d\nu, \tag{3.23}$$

where *M* is the point of the plane (ξ,ν) with the coordinates (ξ_0,ν_0) and *PQ* is the *arc*, where the function ω and its derivatives are known. If *PQ* does not contain the characteristics of equation (3.20), then we have the Cauchy problem, on the other hand, if *PQ* consists of characteristics, then (3.23) defines the solution of the Gorse problem. Finally, we note that formula (3.23) represents an exact solution of the considered equation, which is known, once values for the function $\omega(\xi,\nu)$ and its ξ and ν-derivatives are prescribed along the arc *PQ*. Therefore, Riemann's formula together with (3.21) can be used to construct solutions to different problems.
The next section will list a few applied problems in order of increasing complexity.

4. Physical interpretation and discussions

4.1. ***Example 1***: Consider a rectangular pile of length 2λ and height h_o, held between two walls on a straight chute with inclination angle ζ. Assume that at time t=0 the upper and the lower walls are suddenly removed. In a coordinate system moving with the centre of gravity the motion of the deforming sand pile is described by equations (1.1). It is possible with the help of (2.7) to construct the solution to this spreading problem. We impose the following initial conditions:

$$t=0,\ h=h_0,\ u=0,\ -l<\eta<l$$

with $m=\int_{-l}^{l}h_0d\eta=2lh_0$ as the initial mass (or volume) which is a preserved quantity for all time, see also Fig.1.

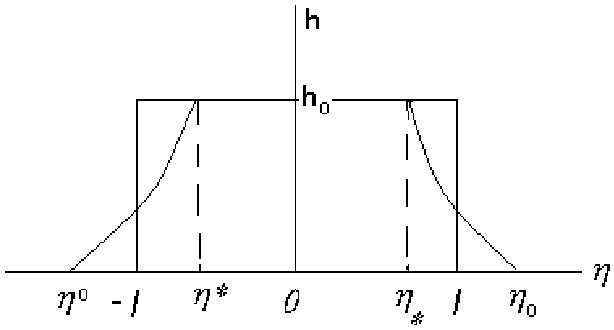

Figure 1: Sketch of the spreading of a granular heap, given in its initial configuration as a rectangle and at a later time $0<t<t^*$.

The construction of the solution to this problem can more easily be understood, if the characteristics to system (1.1) are determined. To this end, let

$$\eta = \eta(t) \tag{4.1}$$

be the equation of such a characteristic line. Along such a line the total time derivatives of h and u are given by

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial \eta}\dot{\eta}; \qquad \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial \eta}\dot{\eta},$$

where $\dot{\eta}$ is the uni-variate derivative of $\eta(t)$. With these expressions, equations (1.1) take the forms

$$\frac{\partial u}{\partial \eta}(u-\dot{\eta}) + \beta\frac{\partial h}{\partial \eta} = -\frac{du}{dt}, \quad \frac{\partial u}{\partial \eta}h + (u-\dot{\eta})\frac{\partial h}{\partial \eta} = -\frac{dh}{dt}, \tag{4.2}$$

which, when viewed as linear equations for $\partial u/\partial\eta$ and $\partial h/\partial\eta$, can be solved for these, viz.,

$$\frac{\partial u}{\partial \eta} = \frac{\beta\frac{dh}{dt} - (u-\dot{\eta})\frac{du}{dt}}{(u-\dot{\eta})^2 - \beta h}; \quad \frac{\partial h}{\partial \eta} = -\frac{(u-\dot{\eta})\frac{dh}{dt} - h\frac{du}{dt}}{(u-\dot{\eta})^2 - \beta h}.$$

Now, along the characteristics, these derivatives are undetermined. This implies that the numerators and denominators on the right-hand sides must independently vanish. This implies

$$(u-\dot{\eta})^2 - \beta h = 0, \quad (u-\dot{\eta})\frac{dh}{dt} - h\frac{du}{dt} = 0, \quad \beta\frac{dh}{dt} - (u-\dot{\eta})\frac{du}{dt} = 0.$$

The first can be written as

$$\dot{\eta} = u \pm \sqrt{\beta h}\,; \tag{4.3}$$

the two others, subject to (4.3), reduce to the single relation $\sqrt{\beta/h}\,dh \pm du = 0$ or, after integration

$$u \pm 2\sqrt{\beta h} = const\,. \tag{4.4}$$

So, we have just proved that the expression (4.4) remains constant along the characteristic lines. However, the first of (2.7) can also be expressed in the form (4.4), viz.,

$$u \pm 2\sqrt{\beta h} = -2C\sqrt{\beta}\ is\ const\,, \tag{4.5}$$

and this equation is valid for all η for which the heap exists. Thus, (4.5) is valid, in particular on the lines $\eta = \eta_*(t)$ and $\eta = \eta_0(t)$ (and similarly for $\eta = \eta^*(t)$ and $\eta = \eta^0(t)$), which, respectively, denote the eroding edge and travelling foot points of the spreading granular mass, see Fig. 1. In terms borrowed from aerodynamics, $\eta_*(t)$ and $\eta^*(t)$ are also called the right and left Mach fronts, respectively. Finally, comparing (4.4) and (4.5) for $\eta_*(t)$ and $\eta^*(t)$ yields $C = -\sqrt{h_0}$, since h=h_o and u=0 at these point. Now, since $u \pm 2\sqrt{\beta h}$ remains constant on these lines, they are characteristics, and therefore in view of (4.3)

$\dfrac{d\eta_*}{dt} = u - \sqrt{\beta h} = -\sqrt{\beta h_0}$, since on η_* $h = h_0,\ u = 0$ and η_* moves in the negative direction;

$\dfrac{d\eta_0}{dt} = u + \sqrt{\beta h} = u$, since on η_0 $h = 0$ and η_0 moves in the positive direction.

