Approximation of Convex Curves by Random Lattice Polygons

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Abstract

It is known that convex polygonal lines on $\mathbb{Z}^2$ with the endpoints fixed at $0 = (0,0)$ and $n = (n_1, n_2)$ and with edges of non-negative slope, have limit shape under the scaling $\mathbb{Z}^2 \mapsto n_1^{-1}\mathbb{Z}^2$ as $n \to \infty$. If $n_2/n_1 \to c$ then the limit shape is identified as a parabolic arc with equation $\sqrt{c(1-u)} + \sqrt{v} = \sqrt{c}$. In probabilistic terms, this result amounts to a functional Law of Large Numbers under the uniform distribution on the set $\mathcal{L}_n$ of such polygons. In the present paper, we consider a converse problem, i.e. that of approximation of convex curves by convex lattice polygons. Let γ be the graph of a strictly convex, increasing C^3-function on $[0,1]$, having non-degenerate curvature. We show that for any such γ, one can construct a probability measure $\mathbb{P}_n^\gamma$ on the space $\mathcal{L}_n$ so that under the law $\mathbb{P}_n^\gamma$, the curve γ is indeed the limit shape of polygons from $\mathcal{L}_n$ as $n \to \infty$.

Key words: convex lattice polygons; limit shape; approximation of convex curves; functional law of large numbers; local limit theorem; Möbius inversion formula

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1. Introduction

Consider a convex polygonal line Γ with vertices at sites of the integer lattice $\mathbb{Z}^2$, starting at the origin and such that the slope of each of its edges is non-negative and does not exceed the angle of 90°. Convexity means that the slope of consecutive edges is strictly increasing. Denote by $\mathcal{L}$ the collection of all such polygons having finitely many edges, and by $\mathcal{L}_n$ the subset of polygons $\Gamma \in \mathcal{L}$ with the right endpoint fixed at $n = (n_1, n_2)$.

Asymptotic properties of the ensemble $\mathcal{L}_n$ as $n \to \infty$ (more precisely, as $n_1, n_2 \to \infty$ and $n_2/n_1 \to c$, where $0 < c < \infty$) were studied by Vershik (1994), Sinai (1994) and Bárány (1995). In particular, it was shown that the limit shape of polygons in $\mathcal{L}_n$ is given by the parabolic arc γ_0 determined by the equation $\sqrt{c(1-u)} + \sqrt{v} = \sqrt{c}$ ($0 \le u, v \le 1$). In probabilistic terms, this result amounts to a functional Law of Large Numbers under the uniform distribution $\mathbb{P}_n$ on $\mathcal{L}_n$, in that scaled polygons $n_1^{-1}\Gamma$ converge in $\mathbb{P}_n$-probability to γ_0, as $n \to \infty$. The arguments in Vershik (1994) and Bárány (1995) were of combinatorial-functional and geometric nature, and were based on the analysis of the corresponding generating function using the multi-dimensional saddle-point method for Cauchy integrals (Vershik 1994) or an appropriate Tauberian theorem (Bárány 1995).

Sinai (1994) suggested a probabilistic approach to the statistics of convex lattice polygons. The idea of the method is to treat the probability distribution $\mathbb{P}_n$ on the space $\mathcal{L}_n$ as a conditional distribution induced by a suitable probability measure $\mathbb{Q}$ defined on the space $\mathcal{L}$ of all convex lattice polygons. In turn, the measure $\mathbb{Q} = \mathbb{Q}_z$ depending on a two-dimensional parameter $z = (z_1, z_2)$ is constructed as the distribution of a certain random field $\nu = \{\nu(x)\}$ defined on the set X of all pairs of co-prime natural numbers $x = (x_1, x_2)$ (i.e., such that their greatest common divisor equals 1). One can check that if the random variables $\nu(x)$ are taken to be independent of each other and have geometric distribution with parameter $z_1^{x_1} z_2^{x_2}$, then the corresponding conditional distribution $\mathbb{P}_n$ on $\mathcal{L}_n$ does not depend on z and moreover, appears to be uniform. Using such a construction, limit theorems about polygons (e.g., a Law of Large Numbers) may first be proved for $\mathbb{Q}_z$ on $\mathcal{L}$ and then transferred onto $\mathbb{P}_n$ on $\mathcal{L}_n$ using a suitable local limit theorem. In so doing, it is natural to choose the parameters z_1, z_2 from the condition that the 'expected' right endpoint of a random polygon $\Gamma \in \mathcal{L}$ would lie at the point $n = (n_1, n_2)$ (see Sinai 1994).

