Hydrodynamic Interaction of Rough Spheres

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Abstract
The approach of two spheres along their line of centres is analyzed assuming that each sphere is covered by a porous layer. The slip of the fluid at the surface of the porous layers then permits the spheres to touch without a singularity in the associated force. In a corresponding analysis for flat surfaces, an analytical formula for the force is obtained. This force is also finite for vanishing gap width. It is shown how the properties of the porous layers can be related to the statistics of spatial distributions of surface asperities. Finally, it is emphasised that a finite force is associated with a finite asperity height.

1. Introduction

When two perfectly smooth spheres are submerged in an incompressible, viscous fluid, the force necessary to slowly push them together along their line of centres grows inversely proportional to the gap distance between them. However, in experiments on spherical particles it has been shown that they can touch (Gelbard *et al* [1], Zeng *et al* [2]), implying that the gap width can be made zero at finite normal relative velocity. While a number of elements in the analysis of two perfectly smooth particles in an infinitely extended Newtonian fluid with no-slip boundary conditions may be unrealistic, it has long been recognised that the assumption of perfect smoothness is a strong approximation and that the singularity in the force is unphysical.

The interaction of rough particles in a Newtonian fluid is important for a range of problems: sedimentation (Davis [3], Zeng *et al* [2]), viscosity estimates of particle aggregates (Wilson and Davis [4]), diffusion (Leighton and Acrivos [5]), Acrivos *et al* [6]) and particle pressure estimates that are useful for the description of migration phenomena (McTigue and Jenkins [7], Nott and Brady [8]).

Da Cunha and Hinch [9] investigated two rough interacting particles in a shear field. They employed the smooth hydrodynamic interaction (Kim and Karrila [10]) until the particles reach a separation at which two asperities on their surfaces touch. The basic assumption they make is that the roughness does not affect the hydraulics. This assumption is also stated by Smart and Leighton [11] in their analysis of two approaching smooth spheres, scantily covered by small hemispherical asperities. Patir and Cheng [12] did a numerical simulation of two rough particles that approach one another. Here the hydraulic phenomena *are* affected by the roughness, however, no analytical insight is obtained in this way of approaching the problem. Patir and Cheng [12] investigated surfaces that are thickly covered with roughness features with a height that is generated with a Gaussian probability density.

In this paper, a model is put forward that yields a finite interactive force at zero gap width. In this model the assumption is made that there is a dense covering of asperities, equivalent to a thin layer of porous material at the surface of each sphere. The presence of the layers permits the fluid external to them, the clear fluid, to slip at their outer surfaces. The slip velocity is related to the gradient of the radial velocity of the clear fluid at the surface of the porous layer in a way determined by Saffman [13]. This permits the determination of the radial pressure gradient in the clear fluid at the surface of the porous layer and, through it, the determination of the relation between the relative velocity along the line of centres and the associated force. The presence of slip at the outer surfaces of the porous layers covering the spheres results in a relation between relative velocity and the force that is not singular.

The procedure to link the slip velocity to the gradient of the radial velocity was employed by Vasin *et al* [14] to describe the motion of single particles with a porous mantle in an infinite fluid. When the limit of a vanishingly thin thickness of the porous layer is taken, the procedure approaches Navier-type slipping boundary conditions with a slip-length parameter on the solid surface of the particle. It is shown below that the latter type of boundary conditions still result in a singular force as the gap width approaches zero, though the singularity is logarithmic for a spherical particle, rather than inversely proportional in the gap width. So, in order to obtain a finite force at zero gap width, the porous layer must have a finite thickness.

A check on the model is obtained by a comparison with Patir and Chang's [12] numerical method. When applied to the special case of rough plane surfaces approaching along their common normal, the model involving the porous layers produces a similar relation between the radial flux and the radial pressure gradient as that obtained in numerical solutions of the lubrication equations within and outside of a dense field of asperities by Patir and Chang [12], provided that appropriate values are taken for the thickness and the permeability of the porous layers.