Let us find u at the foot point η_0, where h=0. From (4.5) with $C = -\sqrt{h_0}$ we find

$$u = 2\sqrt{\beta h_0} - 2\sqrt{\beta h}\,, \qquad \text{valid } \forall h \in [0, h_0], \tag{4.6}$$

which for h=0 yields the desired result: $u\big|_{\eta_o} = 2\sqrt{\beta h_0}$. Integrating the equations for η_* and η_0 and using this result, we find

$$\eta_* = \lambda_* - \sqrt{\beta h_0}\, t\,, \tag{4.7}$$

$$\eta_0 = \lambda_0 + 2\sqrt{\beta h}\, t\,. \tag{4.8}$$

where λ_*, λ_0 are constants of integration which must be equal, since $\eta_*(0) = \eta_0(0) = \lambda$. There still remains the determination of $\Phi(u)$. The expression follows from the second of (2.7), which, with $C = -\sqrt{h_0}$ and when subject to (4.6), takes the form

$$\eta = \Phi\left(2\sqrt{\beta h_0} - 2\sqrt{\beta h}\right) + (2\sqrt{\beta h_0} - 3\sqrt{\beta h})t\,. \tag{4.9}$$

When h=h_o or h=0, this equation must coincide with (4.7) and (4.8), respectively, from which we deduce

$\Phi(0) = \lambda$ and $\Phi(2\sqrt{\beta h_0}) = \lambda$ for $\forall h_0$.

This last condition implies

$$\Phi(u) \equiv \lambda. \tag{4.10}$$

Substituting this into (4.9), solving the resulting equation for h, and copying u from (4.6), we find the solution at the right shoulder of the heap in the form

$$\begin{cases} h = \dfrac{1}{9\beta}\left(2\sqrt{\beta h_0} - \dfrac{\eta - \lambda}{t}\right)^2, \\ u = 2\sqrt{\beta}\,(\sqrt{h_0} - \sqrt{h}), \end{cases} \qquad \eta_* < \eta < \eta_0, \tag{4.11}$$

where η_* and η_0 are given in (4.7) and (7.8).

An analogous analysis, conducted for the left shoulder, yields

$$\begin{cases} h = \dfrac{1}{9\beta}\left(2\sqrt{\beta h_0} + \dfrac{\eta + \lambda}{t}\right)^2, \\ u = 2\sqrt{\beta}\,(\sqrt{h} - \sqrt{h_0}), \end{cases} \qquad \eta^0 < \eta < \eta^*, \tag{4.12}$$

where $\eta^0 = -\lambda - 2\sqrt{\beta h_0}\, t$ *and* $\eta^* = -\lambda + \sqrt{\beta h_0}\, t$. By substituting the first of (4.11) into the second of (4.11) and the first of (4.12) into the second of (4.12), it is also possible to write the velocities in terms of the variables η and t, namely

$$\begin{aligned} u &= \frac{2}{3}\left(\sqrt{\beta h_0} + \frac{\eta - \lambda}{t}\right), \qquad \eta_* < \eta < \eta_0, \\ u &= \frac{2}{3}\left(-\sqrt{\beta h_0} + \frac{\eta + \lambda}{t}\right), \qquad \eta^0 < \eta < \eta^*. \end{aligned} \tag{4.13}$$

With the construction of (4.11) and (4.12) the early time solution of the posed problem is complete. We note also that the examined problem is similar to the dam break problem in the inviscid shallow water equations. Furthermore, the constructed solution must satisfy the global conservation of mass, and indeed it does so, since

$M = \int_{\eta^0}^{\eta^*} h\,d\eta + \int_{\eta^*}^{\eta_*} h_0\,d\eta + \int_{\eta_*}^{\eta_0} h\,d\eta = 2h_0\lambda$, which is constant.

Obviously, the solution can only be correct as long as the left and right eroding edges do not meet. Denoting this time by t^*, the solutions hold for $0<t<t^*$, where t^* follows from the equation

$\eta^*(t) = \eta_*(t)$ or $\lambda - \sqrt{\beta h_0}t^* = \sqrt{\beta h_0}t^* - \lambda$, implying that

$$t^* = \frac{\lambda}{\sqrt{\beta h_0}}. \tag{4.14}$$

Let us next try to find the solution for times $t > t^*$. Now, the new Mach fronts move as reflected waves away from the position $\eta = 0$. To describe their coordinates in time, we use the same symbols as before: $\eta_*(t)$ for the Mach front to the right side $\eta = 0$ and $\eta^*(t)$ for that on the left of $\eta = 0$. Obviously, $\eta_*(t)$ moves in the positive and $\eta^*(t)$ in the negative η-direction. The motions of the shoulders to the right of $\eta_*(t)$ and to the left of $\eta^*(t)$ are described by the same solution principle as for $t<t^*$. To construct them, we only must recall that the functions $\eta_*(t)$ and $\eta^*(t)$ are characteristics. The initial value problems for their motion follows from the condition $\eta^*(t^*) = \eta_*(t^*) = 0$. Therefore

$$\begin{cases} \dfrac{d\eta_*}{dt} = u + \sqrt{\beta h}; \\ \eta_* = 0,\ t = t^*. \end{cases} \qquad \begin{cases} \dfrac{d\eta^*}{dt} = u - \sqrt{\beta h}; \\ \eta^* = 0,\ t = t^*. \end{cases}$$