As pointed out by Sinai (1994), the idea of such an approach is in fact well known in Statistical Mechanics (see Khinchin 1960). In the statistical-mechanical language, the sets $\mathcal{L}_n$ and $\mathcal{L}$ endowed with the (Gibbs) measures $\mathbb{P}_n$ and $\mathbb{Q}_z$ are nothing else but the canonical and grand canonical ensembles, respectively, describing a perfect lattice

Bose gas (of indistinguishable particles) on the phase space X. A deep link between asymptotical combinatorial problems for scalar and vector partitions and problems of statistical physics is discussed in detail in Vershik (1996, 1997).

In this work, we consider a converse problem, i.e. that of approximation of convex curves by polygons $\Gamma \in \mathcal{L}_n$ as $n \to \infty$. To be more precise, let γ be the graph of an increasing, strictly convex function g_γ on $[0,1]$ such that $g_\gamma(0) = 0$. Using Sinai's approach, we show that for any sufficiently smooth γ, one can construct a probability measure $\mathbb{P}_n^\gamma$ on the space $\mathcal{L}_n$ (such measure being in general non-uniform), so that under the probability law $\mathbb{P}_n^\gamma$ the arc γ indeed provides the limit shape of polygons $\Gamma \in \mathcal{L}_n$. In other words, a functional Law of Large Numbers holds (see Theorem 8.2), stating that normalized polygons $n_1^{-1}\Gamma$ converge to γ in $\mathbb{P}_n^\gamma$-probability. The measure $\mathbb{P}_n^\gamma$ is again constructed as a conditional distribution induced on the space $\mathcal{L}_n$ by a suitable 'global' probability measure $\mathbb{Q}_z^\gamma$ on $\mathcal{L}$. It turns out, however, that the parameter $z = (z_1, z_2)$, which specifies the geometric distribution of the auxiliary random field $\nu(x)$, now needs to allow for dependence on $x \in X$. We derive the suitable parametric functions $z_1(x)$ and $z_2(x)$, assuming that they depend on x through the ratio x_2/x_1 only, which is particularly convenient in conjunction with the parametrization of the curve γ using its tangent slope $t = g_\gamma'(u)$. As one would expect, if the curve γ is taken to be the aforementioned parabolic arc γ_0 then the parametric functions $z_1(x)$ and $z_2(x)$ are reduced back to constants and our method recovers the uniform distribution on $\mathcal{L}_n$.

Let us point out that in order to be able to pass over from the measure $\mathbb{Q}_z^\gamma$ to the conditional distribution $\mathbb{P}_n^\gamma$, the crucial part is played by an appropriate local Central Limit Theorem (see Theorem 7.1), which is of interest in its own right (see Zarbaliev 1998; cf. also Vershik *et al.* 1999).

2. Approximating polygons

Let g be a bounded function defined on some interval $[0,a]$, such that $g(0) = 0$, and suppose that g is non-decreasing and (in general, non-strictly) convex in $[0,a]$. Furthermore, assume that the function g is continuous on $[0,a]$ and its derivative g' is continuous everywhere except at a finite set of points. Here we allow the derivative $g'(a)$ to be infinite, $g'(a) \le +\infty$. Note that the function $t = g'(u)$ is non-negative and non-decreasing in its domain, and so $0 \le t_0 \le g'(u) \le t_1 \le \infty$, where

$$t_0 := \inf_u g'(u), \qquad t_1 := \sup_u g'(u). \tag{2.1}$$

Denote by $\gamma_g \equiv \gamma$ the graph of a function g with the above properties, and let $\mathfrak{G}$ be the collection of all such curves. Note that for the above defined spaces $\mathcal{L}_n$, $\mathcal{L}$ of convex lattice polygons, we have $\mathcal{L}_n \subset \mathcal{L} \subset \mathfrak{G}$. If a polygon $\Gamma \in \mathcal{L}_n$ is taken as the 'curve' γ, then the corresponding function $g = g_\Gamma$ is a piecewise linear function defined on $[0, n_1]$ and one has $g_\Gamma(n_1) = n_2$.

