LIMIT THEOREMS FOR RANDOM EXPONENTIALS

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We study the limiting distribution of partial sums $S_N(t) = \sum_{i=1}^N e^{tX_i}$ as $t \to \infty$, $N \to \infty$, where (X_i) is a sequence of i.i.d. random variables. Two cases are naturally distinguished: (A) $\operatorname{ess\,sup} X_i = 0$ and (B) $\operatorname{ess\,sup} X_i = +\infty$. In this paper, the problem is considered under the assumption that the log-tail distribution function $h(x) = -\log \mathsf{P}\{X_i > x\}$ (case B) or $h(x) = -\log \mathsf{P}\{X_i > -1/x\}$ (case A) is regularly varying as $x \to +\infty$, with index ϱ such that $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A). An appropriate scale for the growth of N relative to t is of the form $e^{\lambda H_0(t)}$, where the rate function $H_0(t)$ is a certain asymptotic version of the cumulant generating function $H(t) = \log \mathsf{E}[e^{tX_i}]$ (case B) or $H(t) = -\log \mathsf{E}[e^{tX_i}]$ (case A), provided by the Kasahara–de Bruijn exponential Tauberian theorem. We have found two critical points, $0 < \lambda_1 < \lambda_2 < \infty$, below which the Law of Large Numbers and the Central Limit Theorem, respectively, break down. Below λ_2, we impose a slightly stronger condition of normalized regular variation of h. The limit laws here appear to be stable, with characteristic exponent $\alpha = \alpha(\varrho, \lambda)$ ranging from 0 to 2 and with skewness parameter $\beta = 1$. Limit theorems about extreme values of the sample $e^{tX_1}, \dots, e^{tX_N}$ are also proved.

Contents

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1. Introduction.

1.1. *The problem.* In this work, we are concerned with partial sums of exponentials of the form

$$S_N(t) = \sum_{i=1}^{N} e^{tX_i}, \tag{1.1}$$

where (X_i) is a sequence of independent identically distributed random variables and both t and N tend to infinity. Our goal is to study the limiting distribution of

$S_N(t)$ and to explore possible 'phase transitions' due to various rates of growth of the parameters t and N.

In such analysis, two cases are naturally distinguished according to whether X_i are bounded above (*case A*) or unbounded above (*case B*). In the former case, without loss in generality we may and will assume that the upper edge of the support of X_i is zero, $\operatorname{ess\,sup} X_i = 0$.

One can also expect that the results will heavily depend on the structure of the upper tail of the distribution of X_i. In the present work, we focus on a fairly general class of distributions with the upper tail of the *Weibull/Fréchet* form

$$\mathsf{P}\{X_i > x\} \approx \begin{cases} \exp(-cx^{\varrho}) & \text{as } x \to +\infty \quad \text{(case B)}, \\ \exp(-c(-x)^{-\varrho}) & \text{as } x \to 0- \quad \text{(case A)}, \end{cases} \tag{1.2}$$

where $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A). More precisely, we will be assuming that the function $\log \mathsf{P}\{X_i > x\}$ is regularly varying at the vicinity of $\operatorname{ess\,sup} X_i$ with index $\varrho \in (1, \infty)$ (case B) or $-\varrho \in (-\infty, 0)$ (case A). For example, a normal distribution is contained in this class (case B, $\varrho = 2$).

1.2. *Motivation.*

1.2.1. *Topics in Probability.* One motivation for this study is quite abstract and purely probabilistic. In fact, such a setting provides a natural tool to interpolate between the classical limit theorems concerning the bulk of the sample, i.e. the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT), on the one hand, and limit theorems for extreme values, on the other hand. Indeed, it is clear that the limiting behavior of $S_N(t)$ is largely determined by the relationship between the parameters t and N. If, for instance, one lets N tend to infinity with t fixed or growing very slowly, then, under appropriate (exponential) moment conditions, the usual LLN and CLT should be valid. In contrast, if the growth rate of N is small enough as compared to t, then the asymptotic behavior of the sum $S_N(t)$ is dominated by its maximal term. We will see that when both t and N tend to infinity, a rich intermediate picture emerges made up of various limit regimes.

In this connection, let us mention a recent paper by Schlather (2001) who studied the asymptotics of the l_p-norms of samples of positive i.i.d. random variables, $\|Y_{1n}\|_p = (\sum_{i=1}^n Y_i^p)^{1/p}$, where the norm order $p = p(n)$ grows together with the sample size n. The link with our setting becomes clear if one puts $Y_i = e^{X_i}$, so that $\|Y_{1n}\|_p$ is expressed through an exponential sum of the form (1.1). Schlather (2001) has demonstrated that under a suitable parametrization of the functional relation between p and n, there is a 'homotopy' for the limit distributions of $\|Y_{1n}\|_p$ extending from the CLT to a limit law for extreme values. In fact, the situation where $p = p(n) \to \infty$ as $n \to \infty$, arises in Theorem 2.2 [Schlather (2001), p. 864], where the random variables Y_i are bounded above and, in the sense of extreme value theory, belong to the domain of attraction of the Weibull distribution $\Psi_\alpha(x) = \exp(-(-x)^\alpha)$ ($\alpha > 0$, $x < 0$). [In Theorem 2.3 (p. 865), where Y_i are unbounded and are in the domain of attraction of the Fréchet distribution $\Phi_\alpha(x) = \exp(-x^{-\alpha})$ ($\alpha > 0$, $x > 0$), the parameter p does not depend on n.]

Let us point out that our results are complementary to Schlather's findings, since for random variables X_i with the Weibull/Fréchet tails of the form (1.2), the distribution of the maximum of $e^{X_1}, \dots, e^{X_n}$ can be shown to converge, as $n \to \infty$, to the Gumbel (double exponential) distribution $\Lambda(x) = \exp(-e^{-x})$ ($x \in \mathbb{R}$) (see Proposition 9.7 below). Note that in this case Schlather (2001) has obtained a partial result and only for exponential random variables (Theorem 2.4, p. 867). However, our results corroborate a conjecture in Schlather [(2001), p. 867] related to the case of attraction to Λ. In Section 9.2 below, we will provide more comments on the relationship of our results to the work by Schlather (2001).

1.2.2. *Branching populations.* The second motivation (and in fact the most important one) comes from problems related to the long-time dynamics in random media. In the simplest situation, sums of exponentials arise as the expected (quenched) total population size of a colony of non-interacting branching processes with random branching rates. Indeed, consider a collection of N branching processes $Z_i(t)$ driven by the binary branching rates $X_i = X_i(\omega)$ ($i = 1, \dots, N$). More specifically, for a fixed random branching environment ω (i.e., in a 'quenched' setting), each $Z_i(t)$ is a Markov continuous-time branching process evolving as follows: during infinitesimal time dt, a particle from the ith population, independently of other particles and the past history, with probability $|X_i|dt$ may split into two descendants (if $X_i > 0$) or die (if $X_i < 0$); otherwise, with probability $1 - |X_i|dt$, the particle survives over the time dt. Let $m_i(t) = m_i(t, \omega)$ denote the expected number of particles in the ith population at time t. It is well known that $m_i(t)$ satisfies the differential equation $m_i'(t) = X_i m_i(t)$ [see Athreya and Ney (1972), Ch. III, § 4, p. 108]. Hence, assuming that $Z_i(0) = 1$ we obtain $m_i(t) = e^{tX_i}$, and therefore the quenched mean total population size is given by the sum (1.1).

In more interesting and realistic situations, there is spatial motion of particles and hence interaction between individual populations. However, there are grounds to believe that the long-term dynamics problem can be essentially reduced, in each particular case, to sums involving random exponentials, and therefore various asymptotic regimes that we establish in the present paper will provide a basic building block for the understanding of the new dynamical phase transitions for branching processes in random media. Such sums may also contain additional random weights, thus having the form $S_N(t) = \sum_{i=1}^{N} Y_i(t)\, e^{tX_i}$. Here, the parameter N will characterize the spatial span of the initial population, while the random variables X_i and Y_i will represent the local (spectral) characteristics of the quenched branching process, according to the mechanisms of the dynamical randomness in the medium. Typically, the weights (Y_i) will be mutually independent when conditioned on the (X_i) [e.g., being some functions of (X_i)]. These more difficult questions, including a more general type of the abstract problem, will be addressed elsewhere.

To conclude this set of examples, let us mention that weighted exponential sums emerge already in the above context of non-interacting branching populations, if one considers the total population size rather than its quenched expected value. Indeed, if $X_i > 0$ then it is known that $Y_i(t) := Z_i(t)\, e^{-tX_i}$ converges with probability one, as $t \to \infty$, to a random variable with unit exponential distribution [see Athreya and Ney (1972), Ch. III, § 7, Theorem 1, p. 111 and § 11, p. 128]. Therefore, if *all*

X_i are positive, the total population size is represented as $Z(t) = \sum_{i=1}^{N} Y_i(t)\, e^{tX_i}$, where the coefficients $Y_i(t)$ are i.i.d. random variables (in fact, functions of X_i) with $\mathsf{E}[Y_i(t)] = 1$ and unit exponential distribution in the limit $t \to \infty$.

1.2.3. *Random Energy Model.* A completely different example is provided by the Random Energy Model (REM) introduced by Derrida (1980, 1981) as a simplified version of the mean-field Sherrington–Kirkpatrick model of a spin glass. (At about the same time, a similar model was independently proposed by Lifshitz, Gredeskul and Pastur [(1982), in particular see Eq. (2.11), p. 1372], who studied the transmission of waves through a bundle of channels with random transmission coefficients.) The REM describes a system of size n with 2^n energy levels $E_i = \sqrt{n}\, X_i$ $(i = 1, \dots, 2^n)$, where (X_i) are i.i.d. random variables with standard normal distribution. Thermodynamics of the system is determined by the partition function $\mathcal{Z}_n(\beta) := \sum_{i=1}^{2^n} \exp(\beta\sqrt{n}\, X_i)$, where $\beta > 0$ is the inverse temperature, which exemplifies the exponential sum (1.1) with $N = 2^n$, $t = \beta\sqrt{n}$.

The free energy for the REM, first obtained by Derrida (1981) using heuristic arguments, is given by

$$F(\beta) := \lim_{n\to\infty} \frac{\log \mathcal{Z}_n(\beta)}{n} = \begin{cases} \beta^2/2 + \beta_{\mathrm{c}}^2/2 & \text{if } 0 < \beta \le \beta_{\mathrm{c}}, \\ \beta\beta_{\mathrm{c}} & \text{if } \beta \ge \beta_{\mathrm{c}}, \end{cases} \tag{1.3}$$

where $\beta_{\mathrm{c}} = \sqrt{2\log 2}$. Note that the function $F(\beta)$ is continuously differentiable but its second derivative is discontinuous at point β_{c} [a third-order phase transition, see Eisele (1983)]. Later on, Eisele (1983) and Olivieri and Picco (1984) rigorously derived the limit (1.3) (in probability and also with probability one) and also extended this result to the case where the random variables X_i have the Weibull-type upper tail (1.2) (case B). More precisely, the class of distributions considered in these papers is subject to the condition $x^{-\varrho} h(x) \to c$ as $x \to +\infty$, where $h(x) = -\log \mathsf{P}\{X_i > x\}$ and $1 < \varrho < \infty$ [see Eisele (1983), Theorem 2.3, p. 130], which is more restrictive than our assumption of (normalized) regular variation of h. A similar case was considered by Pastur (1989), where the proof was based on a Tauberian theorem by Minlos and Povzner (1967).

Some attempts to characterize the fluctuations of the partition function were undertaken by Gardner and Derrida (1989) using the statistical moments of $\mathcal{Z}_n(\beta)$ and by Galvez, Martinez and Picco (1989) who studied the finite-size corrections of order $(\log n)/n$ to $\log \mathcal{Z}_n(\beta)$. Recently, a detailed analysis of the limit laws for $\mathcal{Z}_n(\beta)$ in the Gaussian case has been accomplished by Bovier, Kurkova and Löwe (2002). In particular, they have shown that in addition to the first phase transition at the critical point β_{c}, manifested as the LLN breakdown for $\beta > \beta_{\mathrm{c}}$, within the high-temperature phase $\beta < \beta_{\mathrm{c}}$ there is a second phase transition at $\tilde{\beta}_{\mathrm{c}} = \sqrt{\log 2/2} = \frac{1}{2}\beta_{\mathrm{c}}$, in that for $\beta > \tilde{\beta}_{\mathrm{c}}$ the fluctuations of $\mathcal{Z}_n(\beta)$ become non-Gaussian.

In our work, we extend these results to the class of distributions with the Weibull/Fréchet-type tails of the form (1.2). As compared to Bovier, Kurkova and Löwe (2002) who proceeded from extreme value theory, we use methods of theory of summation of independent random variables. Moreover, we show that the non-Gaussian limit laws are in fact stable. We will further discuss some applications of our results to the REM in Section 9.1.

1.2.4. *Risk theory.* Finally, let us point out some possible applications related to insurance. A basic quantity in risk theory is the aggregate claim amount $Y(t) := \sum_{i=1}^{N(t)} U_i$, where (U_i) is a sequence of i.i.d. claim sizes and $N(t)$ is a claim counting process independent of (U_i); the risk reserve process is then given by $R(t) := u + \beta t - Y(t)$, where u is the initial reserve and β is the premium income rate [see Rolski et al. (1999), Sect. 5.1.4]. A common problem is to estimate the moment generating function $m_U(s) := \mathsf{E}[e^{sU_i}]$, in particular for large s. Such a question arises, for example, in connection with the Lundberg bounds for the tail distribution of $Y(t)$ or for the ruin probability $\psi(u) := \mathsf{P}\{\min_{t\ge 0} R(t) < 0\}$. Similar questions are of interest in other areas such as queueing theory [the equilibrium waiting time in the $M/G/1$ queue, see Asmussen (1987), Ch. XII, § 5, p. 269 and Ch. XIII, § 1, p. 281] and storage models [a dam process, see Asmussen (1987), Ch. XIII, § 3, 4].

The Lundberg bounds are constructed using the root γ (called the 'adjustment coefficient') of the equation of the form $m_U(\gamma) = 1/p > 1$ [see Rolski et al. (1999), Sect. 4.5.1, p. 125–126 and Sect. 5.4.1, p. 170–171]. Here the parameter p has the meaning of either the claim arrival rate [for the aggregate claim process $Y(t)$] or the expected aggregate claim per unit time [for the risk reserve process $R(t)$], and hence the case $p \to 0$ (and therefore $\gamma \to \infty$) corresponds to the practically important situations of small 'claim load'. The statistical method for estimating the unknown solution γ can be based on the empirical moment generating function $\hat{m}_U(s) := n^{-1} \sum_{i=1}^{n} e^{sU_i}$, which has similarity with the exponential sum (1.1). A natural estimator $\hat{\gamma}$ defined by the equation $\hat{m}_Y(\hat{\gamma}) = 1/p$, has nice asymptotic properties including a.s.-consistency and asymptotic normality, providing $1/p$ is fixed or bounded [see Rolski et al. (1999), Sect. 4.5.3, Lemma 4.5.1 and Theorem 4.5.3, p. 130]. However, the asymptotic behavior of $\hat{\gamma}$ when both n and s are large does not seem to have been addressed so far.

Let us also mention discounted risk processes, which may provide another, more direct link with our setting. In the simplest case, let a company's portfolio consist of n identical policies over term t each, and assume that claim sizes $(U_i,\ i = 1,\dots,n)$ and claim arrival times $(\tau_i,\ i = 1,\dots,n)$ are sequences of independent random variables (not necessarily independent of each other — for instance, U_i may depend on τ_i), with common distributions F_U and F_τ, respectively. Suppose for simplicity that the inflation rate is constant, so that the inflated monetary unit at time s equals $e^{\kappa s}$. Then the aggregate claim amount is given by $Y(t) = \sum_{i=1}^{n} U_i e^{-\kappa\tau_i}$ [cf. Rolski et al. (1999), Sect. 11.4.2, p. 472]. Let us now note that if the insured term t is large, it is reasonable to assume that the ratio τ_i/t has a non-degenerate limit distribution, and hence each τ_i can be approximated by tX_i, where (X_i) is a sequence of i.i.d. random variables not depending on t. Then the expression for $Y(t)$ is reduced to $Y(t) = \sum_{i=1}^{n} U_i e^{-\kappa t X_i}$ [cf. Gerber (1990), Sect. 1.9, Eq. (1.9.1)], which is a weighted sum of random exponentials mentioned above in Section 1.2.2. Similarly, interpreting U_i as the investor's profit on ith share (payable at time τ_i), one would arrive at the sum with the plus sign in the exponent, $Y(t) = \sum_{i=1}^{n} U_i e^{\kappa t X_i}$.

1.3. *General notations.* We write := for 'is defined by' and =: for 'is denoted by'. Abbreviation 'iff' stands for 'if and only if'. Letters X, Y, ... are used for a generic representative of random variables (X_i), (Y_i), ..., respectively. The indicator of an event A is denoted by $\mathbf{1}_A$. Relation $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$. Convergence in probability, in distribution and with probability one is denoted by $\xrightarrow{p}$, $\xrightarrow{d}$ and $\xrightarrow{\text{a.s.}}$, respectively, and the symbol $o_p(1)$ denotes a random variable converging to zero in probability. The symbol $\stackrel{d}{=}$ means equality in distribution. By $\mathcal{N}(0, \sigma^2)$ we denote the normal distribution on $\mathbb{R}$ with zero mean and variance σ^2; in particular, $\mathcal{N}(0,1)$ stands for the standard normal distribution. If a random variable ζ has distribution $\mathcal{F}$, we write $\log \mathcal{F}$ for the distribution of the random variable $\log \zeta$.

We denote $\omega_X := \operatorname{ess\,sup} X$, that is, $\omega_X = \sup\{x : \mathsf{P}(X > x) > 0\}$. Therefore, the above mentioned cases A and B (see Section 1.1) correspond to $\omega_X = 0$ and $\omega_X = +\infty$, respectively. In view of the above interpretation of the problem using the terminology of branching populations (see Section 1.2.2), this labeling can be mnemonically associated with the terms *annihilation* (case A) and *branching* (case B). Let us also make a special convention that will allow us to consider both cases A and B simultaneously: in the symbols $\pm$, $\mp$, $\gtrless$ and the like, the *upper sign always refers to case B*, whereas the *lower sign corresponds to case A*. The notation $a^{\pm}$ stands for the power $a^{\pm 1}$ (we use this for the sake of brevity and also to avoid confusion with a function's inverse). Finally, $f(x)^a$ is understood as $[f(x)]^a$.

2. Statement of the main results.

2.1. *Regularity and scaling.* Recall that ω_X stands for $\operatorname{ess\,sup} X$, and assume that $\mathsf{P}\{X < \omega_X\} = 1$, that is, X is finite with probability 1 (case B) or there is no atom at point $\omega_X = 0$ (case A). Consider the *log-tail distribution function*

$$h(x) := \begin{cases} -\log \mathsf{P}\{X > x\}, & x \in \mathbb{R} \quad \text{(case B)}, \\ -\log \mathsf{P}\{X > -1/x\}, & x > 0 \quad \text{(case A)}. \end{cases} \tag{2.1}$$

Clearly, in both cases $h(\cdot)$ is non-negative, non-decreasing, and right-continuous; it takes finite values in its domain and $h(x) \to +\infty$ as $x \to +\infty$. According to the above $\pm$-convention (see Section 1.3), the upper tail of the distribution of X can be written down in a united manner as

$$\mathsf{P}\{X > x\} = \exp\{-h(\pm x^{\pm})\}, \qquad x < \omega_X. \tag{2.2}$$

We will be working under the assumption that h *is regularly varying at infinity with index* ϱ (we write $h \in R_\varrho$), where $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A). That is, for any $\kappa > 0$ we have $h(\kappa x)/h(x) \to \kappa^\varrho$ as $x \to +\infty$.

It follows that the *cumulant generating function*

$$H(t) := \pm \log \mathsf{E}[e^{tX}], \qquad t \ge 0, \tag{2.3}$$

is well defined; furthermore, it is non-decreasing and $H(t) \to +\infty$ as $t \to \infty$.

The link between the asymptotics of the functions h and H at infinity is characterized by the combined Kasahara–de Bruijn exponential Tauberian theorem [see Bingham et al. (1989), Theorem 4.12.7, p. 253, and Theorem 4.12.9, p. 254]. We will give a precise formulation of this theorem in Section 3.1 below. Here, it suffices to mention that $h \in R_\varrho$ implies $H \in R_{\varrho'}$, where the index ϱ' is defined by

$$\varrho' := \frac{\varrho}{\varrho \mp 1}. \tag{2.4}$$

Remembering that $1 < \varrho < \infty$ in case B and $0 < \varrho < \infty$ in case A, it follows that

$$\begin{aligned} &1 < \varrho' < \infty \qquad \text{(case B)}, \\ &0 < \varrho' < 1 \qquad \text{(case A)}. \end{aligned} \tag{2.5}$$

According to (2.3), the expected value of the sum $S_N(t)$ is given by

$$\mathsf{E}[S_N(t)] = \sum_{i=1}^{N} \mathsf{E}[e^{tX_i}] = N e^{\pm H(t)},$$

suggesting that the function $H(t)$ sets up an appropriate (exponential) scale of the form $e^{\lambda H(t)}$ for the number of terms $N = N(t)$. In fact, it is technically more convenient to use $H_0(t)$ as a rate function, where H_0 is a certain asymptotic version of H provided by the Kasahara–de Bruijn Tauberian theorem. This makes no difference in the 'crude' Theorems 2.1 and 2.2 below, since $H_0(t) \sim H(t)$ as $t \to \infty$, but it will be crucial for the more delicate Theorems 2.3, 2.5 and 2.6.

The following two values turn out to be critical with respect to the scale $\lambda H_0(t)$,

$$\lambda_1 := \frac{\varrho'}{\varrho}, \qquad \lambda_2 := 2^{\varrho'} \frac{\varrho'}{\varrho}, \tag{2.6}$$

in that the LLN and CLT break down below λ_1 and λ_2, respectively. Let us also introduce another parameter,

$$\alpha \equiv \alpha(\varrho, \lambda) := \left(\frac{\varrho\lambda}{\varrho'}\right)^{1/\varrho'}, \tag{2.7}$$

which will be shown to play the role of characteristic exponent in the limit laws and hence provides their natural parametrization. In particular, note that the critical values of α corresponding to λ_1, λ_2 are given by $\alpha_1 = 1$, $\alpha_2 = 2$, respectively.

Below the critical points, the behavior of the sum $S_N(t)$ becomes increasingly sensitive to subtle details of the upper tail's structure. It turns out, however, that enough control is gained via imposing a slightly stronger condition on regularity of the log-tail distribution function h — that of *normalized regular variation*, $h \in NR_\varrho$ [see Bingham et al. (1989), Sect. 1.3, § 2, p. 15]. This property will be discussed in detail in Section 5.1. A characteristic property of this class is that h is (absolutely) continuous and a.e.-differentiable, and

$$\frac{xh'(x)}{h(x)} \to \varrho \qquad (x \to \infty) \tag{2.8}$$

(see Lemma 5.1 below). As a benefit of this assumption, the relationship between the functions h and H_0 can be characterized explicitly. We will describe this relationship in detail later on (see Section 5.1). Here we note that $H_0(t)$ can be found (for all t large enough) as the unique solution of the equation

$$\varrho' H_0 = \varrho h\big((\varrho' H_0/t)^{\pm}\big). \tag{2.9}$$

Examples illustrating the difference between the functions H and H_0 can be found in Appendix A.1.

2.2. *Statement of the main theorems.* We proceed to state our results. The first two theorems assert that $S_N(t)$ satisfies the Law of Large Numbers and the Central Limit Theorem in their conventional form provided that the number of terms N in $S_N(t)$ grows fast enough relative to t (roughly speaking, $N \gg \exp\{\lambda_1 H_0(t)\}$ for LLN or $N \gg \exp\{\lambda_2 H_0(t)\}$ for CLT). More precisely, denote

$$\lambda := \liminf_{t\to\infty} \frac{\log N}{H_0(t)}. \tag{2.10}$$

THEOREM 2.1 (LLN, $\lambda > \lambda_1$). *Suppose that $\lambda > \lambda_1$. Then*

$$\frac{S_N(t)}{\mathsf{E}[S_N(t)]} \xrightarrow{p} 1 \qquad (t \to \infty).$$

THEOREM 2.2 (CLT, $\lambda > \lambda_2$). *Suppose that $\lambda > \lambda_2$. Then*

$$\frac{S_N(t) - \mathsf{E}[S_N(t)]}{(\mathsf{Var}[S_N(t)])^{1/2}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (t \to \infty).$$

At and below the critical points, we need to specify the growth rate of N more precisely. Namely, we assume that

$$N \sim e^{\lambda H_0(t)} \qquad (t \to \infty), \tag{2.11}$$

where $\lambda > 0$ is a parameter. We also require a few more notations. Let $\mu = \mu(t)$ be a (unique) solution of the equation

$$h\big((\mu H_0(t)/t)^{\pm}\big) = \frac{\lambda\varrho}{\varrho'}\, h\big((\varrho' H_0(t)/t)^{\pm}\big). \tag{2.12}$$

Using that $h \in R_\varrho$ and comparing the asymptotics of both parts of equation (2.12) as $t \to \infty$, one can show (see Lemma 5.11 below) that

$$\lim_{t\to\infty} \mu(t) = \frac{\varrho\lambda}{\alpha}. \tag{2.13}$$

For $x > 0$, let us set

$$\eta_x(t) := \frac{\mu(t)H_0(t)}{t} \pm \frac{\log x}{t}. \tag{2.14}$$

In particular, for $x = 1$ this is reduced to

$$\eta_1(t) = \frac{\mu(t)H_0(t)}{t}. \tag{2.15}$$

Note that the identity (2.9) combined with equation (2.12) and notation (2.15) yields a relation which we call the *Basic Identity*:

$$h(\eta_1(t)^{\pm}) \equiv \lambda H_0(t). \tag{2.16}$$

How the function $\eta_x(t)$ emerges and the importance of the Basic Identity will be heuristically explained in Section 2.3.