(Note that the signs in front of $\sqrt{\beta h}$ are opposite to those in equation (4.6) because now, the Mach fronts move in the opposite directions). Inserting for u and h the expressions in (4.11) and (4.12), respectively, now yields

$$\begin{cases} \dfrac{d\eta_*}{dt} = \dfrac{4}{3}\sqrt{\beta h_0} + \dfrac{\eta - \lambda}{3t}; \\ \eta_* = 0,\ t = t^*. \end{cases} \qquad \begin{cases} \dfrac{d\eta^*}{dt} = -\dfrac{4}{3}\sqrt{\beta h_0} + \dfrac{\eta + \lambda}{3t} \\ \eta^* = 0,\ t = t^* \end{cases}$$

and straightforward integration subject to the initial conditions yields

$$\begin{aligned} &\eta_*(t) - \lambda = 2\sqrt{\beta h_0}\,t - (\lambda + 2\sqrt{\beta h_0}\,t^*)\left(\frac{t}{t^*}\right)^{1/3}, \qquad t \geq t^*; \\ &\eta^*(t) + \lambda = -2\sqrt{\beta h_0}\,t + (\lambda + 2\sqrt{\beta h_0}\,t^*)\left(\frac{t}{t^*}\right)^{1/3}, \qquad t \geq t^* \end{aligned} \tag{4.15}$$

These two solutions must be patched together with the thickness, *h*, and velocity, *u*, profiles in the interior of the interval (η^*, η_*). To construct this interior solution, equation (3.16) for the function ψ must be solved. So, let us first determine the values for ψ on the above characteristics. It follows from (4.11) that the value of $u + 2\sqrt{\beta h} = \lambda$ is constant on the first characteristics, and $\lambda = 2\sqrt{\beta h_0}$, according to (4.11). Similarly, (4.12) implies that $u - 2\sqrt{\beta h} = -2\sqrt{\beta h_0} = -\lambda$ is constant on the second characteristics. According to (3.15) and (3.17) these expressions in the plane (ξ,ν) take the form $\xi=\lambda$ and $\nu=-\lambda$; these are straight lines parallel to the coordinates ν and ξ, respectively, and form the characteristics of equation (3.20).

Next, let us calculate $d\psi/dh$ on the characteristics η_* and η^* respectively. Using (4.11) and (3.14) and recalling the definitions of T and Z, yields

$$\left.\frac{d\psi}{dh}\right|_{\eta_*}=\frac{\partial\psi}{\partial h}+\frac{\partial\psi}{\partial u}\frac{du}{dh}=\left.\frac{\partial\psi}{\partial h}\right|_{\eta_*}-\sqrt{\frac{\beta}{h}}\left.\frac{\partial\psi}{\partial u}\right|_{\eta_*}=\beta t-\sqrt{\frac{\beta}{h}}(ut-\eta)=\lambda\sqrt{\frac{\beta}{h}},$$

in which the very last step follows by substituting u and η from (4.11) and (4.9), respectively. Thus, on the characteristics $\xi=\lambda$ $(\eta=\eta_*(t))$

$$\frac{d\psi}{dh}=\lambda\sqrt{\frac{\beta}{h}}\quad or\quad \left.\psi\right|_{\xi=\lambda}=2\lambda\sqrt{\beta h}+const.$$

Analogously, on the characteristics $\nu=-\lambda$ $(\eta=\eta_*)$

$$\frac{d\psi}{dh}=\lambda\sqrt{\frac{\beta}{h}}\quad or\quad \left.\psi\right|_{\nu=-\lambda}=2\lambda\sqrt{\beta h}+const.$$

The constants of integration are both the same, but may be set to zero, since ψ has potential character. Therefore

$$\left.\psi\right|_{\xi=\lambda}=2\lambda\sqrt{\beta h}\text{ and }\left.\psi\right|_{\nu=-\lambda}=2\lambda\sqrt{\beta h}. \qquad (4.16)$$

Now, from (3.15) and (3.17) we deduce $2\sqrt{\beta h}=0.5(\xi-\nu)$; so,

$$\left.\psi\right|_{\xi=\lambda}=0.5\lambda(\lambda-\eta),\quad \left.\psi\right|_{\nu=-\lambda}=0.5\lambda(\lambda+\xi)$$

and in view of (3.19)

$$\left.\omega\right|_{\xi=\lambda}=0.5\lambda(\lambda-\nu)^{3/2},\quad \left.\omega\right|_{\nu=-\lambda}=0.5\lambda(\lambda+\xi)^{3/2}. \qquad (4.17)$$

From these, the derivatives along the characteristics are given by

$$\left.\frac{\partial\omega}{\partial\eta}\right|_{\xi=\lambda}=-\frac{3}{4}\lambda(\lambda-\nu)^{1/2},\quad \left.\frac{\partial\omega}{\partial\xi}\right|_{\nu=-\lambda}=\frac{3}{4}\lambda(\lambda+\xi)^{1/2}. \qquad (4.18)$$

Relations (4.17) define the boundary values of ω along the characteristics $\xi=\lambda$ and $\nu=-\lambda$ for a solution in the rectangle $[0<\xi<\lambda, 0<\nu<-\lambda]$. This defines a Gorse problem, of which the solution can be constructed with the Riemann formula (3.23). Recalling (3.21), (3.22) and the properties of the hypergeometric functions [5], we have

$$R\big|_{\xi=\lambda} = F\left(\frac{1}{2},\frac{1}{2},1,\frac{(\lambda-\xi_0)(\nu-\nu_0)}{(\xi_0-\nu_0)(\lambda-\nu)}\right),\ R\big|_{\nu=-\lambda} = F\left(\frac{1}{2},\frac{1}{2},1,\frac{(\xi-\xi_0)(-\lambda-\nu_0)}{(\xi_0-\nu_0)(\xi+\lambda)}\right);$$

$$\frac{\partial R}{\partial \xi} = \frac{1}{4}\frac{(\xi_0-\nu)(\nu-\nu_0)}{(\xi_0-\nu_0)(\xi-\nu)^2}F\left(\frac{3}{2},\frac{3}{2},2,\frac{(\xi-\xi_0)(\nu-\nu_0)}{(\xi_0-\nu_0)(\xi-\nu)}\right),$$

$$\frac{\partial R}{\partial \nu} = \frac{1}{4}\frac{(\xi-\xi_0)(\xi-\nu_0)}{(\xi_0-\nu_0)(\xi-\nu)^2}F\left(\frac{3}{2},\frac{3}{2},2,\frac{(\xi-\xi_0)(\nu-\nu_0)}{(\xi_0-\nu_0)(\xi-\nu)}\right);$$

$$R\big|_{\substack{\xi=\xi_0\\ \nu=\nu_0}} = 1,\quad R_P = R\big|_{\substack{\xi=\xi_0\\ \nu=-\lambda}} = 1,\quad R_Q = R\big|_{\substack{\xi=\lambda\\ \nu=\nu_0}} = 1.$$