2. Basic definition of the problem - use of the lubrication limit

Two solid spherical particles are assumed to move relative to one another along their line of centres with relative velocity $2u$. A porous layer with thickness δ, porosity ε, and permeability k covers each sphere. The distance between the surfaces of the permeable layers is $2h$. Figure 1 gives a sketch of the geometry as the surfaces actually are; the dimensions relevant to the model are on the right hand side of the figure. Figure 2 illustrates the model; the permeable layer is enclosed between the dashed lines.

The fluid has viscosity μ. The co-ordinate frame is chosen such that z is the co-ordinate in the direction of the line of centres and r is the radial direction. The lubrication limit is employed for the fluid between the porous surfaces. This requires that (*e.g*. Landau and Lifschitz [15]),

$$v_z << v_r \,; \; \frac{\partial v_r}{\partial r} << \frac{\partial v_r}{\partial z} \; ; \text{and } \frac{\partial p}{\partial z} \approx 0 . \tag{1}$$

In order to take advantage of the symmetry of the problem, the origin of the co-ordinate frame is chosen at the centre of the system. Then $v_r(r,z) = v_r(r,-z)$ and $v_z(r,z) = -v_z(r,-z)$.

The equations of motion in the region of clear fluid are

$$\frac{\partial p}{\partial r} = \mu \frac{\partial^2 v_r}{\partial z^2}\,; \; \frac{\partial p}{\partial z} = \mu \left(\frac{\partial^2 v_z}{\partial z^2} + \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} \right); \tag{2}$$

while those in the permeable regions are

$$\frac{v_z}{\varepsilon} - u = -\frac{k}{\mu\varepsilon}\frac{\partial p}{\partial z}\,; \; v_r = -\frac{k}{\mu}\frac{\partial p}{\partial r}\,. \tag{3}$$

In both types of regions the equation of continuity is

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0\,. \tag{4}$$

In the permeable regions, ***v*** is the superficial velocity.

The boundaries in the problem are described using a function $z = f(r)$, which describes the shape of the surface of the upper particle. At the boundary of the permeable medium and clear fluid, $z = \pm(f(r) + h)$, (a) the fluid pressure p is continuous, and (b) the normal fluid velocity v_z is continuous. At the particle surface, $z = \pm(f(r) + h + \delta)$, (c) the normal fluid velocity relative to the solid matrix vanishes: $v_z/\varepsilon \mp u = 0$.

3. Analysis of the problem

The flow in the clear fluid may be solved first. From the first of Equation (2) it follows that

$$v_r(r,z) = \frac{1}{2\mu}\frac{\partial p}{\partial r}\left[z^2 - (f(r) + h)^2\right] + v_s(r)\,, \tag{5}$$

where $v_s(r)$ is a slip velocity along the interface between the permeable region and the clear fluid. Integrating the continuity equation (4) over *z* between the boundaries of the clear fluid gives

$$2v_z(r, f(r) + h) = -\frac{1}{r}\frac{\partial}{\partial r}\int_{-f(r)-h}^{f(r)+h}\left(\frac{r}{2\mu}\frac{\partial p}{\partial r}\left[z^2 - (f(r)+h)^2\right] + r v_s(r)\right)dz$$

$$= \frac{2}{r}\frac{\partial}{\partial r}\left(\frac{r}{3\mu}\frac{\partial p}{\partial r}(f(r)+h)^3 - r v_s(r)(f(r)+h)\right). \tag{6}$$

The second part of the problem is the solution of the permeable medium flow, which yields a value for $v_z(r, f(r)+h)$. The continuity equation (4) is integrated over the thickness of the permeable layer

$$\int_{h+f(r)}^{h+\delta+f(r)} \frac{\partial v_z}{\partial z} dz = -\frac{1}{r} \int_{h+f(r)}^{h+\delta+f(r)} \frac{\partial (r v_r)}{\partial r} dz . \tag{7}$$

The integral on the left-hand-side is evaluated using the boundary condition (c). For the integral on the right-hand-side the second of the constitutive relation in the permeable region, equation (3), is used. It is recalled that the lubrication limit states that the pressure gradient in the *z*-direction approximately vanishes (equation (1)). It follows that

$$\varepsilon u - v_z(r, f(r)+h) = \frac{\delta k}{\mu r} \frac{\partial}{\partial r}\left(r \frac{\partial p}{\partial r} \right). \tag{8}$$