We are now in a position to state one of our main results.

THEOREM 2.3 (Convergence to a stable law, $\lambda < \lambda_2$). *Let $0 < \lambda < \lambda_2$, and set*

$$B(t) := e^{\pm\mu(t)H_0(t)}, \tag{2.17}$$

$$A(t) := \begin{cases} \mathsf{E}[S_N(t)] & (\lambda_1 < \lambda < \lambda_2), \\ NB_1(t) & (\lambda = \lambda_1), \\ 0 & (0 < \lambda < \lambda_1), \end{cases} \tag{2.18}$$

where $B_1(t)$ is a truncated exponential moment,

$$B_1(t) := \mathsf{E}\big[e^{tX}\mathbf{1}_{\{X \le \pm\eta_1(t)\}}\big]. \tag{2.19}$$

Then, as $t \to \infty$,

$$\frac{S_N(t) - A(t)}{B(t)} \xrightarrow{d} \mathcal{F}_\alpha, \tag{2.20}$$

where $\mathcal{F}_\alpha$ is a stable law with exponent $\alpha \in (0,2)$ defined in (2.7) and skewness parameter $\beta = 1$. The characteristic function of the law $\mathcal{F}_\alpha$ is given by

$$\phi_\alpha(u) = \begin{cases} \exp\left\{-\Gamma(1-\alpha)|u|^\alpha \exp\left(-\frac{i\pi\alpha}{2}\operatorname{sgn} u\right)\right\} & (0 < \alpha < 1) \\ \exp\left\{\frac{\Gamma(2-\alpha)}{\alpha - 1}|u|^\alpha \exp\left(-\frac{i\pi\alpha}{2}\operatorname{sgn} u\right)\right\} & (1 < \alpha < 2) \\ \exp\left\{iu(1-\gamma) - \frac{\pi}{2}|u|\left(1 + i\operatorname{sgn} u \cdot \frac{2}{\pi}\log|u|\right)\right\} & (\alpha = 1) \end{cases} \tag{2.21}$$

where $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx$ is the gamma function, $\operatorname{sgn} u := u/|u|$ for $u \neq 0$ and $\operatorname{sgn} 0 := 0$, *and $\gamma = 0.5772\ldots$ is the Euler constant* [see Gradshteyn and Ryzhik (1994), 8.367, p. 955].

REMARK 2.4. For $1 < \alpha < 2$, expression (2.21) can be reduced to that in the case $0 < \alpha < 1$, using an analytic continuation $\Gamma(1-\alpha) = \Gamma(2-\alpha)/(1-\alpha)$.

Let us now describe what happens at the critical points. In fact, the Law of Large Numbers and the Central Limit Theorem prove to be valid at λ_1 and λ_2, respectively; however the normalizing constants now require some truncation.

THEOREM 2.5 (LLN, $\lambda = \lambda_1$). *If $\lambda = \lambda_1$ then*

$$\frac{S_N(t)}{NB_1(t)} \xrightarrow{p} 1 \qquad (t \to \infty), \tag{2.22}$$

where $B_1(t)$ is given by (2.19).

THEOREM 2.6 (CLT, $\lambda = \lambda_2$). *If $\lambda = \lambda_2$ then*

$$\frac{S_N(t) - \mathsf{E}[S_N(t)]}{(NB_2(t))^{1/2}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (t \to \infty),$$

where $B_2(t)$ *is a truncated exponential moment of 'second order'*,

$$B_2(t) := \mathsf{E}\big[e^{2tX}\mathbf{1}_{\{X\le\pm\eta_1(t)\}}\big]. \tag{2.23}$$

The last set of results refer to the limit distribution of extreme terms of the exponential sample $\{e^{tX_i},\ i=1,\dots,N\}$. Surprisingly enough, it appears that the picture here precisely replicates the classical results in the i.i.d. extreme value theory, known in the case of attraction to the Fréchet distribution. We will work out the extreme value theory for i.i.d. random exponentials in Section 8 below. For illustration, let us state here the simplest result of this kind—for the maximal term $M_{1,N}(t) := \max\{e^{tX_i},\ i=1,\dots,N\}$.

THEOREM 2.7 (Limit distribution of $M_{1,N}$). *Let* $\alpha>0$ *be given by* (2.7) *and* $B(t)$ *defined in* (2.17). *Then for all* $\lambda>0$, *as* $t\to\infty$,

$$\mathsf{P}\left\{\frac{M_{1,N}(t)}{B(t)}\le x\right\}\to\exp(-x^{-\alpha}) =: \Phi_\alpha(x),\qquad x>0. \tag{2.24}$$

(Φ_α is known as the *Fréchet distribution.*)

2.3. *Orientation and comments.* Our results (and in particular Theorems 2.3, 2.5 and 2.6) can be proved using the known methods for sums of independent random variables [see Gnedenko and Kolmogorov (1968) and Petrov (1975)]. However, the proofs are technically quite involved, because we have imposed only minimal smoothness conditions on the distribution of X (regular or normalized regular variation). Nevertheless, it is not difficult to explain heuristically the main points behind the calculations—hopefully, this will give the reader some orientation in what will follow in the proofs. In particular, it is important to clarify the central role and power of the Basic Identity (2.16).

The key step in the proofs is the evaluation of the tail probability

$$\mathsf{P}\big\{e^{tX}>xB(t)\big\}=\mathsf{P}\{X>\pm\eta_x(t)\}=e^{-h(\eta_x(t)^{\pm})}, \tag{2.25}$$

where we used (2.17), (2.14) and (2.2). This needs to be compared to the sample size, $N\sim e^{\lambda H_0(t)}$, and therefore we have to relate the function $h(\eta_x(t)^{\pm})$ to the canonical scale determined by the rate function $H_0(t)$. In so doing, the Basic Identity (2.16) plays the major role, as well as the following formula (cf. Lemma 5.15 below):

$$\lim_{t\to\infty}\big[h(\eta_x(t)^{\pm})-h(\eta_1(t)^{\pm})\big]=\alpha\log x\qquad(x>0). \tag{2.26}$$

An explanation of (2.26) can be as follows: let us apply Taylor's approximation and use the property (2.8) of normalized regular variation of h to obtain

$$h(\eta_x^{\pm})-h(\eta_1^{\pm})\sim h'(\eta_1^{\pm})(\eta_x^{\pm}-\eta_1^{\pm})\sim\varrho h(\eta_1^{\pm})\,\frac{\eta_x^{\pm}-\eta_1^{\pm}}{\eta_1^{\pm}}\qquad(t\to\infty). \tag{2.27}$$

By the Basic Identity (2.16), $h(\eta_1^{\pm})$ can be replaced by $\lambda H_0(t)$. On the other hand, using the definition (2.14) of η_x we get

$$\left(\frac{\eta_x}{\eta_1}\right)^{\pm}-1=\left(1\pm\frac{\log x}{\mu(t)H_0(t)}\right)^{\pm}-1\sim\frac{\log x}{\mu(t)H_0(t)}.$$

Hence, the right-hand side of (2.27) is asymptotically equivalent to $(\varrho\lambda/\mu(t))\log x$, and our claim readily follows using the limit of $\mu(t)$ given by (2.13).

It is now easy to obtain the main ingredient of the limiting infinitely divisible law for $0 < \lambda < \lambda_2$ — the *Lévy-Khinchin spectral function* (see Section 6.1 below)

$$L(x) = -\lim_{t\to\infty} N\,\mathsf{P}\{e^{tX} > xB(t)\} \qquad (x > 0).$$

Indeed, using the scaling assumption (2.11), formula (2.25), the Basic Identity (2.16) and relation (2.26) we get, as $t \to \infty$,

$$N\,\mathsf{P}\{e^{tX} > xB(t)\} \sim e^{\lambda H_0(t)-h(\eta_x^\pm)} = e^{h(\eta_1^\pm)-h(\eta_x^\pm)} \to e^{-\alpha\log x}, \tag{2.28}$$

and hence $L(x) = -x^{-\alpha}$ (see Theorem 6.1). In particular, α is indeed the characteristic exponent of a limit law.

Let us now pay attention to the normalizing function $B(t)$ defined by (2.17). Note that by (2.11), we have $B(t) \sim N^{\pm\mu(t)/\lambda}$ as $t \to \infty$. In particular, (2.13) implies that in case B, N is being raised to the power $\mu(t)/\lambda \sim \varrho/\alpha > 1/\alpha$. This should be compared to classical results in the i.i.d. case [see Ibragimov and Linnik (1971), Theorem 2.1.1, p. 37, 46], where the normalization is essentially of the form $N^{1/\alpha}$. As we see, in case B the sums of random exponentials (1.1) have a limit (stable) distribution by virtue of a non-classical (heavier) normalization. As for case A, we have $B(t) \sim N^{-\mu(t)/\lambda} \to 0$, which has no analogies in the classical theory.

However, another look at the tail probability reveals the mechanism of settling down to a stable law which is in fact quite analogous to that in the i.i.d. situation. Indeed, in order that i.i.d. random variables (Y_i) belong to the domain of attraction of a stable law with characteristic exponent $\alpha > 0$, it is sufficient that

$$\mathsf{P}\{Y > n^{1/\alpha}x\} \sim \frac{1}{nx^\alpha} \qquad (n \to \infty) \tag{2.29}$$

[see Ibragimov and Linnik (1971), Theorem 2.6.1, p. 76]. Note that if we set $Y_i := e^{tX_i}/B(t)N^{-1/\alpha}$ $(i = 1, \dots, N)$, then, according to (2.28), for $x > 0$

$$\mathsf{P}\{Y > N^{1/\alpha}x\} = \mathsf{P}\{e^{tX} > xB(t)\} \sim \frac{1}{Nx^\alpha} \qquad (t \to \infty),$$

which mimics the condition (2.29). Thus, in the normalizing function represented in the form $B(t) = B(t)N^{-1/\alpha} \cdot N^{1/\alpha}$, the factor $B(t)N^{-1/\alpha}$ is responsible for the correct behavior of the distribution tail, while the conventional power $N^{1/\alpha}$ performs averaging towards a stable law with characteristic exponent α.

This simple observation explains heuristically the many similarities between the limit behavior of random exponentials e^{tX_i} and that of the usual i.i.d. random variables — from convergence to a stable law (Theorem 2.3 and Section 6) to the properties of extreme values (Section 8).

Outline. The structure of the paper should be clear from the table of contents. Let us briefly comment on how the remaining part is laid out. In Section 3 we specify our regularity assumption on the distribution tail of the random variables X_i and formulate the Tauberian theorem of Kasahara – de Bruijn. In Section 4 we prove the LLN above λ_1 (Theorem 2.1) and the CLT above λ_2 (Theorem 2.2). In Section 5, the condition of normalized regular variation of the function h is discussed and the Basic

Identity is established (Lemma 5.12). Section 6 is devoted to the proof of Theorem 2.3 ($0 < \lambda < \lambda_2$). We first demonstrate convergence to an infinitely divisible law (Theorem 6.1), which is then reduced to a canonical stable form (Theorem 6.6). In Section 7 we prove the LLN at $\lambda = \lambda_1$ (Theorem 2.5) and the CLT at $\lambda = \lambda_2$ (Theorem 2.6). Section 8 is devoted to characterization of the limit behavior of extremes. In particular, we obtain the limit distribution of the maximal term $M_{1,N}(t)$ (Theorem 2.7) and of the ratio $S_N(t)/M_{1,N}(t)$ (Theorem 8.20). Section 9 contains two applications of our results — to the limit laws for fluctuations of the 'free energy', by analogy with the REM, and to the limit laws for l_t-norms of vectors $(e^{X_i})_{i=1}^N$, in the spirit of Schlather (2001). Appendix A presents some examples, in particular in the model case of the Weibull/Fréchet distribution where a more explicit transcription of the limit theorems is available. Appendix B is devoted to the (quite technical) proof of Lemma 5.16 about asymptotics of truncated exponential moments. Finally, in Appendix C we give two direct proofs of Corollary 8.25 about the expected value of the limiting ratio $S_N(t)/M_{1,N}(t)$ in the case $0 < \lambda < \lambda_1$.

3. Preliminaries.

3.1. *Regularity.* Let us start by making precise our basic assumption on the regularity of the log-tail distribution function h defined in (2.1).

REGULARITY ASSUMPTION. The function h is *regularly varying at infinity with index* ϱ (we write $h \in R_\varrho$), such that $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A). That is, for every constant $\kappa > 0$

$$\lim_{x\to\infty} \frac{h(\kappa x)}{h(x)} = \kappa^\varrho \tag{3.1}$$

[see Bingham et al. (1989), Sect. 1.5, § 2, p. 18].

It is known that $h \in R_\varrho$ iff h admits the *Karamata representation* [see Bingham et al. (1989), Eq. (1.5.2), p. 21]

$$h(x) = c(x) \exp\left\{\int_a^x \frac{\varrho + \varepsilon(u)}{u}\, du\right\} \qquad (x \ge a) \tag{3.2}$$

for some $a > 0$, where $c(\cdot)$, $\varepsilon(\cdot)$ are measurable functions and $c(x) \to c_0 > 0$, $\varepsilon(x) \to 0$ as $x \to \infty$.

The following result, known as the *Uniform Convergence Theorem* (UCT) [see Bingham et al. (1989), Theorem 1.5.2, p. 22], significantly extends the definition of regular variation (3.1) and proves to be extremely useful.

LEMMA 3.1 (UCT). *If* $h \in R_\varrho$ *with* $\varrho > 0$ *then* (3.1) *holds uniformly in* κ *on each interval* $(0, b]$.

3.2. *Exponential Tauberian theorems.* Let the *generalized inverse* of a function f be defined by $f^{\leftarrow}(y) := \inf\{x : f(x) \ge y\}$, with the convention that $\inf\varnothing = +\infty$ [see Resnick (1987), Sect. 0.2, p. 3–4]. In what follows, we will be extensively using the inverse of the log-tail distribution function h (2.2). Thanks to the fact that h is non-decreasing and right-continuous, its inverse $h^{\leftarrow}$ has the following useful property allowing one to handle 'level' inequalities [Resnick (1987), Eq. (0.6c), p. 4]:

$$h^{\leftarrow}(y) \le x \quad \text{iff} \quad y \le h(x). \tag{3.3}$$

The next result [cf. Bingham et al. (1989), Theorem 1.5.12, p. 28] shows that the generalized inverse inherits the property of regular variation and, quite naturally, is an 'asymptotic inverse'.

LEMMA 3.2. *If $f \in R_\varrho$ with $\varrho > 0$, then there exists $g \in R_{1/\varrho}$ such that*

$$g(f(x)) \sim f(g(x)) \sim x \qquad (x \to \infty).$$

Such g is unique to within asymptotic equivalence, and one version is $f^{\leftarrow}$.

For $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A), we define the 'conjugate' index ϱ' by the formula (2.4). Rearranging (2.4), we obtain the useful identities

$$\frac{\varrho'}{\varrho} = \pm(\varrho' - 1), \qquad \frac{\varrho}{\varrho'} = \varrho \mp 1. \tag{3.4}$$

We are now in a position to formulate the exponential Tauberian theorems of Kasahara and de Bruijn [see Bingham et al. (1989), Theorem 4.12.7, p. 253 and Theorem 4.12.9, p. 254], which play a fundamental role in our analysis. We will state both theorems in a unified way and in terms convenient for our purposes.

LEMMA 3.3 (Kasahara–de Bruijn's exponential Tauberian theorem). *Let h be the log-tail distribution function* (2.2) *and H the corresponding cumulant generating function* (2.3). *Suppose that $\varphi \in R_{1/\varrho}$ and put*

$$\psi(u) := u\,\varphi(u)^{\mp} \in R_{1/\varrho'}. \tag{3.5}$$

Then

$$h(x) \sim \frac{1}{\varrho}\,\varphi^{\leftarrow}(x) \quad (x \to \infty) \qquad \textit{iff} \qquad H(t) \sim \frac{1}{\varrho'}\,\psi^{\leftarrow}(t) \quad (t \to \infty). \tag{3.6}$$

In particular, $h \in R_\varrho$ iff $H \in R_{\varrho'}$.

Let us point out that the function

$$H_0(t) := \frac{1}{\varrho'}\,\psi^{\leftarrow}(t) \sim H(t), \tag{3.7}$$

appearing in (3.6), is the rate function H_0 mentioned above in Section 2.1.

3.3. *Some elementary inequalities.* The following inequalities will be useful [see Hardy et al. (1952), Theorem 41, p. 39]: *Let $a > 0$, $b > 0$ and $a \ne b$, then*

$$pa^{p-1}(a-b) < a^p - b^p < pb^{p-1}(a-b) \qquad (0 < p < 1), \tag{3.8}$$

$$pb^{p-1}(a-b) < a^p - b^p < pa^{p-1}(a-b) \qquad (p < 0 \ \text{ or } \ p > 1). \tag{3.9}$$

Let us also record a technical lemma.

LEMMA 3.4. *Consider the function*

$$v_\lambda(x) := \lambda(x-1) \mp (x^{\varrho'} - x), \qquad x \ge 1. \tag{3.10}$$

If $\lambda > \lambda_1$ then there exists $x_0 > 1$ such that $v_\lambda(x) > 0$ for all $x \in (1, x_0)$.

PROOF. One could use inequalities (3.8), (3.9), but we choose to give a shorter, analytic proof. By (2.6) and (3.4), we have $\lambda_1 = \varrho'/\varrho = \pm(\varrho' - 1)$. Note that $v_\lambda(1) = 0$ and $v'_\lambda(x) = \lambda \mp (\varrho' x^{\varrho'-1} - 1)$, so that $v'_\lambda(1) = \lambda \mp (\varrho' - 1) = \lambda - \lambda_1 > 0$, according to the hypothesis of the lemma. Therefore, Taylor's formula yields $v_\lambda(x) = (x-1)\big(v'_\lambda(1) + o(1)\big) > 0$ for all $x > 1$ sufficiently close to 1. □

4. Limit theorems above the critical points. In this section, parameter λ is defined by (2.10). We also recall that λ_1 and λ_2 are given by (2.6).

4.1. *Proof of Theorem* 2.1 (*LLN above* λ_1). Set

$$S_N^*(t) := \frac{S_N(t)}{\mathsf{E}[S_N(t)]} = \frac{1}{N} \sum_{i=1}^{N} e^{tX_i \mp H(t)},$$

so one has to prove that $S_N^*(t) \xrightarrow{p} 1$ as $t \to \infty$. To this end, it suffices to show that $\lim_{t\to\infty} \mathsf{E}|S_N^*(t) - 1|^r = 0$ for some $r > 1$.

By the inequality of von Bahr and Esseen [(1965), Theorem 2, p. 301; see also Petrov (1975), Ch. III, § 5, no. 15, p. 60], for any $r \in [1, 2]$ we have

$$\mathsf{E}|S_N^* - 1|^r \le 2N^{1-r}\,\mathsf{E}|e^{tX \mp H(t)} - 1|^r \le 2N^{1-r}\,\mathsf{E}|e^{tX \mp H(t)} + 1|^r.$$

Applying the elementary inequality $(x+1)^r \le 2^{r-1}(x^r + 1)$ $(x > 0,\ r \ge 1)$, which follows easily from Jensen's inequality, we further obtain

$$\mathsf{E}|S_N^* - 1|^r \le 2^r N^{1-r} e^{\pm H(rt) \mp rH(t)} + 2^r N^{1-r}. \tag{4.1}$$

Since $H \in R_{\varrho'}$ and also using (2.10) and the asymptotic equivalence $H(t) \sim H_0(t)$ [see (3.7)], we get

$$\liminf_{t\to\infty} \left[\frac{(r-1)\log N}{H(t)} \mp \frac{H(rt)}{H(t)} \pm r\right] = \lambda(r-1) \mp r^{\varrho'} \pm r \equiv v_\lambda(r)$$

[see (3.10)]. By Lemma 3.4, we can choose $r > 1$ such that $v_\lambda(r) > 0$, which implies that in the limit $t \to \infty$ the right-hand side of (4.1) is bounded by $e^{-cH(t)} = o(1)$.

4.2. *Proof of Theorem* 2.2 (*CLT above* λ_2). Denote

$$\sigma(t)^2 := \mathsf{Var}[e^{tX}] = \mathsf{E}[e^{2tX}] - \left(\mathsf{E}[e^{tX}]\right)^2 = e^{\pm H(2t)} - e^{\pm 2H(t)}. \tag{4.2}$$

LEMMA 4.1. *As* $t \to \infty$,

$$\sigma(t)^2 = e^{\pm H(2t)}(1 + o(1)) \qquad \text{and} \qquad e^{\pm 2H(t)} = \sigma(t)^2\, o(1). \tag{4.3}$$

PROOF. In view of (4.2) it suffices to prove the first statement. Note that

$$e^{\mp H(2t)}\sigma(t)^2 = 1 - e^{\mp H(2t) \pm 2H(t)}. \tag{4.4}$$

Using that $H \in R_{\varrho'}$ we obtain

$$\lim_{t\to\infty} \left(\pm \frac{H(2t)}{H(t)} \mp 2 \right) = \pm(2^{\varrho'} - 2) = |2^{\varrho'} - 2| > 0$$

[see (2.5)]. Hence, the exponential term on the right-hand side of (4.4) vanishes as $t \to \infty$, and (4.3) follows. □

The following lemma is a variation of Chebyshev's inequality.

LEMMA 4.2. *Let* Y *be an arbitrary non-negative random variable. Then for every* $\tau > 0$ *and all* $k \le m$

$$\mathsf{E}\big[Y^k \mathbf{1}_{\{Y>\tau\}}\big] \le \tau^{k-m}\, \mathsf{E}[Y^m]. \tag{4.5}$$

PROOF. Similarly to the usual proof of Chebyshev's inequality, we write

$$\mathsf{E}[Y^m] \ge \mathsf{E}\big[Y^m \mathbf{1}_{\{Y>\tau\}}\big] = \mathsf{E}\big[Y^{m-k} \cdot Y^k \mathbf{1}_{\{Y>\tau\}}\big] \ge \tau^{m-k}\, \mathsf{E}\big[Y^k \mathbf{1}_{\{Y>\tau\}}\big],$$

whence (4.5) follows. □

PROOF OF THEOREM 2.2. In view of Lemma 4.1, the statement of the theorem may be rewritten as follows:

$$\frac{S_N(t) - N\, e^{\pm H(t)}}{N^{1/2}\, e^{\pm H(2t)/2}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (t \to \infty). \tag{4.6}$$

Denote

$$Y_i \equiv Y_i(t) := \frac{e^{tX_i}}{N^{1/2}\, e^{\pm H(2t)/2}}, \qquad i = 1, 2, \dots \tag{4.7}$$

1) According to the classical results on the Central Limit Theorem for independent summands [see, e.g., Petrov (1975), Ch. IV, § 4, Theorem 18, p. 95], we firstly need to check that for all $\tau > 0$

$$\sum_{i=1}^{N} \mathsf{P}\{Y_i(t) > \tau\} = N\, \mathsf{P}\{Y(t) > \tau\} \to 0 \qquad (t \to \infty).$$

Assuming that $r > 1$, let us apply Chebyshev's inequality (of order $2r$) and recall the definition (4.7) to obtain

$$N\, \mathsf{P}\{Y > \tau\} \le N\tau^{-2r}\, \mathsf{E}\big[Y^{2r}\big] = N^{1-r}\tau^{-2r} e^{\pm H(2rt) \mp rH(2t)}. \tag{4.8}$$

Using that $H \in R_{\varrho'}$ and $H(t) \sim H_0(t)$ as $t \to \infty$, we find

$$\begin{aligned}\liminf_{t\to\infty}\left[\frac{(r-1)\log N}{H(t)} \mp \frac{H(2rt)}{H(t)} \pm \frac{rH(2t)}{H(t)}\right] &= \lambda(r-1) \mp (2r)^{\varrho'} \pm r2^{\varrho'}\\ &= 2^{\varrho'}\Big(2^{-\varrho'}\lambda(r-1) \mp (r^{\varrho'} - r)\Big) \equiv 2^{\varrho'} v_{\lambda'}(r),\end{aligned}$$

where $\lambda' := 2^{-\varrho'}\lambda$ and the function $v_\lambda(\cdot)$ is defined in (3.10). By the theorem's hypothesis, $\lambda' > 2^{-\varrho'}\lambda_2 = \lambda_1$ and hence, by Lemma 3.4, $v_{\lambda'}(r) > 0$ for a suitable $r > 1$. Therefore, the right-hand part of (4.8) tends to zero as $t \to \infty$.

2) Next, we have to verify that for every $\tau > 0$, as $t \to \infty$,

$$\sum_{i=1}^{N}\left\{\mathsf{E}\big[Y_i^2\mathbf{1}_{\{Y_i\le\tau\}}\big] - \Big(\mathsf{E}\big[Y_i\mathbf{1}_{\{Y_i\le\tau\}}\big]\Big)^2\right\} = N\,\mathsf{Var}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big] \to 1. \tag{4.9}$$

By Lemma 4.1, $\mathsf{Var}[Y] \sim 1/N$, so condition (4.9) can be rewritten in the form

$$N\,\mathsf{Var}[Y] - N\,\mathsf{Var}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big] \to 0.$$

Expanding the variances, the left-hand side is represented as

$$N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{Y>\tau\}}\big] - N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y>\tau\}}\big]\,\mathsf{E}\big[Y\,(1+\mathbf{1}_{\{Y\le\tau\}})\big]\,. \tag{4.10}$$

Applying Lemma 4.2 to the first term in (4.10) (with $k = 2$, $m = 2r > 2$) yields

$$N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{Y>\tau\}}\big] \le N\tau^{-2(r-1)}\,\mathsf{E}\big[Y^{2r}\big] = o(1), \tag{4.11}$$

as shown in the first part of the proof. The second term in (4.10) is bounded by $2N(\mathsf{E}[Y])^2 = 2e^{\mp H(2t)\pm 2H(t)}$, which is $o(1)$ by Lemma 4.1. Hence, (4.10) vanishes as $t \to \infty$, and condition (4.9) follows.