Using these in Riemann formula, we have

$$\frac{4}{\lambda}\omega(\xi_0,\nu_0) = (\xi_0+\lambda)^{3/2} + (\lambda-\nu_0)^{3/2} - \int_{\xi_0}^{\lambda}\Phi_1(\xi,\xi_0,\nu_0)d\xi - \int_{-\lambda}^{\nu_0}\Phi_2(\nu,\xi_0,\nu_0)d\nu\ ,\text{(4.19)}$$

where now

$$\Phi_1(\xi,\xi_0,\nu_0) = \frac{3}{2}(\xi+\lambda)^{1/2}F\left(\frac{1}{2},\frac{1}{2},1,\frac{(\xi-\xi_0)(-\lambda-\nu_0)}{(\xi_0-\nu_0)(\xi+\lambda)}\right) +$$
$$+\frac{1}{4}\frac{(\lambda+\nu_0)(\xi_0+\lambda)}{(\xi_0-\nu_0)(\xi+\lambda)^{1/2}}F\left(\frac{3}{2},\frac{3}{2},2,\frac{(\xi-\xi_0)(-\lambda-\nu_0)}{(\xi_0-\nu_0)(\xi+\lambda)}\right),$$

Knowing the function ω it is easy to write, using (3.17) and (3.19), the expression for ψ in the coordinates *(u, y)*:

$$\psi(u,y) = \frac{\lambda}{4\sqrt{2y}}\left[(u+y+\lambda)^{3/2} + (y+\lambda-u)^{3/2} - \int_{y+u}^{\lambda}\Phi_1(\xi,u,y)d\xi - \int_{-\lambda}^{u-y}\Phi_2(\nu,u,y)d\nu\right],\ \text{(4.20)}$$

here

$$\Phi_1(\xi,u,y) = \frac{3}{2}(\xi+\lambda)^{1/2}F\left(\frac{1}{2},\frac{1}{2},1,\frac{(\xi-u-y)(-\lambda-u+y)}{(\xi+\lambda)\sqrt{2y}}\right) +$$
$$+\frac{1}{4}\frac{(\lambda+u-y)(u+y+\lambda)}{\sqrt{2y(\xi+\lambda)}}F\left(\frac{3}{2},\frac{3}{2},2,\frac{(\xi-u-y)(-\lambda-u+y)}{(\xi+\lambda)\sqrt{2y}}\right),$$

$$\Phi_2(\nu,u,y) = \Phi_1(-\nu,-u,y).$$

Formally this completes the construction of the solution to the posed problem: Via differentiation the functions *T* and *Z*, defined in (3.14), can be derived. However, the expressions that are obtained are very complicated and difficult to implement numerically. Therefore, it is desirable to construct an approximate solution to the considered problem.

4.2. *Approximate solution of the problem.*

In order to construct an approximate analytical solution for ψ, let us use the integral method proposed in [6]. According to this method, the profile of the function to be determined is preselected except for a free element, which is determined by integrating the partial differential equation for ψ in one coordinate direction and thus deducing an ordinary differential equation from which the free element can be determined. Since the evolving profile is symmetric with respect to η=0, $h(\eta) = h(-\eta)$; so, (3.16) implies that ψ is also symmetric with respect to u. Therefore, we choose as profile for the function ψ the polynomial

$$\psi = a + b\sigma^2 + c\sigma^4, \quad \sigma = \frac{u}{(\lambda - y)},$$

(notice only the even powers in σ). The coefficients b and c are determined by the values which ψ and $d\psi/du$ take on the characteristic $u = \lambda - y$ ($\sigma = 1$). Using (3.11), (4.11), (4.15) and (4.16), one may easily deduce

$$\psi\big|_{u=\lambda-y} = \lambda y, \quad \left.\frac{\partial\psi}{\partial u}\right|_{u=\lambda-y} = \lambda\left(\sqrt{\frac{\lambda}{y}} - 1\right). \tag{4.21}$$

Using this, the coefficients b and c are obviously functions of y and so

$$\psi = \lambda y + (a - \lambda y)(1-\sigma^2)^2 - \frac{1}{2}\lambda\left(\sqrt{\frac{\lambda}{y}} - 1\right)(\lambda - y)\sigma^2(1-\sigma^2), \tag{4.22}$$

in which *a(y)* remains still unknown, but the derivatives are

$$\frac{\partial\psi}{\partial u} = \lambda\left(\sqrt{\frac{\lambda}{y}} - 1\right)\sigma(2\sigma^2 - 1) - \frac{4(a-\lambda y)}{(\lambda - y)}\sigma(1-\sigma^2), \tag{4.23}$$

$$\begin{aligned}\frac{\partial\psi}{\partial y} = {} & \lambda + (\dot{a} - \lambda)(1-\sigma^2)^2 - \frac{4(a-\lambda y)}{(\lambda - y)}\sigma^2(1-\sigma^2) + \frac{\lambda}{2}\left(\frac{\lambda + y}{2y}\sqrt{\frac{\lambda}{y}} - 1\right)\sigma^2(1-\sigma^2) + \\ & + \lambda\left(\sqrt{\frac{\lambda}{y}} - 1\right)\sigma^2(2\sigma^2 - 1).\end{aligned} \tag{4.24}$$