Combining this with equation (6) to eliminate $v_z(r, f(r)+h)$ yields an equation that may be integrated to determine the pressure:

$$\varepsilon u - \frac{\delta k}{\mu r} \frac{\partial}{\partial r}\left(r \frac{\partial p}{\partial r} \right) = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r}{3\mu}\frac{\partial p}{\partial r}\left(f(r)+h \right)^3 - r v_s(r)\left(f(r)+h \right) \right]. \tag{9}$$

Equation (9) differs from that of the traditional lubrication limit in two ways: (a) in the traditional analysis there is no slip velocity, and (b) the left hand side is simply *u*, rather than the more complex expression given here. The integration of equation (9) results in

$$\tfrac{1}{2}\varepsilon u r^2 + r v_s(r)\left(f(r)+h \right) = \frac{r}{3\mu}\frac{\partial p}{\partial r}\left[\left(f(r)+h \right)^3 + 3\delta k \right] + cnst . \tag{10}$$

The constant is evidently zero as all quantities are finite at $r = 0$, thus

$$\frac{\partial p}{\partial r} = \frac{3\mu\left[\tfrac{1}{2}\varepsilon u r + v_s(r)\left(f(r)+h \right) \right]}{\left(f(r)+h \right)^3 + 3\delta k} . \tag{11}$$

To carry the solution further, another connection between the slip velocity and the pressure gradient needs to be found. From the Brinkman equation Saffman [13] shows that the slip velocity satisfies

$$\left.\frac{\partial v_r}{\partial z}\right|_{z=f(r)+h} = -\frac{\alpha}{k^{1/2}} v_s(r), \tag{12}$$

where α is a coefficient of the order of magnitude of 0.1. From

$$v_r(r,z) = \frac{1}{2\mu}\frac{\partial p}{\partial r}\left[z^2 - \left(f(r)+h\right)^2\right] + v_s(r) \ , \tag{13}$$

it follows that

$$\frac{1}{\mu}\frac{\partial p}{\partial r}\left(f(r)+h\right) = -\frac{\alpha}{k^{1/2}} v_s(r) . \tag{14}$$

Then, from equation (11), there results

$$\frac{\partial p}{\partial r} = \frac{3\mu\varepsilon r u}{2\left[\left(f(r)+h\right)^3 + 3\frac{k^{1/2}}{\alpha}\left(f(r)+h\right)^2 + 3\delta k\right]} . \tag{15}$$

4. Flat surfaces

For flat surfaces the function $f(r) = 0$ and equation (15) simplifies to

$$\frac{\partial p}{\partial r} = \frac{3\mu\varepsilon r u}{2\left(h^3 + 3\frac{k^{1/2}}{\alpha}h^2 + 3\delta k\right)} . \tag{16}$$

Patir and Cheng [12] obtain numerical solutions to the problem of squeeze film between a pair of flat plates that are densely covered with asperities with a height that is generated with a Gaussian probability density. They phrase the problem in terms of the distance between the mean location of the surfaces and focus attention on the behaviour of the radial linear flux q_r defined by

$$q_r = \int_{-h-\delta}^{h+\delta} v_r dz . \tag{17}$$

In their analysis, it is given in terms of a dimensionless flow factor ϕ_r (see also Wells and Tsuji [16])

$$q_r = -\phi_r \frac{\bar{h}^3}{12\mu}\frac{\partial p}{\partial r}. \tag{18}$$

From their numerical solutions, Patir and Cheng [12] obtain the values of ϕ_r at various values of $\bar{h}$.

The question then is whether the simple theory presented here, one that makes use of the slip velocity and employs the expression for the slip velocity that results from the Brinkman equation, is able to reproduce their numerical data. To answer this question, q_r is first evaluated for the flat plates using the simple theory:

$$q_r = -\frac{1}{\mu}\frac{\partial p}{\partial r}\left(\tfrac{2}{3}h^3 + \frac{2h^2k^{1/2}}{\alpha} + 2\delta k\right). \tag{19}$$