3) Finally, we need to show that

$$\sum_{i=1}^{N}\mathsf{E}[Y_i] - \sum_{i=1}^{N}\mathsf{E}\big[Y_i\mathbf{1}_{\{Y_i\le\tau\}}\big] = N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y>\tau\}}\big] \to 0.$$

Indeed, applying Lemma 4.2 with $k = 1$, $m = 2r$ $(r > 1)$, we obtain the estimate

$$N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y>\tau\}}\big] \le N\tau^{1-2r}\,\mathsf{E}\big[Y^{2r}\big] = o(1)$$

[see (4.8), (4.11)], and the proof of the theorem is complete. □

5. Normalized regularity and the Basic Identity.

5.1. *Normalized regular variation.* From now on we impose the following

Normalized Regularity Assumption. The log-tail distribution function h is *normalized regularly varying* at infinity, $h \in NR_\varrho$ (with $1 < \varrho < \infty$ in case B and $0 < \varrho < \infty$ in case A), that is, it can be represented in the form

$$h(x) = c\exp\left\{\int_a^x \frac{\varrho + \varepsilon(u)}{u}\,du\right\} \qquad (x \ge a), \tag{5.1}$$

where $c = \text{const} > 0$ and $\varepsilon(x) \to 0$ as $x \to \infty$ [see Bingham et al. (1989), Sect. 1.3, § 2, p. 15]. That is to say, the function $c(\cdot)$ in the Karamata representation (3.2) is now required to be a constant.

More insight into the property of normalized regular variation is given by the following lemma [cf. Bingham et al. (1989), Sect. 1.3, § 2, p. 15].

LEMMA 5.1. *Let h be a positive (measurable) function. Then $h \in NR_\varrho$ iff h is differentiable (a.e.) and*

$$\frac{xh'(x)}{h(x)} \to \varrho \qquad (x \to \infty). \tag{5.2}$$

PROOF. From representation (5.1) it is seen that h is absolutely continuous, hence the derivative h' exists (a.e.) and

$$h'(x) = c \exp\left\{\int_a^x \frac{\varrho + \varepsilon(u)}{u}\, du\right\} \cdot \frac{\varrho + \varepsilon(x)}{x} = \frac{h(x)\,(\varrho + \varepsilon(x))}{x}. \tag{5.3}$$

Therefore, $xh'(x)/h(x) = \varrho + \varepsilon(x) \to \varrho$ as $x \to \infty$ and (5.2) is fulfilled.

Conversely, let us set

$$\varepsilon(x) := \begin{cases} xh'(x)/h(x) - \varrho & \text{if } h'(x) \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\varepsilon(x) \to 0$ as $x \to \infty$, and integration yields

$$\int_a^x \frac{\varrho + \varepsilon(u)}{u}\, du = \int_a^x \frac{h'(u)}{h(u)}\, du = \log h(x) - \log h(a).$$

Hence, representation (5.1) follows (with $c = h(a)$) and therefore $h \in NR_\varrho$. □

REMARK 5.2. Differentiating the general Karamata representation (3.2) yields

$$\frac{xh'(x)}{h(x)} = \frac{xc'(x)}{c(x)} + \varrho + \varepsilon(x).$$

Hence, (5.2) is equivalent to the condition

$$\frac{xc'(x)}{c(x)} \to 0 \qquad (x \to \infty), \tag{5.4}$$

which looks more general than $c(x) \equiv c = \text{const}$. However, the argument above implies that in fact it is not. This can also be seen directly, as (3.2) may be reduced to the form (5.1) by replacing the original function $\varepsilon(x)$ with $\varepsilon(x) + xc'(x)/c(x)$, which is $o(1)$ as well, due to (5.4).

The following lemma provides another important characterization of normalized regularly varying functions [see Bingham et al. (1989), Theorem 1.5.5, p. 24].

LEMMA 5.3. *A positive (measurable) function h is normalized regularly varying with index ϱ, i.e. $h \in NR_\varrho$, iff for every $\varepsilon > 0$ the function $h(x)/x^{\varrho-\varepsilon}$ is ultimately increasing and the function $h(x)/x^{\varrho+\varepsilon}$ is ultimately decreasing.*

As noted in the proof of Lemma 5.1, the derivative h' may exist only a.e. This presents technical difficulties, as for instance one cannot use the Lagrange mean value theorem, the Laplace asymptotic method etc. However, by exploiting the special structure of a normalized regularly varying function h given by the representation (5.1), it is possible to overcome such difficulties thus avoiding imposing further assumptions like continuity of $\varepsilon(\cdot)$. One preparatory step in this direction is made in the next lemma, which provides a useful integral representation of normalized regularly varying functions.

LEMMA 5.4. *A function $h \in NR_\varrho$ can be written in the form*

$$h(x) = h(a) + \int_a^x \frac{h(u)}{u}\,(\varrho + \varepsilon(u))\,du \qquad (x \geq a), \tag{5.5}$$

where $\varepsilon(x) \to 0$ as $x \to \infty$.

PROOF. Consider the function

$$D(x) := h(x) - h(a) - \int_a^x \frac{h(u)}{u}\,(\varrho + \varepsilon(u))\,du.$$

Obviously, $D(a) = 0$. As already mentioned in the proof of Lemma 5.1, representation (5.1) implies that h is absolutely continuous, and hence $D(\cdot)$ is absolutely continuous as well. Differentiating $D(x)$ and using equation (5.3), we have (a.e.)

$$D'(x) = h'(x) - \frac{h(x)}{x}\,(\varrho + \varepsilon(x)) = 0.$$

Hence, $D(x) \equiv 0$ and (5.5) follows. □

The following lemma can be viewed as a refinement of the UCT of Lemma 3.1 for the case of normalized regular variation.

LEMMA 5.5. *If $h \in NR_\varrho$ ($\varrho > 0$) then, uniformly in κ on each interval $[\kappa_0, \kappa_1] \subset (0, \infty)$,*

$$\frac{h(\kappa x) - h(x)}{h(x)} = (\kappa^\varrho - 1)(1 + o(1)) \qquad (x \to \infty).$$

PROOF. Suppose for definiteness that $\kappa \geq 1$ (the case $0 < \kappa \leq 1$ is considered similarly). Using the representation (5.5), after the substitution $u = xy$ we have

$$\frac{h(\kappa x) - h(x)}{h(x)} = \int_1^\kappa \frac{h(xy)}{h(x)y}\,(\varrho + \varepsilon(xy))\,dy. \tag{5.6}$$

The UCT (Lemma 3.1) implies that the function under the integral sign converges to $\varrho y^{\varrho-1}$ uniformly on $[1, \kappa_1]$ as $x \to \infty$. Therefore, the integral in (5.6) converges, uniformly in $\kappa \in [1, \kappa_1]$, to $\int_1^\kappa \varrho y^{\varrho-1}\,dy = \kappa^\varrho - 1$. □

5.2. *Basic Identity.* Let us now re-examine the application of the Kasahara–de Bruijn Tauberian theorem (Lemma 3.3) to our situation. Note that the function $\varrho h(x)$ is continuous and, by Lemma 5.3, ultimately strictly increasing, and hence its ordinary inverse $\varphi(t) := (\varrho h)^{-1}(t)$ is well defined and strictly increasing for all t large enough. In turn, for all x large enough we have

$$\varphi^{-1}(x) = \varrho h(x). \tag{5.7}$$

It then follows that the function $\psi(t)$ defined by (3.5) is ultimately strictly increasing as well. For suppose $s < t$, then the required inequality $\psi(s) < \psi(t)$ is equivalent to $s\varphi(s)^{\mp} < t\varphi(t)^{\mp}$, or

$$\varphi^{-1}(x)\,x^{\mp} < \varphi^{-1}(y)\,y^{\mp}, \tag{5.8}$$

where $x := \varphi(s)$, $y := \varphi(t)$ and $x < y$. Using (5.7), inequality (5.8) can be rewritten as $h(x)\,x^{-\varrho+\varepsilon} < h(y)\,y^{-\varrho+\varepsilon}$ with $\varepsilon := \varrho \mp 1 > 0$, and the latter holds by Lemma 5.3.

Consequently, the inverse function ψ^{-1} exists and is ultimately increasing. Therefore, formula (3.7) is reduced to

$$\psi^{-1}(t) = \varrho' H_0(t). \tag{5.9}$$

For the sake of notational convenience, let us introduce the function

$$s(t) := \left(\frac{\varrho' H_0(t)}{t}\right)^{\pm}, \qquad t > 0. \tag{5.10}$$

Since $H_0 \in R_{\varrho'}$, we have $s(t) \in R_{\pm(\varrho'-1)} = R_{|\varrho'-1|}$ and hence $s(t) \to \infty$ as $t \to \infty$.

We are now in a position to explicitly characterize the link arising between the regularly varying functions h and H_0 through the Tauberian correspondence. Remarkably, due to normalized regular variation of h, such a relationship has the form of an exact equation, rather than just an asymptotic relation.

LEMMA 5.6. *For all t large enough, the functions h and H_0 satisfy the equation*

$$\varrho' H_0(t) \equiv \varrho h(s(t)). \tag{5.11}$$

REMARK 5.7. Remembering that $s(\cdot)$ is expressed through H_0 [see (5.10)], identity (5.11) can be viewed as a functional equation determining the function H_0.

PROOF OF LEMMA 5.6. Let us apply ψ to (5.9) and use relation (3.5) to obtain

$$t = \psi\big(\varrho' H_0(t)\big) = \varrho' H_0(t)\,\varphi\big(\varrho' H_0(t)\big)^{\mp},$$

which yields

$$\varphi\big(\varrho' H_0(t)\big) = \left(\frac{\varrho' H_0(t)}{t}\right)^{\pm} \equiv s(t).$$

Hence, using (5.7) we get $\varrho' H_0(t) = \varphi^{-1}(s(t)) = \varrho h(s(t))$. $\square$

REMARK 5.8. One can prove an identity dual to (5.11), making the relationship between h and H_0 more symmetric. Namely, for all large enough x one has

$$\varrho h(x) \equiv \varrho' H_0(s^\sharp(x)) \tag{5.12}$$

[cf. (5.11)], where $s^\sharp(x) := \varrho h(x) x^\mp$ [cf. (5.10)]. Formally, (5.12) is obtained by raising (5.11) to the power $\varrho' - 1$. We will not, however, need this relation.

In order to rewrite equation (5.11) in a form suitable for us (to be called 'Basic Identity'), we need to make some technical preparations. Recall that α is defined in (2.7). Conversely, using (3.4) λ is expressed in either of the two forms

$$\lambda = \frac{\varrho' \alpha^{\varrho'}}{\varrho} \equiv \pm(\varrho' - 1)\, \alpha^{\varrho'}. \tag{5.13}$$

LEMMA 5.9. *For large enough s, there exists a unique root $\tilde{\mu}(s)$ of the equation*

$$h\big((\tilde{\mu}/\varrho')^{\pm} s\big) = \alpha^{\varrho'} h(s), \tag{5.14}$$

given by the formula

$$\tilde{\mu}(s)^{\pm} = \frac{(\varrho')^{\pm}}{s}\, h^{-1}\big(\alpha^{\varrho'} h(s)\big). \tag{5.15}$$

In particular, if $\alpha = 1$ then $\tilde{\mu}(s) \equiv \varrho'$.

PROOF. Recall that h is normalized regularly varying and (absolutely) continuous [see (5.1)]. Therefore, by Lemma 5.3 it is strictly increasing in some $[a, \infty)$, so the (usual) inverse h^{-1} exists and is defined on $[h(a), \infty)$. Hence, equation (5.14) can be resolved to yield formula (5.15), which is well defined for all s large enough. The case $\alpha = 1$ follows easily. □

LEMMA 5.10. *The function $\tilde{\mu}(\cdot)$ defined in Lemma 5.9 is ultimately bounded above and below, and furthermore, for all s large enough*

$$\min\{1, \alpha^{\varrho'/2\varrho}\} \le \left(\frac{\tilde{\mu}(s)}{\varrho'}\right)^{\pm} \le \max\{1, \alpha^{2\varrho'/\varrho}\}. \tag{5.16}$$

PROOF. If $\alpha \le 1$ then, due to monotonicity of the function h^{-1},

$$\frac{1}{s}\, h^{-1}\big(\alpha^{\varrho'} h(s)\big) \le \frac{1}{s}\, h^{-1}(h(s)) = 1. \tag{5.17}$$

In the case $\alpha > 1$, we note that for every $\kappa > 1$ and all s large enough

$$\kappa h(s) \le h(\kappa^{2/\varrho} s), \tag{5.18}$$

because $h \in R_\varrho$ and hence $\lim_{s\to\infty} h(\kappa^{2/\varrho} s)/h(s) = \kappa^2 > \kappa$. Applying inequality (5.18) with $\kappa = \alpha^{\varrho'} > 1$, we get

$$\frac{1}{s}\, h^{-1}\big(\alpha^{\varrho'} h(s)\big) \le \frac{1}{s}\, h^{-1}\big(h(\alpha^{2\varrho'/\varrho} s)\big) = \alpha^{2\varrho'/\varrho}. \tag{5.19}$$

Combining (5.17) and (5.19) and using (5.15), the upper bound (5.16) follows.

Similarly, for $\alpha \geq 1$ we obtain

$$\frac{1}{s}\, h^{-1}\big(\alpha^{\varrho'} h(s)\big) \geq \frac{1}{s}\, h^{-1}(h(s)) = 1,$$

whereas for $\alpha < 1$

$$\frac{1}{s}\, h^{-1}\big(\alpha^{\varrho'} h(s)\big) \geq \frac{1}{s}\, h^{-1}\big(h(\alpha^{\varrho'/2\varrho} s)\big) = \alpha^{\varrho'/2\varrho},$$

which is consistent with the lower bound in (5.16). □

LEMMA 5.11. *The function $\tilde{\mu}(s)$ has a finite limit as $s \to \infty$ given by*

$$\lim_{s\to\infty} \tilde{\mu}(s) = \varrho' \alpha^{\varrho'-1}. \tag{5.20}$$

PROOF. Since $\tilde{\mu}(\cdot)$ is bounded by Lemma 5.10, the UCT (Lemma 3.1) implies

$$h\big((\tilde{\mu}(s)/\varrho')^{\pm} s\big) \sim \left(\frac{\tilde{\mu}(s)}{\varrho'}\right)^{\pm\varrho} h(s) \qquad (s \to \infty).$$

Comparing this with equation (5.14), we obtain

$$\left(\frac{\tilde{\mu}(s)}{\varrho'}\right)^{\pm\varrho} \sim \alpha^{\varrho'} \qquad (s \to \infty),$$

whence it follows that the limit (5.20) exists and is given by

$$\lim_{s\to\infty} \tilde{\mu}(s) = \varrho' \alpha^{\pm\varrho'/\varrho} = \varrho' \alpha^{\varrho'-1},$$

in view of the first of the identities (3.4). □

Let us define the function

$$\mu(t) := (\tilde{\mu} \circ s)(t) = \tilde{\mu}(s(t)), \tag{5.21}$$

where $s(t)$ is given by (5.10). From the definition of $\tilde{\mu}(s)$ (see Lemma 5.9), it is clear that for all t large enough the function $\mu(t)$ satisfies the equation

$$h\big((\mu(t)/\varrho')^{\pm} s(t)\big) = \alpha^{\varrho'} h(s(t)). \tag{5.22}$$

Since $s(t) \to \infty$, Lemma 5.11 implies that

$$\lim_{t\to\infty} \mu(t) = \varrho' \alpha^{\varrho'-1}. \tag{5.23}$$

For $\tau > 0$, denote

$$\eta_\tau(t) := \frac{\mu(t) H_0(t) \pm \log \tau}{t}. \tag{5.24}$$

In particular, for $\tau = 1$

$$\eta_1(t) = \frac{\mu(t) H_0(t)}{t} = \frac{\mu(t)}{\varrho'}\, s(t)^{\pm} \tag{5.25}$$

[see (5.10)]. From equations (5.25) and (5.23) it follows

$$\eta_1(t)^{\pm} = \big(\mu(t)/\varrho'\big)^{\pm} s(t) \to \infty \qquad (t \to \infty). \tag{5.26}$$

Furthermore, it is easy to see that

$$\frac{\eta_\tau(t)}{\eta_1(t)} = 1 \pm \frac{\log \tau}{\mu(t) H_0(t)} \to 1 \qquad (t \to \infty). \tag{5.27}$$

Hence, using (5.23) we obtain

$$\frac{t\eta_\tau(t)}{H_0(t)} = \frac{\eta_\tau(t)}{\eta_1(t)}\, \mu(t) \to \varrho' \alpha^{\varrho'-1} \qquad (t \to \infty). \tag{5.28}$$

The following lemma will play the crucial role in our analysis.

LEMMA 5.12 (Basic Identity). *For all t large enough,*

$$h\big(\eta_1(t)^{\pm}\big) \equiv \lambda H_0(t). \tag{5.29}$$

PROOF. From (5.25) and (5.22) it follows

$$h\big(\eta_1(t)^{\pm}\big) = h\big((\mu(t)/\varrho')^{\pm} s(t)\big) = \alpha^{\varrho'} h(s(t)).$$

By Lemma 5.6 and relation (5.13), this coincides with $\lambda H_0(t)$. □

5.3. *Implications of the Basic Identity.* In this section, we prove three useful lemmas concerning the asymptotics of various 'perturbations' of the function $h(\eta_1(t)^{\pm})$. Of particular importance for further calculations will be Lemma 5.15.

LEMMA 5.13. *Let $g(\cdot)$ be such that $tg(t)/H_0(t) \to 0$ as $t \to \infty$. Set $\tilde{\eta}_{\tau,y}(t) := \eta_\tau(t) \mp y g(t)$. Then for each $\tau > 0$ uniformly in y on every finite interval $[y_0, y_1]$*

$$\lim_{t\to\infty} \frac{h(\tilde{\eta}_{\tau,y}(t)^{\pm})}{t\tilde{\eta}_{\tau,y}(t)} = \frac{\alpha}{\varrho}. \tag{5.30}$$

In particular, for $g \equiv 0$ one has

$$\lim_{t\to\infty} \frac{h(\eta_\tau(t)^{\pm})}{t\eta_\tau(t)} = \frac{\alpha}{\varrho}. \tag{5.31}$$

PROOF. Relation (5.28) implies that, uniformly in $y \in [y_0, y_1]$,

$$\kappa_y(t) := \frac{\tilde{\eta}_{\tau,y}(t)}{\eta_1(t)} = 1 \pm \frac{\log \tau}{t\eta_1(t)} \mp \frac{ytg(t)}{H_0(t)} \cdot \frac{H_0(t)}{t\eta_1(t)} \to 1 \qquad (t \to \infty).$$

Therefore, by the UCT (Lemma 3.1), uniformly in y on any finite interval $[y_0, y_1]$

$$h(\tilde{\eta}_{\tau,y}^{\pm}) = h(\kappa_y^{\pm} \eta_1^{\pm}) \sim \kappa_y^{\pm\varrho} h(\eta_1^{\pm}) \sim h(\eta_1^{\pm}).$$

Hence, taking into account Lemma 5.12 and the limit (5.23), we obtain

$$\frac{h(\tilde{\eta}_{\tau,y}^{\pm})}{t\tilde{\eta}_{\tau,y}} \sim \frac{h(\eta_\tau^{\pm})}{t\eta_\tau} = \frac{\lambda H_0(t)}{t\eta_1} = \frac{\lambda}{\mu(t)} \to \frac{\lambda}{\varrho' \alpha^{\varrho'-1}} = \frac{\alpha}{\varrho},$$

in view of formula (2.7). □

LEMMA 5.14. *Under the conditions of Lemma* 5.13, *for each* $\tau > 0$

$$\lim_{t\to\infty} \frac{h(\eta_\tau(t)^\pm) - h(\tilde{\eta}_{\tau,y}(t)^\pm)}{tg(t)} = \alpha y,$$

uniformly in y *on every finite interval* $[y_0, y_1]$.

PROOF. Similarly to the proof of Lemma 5.13 we get

$$\kappa_y(t) := \frac{\tilde{\eta}_{\tau,y}(t)}{\eta_\tau(t)} = 1 \mp \frac{ytg(t)}{H_0(t)} \cdot \frac{H_0(t)}{t\eta_\tau(t)} \to 1 \qquad (t \to \infty)$$

uniformly in $y \in [y_0, y_1]$. Therefore, for all large enough t the function $\kappa_y(t)$ is uniformly bounded, $0 < \kappa_0 \le \kappa_y(t) \le \kappa_1 < \infty$. Applying Lemma 5.5 we have

$$h(\eta_\tau^\pm) - h(\tilde{\eta}_{\tau,y}^\pm) \sim -h(\eta_\tau^\pm)(\kappa_y^{\pm\varrho} - 1) \qquad (t \to \infty). \tag{5.32}$$

Furthermore,

$$\kappa_y^{\pm\varrho} - 1 = \left(1 \mp \frac{yg(t)}{\eta_\tau(t)}\right)^{\pm\varrho} - 1 \sim -\frac{\varrho y g(t)}{\eta_\tau(t)}.$$

Substituting this into (5.32) and using the limit (5.31), we finally obtain

$$h(\eta_\tau^\pm) - h(\tilde{\eta}_{\tau,y}^\pm) \sim h(\eta_\tau^\pm)\frac{\varrho y g(t)}{\eta_\tau(t)} \sim \alpha y t g(t),$$

and the lemma follows. □

LEMMA 5.15. *For each* $\tau > 0$

$$\lim_{t\to\infty} \left[h(\eta_1(t)^\pm) - h(\eta_\tau(t)^\pm)\right] = -\alpha \log \tau.$$

PROOF. Apply Lemma 5.14 with $y = -\log\tau$, $g(t) = 1/t$. □

5.4. *Asymptotics of truncated exponential moments.* The goal of this section is to establish some general estimates for truncated exponential moments, which will be instrumental later on. Recall that the parameter $\alpha > 0$ is defined in (2.7).

LEMMA 5.16. *If* $\tau > 0$ *is a fixed number then*
(i) *for each* $p > \alpha$,

$$\lim_{t\to\infty} e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)}\, \mathsf{E}\left[e^{ptX}\mathbf{1}_{\{X \le \pm\eta_\tau\}}\right] = \frac{\alpha}{p - \alpha};$$

(ii) *for each* $p < \alpha$,

$$\lim_{t\to\infty} e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)}\, \mathsf{E}\left[e^{ptX}\mathbf{1}_{\{X > \pm\eta_\tau\}}\right] = \frac{\alpha}{\alpha - p}.$$

The proof of this lemma is deferred to Appendix B.

In the case $p = \alpha$ not covered by Lemma 5.16, we prove one crude estimate that will nevertheless be sufficient for our purposes below.

LEMMA 5.17. *For $\alpha > 0$, denote*

$$B_\alpha(t) := \mathsf{E}\big[e^{\alpha t X}\mathbf{1}_{\{X\le \pm\eta_1\}}\big]\,, \tag{5.33}$$

where $\eta_1(t)$ is defined in (5.25). *Then*

$$b_\alpha(t) := e^{\mp\alpha t\eta_1 + h(\eta_1^\pm)} B_\alpha(t) \to +\infty \qquad (t\to\infty). \tag{5.34}$$

PROOF. Set $\tilde\eta_1(t) := \eta_\tau(t) \mp g(t)$, $g(t) := t^{-1+\varrho'/2}$. Integration by parts yields

$$\begin{aligned}\mathsf{E}\big[e^{\alpha t X}\mathbf{1}_{\{X\le\pm\eta_1\}}\big] &\ge \mathsf{E}\big[e^{\alpha t X}\mathbf{1}_{\{\pm\tilde\eta_1 < X\le\pm\eta_1\}}\big] = \int_{\pm\tilde\eta_1}^{\pm\eta_1} e^{\alpha t x}\, d\big(1-e^{-h(\pm x^\pm)}\big)\\ &= -\int_{\pm\tilde\eta_1}^{\pm\eta_1} e^{\alpha t x}\, d\big(e^{-h(\pm x^\pm)}\big) \ge -e^{\pm\alpha t\eta_1 - h(\eta_1^\pm)} + \alpha t\int_{\pm\tilde\eta_1}^{\pm\eta_1} e^{\alpha t x - h(\pm x^\pm)}\, dx.\end{aligned} \tag{5.35}$$

Making in the last integral in (5.35) the substitution $\pm x = \eta_1(t) \mp y g(t) =: \tilde\eta_{1,y}(t)$, we obtain

$$b_\alpha(t) \ge -1 + \alpha t g(t)\int_0^1 e^{-\alpha t g(t) y + h(\eta_1^\pm) - h(\tilde\eta_{1,y}^\pm)}\, dy. \tag{5.36}$$

By Lemma 5.14, $h(\eta_1^\pm) - h(\tilde\eta_{1,y}^\pm) = \alpha t g(t) y(1+o(1))$, uniformly in $y\in[0,1]$. So for any $\delta > 0$ and all large enough t we have $h(\eta_1^\pm) - h(\tilde\eta_{1,y}^\pm) \ge \alpha t g(t) y(1-\delta)$. Returning to (5.36) we obtain

$$b_\alpha(t) \ge -1 + \alpha t g(t)\int_0^1 e^{-\alpha t g(t)\delta y}\, dy = -1 + \frac{1}{\delta}\left(1 - e^{-\alpha t g(t)\delta}\right),$$

hence $\liminf_{t\to\infty} b_\alpha(t) \ge (1/\delta) - 1$. But the number $\delta > 0$ can be chosen arbitrarily small, so it follows that $\liminf_{t\to\infty} b_\alpha(t) = +\infty$, as claimed. □

The next lemma provides some additional information in the case $p = \alpha$.