Here $\dot{a}$ is the derivative of the function *a(y)*. To determine this function, consider the partial differential equation (3.16) for ψ and integrate it over u from u=0 to $u = \lambda - y$ (corresponding to an integration from η=0 to η^* (t)). This yields

$$\frac{d^2V}{dy^2} + \frac{1}{y}\frac{dV}{dy} = -3\lambda \tag{4.25}$$

in which $\partial\psi/\partial u\big|_{u=0} = 0$ has been used and where

$$V = \int_{0}^{\lambda - y} \psi \, du \quad . \tag{4.26}$$

So, $V(y = \lambda) = 0$. The solution of (4.25) subject to this initial condition is given by

$$V = \frac{3}{4}\lambda(\lambda^2 - y^2) + C \ln \frac{y}{\lambda} . \tag{4.27}$$

Its constant of integration will be determined lateron. However, when substituting (4.22) into (4.26), the relation

$$V = \frac{(\lambda - y)}{15}\left[8a + \lambda\lambda + 6\lambda y + \lambda\sqrt{\frac{\lambda}{y}}(y - \lambda) \right] \tag{4.28}$$

is obtained. Comparison of the two different expressions for V allows determination of the function $a(y)$. Indeed, from (4.27) and (4.28) we obtain

$$A(y) = 8a(y) = \frac{41}{4}\lambda\lambda + \frac{21}{4}y\lambda + \lambda\sqrt{\frac{\lambda}{y}}(\lambda - y) + \frac{15C}{\lambda - y}\ln\frac{y}{\lambda} , \tag{4.29}$$

from which

$$\dot{A}(y) = \frac{1}{2}\left[\frac{21}{2}\lambda - \lambda\sqrt{\frac{\lambda}{y}}\left(1 + \frac{\lambda}{y}\right) + \frac{30C}{\lambda - y}\left(\frac{1}{y} + \frac{1}{\lambda - y}\ln\frac{y}{\lambda}\right) \right] \tag{4.30}$$

is derived. The expression (4.29) completes the approximate solution, except that the constant C is still not determined. To this end, consider the first of formulas (3.11) in the form

$$t = \frac{1}{\beta}\frac{\partial \psi}{\partial h} = \frac{2}{y}\frac{\partial \psi}{\partial y} . \tag{4.31}$$

If (4.24) is substituted in this formula, then the resulting expression describes implicitly the height h as a function of t and u. At the centre (η=0) one has u=0 (σ=0) due to symmetry, and h assumes its maximum value here. Furthermore, the limit $y \to \lambda$ corresponds to $h \to h_0$, and $h \to h_0$ is also assumed everywhere in the region $[\eta_*, \eta^*]$ as t $t \to t^*$. Thus

$$y \to \lambda \quad (h \to h_0), \quad t \to t^* = \frac{\lambda}{\sqrt{\beta h_0}} = \frac{2\lambda}{\lambda} .$$

This condition can be used to find the constant C. Indeed, from (4.31) we conclude that

$$\frac{2\lambda}{\lambda} = \lim_{y \to \lambda}\frac{2}{y}\frac{\partial \psi}{\partial y}\bigg|_{\sigma = 0} = \frac{1}{4\lambda}\lim_{y \to \lambda}\dot{A} = \frac{1}{8\lambda}\left(\frac{17}{2}\lambda + \frac{15C}{\lambda^2}\right) \quad \text{or}$$

$$C = 0.5\lambda^2 . \tag{4.32}$$

This value determines $A(y)$ and $\dot{A}(y)$ in (4.29) and (4.30) which now become

$$A(y)=\frac{41}{4}\lambda\lambda+\frac{21}{4}y\lambda+\lambda\sqrt{\frac{\lambda}{y}}(\lambda-y)+\frac{15\lambda^2\lambda}{2(\lambda-y)}\ln\frac{y}{\lambda};$$

$$\dot{A}(y)=\frac{1}{2}\left[\frac{21}{2}\lambda-\lambda\sqrt{\frac{\lambda}{y}}\left(1+\frac{\lambda}{y}\right)+\frac{15\lambda^2\lambda}{\lambda-y}\left(\frac{1}{y}+\frac{1}{\lambda-y}\ln\frac{y}{\lambda}\right)\right].$$

They fix ψ and its derivatives in (4.22)-(4.24). The approximate solution of the original problem then follows from the application of (3.11); what obtains reads as follows:

$$\eta=ut-\left[\lambda\left(\sqrt{\frac{\lambda}{y}}-1\right)\sigma(2\sigma^2-1)-\frac{A-8\lambda y}{2(\lambda-y)}\sigma(1-\sigma^2)\right], \tag{4.33}$$

$$t=\frac{2}{y}\left[\lambda+\left(\frac{1}{8}\dot{A}-\lambda\right)(1-\sigma^2)^2-\frac{A-8\lambda y}{2(\lambda-y)}\sigma(1-\sigma^2)\right]+$$
$$+\frac{\lambda}{y}\left[\left(\frac{\lambda+y}{2y}\sqrt{\frac{\lambda}{y}}-1\right)\sigma^2(1-\sigma^2)-2\left(\sqrt{\frac{\lambda}{y}}-1\right)\sigma^2(1-2\sigma^2)\right]. \tag{4.34}$$

It is expedient to adjust the formulas (4.33), (4.34) to the particular needs in the explicit calculations. This will now be done.

1. A practically relevant information is the temporal evolution of the maximum thickness at $\eta=0$. To evaluate this relation, note that u=0 (σ=0) at $\eta=0$; this can be verified in (4.33), whilst (4.34) yields

$$t=\frac{\lambda}{8y_{\max}}\left[\frac{21}{2}-\sqrt{\frac{\lambda}{y_{\max}}}\left(1+\frac{\lambda}{y_{\max}}\right)+\frac{15\lambda^2}{\lambda-y_{\max}}\left(\frac{1}{y_{\max}}+\frac{1}{\lambda-y_{\max}}\ln\frac{y_{\max}}{\lambda}\right)\right], \tag{4.35}$$

which gives t as a function of $y_{\max}$ ($h_{\max}$).