Let us now equip the space $\mathfrak{G}$ with a suitable metric. If the function $g = g_\gamma$ determines a convex curve $\gamma \in \mathfrak{G}$, we set

$$u_\gamma(t) := \sup\{u : g'_\gamma(u) \le t\}, \qquad 0 \le t \le \infty, \tag{2.2}$$

with the convention that $\sup \varnothing = 0$. That is to say, $u_\gamma(t)$ is a generalized inverse of the derivative $t = g'_\gamma(u)$ (cf. Bingham *et al.* 1989, Sect. 1.5). It follows that the function $u_\gamma(\cdot)$ is non-decreasing and right-continuous on $[0, \infty]$, with values in $[0, a]$. Moreover, if t_0, t_1 are the extreme values of the derivative g'_γ (see (2.1)) then $u_\gamma(t) \equiv 0$ for all $t < t_0$ and $u_\gamma(t) = a$ for all $t \ge t_1$.

Let us now denote by $\ell_\gamma(t)$ the length of the part of γ where the tangent slope does not exceed t:

$$\ell_\gamma(t) = \int_0^{u_\gamma(t)} \sqrt{1 + g'_\gamma(u)^2}\, du, \qquad 0 \le t \le \infty. \tag{2.3}$$

Note that according to our assumptions, every curve $\gamma \in \mathfrak{G}$ is rectifiable, that is, its length is well defined and finite:

$$\ell_\gamma(\infty) = \int_0^{u_\gamma(\infty)} \sqrt{1 + g'_\gamma(u)^2}\, du \le \int_0^a (1 + g'_\gamma(u))\, du = a + g_\gamma(a) < \infty.$$

Finally, we define the function $d_1 : \mathfrak{G} \times \mathfrak{G} \to \mathbb{R}_+$ by setting

$$d_1(\gamma_1, \gamma_2) := \sup_{0 \le t \le \infty} |\ell_{\gamma_1}(t) - \ell_{\gamma_2}(t)|, \qquad \gamma_1, \gamma_2 \in \mathfrak{G}. \tag{2.4}$$

Proposition 2.1. *The function $d_1(\cdot, \cdot)$ satisfies all the properties of distance.*

Proof. Obviously, $d_1(\gamma_1, \gamma_2) = d_1(\gamma_2, \gamma_1)$, and $d_1(\gamma, \gamma) = 0$. The triangle axiom follows from the inequality

$$|\ell_{\gamma_1}(t) - \ell_{\gamma_2}(t)| \le |\ell_{\gamma_1}(t) - \ell_{\gamma_3}(t)| + |\ell_{\gamma_3}(t) - \ell_{\gamma_2}(t)|, \qquad 0 \le t \le \infty.$$

So it remains to check that if $d_1(\gamma_1, \gamma_2) = 0$ then $\gamma_1 = \gamma_2$.

Approximating the given curves $\gamma_1, \gamma_2 \in \mathfrak{G}$ by C^2-smooth strictly convex curves γ_1^k, γ_2^k, respectively, we reduce the problem to checking that if γ_1^k, γ_2^k are close to each other in the sense of d_1, then they are also close in the usual Euclidean metric d. That is to say, if $d_1(\gamma_1^k, \gamma_2^k) \to 0$ as $k \to \infty$, then $d(\gamma_1^k, \gamma_2^k) \to 0$.