Secondly, a distance parameter Δ is introduced such that $\bar{h} \equiv 2h + 2\Delta$. Δ is approximately the mean asperity height reckoned from the mean location of the surface. From equations (18) and (19) and the definition of Δ it follows that the form for ϕ_r is

$$\phi_r = 1 + \frac{6\Delta}{\bar{h}}\left(\frac{k^{1/2}}{\alpha\Delta} - 1\right) - \frac{12\Delta^2}{\bar{h}^2}\left(\frac{2k^{1/2}}{\alpha\Delta} - 1\right) - \frac{8\Delta^3}{\bar{h}^3}\left(1 - \frac{3k^{1/2}}{\alpha\Delta} - \frac{3k\delta}{\Delta^3}\right) \tag{20}$$

Patir and Cheng [12] take the two surfaces to have the same statistics and express the parameter $\bar{h}$ in terms of the combined standard deviation σ of the Gaussian distribution of the asperities on the two surfaces. The parameter Δ may be expressed as $\Delta = \sigma/\left(n\sqrt{2}\right)$, where n is an adjustable parameter of order unity that is required to make an association between the equivalent porous continuum and the actual asperities that are present in the paper by Patir and Chang [12]. The thickness δ of the porous layer is approximately equal to twice the asperity height: $\delta = 2\Delta$. The parameters relevant to the comparison with the numerical model are illustrated in Figure 1 on the left-hand-side of the figure.

Now, on introducing the dimensionless quantities $y \equiv k^{1/2}/(\alpha\Delta)$ and $H \equiv \bar{h}/\sigma$, equation (20), which has been derived from hydraulic considerations, may be mapped onto the spatial geometry of random asperities of Patir and Cheng [12] as

$$\phi_r = 1 + \frac{3\sqrt{2}}{nH}(y-1) + \frac{6}{n^2H^2}(1-2y) + \frac{2\sqrt{2}}{n^3H^3}\left(3y + 3\alpha^2y^2n^2 - 1\right). \tag{21}$$

Patir and Cheng [12], themselves fit the value of ϕ_r to the numerical data as

$$\phi_r = 1 - 0.90e^{-0.56H} \tag{22}$$

It is easy to see that in order to obtain similar curves, the condition $1/2 < y < 1$ must hold. Experimentation with the values of *n* and *y* indicates that the two functions can be made very similar, see Figure 3. A direct estimate of the value of *y*, which expresses the value of the permeability of the rough coating in terms of the characteristic length scale of the asperities, is possible using Kozeny-Carman's permeability estimate for a porous medium with mean diameter *D*

$$k = \frac{D^2 \varepsilon^3}{150(1-\varepsilon)^3}. \tag{23}$$

The values are derived from the Patir and Cheng's [12] method of generating the rough surface. They employ a Gaussian distribution, implying that there is as much solid material above the mean as there is below the mean; hence the porosity is estimated as $\varepsilon = 0.5$ and for the length scale *D* the standard deviation of the distribution is used. For such a distribution the parameter *n* is estimated to be in the order of $n = 2$. Choosing $\alpha = 0.1$, this leaves *y* in the order of unity. These estimated values are in the same order of magnitude as the ones for which a reasonable fit is obtained, as illustrated in Figure 3.

From equation (16), the force between the plates may be calculated. For circular plates of radius *R* the force is

$$F = -\frac{3\pi\mu\varepsilon u R^4}{8\left(h^3 + 3\frac{k^{1/2}}{\alpha}h^2 + 3\delta k\right)}. \tag{24}$$

This reduces to the classical result (Landau and Lifschitz [15]) when $h >> k^{1/2}$, $h >> \delta$.

In order to expose the influence of the various length scales in the problem, equation (24) is recast in the non-dimensional form

$$F^* \equiv \frac{8F\delta^3}{3\pi\mu\varepsilon u R^4} = -\frac{\left(\delta^*\right)^2}{\zeta^3\left(\delta^*\right)^2 + 3\zeta^2\delta^*\alpha^{-1} + 3}, \tag{25}$$

where the parameter ζ measures the semi-gap width as a fraction of the thickness of the permeable layer: $h = \zeta\delta$ and the parameter δ^* expresses the permeability in terms of the thickness of the permeable layer: $\delta^* = \delta/\sqrt{k}$. A plot of F^* as a function of ζ for various values of δ^* is presented in Figure 4. Patir and Cheng's case corresponds to a value of δ^* in the order 25. In order to demonstrate Smart and Leighton's [11] case for two flat plates the function $(\zeta+1)^{-3}$ has also been plotted.