2. Another significant relation is the value of $y(h)$ at η^* (or η_*) which are characteristics. It is physically obvious that this value of y is the smallest in the interval [0, η^*]. The formula follows from (4.34) by setting σ=1 ($u=\lambda-y$):

$$y_{\min}=\lambda^{1/3}\left(\frac{2\lambda}{t}\right)^{2/3}. \tag{4.36}$$

[This expression could also be found from (4.11)].

3. Evidently, if t is known, then the function $\eta(y)$ can be found from (4.33), which determines the profile of the free surface in the interval [$y_{\min}, y_{\max}$]. To facilitate the corresponding numerical calculations, the expression (4.34) must be solved for u or σ. As for σ, the equation is biquadratic; so, to find σ is straightforward. To this end, let

$w=1-\sigma^2$ or $\sigma=\sqrt{1-w}$.

Then, from (4.34) we may derive

$$w(y) = (2B_1)^{-1}\left(-B_2 + \sqrt{B_2^2 - 4B_1\left(\lambda\sqrt{\frac{\lambda}{y}} - \frac{1}{2}yt\right)}\right), \tag{4.37}$$

(the second solution with w<0 is meaningless) where

$$B_1 = \dot{A}/8 - \lambda + P(y);\;\; B_2 = -P(y) - \lambda\left(\sqrt{\frac{\lambda}{y}} - 1\right);$$

$$P(y) = \frac{A(y) - 8\lambda y}{2(\lambda - y)} - \frac{\lambda}{2}\left(\frac{\lambda + y}{2y}\sqrt{\frac{\lambda}{y}} - 1\right) + 2\lambda\left(\sqrt{\frac{\lambda}{y}} - 1\right).$$

Finally, (4.33) is expressed in terms of $w(y)$ as follows

$$\eta = \sqrt{1 - w(y)}\left[t(\lambda - y) - \lambda\left(\sqrt{\frac{\lambda}{y}} - 1\right)(1 - 2w(y)) + \frac{A(y) - 8\lambda y}{2(\lambda - y)}w(y)\right] \tag{4.38}$$

which is the inverse relationship to $y(\eta)$.

To summarize: With y= $y = 2\sqrt{\beta h}$ the formulae (4.35) - (4.38) allow the computation of the motion of the pile in the interior region $[\eta_*, \eta^*]$.

Figure 3 depicts the variation of the maximum thickness $y_{\max} = 2\sqrt{\beta h_{\max}}$ as a function of time *t*, whilst Figs. 4 and 5 display time series of the free surface of the granular layer before the two Mach fronts meet (Fig. 4) and after they have met, (Fig. 5). The initial data for these figures are $\lambda = 1;\ \lambda = 1;\ h_0 = 1;\ \beta = 0.25.$ Finally, Fig. 6 shows a comparison of the exact and the approximate solution of the pile thickness at the centre η=0 as a function of time. It is evident that the two curves agree very well with one another.

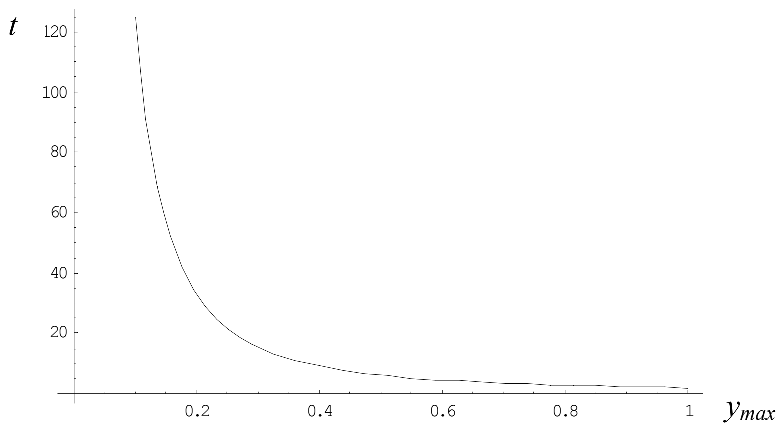

Figure 3. Variation of the layer thickness with time at $\eta = 0$.

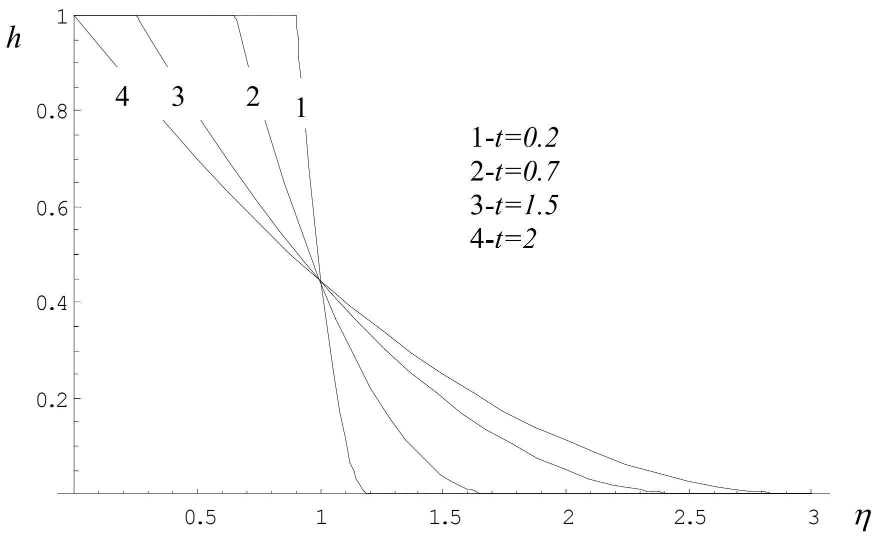

Figure 4. Evolution of the free surface of the layer of the granular material at early times before the fronts coalesce ($t \le t^* = 2$).