Note that for a strictly convex, increasing function $g_\gamma \in C^2[0, a]$, the function $u_\gamma(t)$ defined in (2.2) is given by

$$u_\gamma(t) = \begin{cases} 0, & 0 \le t \le t_0, \\ (g'_\gamma)^{-1}(t), & t_0 \le t \le t_1, \\ a, & t_1 \le t \le \infty, \end{cases} \tag{2.5}$$

where $(g'_\gamma)^{-1}(t)$ is the (ordinary) inverse of the derivative $g'_\gamma(u)$. In particular, the equations $u = u_\gamma(t),\ v = g_\gamma(u_\gamma(t))$ determine a parametrization of the curve γ via the derivative $t = g'_\gamma(u)$. Differentiating formula (2.3) with respect to t, we find

$$\frac{d\ell_\gamma}{dt} = \sqrt{1+t^2}\,\frac{du_\gamma}{dt}\,, \tag{2.6}$$

whence

$$\frac{du_\gamma}{dt} = \frac{1}{\sqrt{1+t^2}}\,\frac{d\ell_\gamma}{dt}\,. \tag{2.7}$$

Therefore,

$$\frac{dv_\gamma}{dt} = \frac{dg_\gamma}{du}\cdot\frac{du_\gamma}{dt} = \frac{t}{\sqrt{1+t^2}}\,\frac{d\ell_\gamma}{dt}\,. \tag{2.8}$$

Integrating equations (2.7), (2.8) and using the initial conditions $u_\gamma(0) = 0$, $v_\gamma(0) = 0$, we obtain

$$\begin{aligned} u_\gamma(t) &= \int_0^t \frac{1}{\sqrt{1+s^2}}\,d\ell_\gamma(s), \\ v_\gamma(t) &= \int_0^t \frac{s}{\sqrt{1+s^2}}\,d\ell_\gamma(s). \end{aligned}$$

Integration by parts yields

$$u_\gamma(t) = \frac{\ell_\gamma(t)}{\sqrt{1+t^2}} + \int_0^t \frac{s\ell_\gamma(s)}{(1+s^2)^{3/2}}\,ds, \tag{2.9}$$

$$v_\gamma(t) = \frac{t\ell_\gamma(t)}{\sqrt{1+t^2}} - \int_0^t \frac{\ell_\gamma(s)}{(1+s^2)^{3/2}}\,ds. \tag{2.10}$$

Note that these equations are linear in ℓ_γ. Hence, setting for γ_1^k, γ_2^k

$$\Delta u_k(t) := u_{\gamma_1^k}(t) - u_{\gamma_2^k}(t), \qquad \Delta v_k(t) := v_{\gamma_1^k}(t) - v_{\gamma_2^k}(t),$$
$$\Delta\ell_k(t) := \ell_{\gamma_1^k}(t) - \ell_{\gamma_2^k}(t),$$

from (2.9) and (2.10) we get

$$\begin{aligned} \Delta u_k(t) &= \frac{\Delta\ell_k(t)}{\sqrt{1+t^2}} + \int_0^t \frac{s\Delta\ell_k(s)}{(1+s^2)^{3/2}}\,ds, \\ \Delta v_k(t) &= \frac{t\Delta\ell_k(t)}{\sqrt{1+t^2}} - \int_0^t \frac{\Delta\ell_k(s)}{(1+s^2)^{3/2}}\,ds. \end{aligned}$$

This implies that if $\Delta\ell_k(t) \to 0$ as $k \to \infty$, uniformly in $t \in [0, \infty]$, then $\Delta u_k(t) \to 0$, $\Delta v_k(t) \to 0$, also uniformly on $[0, \infty]$. Indeed, if $\sup_t |\Delta\ell_k(t)| \le \varepsilon$ then

$$
\begin{aligned}
|\Delta u_k(t)| &\le \frac{\varepsilon}{\sqrt{1+t^2}} + \varepsilon \int_0^t \frac{s}{(1+s^2)^{3/2}}\, ds \\
&= \frac{\varepsilon}{\sqrt{1+t^2}} - \varepsilon \int_0^t \left(\frac{1}{\sqrt{1+s^2}} \right)' ds \\
&= \frac{\varepsilon}{\sqrt{1+t^2}} - \varepsilon \left(\frac{1}{\sqrt{1+t^2}} - 1 \right) \\
&= \varepsilon.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
|\Delta v_k(t)| &\le \frac{\varepsilon t}{\sqrt{1+t^2}} + \varepsilon \int_0^t \frac{1}{(1+s^2)^{3/2}}\, ds \\
&= \frac{\varepsilon t}{\sqrt{1+t^2}} + \varepsilon \int_0^t \left(\frac{s}{\sqrt{1+s^2}} \right)' ds \\
&= \frac{2\varepsilon t}{\sqrt{1+t^2}} \\
&\le 2\varepsilon.
\end{aligned}
$$