5. Spherical surfaces

The function $f(r)$ for a curved surface of a sphere with radius R may be expanded around $r=0$ as $f(r)=\frac{1}{2}r^2/R+O(r^4/R^3)$. The force the particles exert on one another is

$$2\pi\int_0^\infty r^2\frac{\partial p}{\partial r}dr=\pi\int_0^\infty\frac{3\mu\varepsilon r^3u}{\left[\left(h+f(r)\right)^3+3\dfrac{k^{1/2}}{\alpha}\left(h+f(r)\right)^2+3\delta k\right]}dr\,. \tag{26}$$

Introducing $\Delta=\sigma/(n\sqrt{2})$, $\delta\approx 2\Delta$, and $y\equiv k^{1/2}/(\alpha\Delta)$, making the change the integration variable $r^2=2(z-h)R$, and phrasing the result in terms of $\bar{z}=z/h$ yields an alternative expression for the force:

$$2\pi\int_0^\infty r^2\frac{\partial p}{\partial r}dr=\frac{6\pi\mu\varepsilon uR^2}{h}\int_1^\infty\frac{\bar{z}-1}{\left(\bar{z}^3+3\dfrac{y\sigma}{nh\sqrt{2}}\bar{z}^2+\frac{3}{2\sqrt{2}}\dfrac{\alpha^2y^2\sigma^3}{nh^3}\right)}d\bar{z}\,. \tag{27}$$

The function

$$G\left(\frac{h}{\sigma}\right)\equiv\int_1^\infty\frac{\bar{z}-1}{\left(\bar{z}^3+3\dfrac{y\sigma}{nh\sqrt{2}}\bar{z}^2+\dfrac{3}{2\sqrt{2}}\dfrac{\alpha^2y^2\sigma^3}{nh^3}\right)}d\bar{z} \tag{28}$$

is plotted in Figure 5 for various values of n and y for $\alpha=0.1$. As $h/\sigma\to\infty$, G approaches 0.5. This is also the classical value for smooth surfaces. As $h/\sigma\to 0$ G behaves as the product of the derivative G' and h/σ. As a result, the force remains finite for $h/\sigma\to 0$.

6. Pure slipping conditions

Navier-type slipping boundary conditions (Lamb [17]) have a similar form as Equation (12). These conditions hold on a smooth solid surface and employ a slip length L_s, such that at the boundary

$$L_s\frac{\partial v_r}{\partial z}=-v_s\,. \tag{29}$$

Such boundary conditions are believed to hold for continuum descriptions in gap widths that approach the fluid molecular diameter. A recent paper by Thompson and Troian [18] gives the result of molecular simulations of liquids in the vicinity of a solid wall, purporting to demonstrate the validity of this concept. Experimental results of nanometer scale channel flow appear to support this interpretation: Cheng and Giordano [19]. In passing it is noted that an alternative continuum formulation of these simulations entails a non-local fluid rheology, leaving the no-slip boundary conditions concept intact; see Britsanis *et al* [20] and Travis and Gubbins [21] who simulated flow in narrow slits for a variety of choices of wall-wall, wall-fluid and fluid-fluid Leonard Jones potentials.

A point that needs to be made is mere slipping by itself does not lead to a finite force for zero gap width. The force between smooth surfaces with slipping boundary conditions is found by simply taking the limit $\delta \to 0$, $\varepsilon \to 1$ and $k^{1/2}/\alpha \to L_s$ in either equations (24) or (31). For the flat plate one obtains

$$F = -\frac{\pi\mu u R^4}{8L_s h^2} \quad \left(h \to 0\right). \tag{30}$$

For the two spheres the force becomes

$$\pi\int_0^\infty \frac{3\mu r^3 u}{\left[\left(h+\tfrac{1}{2}r^2/R\right)^3 + 3L_s\left(h+\tfrac{1}{2}r^2/R\right)^2\right]}dr = \frac{2\pi\mu u R^2\left(h+3L_s\right)}{3L_s^2}\ln\left(\frac{h+3L_s}{h}\right) - \frac{2\pi\mu u R^2}{L_s}, \tag{31}$$

which for $h \to 0$ behaves as

$$\frac{2\pi\mu u R^2}{L_s}\ln\left(\frac{3L_s}{h}\right). \tag{32}$$

Both expressions (30) and (32) are still singular, though less intensely so than the no-slip smooth boundary cases, which depend on the gap width as h^{-3} (flat plate) and h^{-1} (spheres).