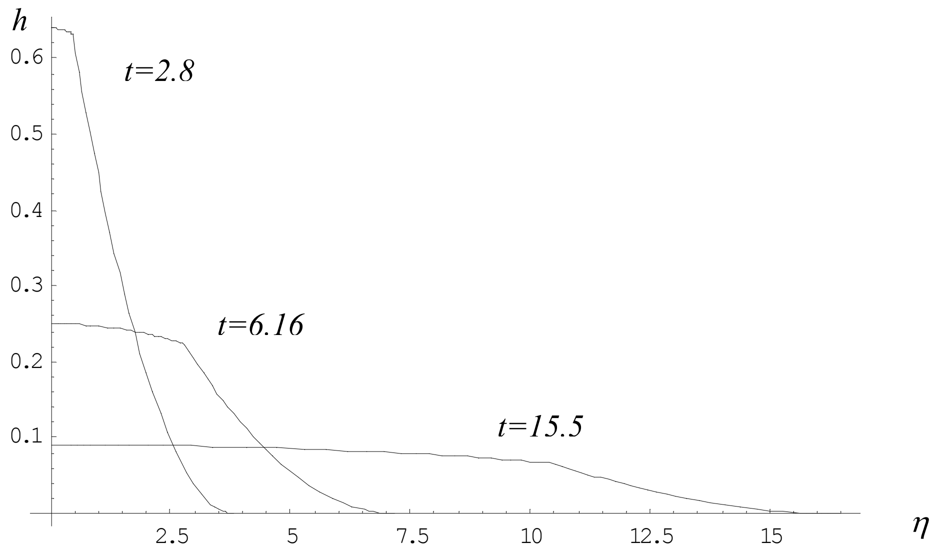

Figure 5. Evolution of the free surface of the layer of the granular material after the fronts did coalesce ($t > t^* = 2$).

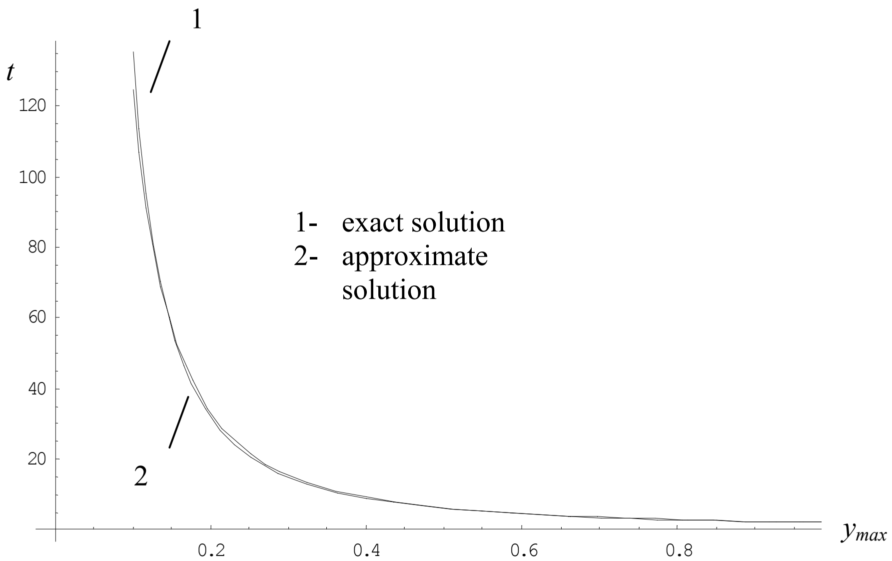

Figure 5. Comparison of the exact solution with the approximate solution in the center $\eta = 0$.

Remark 2: Using the Riemann function, it is possible to construct exact solutions for the spreading motion of granular materials along an incline starting from an arbitrary initial profile at rest**.** •

Remark 3: The above solution of the spreading of a rectangular pile can only be correct as long as the physical velocity at the upper shoulder is in the downhill direction. This is the case so long as

$t > \sqrt{\beta h} / s$.

4.3. ***Example 2***: We now alter the fomulation of Example 1 by confining the material at the upper edge for all time by a wall at rest, see Fig. 2. In the coordinates of the laboratory frame (*x,h*), the wall, located at $x = -\lambda$ remains still during the entire process. Therefore, in the coordinates of the moving frame (η, h), the motion of the wall and its velocity are, respectively, given by

$$\eta_w(t) = -\lambda - \frac{s_0 t^2}{2} \tag{4.39}$$

$$u_w = \dot{\eta}_w = -s_0 t, \tag{4.40}$$

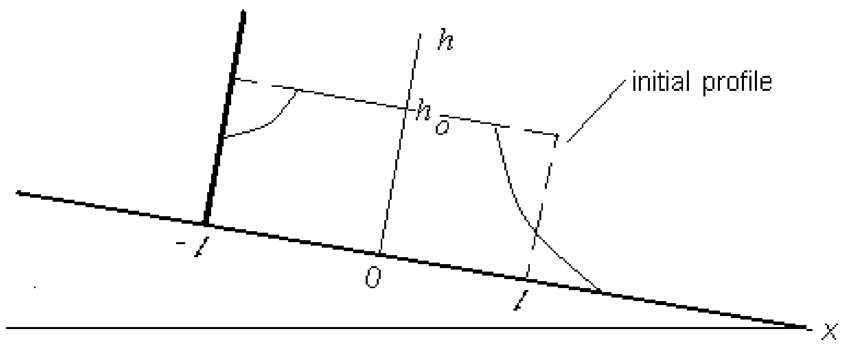

Figure 2. Sketchy image of the scattering of the granular material, originally resting at the motionless wall.