As a result, for all $t \in [0, \infty]$ we have $|\Delta u_k(t)|^2 + |\Delta v_k(t)|^2 \le 5\varepsilon^2$, and hence $d(\gamma_1^k, \gamma_2^k) \le \sqrt{5}\,\varepsilon$. This completes the proof of the proposition. $\square$

From the proof of Proposition 2.1, one can see that the following result holds.

Corollary 2.2. *The metrics d_1 and d are equivalent to each other, and in particular $d_1(\gamma_n, \gamma) \to 0$ if and only if $d(\gamma_n, \gamma) \to 0$, as $n \to \infty$.*

A general problem of approximation of convex arcs by convex lattice polygons can now be set as follows.

Definition 2.1. Let us be given a convex curve $\gamma \in \mathfrak{G}$ determined by a function $g_\gamma(u)$, $0 \le u \le 1$. We shall say that a polygon $\Gamma \in \mathcal{L}_n$, $n = (n_1, n_2)$, is an *ε-approximation to* γ if under the scaling transformation with respect to the origin with the coefficient $1/n_1$, the polygon Γ gets into an ε-vicinity (in the metric d_1) of the curve γ, that is, $d_1(\tilde{\Gamma}_n, \gamma) < \varepsilon$, where $\tilde{\Gamma}_n := n_1^{-1}\Gamma$.

Our goal is to study the statistics of approximating polygons $\Gamma \in \mathcal{L}_n$ as $n \to \infty$ with respect to a suitable family of probability measures $\mathbb{P}_n^\gamma$ on $\mathcal{L}_n$. More precisely, we intend to construct a probability measure $\mathbb{P}_n^\gamma$ in such a way that in the limit $n \to \infty$, with respect to $\mathbb{P}_n^\gamma$ the overwhelming majority of polygons from $\mathcal{L}_n$ are ε-approximations to the γ (for any $\varepsilon > 0$). In other words, a functional Law of Large Numbers should be valid in the form

$$\lim_{n\to\infty} \mathbb{P}_n^\gamma\{d_1(\tilde{\Gamma}_n, \gamma) < \varepsilon\} = 1. \tag{2.11}$$

Assumption 2.2. Note that the right endpoint of the scaled polygon $\tilde{\Gamma}_n$ has the coordinates $(1, c_n)$, where $c_n := n_2/n_1$, whereas the right endpoint of the arc γ lies at the point $(1, c_\gamma)$, where $c_\gamma := g_\gamma(1)$ $(0 < c_\gamma < \infty)$. This suggests that, in order for the relation (2.11) to be true, we need to pass to the limit $n \to \infty$ in such a way that $c_n \to c_\gamma$. In what follows, we will always be assuming that this condition is fulfilled.

Let us point out that the classical result on the limit shape of lattice polygons with uniform distribution on $\mathcal{L}_n$ (see Vershik 1994, Sinai 1994 and Bárány 1995) can be viewed as an approximation result for the particular curve γ_0 determined by the equation

$$\sqrt{c(1-u)} + \sqrt{v} = \sqrt{c}, \qquad 0 \le u \le 1. \tag{2.12}$$

In the present work, we solve the approximation problem for a subclass of $\mathfrak{G}$ consisting of *C^3-smooth strictly convex arcs $\gamma \in \mathfrak{G}$ with non-degenerate curvature.*

3. Construction of the measures $\mathbb{Q}_z^\gamma$ and $\mathbb{P}_n^\gamma$

Let us first describe the construction of the global measure $\mathbb{Q}_z^\gamma$ on the space $\mathcal{L}$. The measure $\mathbb{P}_n^\gamma$ on the space $\mathcal{L}_n$ is then obtained as a conditional distribution induced by fixing the right end of polygons $\Gamma \in \mathcal{L}$ at point $n = (n_1, n_2)$.