For a finite force to occur, therefore, the non-zero thickness of the permeable layer must be accounted for. However, slip at the surface of the porous layer is not necessary. Sherwood [12], for example, solves the flow field for the case of a spherical particle approaching a permeable half-space. He determines the flow fields both within and outside the porous medium, adopting a no-slip boundary condition at the surface and finds the force to be finite.

7. Conclusions

The introduction of porous layers on the surface of two spheres permits their approach along their line of centres until contact without a singularity in the associated force.

The hydraulic model can be placed in the context of a statistical distribution of surface asperities in a relatively straightforward way. The hydraulic model can then be used to study, for example, the trajectories, until solid contact, of two spheres that are driven together by viscous forces in a sheared fluid.

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Figure captions

Figure 1. Illustration of two rough surfaces with the dimensions indicating the permeability model on the right hand side of the figure and those employed for the comparison with Patir and Cheng [12] on the left hand side.

Figure 2. Schematic illustration of the permeability model.

Figure 3. Comparison of Patir and Cheng's [12] numerical determinations of ϕ_r compared to those obtained analytically for a few choices of the parameters n and y.

Figure 4. The non-dimensionalised force between two plat plates as a function of the non-dimensional semi-gap.

Figure 5. The function G for various values of the parameters.

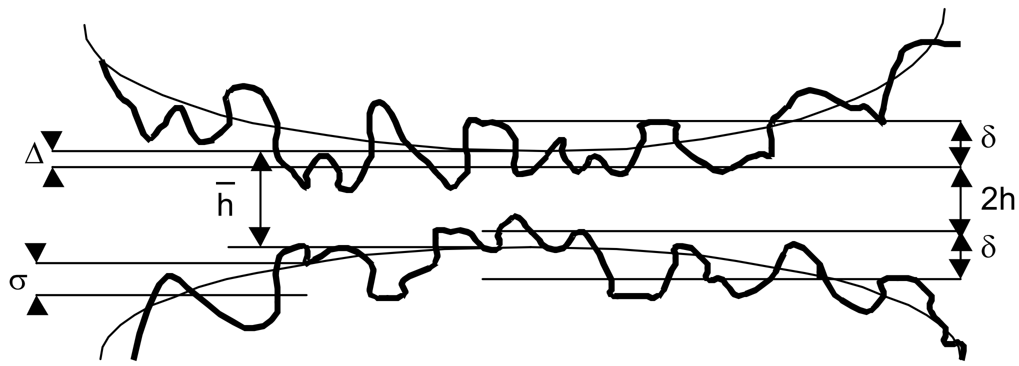


Figure 1

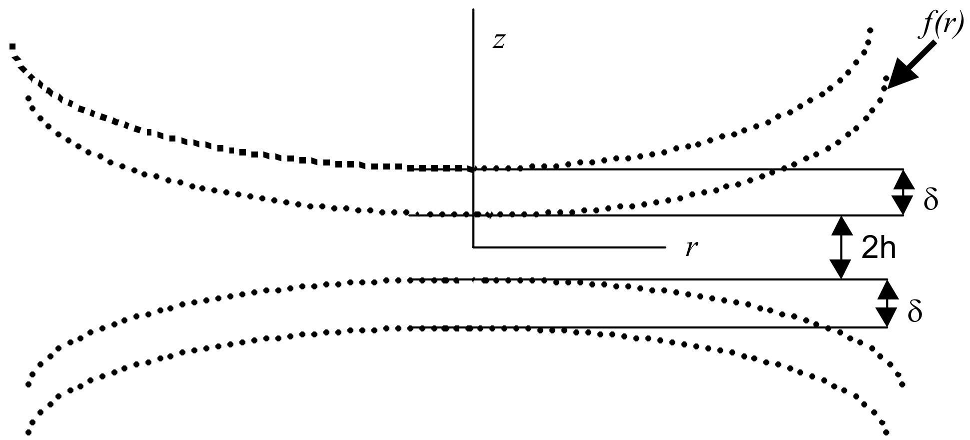
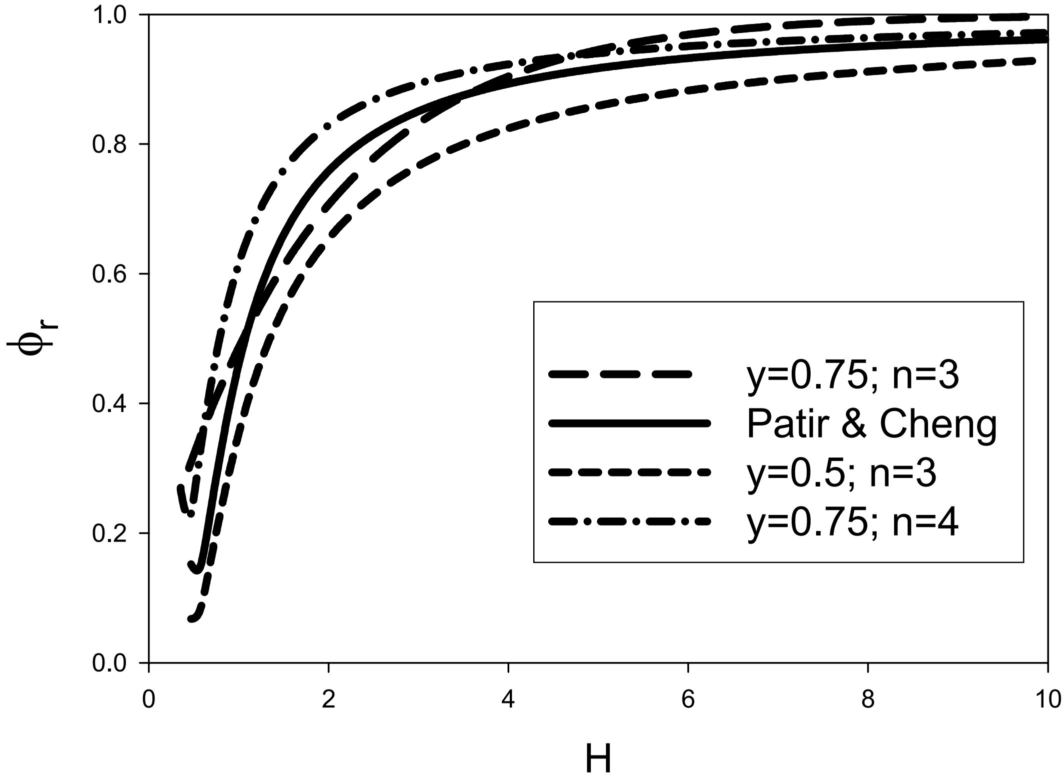


Figure 2



1.0
0.8
0.6
0.4
0.2
0.0
ϕ_r
0
2
4
6
8
10
H
y=0.75; n=3
Patir & Cheng
y=0.5; n=3
y=0.75; n=4