or, when solving for time

$$t = -u_w / s_0, \quad \eta = -\lambda - u_w^2 /(2s_0) \tag{4.41}$$

The origin of the η-coordinates lies at the x-position of the centre of mass of the pile. It is also clear that the solution of the eroding flank at the downstream side, i.e., to the right of η=0 is identical to the corresponding solution of ***Example 1***; it does not need to be repeated here. The moving eroding point on the upstream side, to the left of η=0, is given by $\eta^*(t)$. This Mach front is characteristic; in view of (11), it is given by the differential equation

$$\frac{d\eta^*}{dt} = u + \sqrt{\beta h} = \sqrt{\beta h_0} \quad \text{since } u=0 \text{ and } \eta^*(t) \text{ moves to the right.}$$

Integration yields

$$\eta^*(t) = \sqrt{\beta h_0}\, t - \lambda. \tag{4.42}$$

Besides, the first of (2.7) again yields (4.5) (with the lower sign), implying $C=\sqrt{h_0}$, so that

$$u = 2\sqrt{\beta h} - 2\sqrt{\beta h_0}\ . \tag{4.43}$$

Consequently, the second of (2.7) becomes

$$\eta = \Phi(u) + (1.5u + \sqrt{\beta h_0})t.$$

To find $\Phi(u)$, we write this equation for the wall: $\eta = \eta_w$, $t = t_w$, $u = u_w$, whence

$$\eta_w = \Phi(u_w) + (1.5u_w + \sqrt{\beta h_0})t_w.$$

If in this relation η_w and t_w are replaced by the expressions on the right-hand side of (4.41) and the resulting expression is solved for $\Phi(u_w)$

$$\Phi(u) = -\lambda - \frac{u^2}{2s_0} + \left(\frac{3}{2}u + \sqrt{\beta h_0}\right)\frac{u}{s_0} \tag{4.44}$$

is obtained, in which u_w is replaced by *u*, because (4.44) is simply an identify in its argument. Substituting this in (4.43) and rewriting (4.42) yields

$$\eta = -\lambda + \frac{u^2}{s_0} + \left(\frac{\sqrt{\beta h_0}}{s_0} - \frac{3}{2} t \right) u + \sqrt{\beta h_0}\, t \,, \tag{4.45}$$

$$h = \left(\frac{u}{2\sqrt{\beta}} + \sqrt{h_0} \right)^2 , \quad u_w < u < 0 \,. \tag{4.46}$$

These formulas describe the parametric solution for the upper eroding shoulder. It is possible to write down an explicit formula for h as a function of η and *t*. This form of the solution is obtained by solving the quadratic equation (4.45) for *u* and substituting the resulting expression in (4.46), the result being

$$\begin{aligned} h &= \frac{9}{16\beta} \left[\sqrt{\beta h_0} - 0.5 s_0 t + \frac{1}{3} \sqrt{(\sqrt{\beta h_0} + \frac{3}{2} s_0 t)^2 - 4 s_0 \sqrt{\beta h_0}\, t + 4 s_0 (\eta + \lambda)} \right]^2 , \\ & -\lambda - \frac{s_0 t^2}{2} \le \eta < -\lambda + \sqrt{\beta h_0} \end{aligned} \tag{4.47}$$

As in ***Example 1*** this solution is only valid for early times in the interval *0<t<T*, where *T= =min(*t_o*,* t^**),* in which t_o is the time when the foot point of the upper shoulder at the wall reaches the base, and t^* is the time when the fronts meet. Obviously, t^* follows from the equation $\eta^*(t^*) = \eta_*(t^*)$, implying

$$t^* = \frac{\lambda}{\sqrt{\beta h_0}}$$

and t_o is found from the condition $h(\eta_w, t_0) = 0$. Equating the velocities inferred from (4.40) and (4.43) thus yields

$$t_0 = \frac{2\sqrt{\beta h_0}}{s_0} \,. \tag{4.48}$$

When *t>T*, new Mach fronts appear which differ from one anther according to whether *T=* t_o or *T=* t^*. For the construction of these, equation (2.7) is not helpful and it is necessary to operate as in ***Example 1***.

Remark 4: Experiments in which this solution is reproduced, can be used to determine the parameter β by measuring the time *T*, i. e., t_o or t^*. Its value depends upon which one occurs first from (4.14) and (4.48), respectively, namely

(i) if *T=* t_o, then from (4.48) $\beta = \frac{1}{h_0}\left(\frac{\lambda}{t^*}\right)$

(4.49)

(ii) if $T= t^*$, then from (4.14) $\beta = \frac{1}{h_0}\left(\frac{t_0 s_0}{2}\right)^2$. •

4.4. ***Example 3*****:** Let us return to the partially invariant solution (2.7), and let

$C_2 = 1,\ C_3^2 = 4\beta C_1,\ C_4 = 0$ in (2.12).

Then, the pile shape is given by the equation

$$h = \frac{C_1}{g}\left(1 - \frac{\eta^2}{g^2}\right)$$

and describes an extending parabola. The constant C_1 follows from mass balance

$$M = \int_{-g}^{g} h d\eta = \frac{C_1}{g}\left(\frac{6}{3}g - \frac{2}{3}g\right) = \frac{4}{3}C_1\ , \qquad C_1 = 3M/4$$

and the spreading is given by a1). Thus, summarizing, we have in this case

$$\begin{cases} h = \frac{3M}{4}\left(1 - \frac{\eta^2}{g^2}\right); \\ u = \frac{2}{g}\sqrt{\frac{3M\beta}{4}}\sqrt{\frac{g-1}{g}}\eta; \\ \sqrt{g(g-1)} + \ln(\sqrt{g} + \sqrt{g-1}) = \sqrt{3M\beta}t \end{cases} \qquad (4.50)$$

which recovers the solution given by Savage and Hutter in [1].

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