LEMMA 5.18. *For any $\tau > 0$*

$$\lim_{t\to\infty} e^{\mp\alpha t\eta_1 + h(\eta_1^\pm)}\, \mathsf{E}\big[e^{\alpha t X}\big(\mathbf{1}_{\{X\le\pm\eta_\tau\}} - \mathbf{1}_{\{X\le\pm\eta_1\}}\big)\big] = \alpha\log\tau. \tag{5.37}$$

PROOF. Let us assume for definiteness that $\tau \ge 1$, so that $\pm\eta_\tau(t) \ge \pm\eta_1(t)$. Integrating by parts and using the substitution $x = \pm\eta_1(t) + y/t$, we obtain

$$\begin{aligned}&e^{\mp\alpha t\eta_1 + h(\eta_1^\pm)}\, \mathsf{E}\big[e^{\alpha t X}\mathbf{1}_{\{\pm\eta_1 < X\le\pm\eta_\tau\}}\big]\\ &\qquad = 1 - e^{\alpha\log\tau + h(\eta_1^\pm) - h(\eta_\tau^\pm)} + \alpha\int_0^{\log\tau} e^{\alpha y + h(\eta_1^\pm) - h((\eta_1\pm y/t)^\pm)} dy.\end{aligned} \tag{5.38}$$

By Lemma 5.14 (with $g(t) = -1/t$), we have $h(\eta_1^\pm) - h((\eta_1 \pm y/t)^\pm) \to -\alpha y$ as $t\to\infty$, uniformly in $0\le y\le\log\tau$, and in particular $h(\eta_1^\pm) - h(\eta_\tau^\pm) \to -\alpha\log\tau$. It is then easy to see that the right-hand side of (5.38) tends to $\alpha\log\tau$ as $t\to\infty$. □

For convenience of reference, we record here a few further estimates for truncated moments of the random variable e^{tX} under a certain normalization adopted in this section. Namely, consider the random variables

$$Y_i \equiv Y_i(t) := \frac{e^{tX_i}}{(NB_\alpha(t))^{1/\alpha}}, \tag{5.39}$$

where N is subject to the scaling assumption (2.11) and B_α is defined in (5.33). For $\alpha > 0$ and $\tau > 0$ denote

$$\tilde{\eta}_{\alpha,\tau}(t) := \pm\frac{\log(NB_\alpha(t))}{\alpha t} \pm \frac{\log\tau}{t}. \tag{5.40}$$

From (5.39) it is seen that the inequality $Y(t) > \tau$ is equivalent to $X > \pm\tilde{\eta}_{\alpha,\tau}(t)$.

Recalling representation (5.34) and using the Basic Identity (5.29), we obtain

$$NB_\alpha(t) \sim e^{\lambda H_0(t)\pm\alpha t\eta_1 - h(\eta_1^\pm)}\, b_\alpha(t) = e^{\pm\alpha t\eta_1}\, b_\alpha(t). \tag{5.41}$$

Therefore, formula (5.40) implies

$$\tilde{\eta}_{\alpha,\tau}(t) = \eta_1(t) \pm \frac{\log b_\alpha(t)}{\alpha t} + \frac{O(1)}{t}, \tag{5.42}$$

whence it follows that for all sufficiently large t

$$\pm\tilde{\eta}_{\alpha,\tau}(t) > \pm\eta_1(t). \tag{5.43}$$

LEMMA 5.19. *For any p such that $0 \le p < \alpha$ and each $\tau > 0$*

$$\lim_{t\to\infty} N\,\mathsf{E}\big[Y(t)^p\,\mathbf{1}_{\{Y(t)>\tau\}}\big] = 0. \tag{5.44}$$

In particular, for $p = 0$ this yields

$$\lim_{t\to\infty} N\,\mathsf{P}\{Y(t) > \tau\} = 0. \tag{5.45}$$

PROOF. From (5.39), (5.40) and (5.43) we obtain

$$\mathsf{E}\big[Y^p\,\mathbf{1}_{\{Y>\tau\}}\big] \le (NB_\alpha)^{-p/\alpha}\,\mathsf{E}\big[e^{ptX}\,\mathbf{1}_{\{X>\pm\eta_1\}}\big].$$

Using Lemma 5.16(ii) and relations (2.11), (5.41), (5.29) and (5.34), we get

$$\begin{aligned} \frac{N}{(NB_\alpha)^{p/\alpha}}\,\mathsf{E}\big[e^{ptX}\,\mathbf{1}_{\{X>\pm\eta_1\}}\big] &\sim \frac{e^{\lambda H_0(t)}}{e^{\pm pt\eta_1} b_\alpha^{p/\alpha}}\cdot\frac{\alpha}{\alpha-p}\,e^{\pm pt\eta_1 - h(\eta_1^\pm)}\\ &= \frac{\alpha}{\alpha-p}\,b_\alpha^{-p/\alpha} = o(1). \end{aligned} \tag{5.46}$$

Thus, relation (5.44) is proved. □

Denote

$$y_\alpha \equiv y_\alpha(t) := \frac{e^{\pm t\eta_1(t)}}{(NB_\alpha(t))^{1/\alpha}}, \tag{5.47}$$

so that $Y > y_\alpha$ iff $X > \pm\eta_1$. From (5.41) it follows that $y_\alpha(t) \sim b_\alpha(t)^{-1/\alpha} \to 0$.

LEMMA 5.20. *Suppose that $p>0$. Then for any $\tau>0$*

$$\lim_{t\to\infty} N\,\mathsf{E}\big[Y(t)^p\,\mathbf{1}_{\{y_\alpha(t)<Y(t)\le\tau\}}\big]=0. \tag{5.48}$$

PROOF. Pick a number q such that $0<q<\min\{\alpha,p\}$. Applying Chebyshev's inequality (4.5), we can write

$$N\,\mathsf{E}\big[Y^p\,\mathbf{1}_{\{y_\alpha<Y\le\tau\}}\big]\le N\tau^{p-q}\,\mathsf{E}\big[Y^q\,\mathbf{1}_{\{y_\alpha<Y\}}\big]=\frac{N\,\tau^{p-q}}{(NB_\alpha)^{q/\alpha}}\,\mathsf{E}\big[e^{qtX}\,\mathbf{1}_{\{X>\pm\eta_1\}}\big],$$

and the latter expression is $o(1)$ as shown above [see (5.46)]. □

LEMMA 5.21. *Suppose that $p>\alpha>0$. Then for any $\tau>0$*

$$\lim_{t\to\infty} N\,\mathsf{E}\big[Y(t)^p\,\mathbf{1}_{\{Y(t)\le\tau\}}\big]=0. \tag{5.49}$$

PROOF. Let us write

$$N\,\mathsf{E}\big[Y^p\,\mathbf{1}_{\{Y\le\tau\}}\big]=N\,\mathsf{E}\big[Y^p\,\mathbf{1}_{\{Y\le y_\alpha\}}\big]+N\,\mathsf{E}\big[Y^p\,\mathbf{1}_{\{y_\alpha<Y\le\tau\}}\big]. \tag{5.50}$$

Applying Lemma 5.16(i), one can show, similarly to (5.46), that the first term on the right-hand side of (5.50) is asymptotically equivalent to

$$\frac{e^{\lambda H_0(t)}}{e^{\pm pt\eta_1(t)}b_\alpha(t)^{p/\alpha}}\cdot\frac{\alpha}{p-\alpha}\,e^{\pm pt\eta_1-h(\eta_1^\pm)}=\frac{\alpha}{p-\alpha}\,b_\alpha(t)^{-p/\alpha}=o(1),$$

while the second term on the right of (5.50) is $o(1)$ by Lemma 5.20. □

6. Limit theorems below λ_2.

6.1. *Convergence to an infinitely divisible law,* $0<\lambda<\lambda_2$. As already mentioned in Section 2.2 [see (2.11)], in the case $0<\lambda<\lambda_2$ we impose the following

SCALING ASSUMPTION. The number $N=N(t)$ of terms in the sum $S_N(t)$ satisfies the condition

$$\lim_{t\to\infty} N\,e^{-\lambda H_0(t)}=1, \tag{6.1}$$

where λ is a parameter such that $0<\lambda<\infty$.

Denote

$$Y_i\equiv Y_i(t):=\frac{e^{tX_i}}{B(t)},\qquad i=1,2,\dots, \tag{6.2}$$

where $B(t)$ is defined in (2.17). According to classical theorems on weak convergence of sums of independent random variables [see Petrov (1978), Ch. IV, § 2, Theorem 8, p. 81–82; cf. also Theorem 7, p. 80–81], in order that the sum

$$S_N^*(t):=\sum_{i=1}^N Y_i(t)-A^*(t)$$

converges in distribution to an infinitely divisible law with characteristic function

$$\phi(u) = \exp\left\{ iau - \frac{\sigma^2 u^2}{2} + \int_{|x|>0} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) dL(x) \right\}, \tag{6.3}$$

it is necessary and sufficient that the following three conditions be fulfilled:

1) In all points of its continuity, the function $L(\cdot)$ satisfies

$$L(x) = \begin{cases} \lim\limits_{t\to\infty} N\,\mathsf{P}\{Y \le x\} & \text{for } x < 0, \\ -\lim\limits_{t\to\infty} N\,\mathsf{P}\{Y > x\} & \text{for } x > 0. \end{cases} \tag{6.4}$$

2) The constant σ^2 is given by

$$\sigma^2 = \lim_{\tau\to 0+} \limsup_{t\to\infty} N\,\mathsf{Var}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big] = \lim_{\tau\to 0+} \liminf_{t\to\infty} N\,\mathsf{Var}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big]. \tag{6.5}$$

3) For each $\tau > 0$, the constant a satisfies the identity

$$\lim_{t\to\infty} \big\{ N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big] - A^*(t) \big\} = a + \int_0^\tau \frac{x^3}{1+x^2}\,dL(x) - \int_\tau^\infty \frac{x}{1+x^2}\,dL(x). \tag{6.6}$$

As the first step towards the proof of Theorem 2.3, we establish convergence to an infinitely divisible law.

THEOREM 6.1. *Suppose that* $0 < \lambda < \lambda_2$. *Then*

$$\frac{S_N(t) - A(t)}{B(t)} \xrightarrow{d} \mathcal{F}_\alpha \qquad (t \to \infty),$$

where $B(t)$ *and* $A(t)$ *are defined in* (2.17) *and* (2.18), *respectively, and* $\mathcal{F}_\alpha$ *is an infinitely divisible law with characteristic function*

$$\phi_\alpha(u) = \exp\left\{ iau + \alpha \int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{dx}{x^{\alpha+1}} \right\}, \tag{6.7}$$

where the constant a *is given by*

$$a = \begin{cases} \dfrac{\alpha\pi}{2\cos\frac{\alpha\pi}{2}} & (\alpha \ne 1), \\ 0 & (\alpha = 1). \end{cases} \tag{6.8}$$

6.2. *Proof of Theorem* 6.1. The proof is broken down into several steps according to formulas (6.4), (6.5) and (6.6).

PROPOSITION 6.2. *The function* L *defined in* (6.4) *is given by*

$$L(x) = \begin{cases} 0, & x < 0, \\ -x^{-\alpha}, & x > 0. \end{cases} \tag{6.9}$$

PROOF. Since $Y \geq 0$, it is clear that $L(x) \equiv 0$ for $x < 0$. Henceforth, assume that $x > 0$. Using (6.2), (2.17) and (6.1), we obtain

$$N\,\mathsf{P}\{Y(t) > x\} = N\,\mathsf{P}\big\{X > \pm\eta_x(t)\big\} \sim e^{\lambda H_0(t) - h(\eta_x(t)^{\pm})} \qquad (t \to \infty),$$

where $\eta_x(t)$ is defined in (5.24). Furthermore, by Lemmas 5.12 and 5.15

$$\lambda H_0(t) - h(\eta_x(t)^{\pm}) = h(\eta_1(t)^{\pm}) - h(\eta_x(t)^{\pm}) \to -\alpha \log x \qquad (t \to \infty),$$

so from (6.4) we get $L(x) = -e^{-\alpha \log x} = -x^{-\alpha}$. □

PROPOSITION 6.3. *For σ^2 defined in (6.5), for all $\alpha \in (0,2)$ we have $\sigma^2 \equiv 0$.*

PROOF. Since $0 \leq \mathsf{Var}\big[Y\,\mathbf{1}_{\{Y \leq \tau\}}\big] \leq \mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y \leq \tau\}}\big]$, it suffices to prove that

$$\lim_{\tau \to 0+} \lim_{t \to \infty} N\,\mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y \leq \tau\}}\big] = 0.$$

Let us fix $\tau > 0$. Recalling (6.2) and (2.17) and using condition (6.1), we have

$$N\,\mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y \leq \tau\}}\big] \sim e^{(\lambda \mp 2\mu(t))H_0(t)}\,\mathsf{E}\big[e^{2tX}\,\mathbf{1}_{\{X \leq \pm\eta_\tau\}}\big] \qquad (t \to \infty). \tag{6.10}$$

Application of Lemma 5.16(i) with $p = 2$ and $0 < \alpha < 2$ yields

$$\mathsf{E}\big[e^{2tX}\,\mathbf{1}_{\{X \leq \pm\eta_\tau\}}\big] \sim \frac{\alpha}{2-\alpha}\, e^{\pm 2t\eta_\tau - h(\eta_\tau^{\pm})} \qquad (t \to \infty).$$

Returning to (6.10) and recalling relation (5.24), we conclude that

$$\begin{aligned} N\,\mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y \leq \tau\}}\big] &\sim \frac{\alpha}{2-\alpha}\, e^{\lambda H_0(t) \mp 2\mu(t) H_0(t) \pm 2t\eta_\tau - h(\eta_\tau^{\pm})} \\ &= \frac{\alpha}{2-\alpha}\, e^{\lambda H_0(t) - h(\eta_\tau^{\pm}) + 2\log\tau} \to \frac{\alpha}{2-\alpha}\, e^{(2-\alpha)\log\tau} = \frac{\alpha}{2-\alpha}\,\tau^{2-\alpha}, \end{aligned}$$

where we have also used Lemmas 5.12 and 5.15. Letting now $\tau \to 0+$, we see that $\tau^{2-\alpha} \to 0$, since $2 - \alpha > 0$. □

PROPOSITION 6.4. *Set $A^*(t) := A(t)/B(t)$, where $B(t)$ and $A(t)$ are given by* (2.17) *and* (2.18), *respectively. Then the limit*

$$D_\alpha(\tau) := \lim_{t \to \infty} \big\{N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y \leq \tau\}}\big] - A^*(t)\big\} \tag{6.11}$$

exists for all $\alpha \in (0,2)$ and is given by

$$D_\alpha(\tau) = \begin{cases} \dfrac{\alpha}{1-\alpha}\,\tau^{1-\alpha} & (\alpha \neq 1), \\ \log\tau & (\alpha = 1). \end{cases} \tag{6.12}$$

PROOF. Using expressions (6.2), (2.17) and recalling (5.24) we obtain

$$N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y \leq \tau\}}\big] = N e^{\mp\mu(t) H_0(t)}\,\mathsf{E}\big[e^{tX}\,\mathbf{1}_{\{X \leq \pm\eta_\tau\}}\big]. \tag{6.13}$$

1) Let $0 < \alpha < 1$, then $A^* = 0$. Lemma 5.16(i) with $p = 1$ yields

$$\mathsf{E}\big[e^{tX}\,\mathbf{1}_{\{X \leq \pm\eta_\tau\}}\big] \sim \frac{\alpha}{1-\alpha}\, e^{\pm t\eta_\tau - h(\eta_\tau^{\pm})} \qquad (t \to \infty).$$

Hence, on account of the scaling condition (6.1) the right-hand side of (6.13) is asymptotically equivalent to

$$\frac{\alpha}{1-\alpha}\, e^{\lambda H_0(t)\mp\mu(t)H_0(t)\pm t\eta_\tau - h(\eta_\tau^\pm)} = \frac{\alpha}{1-\alpha}\, e^{\log\tau+\lambda H_0(t)-h(\eta_\tau^\pm)}.$$

Finally, using the Basic Identity (5.29) and Lemma 5.15, we get

$$\log\tau + \lambda H_0(t) - h(\eta_\tau^\pm) \to (1-\alpha)\log\tau \qquad (t\to\infty), \tag{6.14}$$

and (6.12) follows.

2) Let $1<\alpha<2$. Using (6.13), (6.1), (2.17) and (2.18), we obtain

$$\begin{aligned} N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big] - A^*(t) &= Ne^{\mp\mu(t)H_0(t)}\Big(\mathsf{E}\big[e^{tX}\mathbf{1}_{\{X\le\pm\eta_\tau\}}\big] - \mathsf{E}\big[e^{tX}\big]\Big) \\ &= -N e^{\mp\mu(t)H_0(t)}\,\mathsf{E}\big[e^{tX}\mathbf{1}_{\{X>\pm\eta_\tau\}}\big] \sim -\frac{\alpha}{\alpha-1}\, e^{\log\tau+\lambda H_0(t)-h(\eta_\tau^\pm)}, \end{aligned}$$

where we used Lemma 5.16(ii) with $p=1$. Applying (6.14) we arrive at (6.12).

3) Let $\alpha=1$. Similarly as above, we obtain using Lemmas 5.18 and 5.12:

$$\begin{aligned} N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big] - A^*(t) &= Ne^{\mp\mu(t)H_0(t)}\Big(\mathsf{E}\big[e^{tX}\mathbf{1}_{\{X\le\pm\eta_\tau\}}\big] - \mathsf{E}\big[e^{tX}\mathbf{1}_{\{X\le\pm\eta_1\}}\big]\Big) \\ &\sim e^{\lambda H_0(t)\mp\mu(t)H_0(t)}\cdot e^{\pm t\eta_1 - h(\eta_1^\pm)}\log\tau = \log\tau, \end{aligned}$$

and the proof is complete. □

PROPOSITION 6.5. *The parameter a defined in* (6.8) *satisfies the identity* (6.6) *with $L(\cdot)$ specified by* (6.9), *that is,*

$$D_\alpha(\tau) = a + \alpha\int_0^\tau \frac{x^{2-\alpha}}{1+x^2}\,dx - \alpha\int_\tau^\infty \frac{x^{-\alpha}}{1+x^2}\,dx \qquad (\tau>0), \tag{6.15}$$

where $D_\alpha(\tau)$ is given by (6.12).

PROOF. 1) Let $0<\alpha<1$. Observe that

$$\int_0^\tau \frac{x^{2-\alpha}}{1+x^2}\,dx = \frac{1}{1-\alpha}\,\tau^{1-\alpha} - \int_0^\tau \frac{x^{-\alpha}}{1+x^2}\,dx.$$

Taking into account (6.12) and (6.8), we see that equation (6.15) amounts to

$$\int_0^\infty \frac{x^{-\alpha}}{1+x^2}\,dx = \frac{\pi}{2\cos\frac{\alpha\pi}{2}}, \tag{6.16}$$

which is true by a formula in Gradshteyn and Ryzhik [(1994), 3.241(2), p. 340].

2) For $1<\alpha<2$, we note that

$$\int_\tau^\infty \frac{x^{-\alpha}}{1+x^2}\,dx = \frac{\tau^{1-\alpha}}{\alpha-1} - \int_\tau^\infty \frac{x^{2-\alpha}}{1+x^2}\,dx,$$

and hence, in view of (6.12) and (6.8), equation (6.15) is reduced to

$$\frac{\pi}{2\cos\frac{\alpha\pi}{2}} + \int_0^\infty \frac{x^{2-\alpha}}{1+x^2}\,dx = 0, \tag{6.17}$$

which again follows from Gradshteyn and Ryzhik [(1994), 3.241(2), p. 340].

3) Finally, for $\alpha = 1$ equation (6.15) takes the form

$$\log \tau = \int_0^\tau \frac{x}{1+x^2}\,dx - \int_\tau^\infty \frac{1}{(1+x^2)x}\,dx. \tag{6.18}$$

The integrals on the right of (6.18) are easily computed to yield

$$\frac{1}{2}\log(1+x^2)\Big|_0^\tau - \frac{1}{2}\log\frac{x^2}{1+x^2}\Big|_\tau^\infty = \log\tau,$$

and this completes the proof of Proposition 6.5. □

PROOF OF THEOREM 6.1. Gathering the results of Propositions 6.2, 6.3, 6.4 and 6.5, which identify the ingredients of the limit characteristic function ϕ_α, we conclude that Theorem 6.1 is true. □

6.3. *Stability of the limit law.* In this section, we show that an infinitely divisible law $\mathcal{F}_\alpha$ with the characteristic function (6.7) is in fact stable.

THEOREM 6.6. *The characteristic function ϕ_α determined by Theorem* 6.1 *corresponds to a stable probability law with exponent $\alpha \in (0,2)$ and skewness parameter $\beta = 1$, and can be represented in a canonical form* (2.21).

REMARK 6.7. Formula (6.9) and Proposition 6.3 imply that ϕ_α corresponds to a stable law [see Ibragimov and Linnik (1971), Theorem 2.2.1, p. 39–40]. We give a direct proof of this fact via reducing ϕ_α to the canonical form (2.21), which allows us to explicitly identify all the parameters.

PROOF OF THEOREM 6.6. According to general theory [see Zolotarev (1957), p. 441; Hall (1981), p. 24], the characteristic function of a stable law with exponent $\alpha \in (0,2)$ admits a canonical representation

$$\phi_\alpha(u) = \begin{cases} \exp\left\{i\mu u - b|u|^\alpha\Big(1 - i\beta \operatorname{sgn} u \cdot \tan\dfrac{\pi\alpha}{2}\Big)\right\} & (\alpha \neq 1), \\ \exp\left\{i\mu u - b|u|\Big(1 + i\beta \operatorname{sgn} u \cdot \dfrac{2}{\pi}\,\log|u|\Big)\right\} & (\alpha = 1), \end{cases} \tag{6.19}$$

where μ is a real constant, $b > 0$ and $-1 \le \beta \le 1$.

1) Suppose that $0 < \alpha < 1$. It is easy to verify that, due to formula (6.8) and identity (6.16), the characteristic function (6.7) can be rewritten in the form

$$\phi_\alpha(u) = \exp\left\{\alpha \int_0^\infty \frac{e^{iux}-1}{x^{\alpha+1}}\,dx\right\}. \tag{6.20}$$

The integral in (6.20) can be computed [see Ibragimov and Linnik (1971), p. 43–44]:

$$\int_0^\infty \frac{e^{iux}-1}{x^{\alpha+1}}\,dx = -\frac{\Gamma(1-\alpha)}{\alpha}\,|u|^\alpha\, e^{-(i\pi\alpha/2)\operatorname{sgn} u},$$

and (6.19) follows with $\mu = 0$, $b = \Gamma(1-\alpha)\cos(\pi\alpha/2) > 0$, $\beta = 1$ [cf. (2.21)].

2) Let now $1 < \alpha < 2$. Using relation (6.17), we can rewrite (6.7) in the form

$$\phi_\alpha(u) = \exp\left\{\alpha \int_0^\infty (e^{iux} - 1 - iux)\, \frac{dx}{x^{\alpha+1}}\right\}. \tag{6.21}$$

The integral in (6.21) is given by [see Ibragimov and Linnik (1971), p. 44–45]

$$\int_0^\infty (e^{iux} - 1 - iux)\, \frac{dx}{x^{\alpha+1}} = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}\, |u|^\alpha\, e^{(i\pi\alpha/2)\operatorname{sgn} u},$$

and (6.19) is satisfied with $\mu = 0$, $b = -\Gamma(2-\alpha)/(\alpha-1)\cdot\cos(\pi\alpha/2) > 0$, $\beta = 1$.

3) If $\alpha = 1$, from (6.8) and (6.7) we get by the substitution $y = |u|x$ (for $u \neq 0$)

$$\phi_1(u) = \exp\left\{-|u| \int_0^\infty \frac{1-\cos y}{y^2}\, dy - iu \int_0^\infty \left(\sin y - \frac{u^2 y}{u^2+y^2}\right) \frac{dy}{y^2}\right\}. \tag{6.22}$$

It is well known [see Gradshteyn and Ryzhik (1994), 3.782(2), p. 470] that

$$\int_0^\infty \frac{1-\cos y}{y^2}\, dy = \frac{\pi}{2}. \tag{6.23}$$

To evaluate the second integral in (6.22), let us represent it in the form

$$\int_0^\infty \left(\frac{\sin y}{y} - \frac{1}{1+y}\right) \frac{dy}{y} + \int_0^\infty \left(\frac{1}{1+y} - \frac{u^2}{u^2+y^2}\right) \frac{dy}{y}. \tag{6.24}$$

It is known that [see Gradshteyn and Ryzhik (1994), 3.781(1), p. 470]

$$\int_0^\infty \left(\frac{\sin y}{y} - \frac{1}{1+y}\right) \frac{dy}{y} = 1 - \gamma, \tag{6.25}$$

where γ is the Euler constant. Furthermore, note that

$$\int_0^\infty \left(\frac{1}{1+y} - \frac{u^2}{u^2+y^2}\right) \frac{dy}{y} = \frac{1}{2} \log \frac{u^2+y^2}{(1+y)^2}\bigg|_0^\infty = -\log|u|. \tag{6.26}$$

Returning to (6.24), from (6.25) and (6.26) we get

$$\int_0^\infty \left(\sin y - \frac{u^2 y}{u^2+y^2}\right) \frac{dy}{y^2} = 1 - \gamma - \log|u|. \tag{6.27}$$

Therefore, substituting expressions (6.23) and (6.27) into (6.22), we obtain a required canonical form (6.19) with $\mu = 1-\gamma$, $b = \pi/2$, $\beta = 1$. □

7. Limit theorems at the critical points.

7.1. *Proof of Theorem* 2.5 (*LLN at* λ_1). For $\alpha = 1$, notation (5.39) amounts to

$$Y_i(t) = \frac{e^{tX_i}}{NB_1(t)}, \tag{7.1}$$

where [see (5.33), (5.34)]

$$B_1(t) := \mathsf{E}\big[e^{tX}\mathbf{1}_{\{X \le \pm\eta_1\}}\big] = e^{\pm t\eta_1 - h(\eta_1^\pm)}\, b_1(t), \tag{7.2}$$

with $b_1(t) \to +\infty$ as $t \to \infty$. Note that η_1, defined in (5.25), is reduced to $\eta_1(t) = \varrho' H_0(t)/t$, since by Lemma 5.9 in the case $\alpha = 1$ one has $\mu(t) \equiv \varrho'$. Recalling the Basic Identity (5.29) we get

$$\pm t\eta_1 - h(\eta_1^{\pm}) = \pm\varrho' H_0(t) \mp (\varrho' - 1)H_0(t) = \pm H_0(t),$$

so that

$$B_1(t) = e^{\pm H_0(t)}\, b_1(t). \tag{7.3}$$

Let us find the median m_X of X. According to the definition and the 'tail' formula (2.2), we have

$$m_X : = \inf\{x : \mathsf{P}(X \le x) \ge 1/2\} = \inf\{x : h(\pm x^{\pm}) \ge \log 2\}.$$

Using the property (3.3), this is rewritten as

$$m_X = \inf\{x : \pm x^{\pm} \ge h^{\leftarrow}(\log 2)\} = \pm h^{\leftarrow}(\log 2)^{\pm}.$$

It is easy to see that there exists a (large enough) number $M > 0$ such that

$$m_X = \pm h^{\leftarrow}(\log 2)^{\pm} \le \pm M^{\pm}. \tag{7.4}$$

This amounts to saying that $m_X < +\infty$ in case B and $m_X < 0$ in case A, which is obvious since, by assumption, the distribution of X does not charge the upper edge of its support (see Section 2.1). More formally, the condition (7.4) is equivalent to $h^{\leftarrow}(\log 2) \le M$, which is obviously satisfied for M large enough.