Consider the set X of all pairs of co-prime non-negative integer numbers,

$$X := \{x = (x_1, x_2) \in \mathbb{Z}^2 :\ x_1 > 0,\ x_2 \ge 0,\ \text{and g.c.d.}\,(x_1, x_2) = 1\}.$$

In particular, the pair $(1, 0)$ is included in this set, while $(0, 1)$ is not. In what follows, we denote by $\tau(x) := x_2/x_1$ the slope of the vector $x = (x_1, x_2)$.

Let $\Phi(X) := \{\varphi\colon X \to \mathbb{Z}_+\} = (\mathbb{Z}_+)^X$ be the space of functions on X with non-negative integer values. Denote by $\operatorname{supp}\varphi := \{x \in X : \varphi(x) > 0\}$ the support of the function $\varphi \in \Phi(X)$ and consider the subspace $\Phi_0(X) := \{\varphi \in \Phi(X) :\ \#(\operatorname{supp}\varphi) < \infty\}$ of functions with a *finite* support. It is easy to see that functions $\varphi \in \Phi_0(X)$ are in a one-to-one correspondence with finite polygons $\Gamma \in \mathcal{L}$. Indeed, let us arrange the points $x = (x_1, x_2) \in \operatorname{supp}\varphi$ according to the increase of their slope $\tau(x)$. Multiplying the vector (x_1, x_2) by the corresponding value $\varphi(x) > 0$, we obtain a collection of consecutive edges of some convex finite polygon Γ. The converse mapping is constructed similarly. Note that the function $\varphi(x) \equiv 0$ formally corresponds to the 'trivial' polygon with coinciding endpoints. In the sequel, we will identify the spaces $\mathcal{L}$ and $\Phi_0(X)$.

Following Bogachev and Zarbaliev (1999a), let us introduce on $\varPhi_0(X)$ a probability measure $\mathbb{Q}_z^\gamma$ by setting

$$\mathbb{Q}_z^\gamma(\nu) := \prod_{x\in X} (z_1^{x_1} z_2^{x_2})^{\nu(x)} (1 - z_1^{x_1} z_2^{x_2}) = \prod_{x\in X} (z_1^{x_1} z_2^{x_2})^{\nu(x)} \prod_{x\in X}(1 - z_1^{x_1} z_2^{x_2}), \tag{3.1}$$

where $z_1 = z_1(x)$, $z_2 = z_2(x)$ are parameters (parametric functions), such that $0 \le z_i(x) < 1$ $(x \in X)$. Their explicit form, determined by the given curve $\gamma \in \mathfrak{G}$, will be specified later on. So far, we only assume that the following condition is satisfied:

$$\prod_{x\in X}(1 - z_1^{x_1} z_2^{x_2}) > 0, \tag{3.2}$$

which guarantees that the normalization in (3.1) is well defined.

Let us point out that according to (3.1), the random variables $\{\nu(x)\}_{x\in X}$ are independent of each other and have geometric distribution with parameter $z_1^{x_1} z_2^{x_2}$:

$$\mathbb{Q}_z^\gamma\{\nu(x) = k\} = (z_1^{x_1} z_2^{x_2})^k (1 - z_1^{x_1} z_2^{x_2}), \quad k = 0, 1, 2, \ldots \tag{3.3}$$

Note that the measure $\mathbb{Q}_z^\gamma$ can be extended in a standard way to a measure on the space $\varPhi(X)$ of all non-negative integer-valued functions on X (cf. Bogachev and Zarbaliev 1999b, 2003). However, $\mathbb{Q}_z^\gamma$ is in fact concentrated on the subset $\varPhi_0(X) \subset \varPhi(X)$ consisting of all finite configurations $\nu(\cdot)$.