Figure 3

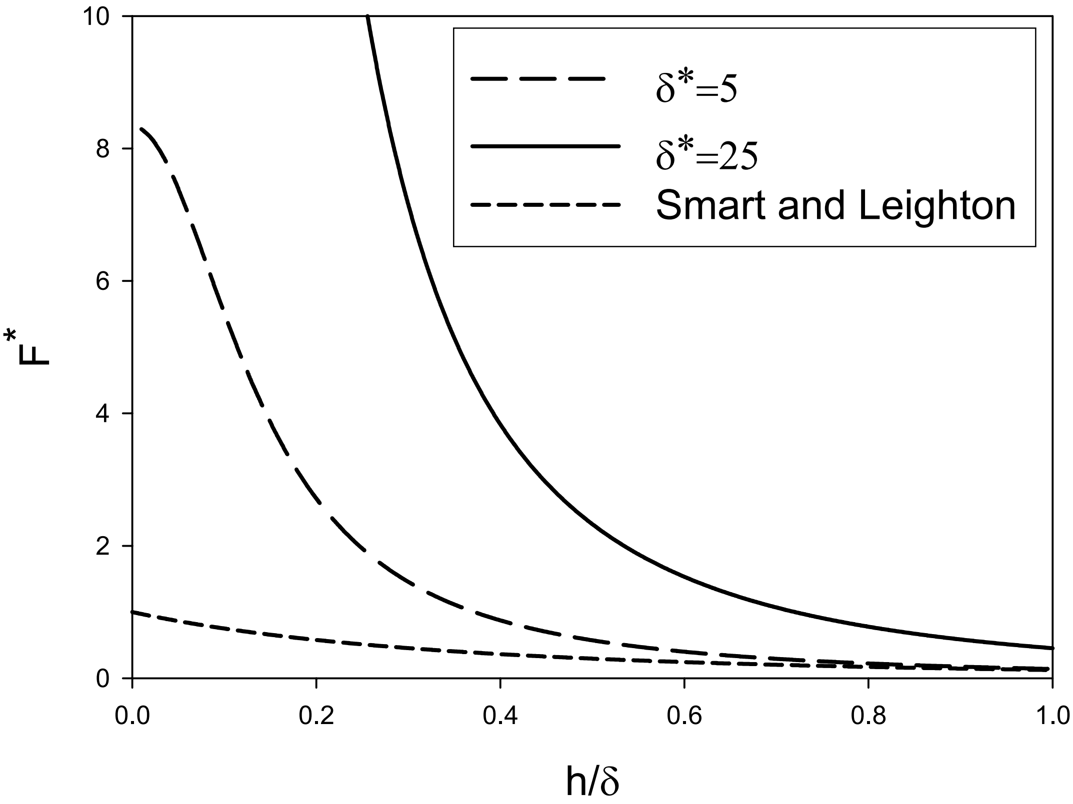


Fig 4.

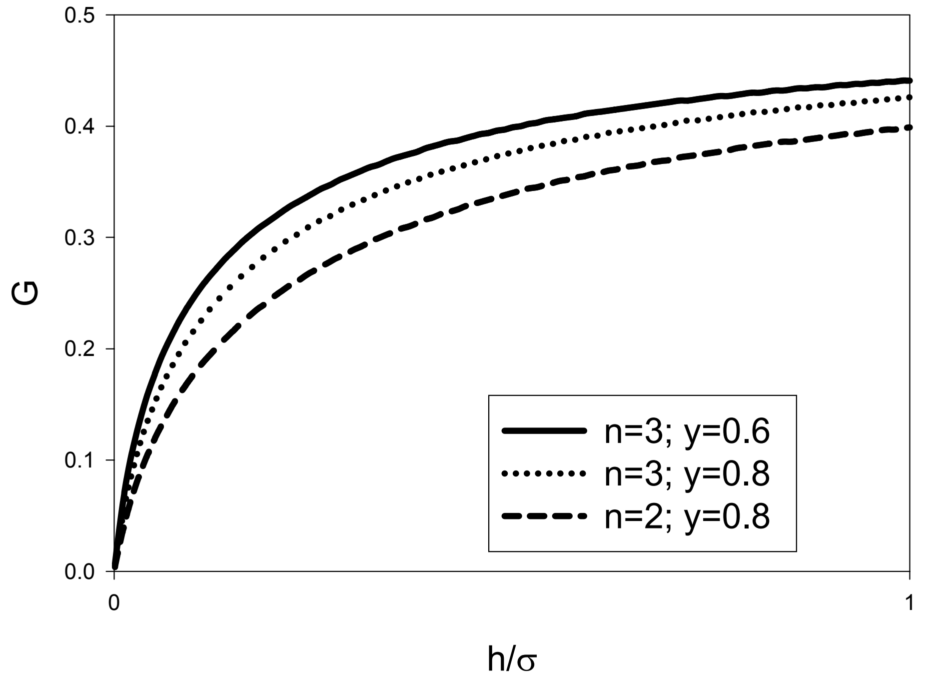


Figure 5