The median of the random variables Y_i defined in (7.1) is expressed as

$$m_Y \equiv m_Y(t) = \frac{e^{t m_X}}{N B_1(t)}. \tag{7.5}$$

Let us show that $m_Y(t) \to 0$ as $t \to \infty$, and moreover

$$\lim_{t\to\infty} N m_Y(t) = 0. \tag{7.6}$$

Indeed, from (7.5), (7.3) and (7.4) we obtain

$$N m_Y(t) = \frac{e^{t m_X}}{e^{\pm H_0(t)} b_1(t)} \le \frac{e^{\pm(tM^{\pm} - H_0(t))}}{b_1(t)}. \tag{7.7}$$

Using that $H_0 \in R_{\varrho'}$, where $\varrho' > 1$ (case B) or $\varrho' < 1$ (case A), it is easy to check that $\pm(tM^{\pm} - H_0(t)) \to -\infty$. On the other hand, $b_1(t) \to \infty$ [see (5.34)], and hence the right-hand side of (7.7) tends to zero, which proves (7.6).

Denote $\widehat{Y}(t) := Y(t) - m_Y(t)$. According to classical theorems on the LLN for sums of independent random variables [see Petrov (1975), Ch. IX, § 1, Theorem 1, p. 258], we have to check that for any $\tau > 0$ the following three conditions hold:

$$\lim_{t\to\infty} N\,\mathsf{P}\{|\hat{Y}(t)| > \tau\} = 0, \tag{7.8}$$

$$\lim_{t\to\infty} N\,\mathsf{E}\Big[\hat{Y}(t)^2\,\mathbf{1}_{\{|\hat{Y}(t)| \le \tau\}}\Big] = 0, \tag{7.9}$$

$$\lim_{t\to\infty} N\Big(m_Y(t) + \mathsf{E}\Big[\hat{Y}(t)\,\mathbf{1}_{\{|\hat{Y}(t)| \le \tau\}}\Big]\Big) = 1. \tag{7.10}$$

Since $m_Y(t) = o(1)$, we may and will assume that $|m_Y(t)| \le \varepsilon\tau$ for all sufficiently large t, where ε is a number such that $0 < \varepsilon < 1$. Therefore, if $|\hat{Y}(t)| > \tau$ then

$$Y(t) \ge |\hat{Y}(t)| - |m_Y(t)| > \tau - \varepsilon\tau = (1-\varepsilon)\tau. \tag{7.11}$$

Hence, $\mathsf{P}\{|\hat{Y}(t)| > \tau\} \le \mathsf{P}\{Y(t) > (1-\varepsilon)\tau\}$, and Lemma 5.19 [see (5.45)] yields

$$N(t)\,\mathsf{P}\{|\hat{Y}(t)| > \tau\} \le N(t)\,\mathsf{P}\{Y(t) > (1-\varepsilon)\tau\} \to 0 \qquad (t\to\infty),$$

which proves condition (7.8). Analogously, if $|\hat{Y}(t)| \le \tau$ then

$$Y(t) = \hat{Y}(t) + m_Y(t) \le |\hat{Y}(t)| + |m_Y(t)| \le (1+\varepsilon)\tau. \tag{7.12}$$

Also note that $\hat{Y}(t)^2 = (Y(t) - m_Y(t))^2 \le 2Y(t)^2 + 2m_Y(t)^2$. Therefore,

$$\begin{aligned} N\,\mathsf{E}\Big[\hat{Y}^2\mathbf{1}_{\{|\hat{Y}(t)|\le\tau\}}\Big] &\le 2N\,\mathsf{E}\Big[Y^2\mathbf{1}_{\{|\hat{Y}(t)|\le\tau\}}\Big] + 2N\,\mathsf{E}\Big[m_Y^2\mathbf{1}_{\{|\hat{Y}(t)|\le\tau\}}\Big] \\ &\le 2N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\le(1+\varepsilon)\tau\}}\big] + 2Nm_Y^2 = o(1), \end{aligned} \tag{7.13}$$

where we used Lemma 5.21 and relation (7.6), and hence condition (7.9) follows.

In view of (7.6) the condition (7.10) is reduced to

$$\lim_{t\to\infty} N\,\mathsf{E}\Big[Y(t)\mathbf{1}_{\{|\hat{Y}(t)|\le\tau\}}\Big] = 1. \tag{7.14}$$

Inequalities (7.11) and (7.12) imply

$$\mathsf{E}\big[Y\mathbf{1}_{\{Y\le(1-\varepsilon)\tau\}}\big] \le \mathsf{E}\Big[Y\mathbf{1}_{\{|\hat{Y}|\le\tau\}}\Big] \le \mathsf{E}\big[Y\mathbf{1}_{\{Y\le(1+\varepsilon)\tau\}}\big].$$

We also note that according to (7.1), (7.2) and (5.47) (with $\alpha = 1$)

$$N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y\le y_1\}}\big] = \frac{1}{B_1}\,\mathsf{E}\big[e^{tX}\mathbf{1}_{\{X\le\pm\eta_1\}}\big] \equiv 1.$$

Hence, in order to prove (7.14), it suffices to check that for any $\tau' > 0$

$$N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y\le\tau'\}}\big] - N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y\le y_1\}}\big] = N\,\mathsf{E}\big[Y\mathbf{1}_{\{y_1<Y\le\tau'\}}\big] \to 0.$$

It remains to notice that the latter property follows from Lemma 5.20.

Thus, the proof of Theorem 2.5 is complete.

REMARK 7.1. Let us point out that Theorem 2.5 follows from Theorem 2.3 (for $\alpha = 1$). Indeed, according to (2.18) and (7.2), $A(t) = NB_1(t) = Ne^{\pm t\eta_1 - h(\eta_1^\pm)}\,b_1(t)$. Furthermore, (2.17), (6.1), (5.29) and (5.34) imply

$$A^*(t) := \frac{A(t)}{B(t)} \sim e^{\lambda H_0(t) - h(\eta_1^\pm)}\,b_1(t) = b_1(t) \to \infty \qquad (t\to\infty). \tag{7.15}$$

Therefore, dividing (2.20) by $A^*(t) \to \infty$ we obtain $S_N(t)/A(t) = 1 + o_p(1)$, which is in agreement with (2.22). However, we have chosen to give a direct independent proof of this result, which may be helpful for further applications.

7.2. *Proof of Theorem* 2.6 (*CLT at* λ_2). Denote

$$Y_i \equiv Y_i(t) := \frac{e^{tX_i}}{(NB_2(t))^{1/2}}, \qquad i = 1, \dots, N, \tag{7.16}$$

where $B_2(t)$ is defined in (5.33). According to a classical CLT for independent summands [see Petrov (1975), Ch. IV, § 4, Theorem 18, p. 95)], it suffices to check that for any $\tau > 0$ the following three conditions are satisfied as $t \to \infty$:

$$N\,\mathsf{P}\{Y(t) > \tau\} \to 0, \tag{7.17}$$

$$N\Big(\mathsf{E}\big[Y(t)^2 \mathbf{1}_{\{Y(t)\le\tau\}}\big] - \big(\mathsf{E}\big[Y(t)\mathbf{1}_{\{Y(t)\le\tau\}}\big]\big)^2\Big) \to 1, \tag{7.18}$$

$$N\,\mathsf{E}\big[Y(t)\mathbf{1}_{\{Y(t)>\tau\}}\big] \to 0. \tag{7.19}$$

Firstly, note that condition (7.17) is guaranteed by (5.45). Next, let us show that

$$\frac{\big(\mathsf{E}[Y\mathbf{1}_{\{Y\le\tau\}}]\big)^2}{\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\le\tau\}}\big]} = \frac{\big(\mathsf{E}[e^{tX}\mathbf{1}_{\{X\le\pm\tilde{\eta}_{2,\tau}\}}]\big)^2}{\mathsf{E}\big[e^{2tX}\mathbf{1}_{\{X\le\pm\tilde{\eta}_{2,\tau}\}}\big]} \to 0 \qquad (t\to\infty). \tag{7.20}$$

Indeed, taking into account inequality (5.43) and representation (5.34), the ratio in (7.20) is estimated from above by

$$\frac{\big(\mathsf{E}[e^{tX}]\big)^2}{\mathsf{E}\big[e^{2tX}\mathbf{1}_{\{X\le\pm\eta_1\}}\big]} = \frac{e^{\pm 2H(t)}}{B_2(t)} = \frac{e^{\pm 2H(t)+h(\eta_1^{\pm})\mp 2t\eta_1}}{b_2(t)}. \tag{7.21}$$

Using the Basic Identity (5.29) and the limit (5.28), we have

$$\frac{\pm 2H(t) + h(\eta_1^{\pm}) \mp 2t\eta_1(t)}{H_0(t)} \to \pm 2 \pm (\varrho'-1)\,2^{\varrho'} \mp \varrho' 2^{\varrho'} = \pm(2-2^{\varrho'}) < 0,$$

and hence the numerator on the right of (7.21) tends to zero. Moreover, $b_2(t) \to \infty$ [see (5.34)], and therefore (7.20) is validated. Hence, condition (7.18) amounts to

$$N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\le\tau\}}\big] \to 1. \tag{7.22}$$

Noting that, according to (7.16), (5.47) and (2.23),

$$N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\le y_2\}}\big] = \frac{1}{B_2}\,\mathsf{E}\big[e^{2tX}\mathbf{1}_{\{X\le\pm\eta_1\}}\big] \equiv 1,$$

we can rewrite (7.22) in the form $N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{y_2<Y\le\tau\}}\big] \to 0$. But this is true by Lemma 5.20, and (7.18) follows.

Finally, condition (7.19) is fulfilled by Lemma 5.19 (with $p = 1 < 2 = \alpha$).

8. Limit distribution of extremes. Throughout this section, we assume that the log-tail distribution function h is normalized regularly varying, $h \in NR_\varrho$, and that the scaling condition (6.1) is fulfilled. Let us also recall that the parameter α is given by $\alpha = (\varrho\lambda/\varrho')^{1/\varrho'}$ [see (2.7)], and the normalizing function $B(t)$ has the form $B(t) = \exp\{\pm\mu(t)H_0(t)\}$ [see (2.17)], with $\mu(t)$ defined in Section 5.2.

8.1. *Limit theorems for extreme values.* Let us arrange the random variables $e^{tX_1}, \dots, e^{tX_N}$ in the non-increasing order, $M_{1,N} \ge M_{2,N} \ge \cdots \ge M_{N,N}$, so that $M_{k,N} = M_{k,N}(t)$ are the order statistics of the sample $(e^{tX_i})_{i=1}^N$. In particular, $M_{1,N}(t) = \max\{e^{tX_i},\ i = 1, \dots, N\}$ is the maximal term. Note that $M_{k,N}$ can be represented in the form

$$M_{k,N} = e^{tX_{k,N}}, \qquad k = 1, \dots, N, \tag{8.1}$$

where $X_{k,N}$ are the (non-increasing) order statistics of the sample $(X_i)_{i=1}^N$.

THEOREM 8.1 (Limit distribution of $M_{1,N}$; cf. Theorem 2.7). *For all $\lambda > 0$, as $t \to \infty$, $M_{1,N}(t)/B(t)$ converges in distribution to the Fréchet law Φ_α [see (2.24)].*

REMARK 8.2. The Fréchet distribution Φ_α represents one of the three types of possible weak limits for maxima of i.i.d. random variables [see Galambos (1978), Theorem 2.4.1, p. 71]. However, the known general theorems about convergence to Φ_α are not directly applicable in our case.

PROOF OF THEOREM 8.1. Using (2.17), (5.24) and (2.2), for $x > 0$ we have

$$\begin{aligned} \mathsf{P}\{M_{1,N} \le xB(t)\} &= \mathsf{P}\{X_{1,N} \le \pm\eta_x\} = \left(1 - e^{-h(\eta_x^\pm)}\right)^N \\ &= \exp\left[-Ne^{-h(\eta_x^\pm)}(1 + o(1))\right]. \end{aligned} \tag{8.2}$$

Furthermore, recalling (6.1) and (5.29) we obtain

$$Ne^{-h(\eta_x^\pm)} \sim e^{h(\eta_1^\pm) - h(\eta_x^\pm)} \to e^{-\alpha \log x} \qquad (t \to \infty),$$

according to Lemma 5.15. Hence, returning to (8.2) we get

$$\lim_{t\to\infty} \mathsf{P}\{M_{1,N}(t)/B(t) \le x\} = \exp\left(-e^{-\alpha \log x}\right) = e^{-x^{-\alpha}},$$

and the theorem is proved. □

This result implies the following Law of Large Numbers for the maximum.

COROLLARY 8.3 (Log-LLN for $M_{1,N}$). *For all $\lambda > 0$,*

$$\frac{\log M_{1,N}(t)}{H_0(t)} \xrightarrow{p} \pm\varrho'\alpha^{\varrho'-1} \qquad (t \to \infty).$$

PROOF. From Theorem 2.7 we deduce that

$$\frac{\log M_{1,N}(t)}{H_0(t)} = \pm\mu(t) + o_p(1) \qquad (t \to \infty),$$

whence, recalling the limit (5.23), our claim follows. □

REMARK 8.4. Comparing Theorem 8.3 and Theorem 9.1 which we will prove in Section 9.1 below, we note that in the case $0 < \lambda \le \lambda_1$

$$\frac{\log M_{1,N}(t)}{\log S_N(t)} \xrightarrow{p} 1 \qquad (t \to \infty), \tag{8.3}$$

which indicates that the contribution of the maximal term $M_{1,N}(t)$ to the sum $S_N(t)$ is logarithmically equivalent to the whole sum. In the opposite case where $\lambda > \lambda_1$, the limit in (8.3) is strictly less than 1, so that $M_{1,N}(t)$ is negligible as compared to $S_N(t)$. This observation is supported by the LLN being valid for $\lambda \ge \lambda_1$ (see Theorems 2.1 and 2.5), and is further evidenced by Theorem 8.20 characterizing the limiting distribution of the ratio $S_N(t)/M_{1,N}(t)$ in the case $\lambda < \lambda_1$.

In a way similar to the proof of Theorem 8.1, one can obtain the asymptotic distribution of subsequent extremes.

THEOREM 8.5 (Limit distribution of $M_{k,N}$). *For all $\lambda > 0$ and each $k \in \mathbb{N}$,*

$$\lim_{t\to\infty} \mathsf{P}\left\{\frac{M_{k,N}(t)}{B(t)} \le x\right\} = e^{-x^{-\alpha}} \sum_{j=0}^{k-1} \frac{x^{-\alpha j}}{j!} \qquad (x > 0). \tag{8.4}$$

PROOF. Using (8.1), we have $\mathsf{P}\{M_{k,N} \le xB(t)\} = \mathsf{P}\{X_{k,N} \le \pm\eta_x\}$ [cf. (8.2)]. The distribution of the kth order statistic $X_{k,N}$ is given by the following known formula [see Galambos (1978), Sect. 2.8, Eq. (135), p. 102]:

$$\mathsf{P}\{X_{k,N} \le u\} = \sum_{j=0}^{k-1} \binom{N}{j} [1 - F(u)]^j \, [F(u)]^{N-j},$$

where $F(u)$ is the common distribution function of the random variables X_i:

$$F(u) := \mathsf{P}\{X \le u\} = 1 - e^{-h(\pm u^{\pm})}.$$

Setting $u = \pm\eta_x(t)$, similarly to the proof of Theorem 8.1 we obtain

$$\begin{aligned}
\mathsf{P}\{X_{k,N} \le \pm\eta_x(t)\} &= \sum_{j=0}^{k-1} \binom{N}{j} e^{-jh(\eta_x^{\pm})} \Big(1 - e^{-h(\eta_x^{\pm})}\Big)^{N-j} \\
&\sim \Big(1 - e^{-h(\eta_x^{\pm})}\Big)^N \sum_{j=0}^{k-1} \frac{N^j}{j!} e^{-jh(\eta_x^{\pm})} \\
&\sim \exp\Big(-e^{h(\eta_1^{\pm}) - h(\eta_x^{\pm})}\Big) \sum_{j=0}^{k-1} \frac{1}{j!} e^{j\left(h(\eta_1^{\pm}) - h(\eta_x^{\pm})\right)},
\end{aligned}$$

and (8.4) follows by Lemma 5.15. □

8.2. *Joint distribution of extremes.* One can also derive the limiting form of the joint distribution for a finite number of upper extremes. For instance, let us prove the following assertion.

THEOREM 8.6 (Joint limit of $M_{r,N}$ and $M_{s,N}$). *For any $r < s$ and $x, y > 0$*

$$\lim_{t\to\infty} \mathsf{P}\left\{\frac{M_{r,N}}{B(t)} \le x,\ \frac{M_{s,N}}{B(t)} \le y)\right\}$$

$$= \begin{cases} e^{-y^{-\alpha}} \displaystyle\sum_{j=0}^{r-1}\sum_{k=j}^{s-1} \frac{x^{-\alpha j}\,(y^{-\alpha}-x^{-\alpha})^{k-j}}{j!\,(k-j)!} & \text{if } x > y, \\ e^{-x^{-\alpha}} \displaystyle\sum_{j=0}^{r-1} \frac{x^{-\alpha j}}{j!} & \text{if } x \le y. \end{cases} \tag{8.5}$$

PROOF. For $x \le y$ the inequality $M_{r,N} \le x$ implies $M_{s,N} \le y$, so that

$$\mathsf{P}\{M_{r,N} \le xB(t),\ M_{s,N} \le yB(t)\} = \mathsf{P}\{M_{r,N} \le xB(t)\},$$

and (8.5) follows by Theorem 8.4. For $x > y$, using (8.1) we obtain

$$\mathsf{P}\{M_{r,N} \le xB(t),\ M_{s,N} \le yB(t)\} = \mathsf{P}\{X_{r,N} \le \pm\eta_x(t),\ X_{s,N} \le \pm\eta_y(t)\}.$$

The joint distribution of the order statistics $X_{r,N}$ and $X_{s,N}$ (with $r < s$, $u > v$) is given by [cf. David (1981), Sect. 2.2, p. 11]

$$\mathsf{P}\{X_{r,N} \le u,\ X_{r,N} \le v\}$$
$$= \sum_{j=0}^{r-1}\sum_{k=j}^{s-1} \frac{N!}{j!\,(k-j)!\,(N-j)!}[1-F(v)]^j\,[F(u)-F(v)]^{k-j}\,[F(u)]^{N-k}.$$

Taking $u = \pm\eta_x(t)$, $v = \pm\eta_y(t)$, similarly as above we arrive at (8.5). □

This theorem can be extended to the case of any given number of upper extremes. But it is more instructive to characterize the limit distribution of extreme values in a different way. Let us consider the random measure μ_N (corresponding to the *empirical extremal process*) which counts the order statistics 'from the right':

$$\mu_N(x,\infty) := \sum_{k=1}^{N} \mathbf{1}_{\{M_{k,N} > xB(t)\}}, \qquad x > 0. \tag{8.6}$$

The following theorem reveals a Poisson asymptotic structure as $t \to \infty$.

THEOREM 8.7 (Poisson limit theorem for order statistics). *In the sense of convergence of all finite-dimensional distributions, μ_N converges to a Poisson random measure on $(0,\infty)$ with mean measure $\nu(x,\infty) = x^{-\alpha}$. That is, for any points $x_1 > \cdots > x_n > 0$ and any integers $m_1, \dots, m_n \ge 0$*

$$\lim_{t\to\infty} \mathsf{P}\{\mu_N(\Delta_i) = m_i,\ i = 1,\dots,n\} = \prod_{i=1}^{n} \frac{[\nu(\Delta_i)]^{m_i}}{m_i!}\, e^{-\nu(\Delta_i)}, \tag{8.7}$$

where $\Delta_i := (x_i, x_{i-1}]$, $x_0 := \infty$, *and* $\nu(\Delta_i) = x_i^{-\alpha} - x_{i-1}^{-\alpha}$.

REMARK 8.8. The same answer was established by Weissman (1975) for the counting measure of extremes in the case of i.i.d. random variables in the domain of attraction of the Fréchet distribution Φ_α. Analogous results were also obtained in Weissman (1975) for the cases of attraction to the Weibull distribution Ψ_α and the Gumbel distribution Λ. See also David [(1981), Sect. 9.4, p. 266].

PROOF OF THEOREM 8.7. In view of (8.1) and (8.6), the condition $\mu_N(\Delta_i) = m_i$ means that the interval $(\pm\eta_{x_i}, \pm\eta_{x_{i-1}}]$ contains exactly m_i points out of $X_1, \dots, X_N$. Hence, putting $m_{n+1} := N - m_1 - \cdots - m_n$, by the multinomial formula we obtain

$$\begin{aligned}
\mathsf{P}\{\mu_N(\Delta_i) = m_i,\ i = 1, \dots, n\}
&= N! \prod_{i=1}^{n} \frac{\left[F(\pm\eta_{x_{i-1}}) - F(\pm\eta_{x_i})\right]^{m_i}}{m_i!} \cdot \frac{\left[F(\pm\eta_{x_n})\right]^{m_{n+1}}}{m_{n+1}!} \\
&= N! \prod_{i=1}^{n} \frac{\left[e^{-h(\eta_{x_i}^{\pm})} - e^{-h(\eta_{x_{i-1}}^{\pm})}\right]^{m_i}}{m_i!} \cdot \frac{\left(1 - e^{-h(\eta_{x_n}^{\pm})}\right)^{m_{n+1}}}{m_{n+1}!} \\
&\sim N^{m_1 + \cdots + m_n} \prod_{i=1}^{n} \frac{\left[e^{-h(\eta_{x_i}^{\pm})} - e^{-h(\eta_{x_{i-1}}^{\pm})}\right]^{m_i}}{m_i!} \cdot \left(1 - e^{-h(\eta_{x_n}^{\pm})}\right)^{N} \\
&\sim \prod_{i=1}^{n} \frac{\left[e^{h(\eta_1^{\pm}) - h(\eta_{x_i}^{\pm})} - e^{h(\eta_1^{\pm}) - h(\eta_{x_{i-1}}^{\pm})}\right]^{m_i}}{m_i!} \cdot \exp\left\{e^{h(\eta_1^{\pm}) - h(\eta_{x_n}^{\pm})}\right\}.
\end{aligned}$$

Using that, by Lemma 5.15, $h(\eta_1^{\pm}) - h(\eta_{x_i}^{\pm}) \to -\alpha \log x_i$, we arrive at (8.7). □

8.3. *Some representations of extreme values.* In this section, we record a few (basically well known) representations in distribution for extreme values $M_{k,N}$ and hence for the sum $S_N(t)$. These representations are expressed in terms of auxiliary sequences of i.i.d. random variables with either exponential or uniform distribution. In particular, they will be used to study the asymptotic behavior of $S_N(t)$ in the case $0 < \lambda < \lambda_1$. The advantage of such an approach (usually called the 'method of common probability space') is that the random variables of interest will have a limit *with probability one*, rather than just in distribution.

Consider the random variable $\Xi := \pm h^{\leftarrow}(\xi)^{\pm}$, where ξ has the unit exponential distribution, that is, $\mathsf{P}\{\xi > x\} = e^{-x}$ $(x > 0)$. A key observation is that $\Xi \stackrel{d}{=} X$. Indeed, since h is right-continuous, we can use the property (3.3) to obtain

$$\begin{aligned}
\mathsf{P}\{\Xi \le x\} &= \mathsf{P}\{\pm h^{\leftarrow}(\xi)^{\pm} \le x\} = \mathsf{P}\{h^{\leftarrow}(\xi) \le \pm x^{\pm}\} \\
&= \mathsf{P}\{\xi \le h(\pm x^{\pm})\} = 1 - e^{-h(\pm x^{\pm})} = \mathsf{P}\{X \le x\},
\end{aligned}$$

according to (2.2). Furthermore, if $\xi_1, \dots, \xi_N$ are i.i.d. random variables with unit exponential distribution, then the random variables $\Xi_i := \pm h^{\leftarrow}(\xi_i)^{\pm}$ are also independent and hence the random vector $(\Xi_i)_{i=1}^N$ has the same distribution as $(X_i)_{i=1}^N$.

In particular, the joint distribution of the order statistics $\Xi_{1,N} \geq \ldots \geq \Xi_{N,N}$ coincides with that of the order statistics $X_{1,N} \geq \ldots \geq X_{N,N}$. We also note that since the function $\pm h^{\leftarrow}(x)^{\pm}$ is non-decreasing in its domain, the order statistics $\Xi_{k,N}$ can be represented through the underlying exponential order statistics $\xi_{1,N} \geq \ldots \geq \xi_{N,N}$ as $\Xi_{k,N} = \pm h^{\leftarrow}(\xi_{k,N})^{\pm}$ $(k = 1, \ldots, N)$.

Furthermore, let us recall the following representation in distribution of the exponential order statistics [see David (1981), Sect. 2.7, p. 20–21].