Lemma 3.1. *Condition* (3.2) *is necessary and sufficient in order that*

$$\mathbb{Q}_z^\gamma\{\nu \in \varPhi_0(X)\} = 1.$$

Proof. According to (3.3), we have $\mathbb{Q}_z^\gamma\{\nu(x) > 0\} = z_1^{x_1} z_2^{x_2}$. Hence,

$$\sum_{x\in X} \mathbb{Q}_z^\gamma\{\nu(x) > 0\} = \sum_{x\in X} z_1^{x_1} z_2^{x_2}.$$

Note that convergence of this series is equivalent to convergence of the infinite product (3.2). Therefore, Borel–Cantelli's lemma implies that, with $\mathbb{Q}_z^\gamma$-probability 1, there occur only finitely many events $\{\nu(x) > 0\}$. That is, $\mathbb{Q}_z^\gamma(\varPhi_0(X)) = 1$, and the lemma is proved. □

As a result, with $\mathbb{Q}_z^\gamma$-probability 1 a realization of the random field ν determines a (random) convex polygon $\Gamma \in \mathcal{L}$. Denote by $\xi = (\xi_1, \xi_2)$ the right endpoint of Γ, so that its coordinates ξ_1, ξ_2 are given by

$$\xi_1 = \sum_{x\in X} x_1 \nu(x), \qquad \xi_2 = \sum_{x\in X} x_2 \nu(x). \tag{3.4}$$

The space $\mathcal{L}_n$ of polygons ending up at a fixed point $n = (n_1, n_2)$ is then represented as a 'slice' of $\mathcal{L}$ obtained by the condition $\xi = n$. Accordingly, the probability measure $\mathbb{Q}_z^\gamma$ induces the conditional distribution $\mathbb{P}_n^\gamma$ on $\mathcal{L}_n$ by the formula

$$\mathbb{P}_n^\gamma(\Gamma) := \mathbb{Q}_z^\gamma\{\Gamma \,|\, \xi = n\} = \frac{\mathbb{Q}_z^\gamma(\Gamma)}{\mathbb{Q}_z^\gamma\{\xi = n\}}, \qquad \Gamma \in \mathcal{L}_n. \tag{3.5}$$

4. The choice of the parametric functions $z_1(x)$, $z_2(x)$

Consider a fixed convex curve $\gamma \in \mathfrak{G}$, represented as the graph of an increasing, convex function g_γ, which for definiteness is assumed to be defined on the interval $[0,1]$. To be more specific, in this section we will be working under the following

Assumption 4.1. The function g_γ is strictly increasing and strictly convex on $[0,1]$, and $g_\gamma \in C^2[0,1]$. In particular, $g_\gamma'(u) \ge 0$, $g_\gamma''(u) \ge 0$ for all $u \in [0,1]$. Moreover, the curvature κ_γ of the curve γ, given by the formula

$$\kappa_\gamma(u) = \frac{g_\gamma''(u)}{(1 + g_\gamma'(u)^2)^{3/2}}, \qquad 0 \le u \le 1, \tag{4.1}$$

is uniformly bounded from below by a positive constant,

$$\inf_{u \in [0,1]} \kappa_\gamma(u) \ge K_0 > 0. \tag{4.2}$$

The meaning of the latter assumption is that the curve γ is required to be not 'too flat'.

As mentioned in the proof of Proposition 2.1, the graph γ of the function g_γ can be parametrized by the derivative $t = g_\gamma'(u)$ via the equations $u = u_\gamma(t)$, $v = g_\gamma(u_\gamma(t))$, where $u_\gamma(t)$ is given by (2.5). Note that the expression (4.1) for the curvature is then reduced to

$$\kappa_\gamma(t) = \frac{g_\gamma''(u_\gamma(t))}{(1 + t^2)^{3/2}}, \qquad t_0 \le t \le t_1, \tag{4.3}$$

where $t_0 = \inf_u g_\gamma'(u)$, $t_1 = \sup_u g_\gamma'(u)$ (see (2.1)).

In the above construction, the measure $\mathbb{P}_z^\gamma$ depends on the parameters $z_1(x)$, $z_2(x)$ ($x \in X$). So far, these functions were only assumed to guarantee convergence of the infinite product (3.2). Let us now adjust them to the given curve γ.