LEMMA 8.9. *For each $k = 1, \ldots, N$, the distribution of the kth order statistic $\xi_{k,N}$ coincides with the distribution of the random variable*

$$T_{k,N} := \sum_{i=k}^{N} \frac{\zeta_i}{i}, \tag{8.8}$$

where (ζ_i) is an auxiliary sequence of i.i.d. random variables, each having the unit exponential distribution. Moreover, $(\xi_{1,N}, \ldots, \xi_{N,N}) \stackrel{d}{=} (T_{1,N}, \ldots, T_{N,N})$.

As a result, the joint distribution of the order statistics $M_{k,N} = e^{tX_{k,N}}$ coincides with the joint distribution of the random variables $e^{\pm th^{\leftarrow}(T_{k,N})^{\pm}}$. Hence, the sum $S_N(t) = \sum_{i=1}^{N} e^{tX_i} = \sum_{k=1}^{N} e^{tX_{k,N}}$ has the same distribution as the sum

$$\sum_{k=1}^{N} e^{\pm th^{\leftarrow}(T_{k,N})^{\pm}} \tag{8.9}$$

with $T_{k,N}$ given by (8.8). It is convenient to rewrite $T_{k,N}$ as

$$T_{k,N} = \sum_{i=k}^{N} \left(\frac{\zeta_i}{i} - \log \frac{i+1}{i} \right) + \log \frac{N+1}{k}. \tag{8.10}$$

LEMMA 8.10. *The series*

$$\sum_{i=1}^{\infty} \left(\frac{\zeta_i}{i} - \log \frac{i+1}{i} \right) \tag{8.11}$$

converges with probability one.

PROOF. Let us represent the series (8.11) as

$$\sum_{i=1}^{\infty} \left(\frac{\zeta_i}{i} - \log \frac{i+1}{i} \right) = \sum_{i=1}^{\infty} \frac{\zeta_i - 1}{i} + \sum_{i=1}^{\infty} \left(\frac{1}{i} - \log \frac{i+1}{i} \right) \tag{8.12}$$

and show that both series on the right are convergent. Recalling that ζ_i are exponentially distributed with mean 1, we have $\mathsf{E}[(\zeta_i - 1)/i] = 0$ and

$$\sum_{i=1}^{\infty} \mathsf{Var}\left(\frac{\zeta_i - 1}{i} \right) = \sum_{i=1}^{\infty} \frac{\mathsf{Var}[\zeta_i]}{i^2} = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.$$

Since (ζ_i) are independent, this implies a.s.-convergence of the series $\sum_i(\zeta_i - 1)/i$ [see Petrov (1975), Ch. IX, § 2, Lemma 8, p. 266]. Further, note that

$$\frac{1}{i} - \log\left(1 + \frac{1}{i}\right) = O(i^{-2}) \qquad (i \to \infty),$$

hence the last series in (8.12) converges. □

REMARK 8.11. A similar approach was used by Hall (1978) to obtain a canonical representation for limiting extreme values in the i.i.d. scheme, following the ideas suggested by Rényi (1953) [see also Rényi (1970), Chapter VIII, § 9].

Lemma 8.10 implies that for each $k \geq 1$ the sum of the series

$$Z_k := \sum_{i=k}^{\infty} \left(\frac{\zeta_i}{i} - \log\frac{i+1}{i}\right) \tag{8.13}$$

is finite with probability one. It is not difficult to find the (joint) distribution of Z_k.

LEMMA 8.12. *Let us set $\tau_k := ke^{-Z_k}$ $(k = 1, 2, \dots)$. Then $(\tau_k) \stackrel{d}{=} (\sigma_k)$, where $\sigma_k := \zeta_1' + \cdots + \zeta_k'$ and (ζ_i') is a sequence of independent exponential random variables with mean* 1. *In particular, τ_k has the gamma distribution with mean k.*

REMARK 8.13. The random variables (τ_k) are distributed as a sequence of arrival times of a Poisson process with unit rate.

PROOF OF LEMMA 8.12. According to the definitions (8.13) and (8.8), we have

$$Z_k = \lim_{N\to\infty} \sum_{i=k}^{N} \left(\frac{\zeta_i}{i} - \log\frac{i+1}{i}\right) = \lim_{N\to\infty} (T_{k,N} - \log N) + \log k.$$

Hence,

$$\tau_k = e^{-Z_k + \log k} = \lim_{N\to\infty} \left(Ne^{-T_{k,N}}\right). \tag{8.14}$$

Recall that by Lemma 8.9, the random variable $T_{k,N}$ has the same distribution as the kth (decreasing) order statistic $\xi_{k,N}$ of N independent exponential random variables $(\xi_i)_{i=1}^N$ (with mean 1). From this, one could derive the distribution of τ_k using the known limit results for the exponential order statistics [see Galambos (1978), Example 1.3.1, p. 12, for $k = 1$ and Theorem 2.8.1, p. 102–103, for $k \geq 1$].

However, a more neat proof is possible that requires almost no calculations and simultaneously allows one to establish independence of the successive differences $\tau_{k+1} - \tau_k$. Namely, observe that the random variables $U_i := e^{-\xi_i}$ $(i = 1, \dots, N)$ are independent and uniformly distributed in $[0, 1]$. Therefore, $(e^{-\xi_{k,N}})_{k=1}^N \stackrel{d}{=} (U_{k,N})_{k=1}^N$, where $U_{1,N} \leq \cdots \leq U_{N,N}$ are the order statistics of $(U_i)_{i=1}^N$. In turn, it is well known [see Feller (1971), Ch. III, § 3] that

$$(U_{1,N}, \dots, U_{N,N}) \stackrel{d}{=} \left(\frac{\sigma_1}{\sigma_{N+1}}, \dots, \frac{\sigma_N}{\sigma_{N+1}}\right), \tag{8.15}$$

where σ_k are described in the lemma. Returning to (8.14), the distribution of the vector $(\tau_1, \ldots, \tau_k)$ for each $k \ge 1$ can be computed as the weak limit

$$\left(\frac{N\sigma_k}{\sigma_{N+1}}, \ldots, \frac{N\sigma_k}{\sigma_{N+1}}\right) \xrightarrow{d} (\sigma_1, \ldots, \sigma_k) \qquad (N \to \infty),$$

where we used that, due to the Law of Large Numbers, $\sigma_{N+1}/N \xrightarrow{p} 1$. □

8.4. *Preparatory estimates.* The main goal of this section is to establish, with probability one, a suitable uniform upper bound for the terms of the sum (8.9) (see Lemma 8.16 below), which will allow us to pass to the limit in (8.9) as $t \to \infty$.

The next elementary lemma provides a convenient criterion of convergence in terms of 'level' inequalities.

LEMMA 8.14. *A number sequence (a_n) has a limit $a \in \mathbb{R}$ iff for each $c \neq a$*

$$\lim_{n\to\infty} \mathbf{1}_{\le c}(a_n) = \mathbf{1}_{\le c}(a),$$

where $\mathbf{1}_{\le c}(\cdot)$ denotes the indicator function of the interval $(-\infty, c]$.

LEMMA 8.15. *For each fixed $k \ge 1$, with probability one,*

$$\lim_{t\to\infty} \frac{\exp\{\pm t h^{\leftarrow}(T_{k,N})^{\pm}\}}{B(t)} = k^{-1/\alpha} e^{Z_k/\alpha}, \tag{8.16}$$

where $B(t)$ is given by (2.17) and Z_k is defined in (8.13).

PROOF. Let us fix an arbitrary number $c > 0$ and consider the inequality

$$\frac{\exp\{\pm t h^{\leftarrow}(T_{k,N})^{\pm}\}}{B(t)} \le c, \tag{8.17}$$

which, in view of notations (2.17) and (5.24), amounts to $h^{\leftarrow}(T_{k,N}) \le \eta_c(t)^{\pm}$. Furthermore, due to the property (3.3) the last inequality can be rewritten as

$$T_{k,N} \le h(\eta_c(t)^{\pm}). \tag{8.18}$$

Our next step is to use the identity (5.29) and represent inequality (8.18) as

$$T_{k,N} - \log N \le h(\eta_c(t)^{\pm}) - h(\eta_1(t)^{\pm}) + \lambda H_0(t) - \log N. \tag{8.19}$$

According to (8.10) and (8.13), the left-hand side of (8.19) amounts to

$$\sum_{i=k}^{N} \left(\frac{\zeta_i}{i} - \log\frac{i+1}{i}\right) + \log\frac{N+1}{N} - \log k \to Z_k - \log k \qquad (N \to \infty), \tag{8.20}$$

while on the right, using Lemma 5.15 and the scaling relation (6.1), we have

$$h(\eta_c(t)^{\pm}) - h(\eta_1(t)^{\pm}) + \lambda H_0(t) - \log N \to \alpha \log c \qquad (t \to \infty).$$

Therefore, in the limit $t \to \infty$, $N \to \infty$ inequality (8.19) takes the form $Z_k - \log k \le \alpha \log c$, or equivalently, $k^{-1/\alpha} e^{Z_k/\alpha} \le c$. Comparing this inequality with (8.17) and applying Lemma 8.14, we obtain (8.16). □

Let us note that if $\sum_i a_i$ is a convergent series, then its partial sums $\sum_{i=k}^n a_i$ are uniformly bounded. Indeed, set $s_n := \sum_{i=1}^n a_i$, $s_0 := 0$, then $s^* := \sup_n s_n < +\infty$, $s_* := \inf_n s_n > -\infty$ and $\left|\sum_{i=k}^n a_i\right| = |s_n - s_{k-1}| \le s^* - s_* < \infty$ for all n and $k \le n$.

Therefore, since by Lemma 8.10 the series (8.11) is convergent, there exists a proper random variable Z^* such that, with probability one,

$$\sum_{i=k}^{N}\left(\frac{\zeta_i}{i} - \log\frac{i+1}{i}\right) \le Z^* \qquad \text{for all } N \text{ and } 1 \le k \le N. \tag{8.21}$$

Using (8.10) it follows that with probability one for all N and $1 \le k \le N$

$$T_{k,N} \le Z^* + \log\frac{N+1}{k}. \tag{8.22}$$

LEMMA 8.16. *Let $B(t)$ be given by (2.17). Then for any $\tilde\alpha > \alpha$ and each $\varepsilon > 0$, with probability one for all large enough t and uniformly in $k \le N$*

$$\frac{\exp(\pm t h^{\leftarrow}(T_{k,N})^{\pm})}{B(t)} \le c_k := k^{-1/\tilde\alpha}\exp\left(\frac{Z^*+\varepsilon}{\alpha}\right). \tag{8.23}$$

PROOF. Similarly to (8.17), we can rewrite (8.23) in the form $T_{k,N} \le h(\eta_{c_k}(t)^{\pm})$ [cf. (8.18)]. Furthermore, on account of (8.22) it suffices to check that

$$Z^* - \log k + \log(N+1) \le h(\eta_{c_k}(t)^{\pm}), \qquad k = 1, \dots, N. \tag{8.24}$$

Note that with probability one the ratio

$$\varkappa_k(t) := \frac{\eta_{c_k}(t)}{\eta_1(t)} = 1 \pm \frac{Z^*+\varepsilon}{\alpha t \eta_1(t)} \mp \frac{\log k}{\tilde\alpha t \eta_1(t)}$$

is ultimately bounded above and separated from 0, uniformly in $k \le N$. Indeed, $\varkappa_k(t)^{\pm} \le \varkappa_1(t)^{\pm} \to 1$ as $t \to \infty$. On the other hand, using the scaling (6.1) and relations (5.28), (5.13), and also recalling that $\tilde\alpha > \alpha$, we get

$$\varkappa_k(t)^{\pm} \ge \varkappa_N(t)^{\pm} \to \left(1 \mp \frac{\alpha}{\tilde\alpha\varrho}\right)^{\pm} \ge \left(1 \mp \frac{1}{\varrho}\right)^{\pm} > 0.$$

Hence, we can apply Lemma 5.5 and, using inequality (3.9) and the limit (5.31), obtain, uniformly in $k \le N$,

$$\begin{aligned} h(\eta_{c_k}^{\pm}) - h(\eta_1^{\pm}) &\sim h(\eta_1^{\pm})(\varkappa_k^{\pm\varrho} - 1) \ge \pm\varrho h(\eta_1^{\pm})(\varkappa_k - 1) \\ &= \frac{\varrho h(\eta_1^{\pm})}{\alpha t \eta_1}\left(Z^* + \varepsilon - \frac{\alpha\log k}{\tilde\alpha}\right) \sim Z^* + \varepsilon - \frac{\alpha \log k}{\tilde\alpha}. \end{aligned} \tag{8.25}$$

We also note that, by (6.1) and (5.29),

$$\log(N+1) = h(\eta_1^{\pm}) + o(1) \qquad (t \to \infty). \tag{8.26}$$

The estimates (8.25) and (8.26) imply that inequality (8.24) will be proved once we check that

$$Z^* - \log k + o(1) \le \left(Z^* + \varepsilon - \frac{\alpha\log k}{\tilde\alpha}\right)(1 + o(1)),$$

where all $o(1)$ are uniform in $k \le N$. Rearranging, this is reduced to the inequality

$$0 \le \varepsilon + \Big(1 - \frac{\alpha}{\tilde\alpha} + o(1)\Big) \log k + o(1),$$

which holds as $t \to \infty$, since $\varepsilon > 0$, $1 - \alpha/\tilde\alpha > 0$ and $\log k \ge 0$. □

8.5. *Limit theorems for sums via order statistics.* We are now in a position to prove the following theorem.

THEOREM 8.17. *Assume that* $0 < \alpha < 1$. *Then, as* $t \to \infty$,

$$\frac{1}{B(t)} \sum_{k=1}^{N} e^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} \xrightarrow{\text{a.s.}} V_\alpha := \sum_{k=1}^{\infty} k^{-1/\alpha} e^{Z_k/\alpha}, \tag{8.27}$$

where the last series converges with probability one.

PROOF. Denote $v_k(t) := (1/B(t))\, e^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}}$ for $k \le N(t)$ and $v_k(t) := 0$ otherwise. By Lemma 8.15, $v_k(t) \xrightarrow{\text{a.s.}} k^{-1/\alpha} e^{Z_k/\alpha}$ as $t \to \infty$. Let us pick some $\varepsilon > 0$ and for a given $\alpha \in (0,1)$ choose a number $\tilde\alpha$ such that $\alpha < \tilde\alpha < 1$. Then, according to Lemma 8.16 [see (8.23)], with probability one for all t large enough we have $v_k(t) \le v_k^* := k^{-1/\tilde\alpha} e^{(Z^*+\varepsilon)/\alpha}$, $k = 1, 2, \ldots$ Since $1/\tilde\alpha > 1$, the series $\sum_k v_k^*$ is convergent, so Lebesgue's dominated convergence theorem yields

$$\frac{1}{B(t)} \sum_{k=1}^{N} e^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} = \sum_{k=1}^{\infty} v_k(t) \xrightarrow{\text{a.s.}} \sum_{k=1}^{\infty} k^{-1/\alpha} e^{Z_k/\alpha} \equiv V_\alpha \qquad (t \to \infty),$$

also implying a.s.-convergence of the limiting series. □

REMARK 8.18. The representation of the limit (8.27) can be rewritten as

$$V_\alpha = \sum_{k=1}^{\infty} \tau_k^{-1/\alpha}, \tag{8.28}$$

where (τ_k) are defined in Lemma 8.12. Note that convergence of the series (8.28) is obvious, since a.s. $\tau_k \sim k$ as $k \to \infty$ (by the strong LLN) and $\sum_k k^{-1/\alpha} < \infty$.

Recalling that $S_N(t)$ has the same distribution as the sum (8.9), from Theorem 8.17 it follows that the distribution of $S_N(t)/B(t)$ weakly converges, as $t \to \infty$, to the distribution of the random variable V_α. Comparing this result with Theorems 6.1 and 6.6, we arrive at the following assertion.

THEOREM 8.19. *For* $0 < \alpha < 1$, *the random variable* V_α *defined in* (8.27) *has the stable distribution* $\mathcal{F}_\alpha$ *with characteristic function* ϕ_α *given by formula* (2.21).

This theorem can be viewed as a series representation of the stable distribution $\mathcal{F}_\alpha$ (with $0 < \alpha < 1$ and $\beta = 1$). However, being rewritten in the form (8.28) this representation amounts to one of the known formulas [cf. Samorodnitsky and Taqqu (1994), Theorem 1.4.5, p. 28].

THEOREM 8.20. *For $\alpha \in (0,1)$, the ratio $S_N(t)/M_{1,N}(t)$ has a proper limiting distribution, which can be represented via the random variable*

$$W_\alpha := e^{-Z_1/\alpha} V_\alpha \equiv 1 + \sum_{k=1}^{\infty} \exp\biggl(-\frac{1}{\alpha}\sum_{i=1}^{k}\frac{\zeta_i}{i}\biggr), \tag{8.29}$$

where (ζ_i) is a sequence of independent exponential random variables with mean 1 involved in the representation (8.13).

PROOF. As stated after Lemma 8.9, the joint distribution of $M_{1,N}(t) = e^{tX_{1,N}}$ and $S_N(t) = \sum_{k=1}^N e^{tX_{k,N}}$ coincides with that of the pair $\exp\{\pm t h^{\leftarrow}(T_{1,N})^{\pm}\}$ and $\sum_{k=1}^N \exp\{\pm t h^{\leftarrow}(T_{k,N})^{\pm}\}$. In particular,

$$\frac{S_N(t)}{M_{1,N}(t)} \overset{d}{=} \frac{\sum_{k=1}^N \exp\{\pm t h^{\leftarrow}(T_{k,N})^{\pm}\}}{\exp\{\pm t h^{\leftarrow}(T_{1,N})^{\pm}\}}. \tag{8.30}$$

Dividing both the numerator and denominator by the function $B(t)$ defined in (2.17) and applying Lemma 8.15 (with $k = 1$) and Theorem 8.17, we deduce that the right-hand side of (8.30) with probability one converges to

$$V_\alpha e^{-Z_1/\alpha} = 1 + \sum_{k=2}^{\infty} k^{-1/\alpha} e^{-(Z_1 - Z_k)/\alpha}. \tag{8.31}$$

From (8.13) it follows that for $k \ge 2$

$$Z_1 - Z_k = \sum_{i=1}^{k-1}\left(\frac{\zeta_i}{i} - \log\frac{i+1}{i}\right) = \sum_{i=1}^{k-1}\frac{\zeta_i}{i} - \log k. \tag{8.32}$$

Hence, (8.31) is reduced to the expression

$$1 + \sum_{k=2}^{\infty}\exp\left(-\frac{1}{\alpha}\sum_{i=1}^{k-1}\frac{\zeta_i}{i}\right),$$

which is the same as the right-hand part of (8.29). □

REMARK 8.21. The random variable (8.29) can be represented as [cf. (8.28)]

$$W_\alpha = \sum_{k=1}^{\infty}\left(\frac{\tau_1}{\tau_k}\right)^{1/\alpha}, \tag{8.33}$$

where (τ_k) are defined in Lemma 8.12.

THEOREM 8.22. *The random variable W_α defined in (8.29) has the characteristic function given by*

$$f_\alpha(u) = \frac{e^{iu}}{1 - \alpha\displaystyle\int_0^1 (e^{iux} - 1)\,\frac{dx}{x^{\alpha+1}}}. \tag{8.34}$$

REMARK 8.23. Remembering that W_α has emerged in Theorem 8.20 in relation to the limit of $S_N(t)/M_{1,N}(t)$, it is worthwhile to compare Theorem 8.22 with the analogous result by Darling [(1952), Theorem 5.1, p. 103; see also Arov and Bobrov (1960), Corollary 4, p. 389], asserting that if (Y_i) is a sequence of i.i.d. random variables with distribution belonging to the domain of attraction of a stable law with exponent $0<\alpha<1$, then for $S_n := Y_1+\cdots+Y_n$ and $M_{1,n} := \max\{Y_1,\dots,Y_n\}$ the ratio $S_n/M_{1,n}$ has the limit distribution with characteristic function (8.34).

PROOF OF THEOREM 8.22. Using the observation in Remark 8.23, we will prove the theorem's statement indirectly, via establishing a representation analogous to (8.29) for the limit of the ratio $S_n/M_{1,n}$. Clearly, it suffices to do this with a suitable choice of random variables Y_i. Let us set $Y_i = e^{\xi_i/\alpha}$, where (ξ_i) is an i.i.d. sequence of exponentially distributed random variables with mean 1. Note that $Y_i > 1$ and

$$\mathsf{P}\{Y_i > x\} = \mathsf{P}\{\xi_i > \alpha \log x\} = e^{-\alpha \log x} = x^{-\alpha} \qquad (x > 1).$$

Hence [see Ibragimov and Linnik (1971), Theorem 2.6.1, p. 76], the distribution of Y_i is in the domain of attraction of a stable law $\mathcal{F}_\alpha$ (with $\beta = 1$) and

$$\frac{S_n}{n^{1/\alpha}} = \frac{e^{\xi_1/\alpha} + \cdots + e^{\xi_n/\alpha}}{n^{1/\alpha}} \xrightarrow{d} \mathcal{F}_\alpha \qquad (n\to\infty).$$

Passing to the order statistics $\xi_{1,n} \ge \cdots \ge \xi_{n,n}$, in a way similar to the above we represent the sum S_n as

$$S_n = \sum_{k=1}^{n} e^{\xi_{k,n}/\alpha} \stackrel{d}{=} \sum_{k=1}^{n} e^{T_{k,n}/\alpha},$$

where $T_{k,n}$ are given by (8.8). Analogously to (8.19) and (8.20), we obtain

$$\frac{S_n}{n^{1/\alpha}} \stackrel{d}{=} \sum_{k=1}^{n} e^{(T_{k,n}-\log n)/\alpha} \xrightarrow{\text{a.s.}} \sum_{k=1}^{\infty} k^{-1/\alpha} e^{Z_k/\alpha} \equiv V_\alpha \qquad (n\to\infty), \tag{8.35}$$

where a term-by-term passing to the limit can be justified as before, using (8.22).

Similarly, one shows that

$$\frac{M_{1,n}}{n^{1/\alpha}} \stackrel{d}{=} e^{(T_{1,n}-\log n)/\alpha} \xrightarrow{\text{a.s.}} e^{Z_1/\alpha} \qquad (n\to\infty). \tag{8.36}$$

Hence, dividing (8.35) by (8.36) we obtain [cf. (8.29)]

$$\frac{S_n}{M_{1,n}} \xrightarrow{d} e^{-Z_1/\alpha} V_\alpha \equiv W_\alpha.$$

Comparing this with the result by Darling (1952) mentioned above, we conclude that W_α has distribution with the characteristic function (8.34). □

REMARK 8.24. It would be interesting to derive formula (8.34), or otherwise characterize the distribution of W_α, directly from representation (8.29) [or (8.33)].

The following result, being a direct consequence of Theorem 8.22, is of interest due to the striking simplicity of the answer.

COROLLARY 8.25. *For $0 < \alpha < 1$, the expected value of W_α is given by*

$$\mathsf{E}[W_\alpha] = \frac{1}{1-\alpha}. \tag{8.37}$$

PROOF. Differentiate formula (8.34) at $u = 0$. □

REMARK 8.26. In Appendix C, we will give three alternative proofs of the identity (8.37) based on the direct use of the representations (8.29) and (8.33).

REMARK 8.27. Taking expectation of (8.29) [see (C.1) below] and comparing with (8.37), we arrive at the following curious identity:

$$1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \left(1 + \frac{1}{i\alpha}\right)^{-1} = \frac{1}{1-\alpha} \qquad (0 < \alpha < 1).$$

The next assertion highlights an increasingly overwhelming role played by the maximal term in the sum $S_N(t)$, as α tends to zero.

PROPOSITION 8.28. *With probability one, $W_\alpha \to 1$ as $\alpha \to 0+$.*

PROOF. As we know, the series (8.29) is a.s.-convergent for all $0 < \alpha < 1$. Since $\zeta_i \geq 0$, its terms are non-decreasing functions of α, so that for $0 < \alpha \leq \alpha_0 < 1$ each one is dominated by the respective value with $\alpha = \alpha_0$. Moreover, with probability one each term is $o(1)$ as $\alpha \to 0+$, so the dominated convergence theorem implies that the series in (8.29) almost surely vanishes and hence $W_\alpha \to 1$. □

Using the series representation provided by Theorem 8.19, one can easily derive a limit theorem for the stable distribution $\mathcal{F}_\alpha$ as its parameter α tends to zero.

PROPOSITION 8.29. *Let a random variable ζ_α have the stable distribution $\mathcal{F}_\alpha$ determined by* (2.21), *with parameters $0 < \alpha < 1$ and $\beta = 1$. Then, as $\alpha \to 0+$, the distribution of $\alpha \log \zeta_\alpha$ weakly converges to the double exponential distribution,*

$$\lim_{\alpha \to 0+} \mathsf{P}(\alpha \log \zeta_\alpha \leq x) = \exp(-e^{-x}), \qquad x \in \mathbb{R}. \tag{8.38}$$

REMARK 8.30. A general result of this kind was proved purely analytically by Zolotarev [(1957), Theorem 5, p. 447–448 and (1986), Theorem 2.9.1, p. 160].

PROOF OF PROPOSITION 8.29. By Theorem 8.19, the random variable V_α has the distribution $\mathcal{F}_\alpha$, so it suffices to verify (8.38) for V_α. From (8.29) and using Proposition 8.28 we have $\alpha \log V_\alpha = Z_1 + \log W_\alpha \xrightarrow{\text{a.s.}} Z_1$ as $\alpha \to 0+$. To find the distribution of Z_1, let us note that, according to the definition of τ_k in Lemma 8.12, we have the representation $Z_1 = -\log \tau_1$, where τ_1 has the exponential distribution with mean 1. Hence, $\mathsf{P}\{Z_1 \leq x\} = \mathsf{P}\{\tau_1 \geq e^{-x}\} = \exp(-e^{-x})$. □

We conclude this section by stating a series representation theorem for the limit in the case $\lambda_1 \leq \lambda < \lambda_2$, analogous to Theorem 8.17.

THEOREM 8.31. (a) *For* $\lambda_1 < \lambda < \lambda_2$,

$$\frac{1}{B(t)} \sum_{k=1}^{N} \left(e^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} - \mathsf{E}\left[e^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} \right] \right) \xrightarrow{\text{a.s.}} \sum_{k=1}^{\infty} \frac{e^{Z_k/\alpha} - \mathsf{E}\left[e^{Z_k/\alpha} \right]}{k^{1/\alpha}}.$$

(b) *For* $\lambda = \lambda_1$,

$$\frac{1}{B(t)} \sum_{k=1}^{N} \left(e^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} - \mathsf{E}\left[e^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} \mathbf{1}_{\{T_{k,N} \le h(\eta_1^{\pm})\}} \right] \right)$$

$$\xrightarrow{\text{a.s.}} \sum_{k=1}^{\infty} \frac{e^{Z_k/\alpha} - \mathsf{E}\left[e^{Z_k/\alpha} \mathbf{1}_{\{Z_k \le \log k\}} \right]}{k^{1/\alpha}}.$$

Here the limiting series converge with probability one.

This theorem can be proved along the same lines as in the case $0 < \lambda < \lambda_1$ above. However, the proof is technically more involved and will be presented elsewhere.

9. Complements and applications. In this section, we derive a few implications of our limit theorems for $S_N(t)$ in the context of the two important examples mentioned in the Introduction—the REM (see Section 1.2.3) and the l_p-norms of positive i.i.d. samples (see Section 1.2.1). We assume throughout that the random variables (X_i) satisfy the Normalized Regularity Assumption of Section 5.1 and N is subject to the scaling condition (6.1).

9.1. *Limit theorems for the 'free energy'.* Our goal here is to study the asymptotic behavior of $\log S_N(t)$. More precisely, consider the quantity

$$F_N(t;\lambda) := \frac{\log S_N(t)}{H_0(t)},$$

which can be called the 'free energy', by analogy with the REM [see (1.3)].

9.1.1. *The limiting free energy.* Using the theorems about weak convergence of the sums $S_N(t)$, it is easy to show that $F_N(t;\lambda)$ is a 'self-averaging' quantity.

THEOREM 9.1. *For all* $\lambda > 0$, *the limit* $F(\lambda) := \lim_{t\to\infty} F_N(t;\lambda)$ *exists in the sense of convergence in probability and is given by*

$$F(\lambda) = \begin{cases} \lambda \pm 1 & (\lambda \ge \lambda_1), \\ \pm \varrho' \left(\dfrac{\varrho\lambda}{\varrho'} \right)^{\pm 1/\varrho} & (0 < \lambda \le \lambda_1). \end{cases} \tag{9.1}$$

PROOF. If $\lambda > \lambda_1$, Theorem 2.1 implies that $S_N(t) = N e^{\pm H(t)}(1 + o_p(1))$ as $t \to \infty$. Hence, using (6.1) and the asymptotic equivalence $H(t) \sim H_0(t)$ we get

$$\frac{\log S_N(t)}{H_0(t)} = \frac{\log N}{H_0(t)} \pm \frac{H(t)}{H_0(t)} + o_p(1) = \lambda \pm 1 + o_p(1).$$

For $\lambda = \lambda_1$, Theorem 2.5 yields $S_N(t) = NB_1(t)(1 + o_p(1))$ as $t \to \infty$, and so we only need to check that

$$\lim_{t\to\infty} \frac{\log B_1(t)}{H_0(t)} = \pm 1. \tag{9.2}$$

From (7.2) we note that $B_1(t) = \mathsf{E}\big[e^{tX}\mathbf{1}_{\{X\le\pm\eta_1\}}\big] \le \mathsf{E}\big[e^{tX}\big] = e^{\pm H(t)}$, whence

$$\limsup_{t\to\infty} \frac{\log B_1(t)}{H_0(t)} \le \lim_{t\to\infty} \frac{\pm H(t)}{H_0(t)} = \pm 1.$$

On the other hand, by (7.3) we have $B_1(t) = e^{\pm H_0(t)} b_1(t)$ with $b_1(t) \to +\infty$. Hence,

$$\liminf_{t\to\infty} \frac{\log B_1(t)}{H_0(t)} = \pm 1 + \liminf_{t\to\infty} \frac{\log b_1(t)}{H_0(t)} \ge \pm 1,$$

and (9.2) follows.

If $0 < \lambda < \lambda_1$, Theorem 6.1 implies that $S_N(t)\, e^{\mp\mu(t)H_0(t)}$ converges weakly to a proper random variable. Passing to logarithms and dividing by $H_0(t) \to \infty$ we get

$$\frac{\log S_N(t)}{H_0(t)} = \pm\mu(t) + o_p(1) \qquad (t \to \infty).$$

It remains to note that, according to (5.23) and (2.7),

$$\lim_{t\to\infty} \mu(t) = \varrho' \alpha^{\varrho'-1} = \varrho' \left(\frac{\varrho\lambda}{\varrho'}\right)^{\pm 1/\varrho}. \tag{9.3}$$

Thus, the theorem is proved. □

REMARK 9.2. It is easy to check that the function $F(\lambda)$ given by (9.1) is continuous and continuously differentiable everywhere including the critical point $\lambda = \lambda_1$, but its second derivative has a jump at this point. This corresponds to a phase transition of 'third order' [see Eisele (1983)].

9.1.2. *Fluctuations of* $\log S_N(t)$. First, let us prove a general lemma.

LEMMA 9.3. *Let* $\{S(t),\ t \ge 0\}$ *be a family of positive random variables. Assume that for some (non-negative) functions* $A(t)$ *and* $B(t)$,

$$S^*(t) := \frac{S(t) - A(t)}{B(t)} \xrightarrow{d} \mathcal{F} \qquad (t \to \infty), \tag{9.4}$$

and set $A^*(t) := A(t)/B(t)$.

(a) *If* $A^*(t) \to \infty$ *then*

$$A^*(t) \log \frac{S(t)}{A(t)} \xrightarrow{d} \mathcal{F} \qquad (t \to \infty).$$

(b) *If* $A(t) \equiv 0$ *then*

$$\log \frac{S(t)}{B(t)} \xrightarrow{d} \log \mathcal{F} \qquad (t \to \infty).$$

PROOF. (a) Note that $S(t)$ can be represented as

$$S(t) = A(t)\left(1 + \frac{S^*(t)}{A^*(t)}\right). \tag{9.5}$$

The condition $A^*(t) \to \infty$ implies that $S^*(t)/A^*(t) = o_p(1)$ and hence

$$\log \frac{S(t)}{A(t)} = \frac{S^*(t)}{A^*(t)}\,(1 + o_p(1)).$$

Therefore,

$$A^*(t) \log \frac{S(t)}{A(t)} = S^*(t)(1 + o_p(1)),$$

where the right-hand side has the same weak limit as $S^*(t)$, that is $\mathcal{F}$.

(b) If $A(t) \equiv 0$ then

$$\log \frac{S(t)}{B(t)} = \log S^*(t) \xrightarrow{d} \log \mathcal{F},$$

as claimed. □

Let us now prove a limit theorem for the distribution of $\log S_N(t)$. The corresponding result about the fluctuations of the free energy $F_N(t;\lambda)$ can then be easily deduced (which is left to the interested reader). Recall that $\mathcal{F}_\alpha$ is a stable distribution with characteristic function (2.21).

THEOREM 9.4. (a) *For* $\lambda \ge \lambda_2$,

$$\sqrt{\frac{N}{\tilde{B}(t)}}\, e^{\pm H(t)} \log \frac{S_N(t)}{N e^{\pm H(t)}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (t \to \infty),$$

where

$$\tilde{B}(t) := \begin{cases} e^{\pm H(2t)} & (\lambda > \lambda_2), \\ B_2(t) & (\lambda = \lambda_2), \end{cases} \tag{9.6}$$

and $B_2(t)$ *is defined in* (2.23).

(b) *For* $\lambda_1 \le \lambda < \lambda_2$,

$$\frac{A(t)}{B(t)} \log \frac{S_N(t)}{A(t)} \xrightarrow{d} \mathcal{F}_\alpha \qquad (t \to \infty),$$

where $B(t)$ *is given by* (2.17) *and*

$$A(t) = \begin{cases} N e^{\pm H(t)} & (\lambda_1 < \lambda < \lambda_2), \\ N B_1(t) & (\lambda = \lambda_1), \end{cases} \tag{9.7}$$

with $B_1(t)$ *defined in* (2.19).

(c) *For* $0 < \lambda < \lambda_1$,

$$\log \frac{S_N(t)}{B(t)} \xrightarrow{d} \log \mathcal{F}_\alpha \qquad (t \to \infty),$$

where $B(t)$ *is given by* (2.17).

PROOF. In view of Lemma 9.3, the assertions of the theorem will follow from the limit theorems for the sum $S_N(t)$ obtained in Sections 3.2, 6.1 and 7.2, according as the function $A^*(t) = A(t)/B(t)$ tends to infinity or vanishes.

(a) For $\lambda \ge \lambda_2$, the CLT is valid (see Theorems 2.2 and 2.6), so we have weak convergence of the form (9.4) with $A(t) = Ne^{\pm H(t)}$ and $B(t) = (N\tilde{B}(t))^{1/2}$, where $\tilde{B}(t)$ is defined in (9.6). Clearly, $B_2(t) \le \mathsf{E}[e^{2tX}] = e^{\pm H(2t)}$, and hence for all $\lambda \ge \lambda_2$

$$\begin{aligned}\liminf_{t\to\infty} \frac{\log A^*(t)}{H_0(t)} &\ge \liminf_{t\to\infty} \frac{\log N \pm 2H(t) \mp H(2t)}{2H_0(t)} \\ &= \frac{\lambda}{2} \pm 1 \mp 2^{\varrho'-1} \\ &\ge \pm(\varrho'-1)2^{\varrho'-1} \pm 1 \mp 2^{\varrho'-1} \\ &= \pm\varrho' 2^{\varrho'-1} \mp (2^{\varrho'}-1) > 0,\end{aligned} \tag{9.8}$$

where the last inequality follows from (3.9) (for $\varrho' > 1$) or (3.8) (for $0 < \varrho' < 1$) with $a = 2$, $b = 1$ and $p = \varrho'$. Now, (9.8) implies that $A^*(t) \to \infty$, and the application of Lemma 9.3(a) completes the proof of part (a).

(b) If $\lambda_1 < \lambda < \lambda_2$ then, according to Theorem 2.3, we have (9.4) with $A(t)$ and $B(t)$ given by (2.18) and (2.17), respectively. Using (5.13) and (5.28), we obtain

$$\begin{aligned}\lim_{t\to\infty} \frac{\log A^*(t)}{H_0(t)} &= \lambda \pm 1 \mp \varrho'\alpha^{\varrho'-1} \\ &= \pm(\varrho'-1)\,\alpha^{\varrho'} \pm 1 \mp \varrho'\alpha^{\varrho'-1} \\ &= \pm\varrho'\alpha^{\varrho'-1}(\alpha-1) \mp (\alpha^{\varrho'}-1) > 0,\end{aligned}$$

where the last inequality follows from (3.9) (for $\varrho' > 1$) or (3.8) (for $0 < \varrho' < 1$) with $a = \alpha$, $b = 1$ and $p = \varrho'$. Hence, $A^*(t) \to \infty$ and Lemma 9.3(a) applies.

For $\lambda = \lambda_1$, we have already checked that $A^*(t) \to \infty$ [see (7.15)], and so again it remains to use Lemma 9.3(a).

(c) In the case $0 < \lambda < \lambda_1$, the assertion of the theorem readily follows from Lemma 9.3(b) and Theorem 2.3. □

9.2. *Limit theorems for l_t-norms of exponential samples.* In this section, we obtain the limit distribution of the random variable $R_N(t) := S_N(t)^{1/t}$.

9.2.1. *Fluctuations of $R_N(t)$.* By analogy with Section 9.1.2, we first prove a general lemma tailored to this situation.

LEMMA 9.5. *Let the hypotheses of Lemma 9.3 hold, and set $R(t) := S(t)^{1/t}$.*

(a) *If $A^*(t) \to \infty$ as $t \to \infty$, then*

$$tA^*(t)\left(\frac{R(t)}{A(t)^{1/t}} - 1\right) \xrightarrow{d} \mathcal{F} \qquad (t \to \infty).$$

(b) *If $A(t) \equiv 0$ then*

$$t\left(\frac{R(t)}{B(t)^{1/t}} - 1\right) \xrightarrow{d} \log \mathcal{F} \qquad (t \to \infty).$$

PROOF. (a) From (9.5) we get

$$R(t) = A(t)^{1/t} \exp\left(\frac{1}{t}\log\left(1 + \frac{S^*(t)}{A^*(t)}\right)\right). \tag{9.9}$$

The condition $A^*(t) \to \infty$ implies that $S^*(t)/A^*(t) = o_p(1)$, hence

$$\begin{aligned}\exp\left(\frac{1}{t}\log\left(1 + \frac{S^*(t)}{A^*(t)}\right)\right) &= \exp\left(\frac{S^*(t)}{tA^*(t)}\,(1 + o_p(1))\right)\\ &= 1 + \frac{S^*(t)}{tA^*(t)}\,(1 + o_p(1)).\end{aligned}$$

Substituting this into (9.9) yields

$$tA^*(t)\left(\frac{R(t)}{A(t)^{1/t}} - 1\right) = S^*(t)(1 + o_p(1)) \xrightarrow{d} \mathcal{F} \qquad (t \to \infty).$$

(b) We have $S(t) = S^*(t)B(t)$, whence

$$\frac{R(t)}{B(t)^{1/t}} = \exp\left(\frac{\log S^*(t)}{t}\right) = 1 + \frac{\log S^*(t)}{t}\,(1 + o_p(1)).$$

Therefore,

$$t\left(\frac{R(t)}{B(t)^{1/t}} - 1\right) = \log S^*(t)\,(1 + o_p(1)),$$

which converges weakly to $\log \mathcal{F}$. □

Applying this lemma to the sums $S_N(t)$, similarly to Theorem 9.4 we obtain

THEOREM 9.6. (a) *For* $\lambda \geq \lambda_2$,

$$\sqrt{\frac{N}{\tilde{B}(t)}}\; t e^{\pm H(t)}\left(\frac{R_N(t)}{N^{1/t}e^{\pm H(t)/t}} - 1\right) \xrightarrow{d} \mathcal{N}(0,1) \qquad (t \to \infty),$$

with $\tilde{B}(t)$ *given by* (9.6).

(b) *For* $\lambda_1 \leq \lambda < \lambda_2$,

$$\frac{tA(t)}{B(t)}\left(\frac{R_N(t)}{A(t)^{1/t}} - 1\right) \xrightarrow{d} \mathcal{F}_\alpha \qquad (t \to \infty),$$

where $B(t)$ *and* $A(t)$ *are given by* (2.17) *and* (2.18), *respectively.*

(c) *For* $0 < \lambda < \lambda_1$,

$$t\left(\frac{R_N(t)}{B(t)^{1/t}} - 1\right) \xrightarrow{d} \log \mathcal{F}_\alpha \qquad (t \to \infty).$$

9.2.2. *Comparison with Schlather's* (2001) *results.* Note that $R_N(t) = S_N(t)^{1/t}$ can be viewed as an l-norm (of order t) of the vector $(e^{X_1}, \dots, e^{X_N})$ [cf. Schlather (2001)]. In order to clarify the link with the setting in Schlather (2001), let us show that under our conditions on X_i, the random variables $Y_i = e^{X_i}$ belong to the domain of attraction of the Gumbel (double exponential) distribution Λ.

PROPOSITION 9.7. *Denote* $Y_i := e^{X_i}$, $Y_{1,n} := \max\{Y_1, \dots, Y_n\}$, *and set*

$$a_n := e^{\pm h^{\leftarrow}(\log n)^{\pm}}, \qquad b_n := \pm \frac{a_n \log a_n}{\varrho \log n}. \tag{9.10}$$

Then

$$\lim_{n\to\infty} \mathsf{P}\left\{ \frac{Y_{1,n} - a_n}{b_n} \le x \right\} = \exp(-e^{-x}), \qquad x \in \mathbb{R}. \tag{9.11}$$

PROOF. It is not difficult to verify available sufficient conditions for convergence of the maximum's distribution to Λ [see Galambos (1978), Theorem 2.1.3, p. 52]. However, it appears even simpler to prove (9.11) directly. Let us denote $L_n(x) := \pm \log(a_n + x b_n)^{\pm}$, then we have

$$\mathsf{P}\left\{ \frac{Y_{1,n} - a_n}{b_n} \le x \right\} = \Big(\mathsf{P}\{X \le \log(a_n + x b_n)\} \Big)^n = \Big(1 - e^{-h(L_n(x))} \Big)^n. \tag{9.12}$$

Note that, according to (9.10),

$$L_n(0) = \pm \log(a_n)^{\pm} = h^{\leftarrow}(\log n) \to +\infty \qquad (n \to \infty) \tag{9.13}$$

and

$$\frac{b_n}{a_n} = \pm \frac{\log a_n}{\varrho \log n} = \frac{h^{\leftarrow}(\log n)^{\pm}}{\varrho \log n} \to 0 \qquad (n \to \infty),$$

since $h^{\leftarrow}(x) \in R_{1/\varrho}$ and $h^{\leftarrow}(x)^{\pm}/x \in R_{\pm 1/\varrho - 1}$ with $\pm 1/\varrho - 1 < 0$. Moreover,

$$\frac{b_n}{a_n \log a_n} = \pm \frac{1}{\varrho \log n} \to 0 \qquad (n \to \infty).$$

Therefore, as $n \to \infty$

$$\kappa_n(x) := \frac{L_n(x)}{L_n(0)} = \left(1 + \frac{\log(1 + x b_n / a_n)}{\log a_n} \right)^{\pm} \sim 1 \pm \frac{x b_n}{a_n \log a_n} \to 1. \tag{9.14}$$

Recalling (9.12), it is then easy to see that (9.11) is reduced to

$$h(L_n(x)) - \log n \to x \qquad (n \to \infty). \tag{9.15}$$

Since $h \in NR_\varrho$, a usual inverse h^{-1} exists (see Lemma 5.3) and so (9.13) implies that $h(L_n(0)) = \log n$. Hence, (9.15) takes the form

$$h(L_n(x)) - h(L_n(0)) \to x \qquad (n \to \infty). \tag{9.16}$$

Using Lemma 5.4, the left-hand part of (9.16) can be represented as

$$\int_{L_n(0)}^{L_n(x)} \frac{h(u)}{u} \left(\varrho + \varepsilon(u)\right) du = \int_1^{\kappa_n(x)} \frac{h(L_n(0) y)}{y} \left(\varrho + \varepsilon(L_n(0) y)\right) dy,$$

via the substitution $u = L_n(0)y$. The UCT (Lemma 3.1) implies that, uniformly in y, the function under the last integral is asymptotically equivalent, as $n \to \infty$, to $\varrho y^{\varrho-1} h(L_n(0)) = \varrho y^{\varrho-1} \log n$. Hence, using (9.14), we obtain

$$h(L_n(x)) - h(L_n(0)) \sim \bigl(\kappa_n(x)^\varrho - 1\bigr) \log n \sim \pm \frac{x b_n}{a_n \log a_n}\, \varrho \log n = x,$$

according to the choice of b_n [see (9.10)]. Thus, (9.16) is proved. □

Let us point out that in the case of attraction to Λ, Schlather [(2001), Theorem 2.4, p. 867] has obtained only a partial result for a particular case where the random variables $Y_i = e^{X_i}$ have the unit exponential distribution. More precisely, in our notation he has shown that under the scaling $N = e^{\lambda t}$, the limit distribution of $R_N(t)$ is Gaussian if $\lambda > 2$ and non-Gaussian if $2 \log 2 < \lambda < 2$. Note that our work does not cover this case, since the tail of the form $\mathsf{P}\{X > x\} = \exp(-e^x)$ would heuristically correspond to $\varrho = \infty$ in (1.2) (case B). However, our results corroborate a general conjecture by Schlather [(2001), p. 867], asserting (in our terms) that in the case of attraction to Λ, there exist functions $a(t)$, $b(t)$ such that, under an appropriate scaling $t = cp(N)$, $a(t)/b(t) = p(N)$, the distribution of $\bigl(R_N(t) - a(t)\bigr)/b(t)$ weakly converges to a distribution which, in turn, tends to Λ as $c \to +\infty$ and, properly recentered and renormalized, to $\mathcal{N}(0,1)$ as $c \to 0+$.

Indeed, comparing this conjecture with our Theorem 9.6(c), one can see that for $0 < \lambda < \lambda_1$ the role of c is played by $1/\alpha$, and in particular $c \to +\infty$ is equivalent to $\alpha \to 0+$. In Schlather's terms, Theorem 9.6(c) should be rewritten as

$$\frac{R_N(t) - B(t)^{1/t}}{B(t)^{1/t}/(\alpha t)} \xrightarrow{d} \alpha \log \zeta_\alpha \qquad (t \to \infty),$$

where ζ_α has the distribution $\mathcal{F}_\alpha$. It remains to note that, by Proposition 8.29, $\alpha \log \zeta_\alpha \xrightarrow{d} \Lambda$ as $\alpha \to 0+$, in accord with the above conjecture. On the other hand, normality in the limit $\alpha \to \infty$ is obvious from Theorem 9.6(a).

Another result being of relevance to our setting is Theorem 2.2 in Schlather [(2001), p. 864], where the random variables $Y_i = e^{X_i}$ are bounded above (with, say, $\operatorname{ess\,sup} Y = 1$) and belong to the domain of attraction of the Weibull distribution Ψ_α. According to the general extreme value theory [see Galambos (1975), Theorem 2.1.2, p. 51], this implies that $\bar{F}_Y(x) := \mathsf{P}\{Y > 1 - 1/x\} \in R_{-\alpha}$ or, equivalently, $\bar{F}_X(x) := \mathsf{P}\{X > -1/x\} \in R_{-\alpha}$. Hence, $h(x) = -\log \mathsf{P}\{X > -1/x\} \sim \alpha \log x$ $(x \to \infty)$, which may be heuristically interpreted as having a 'boundary' value $\varrho = 0$ in (1.2) (case A).

Similar considerations show that for $Y_i = e^{X_i}$ in the domain of attraction of the Fréchet distribution Φ_α [Schlather (2001), Theorem 2.3, p. 865], we have $h(x) = -\log \mathsf{P}\{X > x\} \sim \alpha x$ $(x \to \infty)$, which corresponds to another boundary value $\varrho = 1$ in (1.2) (case B). As already mentioned, in this situation the norm order does not depend on n, $p(n) \equiv c$, which in our terms implies that t does not grow to infinity. Therefore, the behavior of the sum $S_N(t)$ is determined by a few extreme terms. Let us point out that the parameter c introduces an appropriate growth scale at 'point' $\varrho = 1$, which resembles our parameter λ in that there are two 'phase transition' with respect to c [see further details in Schlather (2001), p. 865–866].

APPENDICES

A. Some model examples.

A.1. *Functions H and H_0.* Let us consider two examples to illustrate the difference between the cumulant generating function H [see (2.3)] and the rate function H_0 provided by the Kasahara–de Bruijn Tauberian theorem [see (3.7)].

EXAMPLE A.1 (Weibull/Fréchet's distribution). Suppose that

$$\mathsf{P}(X > x) = \exp\{-h(\pm x^{\pm})\}, \qquad x \gtrless 0, \tag{A.1}$$

with the log-tail distribution function $h(x) = x^{\varrho}/\varrho$ ($x \geq 0$), where $1 < \varrho < \infty$ in case B (Weibull's distribution) and $0 < \varrho < \infty$ in case A (Fréchet's distribution). The density is given by $f_X(x) = (\pm x)^{\pm\varrho-1} \exp(-(\pm x)^{\pm\varrho}/\varrho)$ ($x \gtrless 0$), and hence

$$\mathsf{E}[e^{tX}] = \pm \int_0^{\pm\infty} e^{tx} f_X(x)\, dx = t^{\varrho'} \int_0^{\infty} y^{\varrho-1} \exp\{t^{\varrho'}(\pm y^{\pm} - y^{\varrho}/\varrho)\}\, dy, \tag{A.2}$$

using the substitution $x = \pm t^{\varrho'-1} y^{\pm}$ and relation (2.4). Note that the function $g(y) := \pm y^{\pm} - y^{\varrho}/\varrho$ has a (unique) regular maximum at point $y = 1$, with $g(1) = \pm 1 - 1/\varrho = \pm 1/\varrho'$, $g'(1) = 0$, $g''(1) = -(\varrho \mp 1) < 0$. Therefore, the asymptotic Laplace method yields

$$H(t) = \pm \log \mathsf{E}\,[e^{tX}] = \frac{t^{\varrho'}}{\varrho'} \pm \frac{\varrho'}{2} \log t \pm \frac{1}{2} \log\left(\frac{2\pi}{\varrho \mp 1}\right) + o(1) \qquad (t \to \infty).$$

From equation (2.9), using (2.4), it is easy to find $H_0(t) = t^{\varrho'}/\varrho'$.

EXAMPLE A.2 (Normal distribution). Let X have the standard normal distribution $\mathcal{N}(0,1)$. Here $\varrho = \varrho' = 2$ and the function h is given by

$$h(x) = -\log\left(\frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2}\, dy\right) = \frac{x^2}{2} + \log x + \frac{1}{2}\log(2\pi) + o(1) \qquad (x \to \infty),$$

which can be shown to be normalized regularly varying. Note that $\mathsf{E}[e^{tX}] = e^{t^2/2}$, whence $H(t) = t^2/2$. Equation (2.9) can be solved asymptotically as $t \to \infty$. For $\lambda \neq \lambda_1, \lambda_2$ one only needs to find $H_0(t)$ to within $o(1)$,

$$H_0(t) = \frac{t^2}{2} - \log t - \frac{1}{2}\log(2\pi) + o(1) \qquad (t \to \infty).$$

The case of the critical points is more delicate but is perfectly tractable as well.

A.2. *Main results: the model case of the Weibull/Fréchet distribution.* As shown in Example A.1, for the Weibull/Fréchet distribution (A.1) we have $H_0(t) = t^{\varrho'}/\varrho'$, and it is easy to verify that $\mu(t)$, the root of equation (2.12), is given by $\mu(t) \equiv \varrho\lambda/\alpha$ [cf. (2.13)]. From (2.17) using (2.7) we get $B(t) = \exp(\pm\alpha^{\varrho'-1} t^{\varrho'})$. Furthermore, according to (2.15) we have

$$\eta_1(t) = (\alpha t)^{\varrho'-1}. \tag{A.3}$$

If $\alpha = \alpha_1 = 1$ then (A.3) yields $\eta_1(t) = t^{\varrho'-1}$, so for the function $B_1(t)$ defined in (2.19) we obtain similarly to (A.2)

$$\text{(A.4)} \quad B_1(t) = \int_{x \le \pm t^{\varrho'-1}} e^{tx} f_X(x)\, dx = t^{\varrho'} \int_0^1 y^{\varrho-1} \exp\{t^{\varrho'}(\pm y - y^{\pm\varrho}/\varrho)\}\, dy,$$

again employing the substitution $x = \pm t^{\varrho'-1} y^{\pm}$. As already mentioned in Example A.1, the function $g(y) = \pm y - y^{\pm\varrho}/\varrho$ has a regular maximum at point $y = 1$, which happens to be the right endpoint of the integration interval in (A.4). Hence, the Laplace method implies that, asymptotically, $B_1(t)$ makes up exactly one half of the full integral (A.2), that is,

$$\text{(A.5)} \qquad B_1(t) \sim \frac{1}{2}\, \mathsf{E}[e^{tX}] \qquad (t \to \infty).$$

Similarly, from (A.3) with $\alpha = \alpha_2 = 2$ we have $\eta_1(t) = (2t)^{\varrho'-1}$. Hence, the function $B_2(t)$ defined in (2.23) is represented as

$$B_2(t) = (2t)^{\varrho'} \int_0^1 y^{\varrho-1} \exp\{(2t)^{\varrho'}(\pm y - y^{\pm\varrho}/\varrho)\}\, dy$$

[via the substitution $x = \pm(2t)^{\varrho'-1} y^{\pm}$], and exactly the same argument as before shows that

$$\text{(A.6)} \qquad B_2(t) \sim \frac{1}{2}\, \mathsf{E}[e^{2tX}] \sim \frac{1}{2}\, \mathsf{Var}[e^{tX}] \qquad (t \to \infty).$$

As a result, using (A.5) we can combine the LLN of Theorems 2.1 and 2.5 as follows: *If the random variables X_i have the Weibull/Fréchet distribution* (A.1) *then, as $t \to \infty$,*

$$\text{(A.7)} \qquad \frac{S_N(t)}{\mathsf{E}[S_N(t)]} \xrightarrow{p} \begin{cases} 1 & (\lambda > \lambda_1), \\ \frac{1}{2} & (\lambda = \lambda_1), \\ 0 & (0 < \lambda < \lambda_1). \end{cases}$$

The last statement in (A.7) (for $0 < \lambda < \lambda_1$, i.e., $0 < \alpha < 1$) readily follows from Theorem 2.3 using that $\mathsf{E}[S_N(t)]/B(t) \to \infty$ as $t \to \infty$. Indeed, we have

$$\frac{\mathsf{E}[S_N(t)]}{B(t)} = \exp\{\log N \pm H(t) \mp \alpha^{\varrho'-1} t^{\varrho'}\},$$

and we note, using $H(t) \sim H_0(t) = t^{\varrho'}/\varrho'$ and the scaling assumption (6.1), that

$$\lim_{t\to\infty} \frac{\log N \pm H(t) \mp \alpha^{\varrho'-1} t^{\varrho'}}{H_0(t)} = \lambda \pm 1 \mp \varrho' \alpha^{\varrho'-1} > 0,$$

where the last bound follows from the inequality $x^{\varrho'} - 1 \gtrless \varrho'(x-1)$ $[x > 1,\ \varrho' \gtrless 1;$ see (3.8), (3.9)], by expressing λ via (5.13) and using the substitution $x = 1/\alpha$.

Analogously, Theorems 2.2 and 2.6, using (A.6), yield the following united assertion: *If the random variables X_i have the Weibull/Fréchet distribution* (A.1) *then, as $t \to \infty$,*

$$\frac{S_N(t) - \mathsf{E}[S_N(t)]}{(\mathsf{Var}[S_N(t)])^{1/2}} \xrightarrow{d} \begin{cases} \mathcal{N}(0,1) & (\lambda > \lambda_2), \\ \mathcal{N}(0,\frac{1}{2}) & (\lambda = \lambda_2). \end{cases}$$

Finally, Theorem 2.3 takes the following form: *For X_i with the Weibull/Fréchet distribution* (A.1), *we have*

$$\frac{S_N(t)-A(t)}{\exp\left(\pm\alpha^{\varrho'-1}t^{\varrho'}\right)}\xrightarrow{d}\mathcal{F}_\alpha\qquad(t\to\infty),$$

where the stable law $\mathcal{F}_\alpha$ is described in Theorem 2.3 *and $A(t)$ is defined in* (2.18),

$$A(t)=\begin{cases}Ne^{\pm H(t)} & (\lambda_1<\lambda<\lambda_2),\\ NB_1(t) & (\lambda=\lambda_1),\\ 0 & (0<\lambda<\lambda_1),\end{cases}$$

with $B_1(t)$ given by (A.4).

B. Proof of Lemma 5.16.

B.1. *Proof of part* (i). 1) We start by showing that

$$\text{(B.1)}\qquad \lim_{t\to\infty} e^{\mp pt\eta_\tau+h(\eta_\tau^\pm)}\,\mathsf{E}\bigl[e^{ptX}\mathbf{1}_{\{X\le\pm\theta^\pm\eta_\tau\}}\bigr]=0,$$

where $\theta\in(0,1)$. Since $\mathsf{E}\bigl[e^{ptX}\mathbf{1}_{\{X\le\pm\theta^\pm\eta_\tau\}}\bigr]\le e^{\pm pt\theta^\pm\eta_\tau}$, it suffices to check that

$$\text{(B.2)}\qquad e^{\mp pt\eta_\tau+h(\eta_\tau^\pm)\pm pt\theta^\pm\eta_\tau}=e^{\pm(\theta^\pm-1)pt\eta_\tau+h(\eta_\tau^\pm)}\to 0\qquad(t\to\infty).$$

Using the limit (5.31) of Lemma 5.13, we have

$$\text{(B.3)}\qquad \frac{\pm(\theta^\pm-1)pt\eta_\tau+h(\eta_\tau^\pm)}{t\eta_\tau}\to\pm(\theta^\pm-1)p+\frac{\alpha}{\varrho}\qquad(t\to\infty).$$

Since $t\eta_\tau(t)\sim\mu(t)H_0(t)\to+\infty$ as $t\to\infty$, the limit (B.2) will follow if there exists $\theta\in(0,1)$ such that the right-hand side of (B.3) is negative. The latter is guaranteed by the fact that $0<(1\mp\alpha/p\varrho)^\pm<1$, which can be easily verified using that $p>\alpha>0$ and $\varrho>1$ (case B) or $\varrho>0$ (case A).

2) Similarly to (5.35), integration by parts yields

$$\text{(B.4)}\qquad \begin{aligned}\mathsf{E}\bigl[e^{ptX}\mathbf{1}_{\{\pm\theta^\pm\eta_\tau<X\le\pm\eta_\tau\}}\bigr]&=-\,e^{\pm pt\eta_\tau-h(\eta_\tau^\pm)}+e^{\pm pt\theta^\pm\eta_\tau-h(\theta^\pm\eta_\tau^\pm)}\\&\quad+pt\int_{\pm\theta^\pm\eta_\tau}^{\pm\eta_\tau}e^{ptx-h(\pm x^\pm)}\,dx.\end{aligned}$$

Using that $h(\cdot)\ge 0$, we have

$$\text{(B.5)}\qquad e^{\pm pt\theta^\pm\eta_\tau-h(\theta^\pm\eta_\tau^\pm)}\le e^{\pm pt\theta^\pm\eta_\tau}=o(1)\,e^{\pm pt\eta_\tau-h(\eta_\tau^\pm)}\qquad(t\to\infty),$$

as shown above [see (B.2)].

3) Let us set $\tilde\eta_\tau(t):=\eta_\tau(t)\mp g(t)$, where $g(t):=t^{-1+\varrho'/2}$. Using that $\eta_\tau\in R_{\varrho'-1}$, we get $\tilde\eta_\tau/\eta_\tau\to 1$ $(t\to\infty)$ and so for all t large enough, $\pm\theta^\pm\eta_\tau\le\pm\tilde\eta_\tau\le\pm\eta_\tau$.

Let us now show that for any $x\in[\pm\theta^\pm\eta_\tau,\pm\tilde\eta_\tau]$ and all t large enough,

$$\text{(B.6)}\qquad ptx-h(\pm x^\pm)\le\pm pt\tilde\eta_\tau-h(\tilde\eta_\tau^\pm).$$

Setting $\kappa_\tau(t) := \pm x^\pm/\tilde\eta_\tau^\pm$, we have

$$1 \ge \kappa_\tau(t) \ge \theta\left(\frac{\eta_\tau}{\tilde\eta_\tau}\right)^{\pm} \to \theta \qquad (t\to\infty),$$

so by Lemma 5.5 we can write

$$h(\pm x^\pm) - h(\tilde\eta_\tau^\pm) = h(\tilde\eta_\tau^\pm)(\kappa_\tau^\varrho - 1)(1+o(1)) \qquad (t\to\infty), \tag{B.7}$$

uniformly in $x \in [\pm\theta^\pm\eta_\tau, \pm\tilde\eta_\tau]$. Furthermore, inequality (3.9) yields

$$\kappa_\tau^\varrho - 1 = \left(\frac{\pm x}{\tilde\eta_\tau}\right)^{\pm\varrho} - 1 \ge (\pm\varrho)\left(\frac{\pm x}{\tilde\eta_\tau} - 1\right) = \frac{\varrho}{\tilde\eta_\tau}\,(x \mp \tilde\eta_\tau). \tag{B.8}$$

Combining (B.7) and (B.8) and using Lemma 5.13, we obtain that for all t large enough, uniformly in x,

$$\begin{aligned} h(\pm x^\pm) - h(\tilde\eta_\tau^\pm) &\ge \frac{h(\tilde\eta_\tau^\pm)}{\tilde\eta_\tau}\,\varrho(x\mp\tilde\eta_\tau)(1+o(1)) \\ &= \alpha t\,(x\mp\tilde\eta_\tau)(1+o(1)) \\ &\ge pt\,(x\mp\tilde\eta_\tau), \end{aligned} \tag{B.9}$$

since $x \mp \tilde\eta_\tau \le 0$ and $\alpha < p$. Hence, inequality (B.6) follows.

4) We now want to prove that, as $t\to\infty$,

$$I(t) := pt\,e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)} \int_{\pm\theta^\pm\eta_\tau}^{\pm\tilde\eta_\tau} e^{ptx - h(\pm x^\pm)}\,dx \to 0. \tag{B.10}$$

Applying the estimate (B.6) we get

$$I(t) \le pt\,e^{-ptg(t) + h(\eta_\tau^\pm) - h(\tilde\eta_\tau^\pm)}\big[\pm(1-\theta^\pm)\eta_\tau - g(t)\big]. \tag{B.11}$$

Recalling that $g(t)\ge 0$ and $0<\theta<1$, it is easy to check that $\pm(1-\theta^\pm)\eta_\tau - g(t) \le \eta_\tau(1-\theta)/\theta$. Therefore, from (B.11) it follows

$$I(t) \le \frac{p(1-\theta)}{\theta}\,t\eta_\tau\,e^{-ptg(t) + h(\eta_\tau^\pm) - h(\tilde\eta_\tau^\pm)}. \tag{B.12}$$

It remains to observe that the pre-exponential factor in (B.12) grows only polynomially, since $t\eta_\tau(t) \sim \text{const}\cdot H_0(t) \in R_{\varrho'}$, while by Lemma 5.14, $-ptg(t) + h(\eta_\tau^\pm) - h(\tilde\eta_\tau^\pm) \sim -(p-\alpha)tg(t)$, where $p-\alpha>0$ and $tg(t) = t^{\varrho'/2}$. Hence the right-hand side of (B.12) is exponentially small as $t\to\infty$, and (B.10) follows.

5) Let us check that

$$J(t) := pt\,e^{\mp pt\eta_\tau + h(\eta_\tau^\pm)} \int_{\pm\tilde\eta_\tau}^{\pm\eta_\tau} e^{ptx - h(\pm x^\pm)}\,dx \to \frac{p}{p-\alpha} \qquad (t\to\infty). \tag{B.13}$$

By the substitution $\pm x = \eta_\tau(t) \mp yg(t) =: \tilde\eta_{\tau,y}(t)$, the left-hand side of (B.13) is rewritten in the form

$$J(t) = ptg(t)\int_0^1 e^{-ptg(t)y + h(\eta_\tau^\pm) - h(\tilde\eta_{\tau,y}^\pm)}\,dy. \tag{B.14}$$

Note that by Lemma 5.14, $h(\eta_\tau^{\pm})-h(\tilde{\eta}_{\tau,y}^{\pm})=\alpha t g(t) y(1+o(1))$ as $t\to\infty$, uniformly in $y\in[0,1]$. Therefore, given any ε such that $0<\varepsilon<p-\alpha$, for all large enough t and all $y\in[0,1]$ we have

$$(\alpha-\varepsilon)tg(t)y \le h(\eta_\tau^{\pm})-h(\tilde{\eta}_{\tau,y}^{\pm}) \le (\alpha+\varepsilon)tg(t)y.$$

Substituting these estimates into (B.14) and computing the integral, we obtain

$$J(t)\le ptg(t)\int_0^1 e^{-(p-\alpha-\varepsilon)tg(t)y}\,dy=\frac{p(1-e^{-(p-\alpha-\varepsilon)tg(t)})}{p-\alpha-\varepsilon}$$

and similarly

$$J(t)\ge ptg(t)\int_0^1 e^{-(p-\alpha+\varepsilon)tg(t)y}\,dy=\frac{p(1-e^{-(p-\alpha+\varepsilon)tg(t)})}{p-\alpha+\varepsilon}.$$

Using that $p-\alpha\pm\varepsilon>0$ and $tg(t)\to\infty$, in the limit as $t\to\infty$ we get

$$\frac{p}{p-\alpha+\varepsilon}\le\liminf_{t\to\infty}J(t)\le\limsup_{t\to\infty}J(t)\le\frac{p}{p-\alpha-\varepsilon},$$

and since $\varepsilon>0$ is arbitrary, it follows that $\lim_{t\to\infty}J(t)=p/(p-\alpha)$, as required.

6) Finally, gathering formulas (B.1), (B.4), (B.5), (B.10) and (B.13), we obtain

$$\lim_{t\to\infty}e^{\mp pt\eta_\tau+h(\eta_\tau^{\pm})}\,\mathsf{E}\big[e^{ptX}\mathbf{1}_{\{X\le\pm\eta_\tau\}}\big]=-1+\frac{p}{p-\alpha}=\frac{\alpha}{p-\alpha}.$$

B.2. *Proof of part* (*ii*). The proof follows the similar steps as above.

1′) Let us start by showing that if $p<\alpha$ then for any $\theta>1$

$$\lim_{t\to\infty}e^{\mp pt\eta_\tau+h(\eta_\tau^{\pm})}\,\mathsf{E}\big[e^{ptX}\mathbf{1}_{\{X>\pm\theta^{\pm}\eta_\tau\}}\big]=0. \tag{B.15}$$

Note that Lemma 4.2 (with $k=p$, $m=\alpha$) yields

$$\mathsf{E}\big[e^{ptX}\mathbf{1}_{\{X>\pm\theta^{\pm}\eta_\tau\}}\big]\le\mathsf{E}\big[e^{\alpha tX}\big]\cdot e^{\mp\theta^{\pm}(\alpha-p)t\eta_\tau}=e^{\pm H(\alpha t)\mp\theta^{\pm}(\alpha-p)t\eta_\tau}.$$

Hence, it suffices to check that

$$e^{\mp pt\eta_\tau+h(\eta_\tau^{\pm})}\cdot e^{\pm H(\alpha t)\mp\theta^{\pm}(\alpha-p)t\eta_\tau}=o(1)\qquad(t\to\infty). \tag{B.16}$$

To this end, recall that $H\sim H_0\in R_{\varrho'}$ and use (5.28), (5.31) and (3.4) to obtain

$$\begin{aligned}\lim_{t\to\infty}&\frac{\pm H(\alpha t)\mp(p+\theta^{\pm}(\alpha-p))\,t\eta_\tau+h(\eta_\tau^{\pm})}{H_0(t)}\\&=\pm\alpha^{\varrho'}\mp(p+\theta^{\pm}(\alpha-p))\,\varrho'\alpha^{\varrho'-1}+\frac{\alpha}{\varrho}\,\varrho'\alpha^{\varrho'-1}\\&=\pm(1-\theta^{\pm})(\alpha-p)\,\varrho'\alpha^{\varrho'-1}<0,\end{aligned}$$

since $\theta>1$ and $\alpha>p$. Hence, the limit (B.16) follows.

2′) Similarly to (B.4), integration by parts yields

$$\begin{aligned}\mathsf{E}\big[e^{ptX}\mathbf{1}_{\{\pm\eta_\tau<X\le\pm\theta^{\pm}\eta_\tau\}}\big]=&-e^{\pm pt\theta^{\pm}\eta_\tau-h(\theta\eta_\tau^{\pm})}+e^{\pm pt\eta_\tau-h(\eta_\tau^{\pm})}\\&+pt\int_{\pm\eta_\tau}^{\pm\theta^{\pm}\eta_\tau}e^{ptx-h(\pm x^{\pm})}\,dx.\end{aligned}$$

Let us check here that

$$e^{\pm(\theta^{\pm}-1)pt\eta_\tau-[h(\theta\eta_\tau^{\pm})-h(\eta_\tau^{\pm})]}=o(1)\qquad(t\to\infty). \tag{B.17}$$

Recalling that $h\in R_\varrho$ and using the limit (5.31), we obtain

$$h(\theta\eta_\tau^{\pm})-h(\eta_\tau^{\pm})\sim(\theta^\varrho-1)\,h(\eta_\tau^{\pm})\sim\frac{(\theta^\varrho-1)\,\alpha}{\varrho}\,t\eta_\tau.$$

Hence,

$$\frac{\pm(\theta^{\pm}-1)pt\eta_\tau-[h(\theta\eta_\tau^{\pm})-h(\eta_\tau^{\pm})]}{t\eta_\tau}\to\pm p(\theta^{\pm}-1)-\frac{(\theta^\varrho-1)\,\alpha}{\varrho}. \tag{B.18}$$

Inequality (3.9) gives $\theta^\varrho-1=(\theta^{\pm})^{\pm\varrho}-1\ge\pm\varrho(\theta^{\pm}-1)$, so the right-hand side of (B.18) is estimated from above by $\pm p(\theta^{\pm}-1)\mp\alpha(\theta^{\pm}-1)=\pm(\theta^{\pm}-1)(p-\alpha)<0$, because $\theta>1$ and $p<\alpha$. Hence, the limit (B.17) follows.

3′) Let us set $\tilde\eta_\tau(t):=\eta_\tau(t)\pm g(t)$, where the function g is as in step 3, and check that for $x\in[\pm\tilde\eta_\tau,\pm\theta^{\pm}\eta_\tau]$ and all sufficiently large t

$$ptx-h(\pm x^{\pm})\le\pm pt\tilde\eta_\tau-h(\tilde\eta_\tau^{\pm}).$$

To this end, similarly to (B.9) we show that

$$h(\pm x^{\pm})-h(\tilde\eta_\tau^{\pm})\ge\alpha t\,(x\mp\tilde\eta_\tau)(1+o(1))\ge pt\,(x\mp\tilde\eta_\tau),$$

using that $x\mp\tilde\eta_\tau\ge0$ and $\alpha>p$.

4′) The goal here is to prove that, as $t\to\infty$,

$$I(t):=pt\,e^{\mp pt\eta_\tau+h(\eta_\tau^{\pm})}\int_{\pm\tilde\eta_\tau}^{\pm\theta^{\pm}\eta_\tau}e^{ptx-h(\pm x^{\pm})}\,dx\to0.$$

Using the estimate from step 3′, one obtains

$$I(t)\le pt\,e^{ptg(t)+h(\eta_\tau^{\pm})-h(\tilde\eta_\tau^{\pm})}\left(\pm\theta^{\pm}\eta_\tau\mp\tilde\eta_\tau\right)\le p(\theta-1)t\eta_\tau\,e^{ptg(t)+h(\eta_\tau^{\pm})-h(\tilde\eta_\tau^{\pm})}.$$

One now applies the same argument as in step 4 above, using that

$$ptg(t)+h(\eta_\tau^{\pm})-h(\tilde\eta_\tau^{\pm})\sim-(\alpha-p)\,tg(t)\qquad(t\to\infty).$$

5′) Similarly as in step 5 above [cf. (B.13)], one proves that

$$\lim_{t\to\infty}pt\,e^{\mp pt\eta_\tau+h(\eta_\tau^{\pm})}\int_{\pm\eta_\tau}^{\pm\tilde\eta_\tau}e^{ptx-h(\pm x^{\pm})}\,dx=\frac{p}{\alpha-p}.$$

In so doing, the suitable substitution in the integral is of the form $\pm x=\eta_\tau(t)\pm yg(t)$, and an auxiliary ε involved in the estimation is taken to satisfy $0<\varepsilon<\alpha-p$.

6′) Finally, combining the limit formulas obtained in steps 1′–5′ we obtain

$$\lim_{t\to\infty}e^{\mp pt\eta_\tau+h(\eta_\tau^{\pm})}\,\mathsf{E}\big[e^{ptX}\mathbf{1}_{\{X>\pm\eta_\tau\}}\big]=1+\frac{p}{\alpha-p}=\frac{\alpha}{\alpha-p}.$$

C. Direct proofs of Corollary 8.25.

C.1. *Proof using representation* (8.29). From (8.29) it follows

$$\mathsf{E}[W_\alpha] = 1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \mathsf{E}\Big[e^{-\zeta_i/(i\alpha)}\Big] = 1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{i}{\alpha^{-1}+i}. \tag{C.1}$$

Note that the right-hand side of (C.1) coincides with the hypergeometric function [see Gradshteyn and Ryzhik (1994), 9.100, p. 1065]

$$F(a,b;c;z) := 1 + \sum_{k=1}^{\infty} z^k \prod_{i=0}^{k-1} \frac{(a+i)(b+i)}{(c+i)(1+i)}$$

taken at the values $a=1,\ b=1,\ c=1+\alpha^{-1},\ z=1$. Furthermore, it is known [see Gradshteyn and Ryzhik (1994), 9.122(1), p. 1068] that for $z=1$ one has

$$F(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad (c > a+b).$$

For the above specific values of the parameters this yields

$$F(1,1;1+\alpha^{-1};1) = \frac{\Gamma(1+\alpha^{-1})\,\Gamma(\alpha^{-1}-1)}{\Gamma(\alpha^{-1})^2}. \tag{C.2}$$

Using that $\Gamma(1+x) = x\Gamma(x)$, we obtain

$$\Gamma(1+\alpha^{-1})\,\Gamma(\alpha^{-1}-1) = \alpha^{-1}\,\Gamma(\alpha^{-1}) \cdot \frac{\Gamma(\alpha^{-1})}{\alpha^{-1}-1} = \frac{\Gamma(\alpha^{-1})^2}{1-\alpha},$$

so substituting this into (C.2) we get (8.37).

C.2. *Another proof.* The proof above may not seem quite satisfactory, as it relies substantially on the 'external' analytic aid from tables of formulas. Here we give a simpler, self-contained proof based on another representation of W_α given by equation (8.33). Namely, using Lemma 8.12 and relation (8.15) we deduce from (8.33) that $W_\alpha \stackrel{d}{=} 1+\sum_{k=1}^{\infty} U_{1,k}^{1/\alpha}$, where $U_{1,k}$ is the minimum of independent random variables $U_1,\dots,U_k$ with uniform distribution on $[0,1]$. Therefore,

$$\mathsf{E}[W_\alpha] = 1 + \sum_{k=1}^{\infty} \mathsf{E}\big[U_{1,k}^{1/\alpha}\big]. \tag{C.3}$$

Note that $\mathsf{P}\{U_{1,k} > x\} = (1-x)^k$ $(0 \le x \le 1)$, and hence

$$\mathsf{E}\big[U_{1,k}^{1/\alpha}\big] = \int_0^1 x^{1/\alpha}\, k(1-x)^{k-1}\, dx.$$

Substituting this expression into (C.3), we obtain

$$\mathsf{E}[W_\alpha] = 1 + \int_0^1 x^{1/\alpha} \sum_{k=1}^{\infty} k(1-x)^{k-1}\, dx = 1 + \int_0^1 x^{1/\alpha-2}\, dx = \frac{1}{1-\alpha},$$

and (8.37) follows.

C.3. *Yet another proof.* Finally, we have been able to find a most elementary proof of the identity (8.37) proceeding directly from representation (C.1). Namely, for $0 < \alpha < 1$ let us set

$$a_1 := \frac{1}{\alpha^{-1} - 1}, \qquad a_i := \frac{i}{\alpha^{-1} + i - 1} \qquad (i \geq 2),$$

$$A_k := \prod_{i=1}^{k} a_i \qquad (k \geq 1).$$

It is then easy to check that for all $k \geq 1$

$$\prod_{i=1}^{k} \frac{i}{\alpha^{-1} + i} = \prod_{i=1}^{k} a_i \cdot (1 - a_{k+1}) = A_k - A_{k+1}.$$

Since $A_k \to 0$ as $k \to \infty$, it follows that the series on the right-hand side of (C.1) is reduced to

$$1 + \sum_{k=1}^{\infty} (A_k - A_{k+1}) = 1 + a_1 = \frac{1}{1 - \alpha},$$

and formula (8.37) is proved.

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