Recall that $c_n = n_2/n_1$, $c_\gamma = g_\gamma(1)$ (see Assumption 2.2), and set $\rho \equiv \rho_n := c_\gamma/c_n$. According to Assumption 2.2 we have $c_n \to c_\gamma$ and hence $\rho \to 1$ as $n \to \infty$. For $x = (x_1, x_2)$, let us set $\tilde{x} := (x_1, \rho x_2)$. We will seek the functions $z_1(x)$, $z_2(x)$ in the form

$$z_1(x) = \exp\{-\alpha\delta_1(\tau(\tilde{x}))\}, \qquad z_2(x) = \exp\{-\alpha\rho\delta_2(\tau(\tilde{x}))\}, \tag{4.4}$$

where

$$\alpha \equiv \alpha_n := (\rho n_1)^{-1/3} \to 0 \qquad (n \to \infty) \tag{4.5}$$

and $\tau(\tilde{x}) = \tilde{x}_2/\tilde{x}_1 = \rho x_2/x_1$. Here $\delta_1(\cdot)$ and $\delta_2(\cdot)$ are certain functions on $[0,\infty]$ such that

$$\inf_{0\le t\le\infty} \delta_i(t) \ge \delta_* > 0, \qquad i = 1,2. \tag{4.6}$$

Let $\Gamma(t)$ denote the part of the polygon Γ such that the slope of each of its edges does not exceed $t \in [0,\infty]$. Set $X(t) := \{x \in X : \tau(x) \le t\}$. Recalling the association $\Gamma \leftrightarrow \nu$ described in Section 3, the polygon $\Gamma(t)$ is determined by the truncated configuration $\mathbf{1}_{X(t)}(x)\,\nu(x)$. Denote by $\ell_\Gamma(t)$ the length of $\Gamma(t)$:

$$\ell_\Gamma(t) = \sum_{x\in X(t)} |x|\nu(x), \tag{4.7}$$

where $|x| = \sqrt{x_1^2 + x_2^2}$. Note that under the scaling transformation $\Gamma \mapsto \tilde{\Gamma}_n \equiv n_1^{-1}\Gamma$ we have

$$\ell_{\tilde{\Gamma}_n}(t) = n_1^{-1}\ell_\Gamma(t), \qquad 0 \le t \le \infty.$$

The condition we are going to impose on the choice of z_1, z_2 is that the expected value of the function $\ell_{\tilde{\Gamma}_n}(t)$ in the limit $n \to \infty$ coincides with the corresponding value of the function $\ell_\gamma(t)$ associated with the given curve γ:

$$\lim_{n\to\infty} n_1^{-1}\,\mathbb{E}_z^\gamma[\ell_\Gamma(t)] = \ell_\gamma(t), \qquad 0 \le t \le \infty, \tag{4.8}$$

where $\mathbb{E}_z^\gamma$ stands for expectation with respect to the measure $\mathbb{Q}_z^\gamma$.

Theorem 4.1. *Suppose that the functions $\delta_1(t)$, $\delta_2(t)$ satisfy the condition (4.6). Then, in order that the limiting relation (4.8) is fulfilled for all $t \in [0,\infty]$, it is necessary and sufficient that*

$$\delta_i(t) \equiv +\infty \quad (i = 1,2), \qquad\qquad t < t_0, \quad t > t_1, \tag{4.9}$$

$$\delta_1(t) + t\delta_2(t) = \varkappa g_\gamma''(u_\gamma(t))^{1/3}, \qquad\qquad t_0 < t < t_1, \tag{4.10}$$

where $\varkappa := (2\zeta(3)/\zeta(2))^{1/3}$, $\zeta(s) := \sum_{k=1}^\infty 1/k^s$ is the Riemann zeta-function, and the function $u_\gamma(t)$ is given by (2.5).

Proof. Let us evaluate the expectation of the random function $\ell_\Gamma(t)$. Recalling that the random variable $\nu(x)$ has the geometric distribution (3.3), we obtain

$$\mathbb{E}_z^\gamma[\nu(x)] = \frac{z_1^{x_1} z_2^{x_2}}{1 - z_1^{x_1} z_2^{x_2}} = \sum_{k=1}^\infty (z_1^{x_1} z_2^{x_2})^k.$$

Hence, by (4.4) and (4.7) this yields

$$\mathbb{E}_z^\gamma[\ell_\Gamma(t)] = \sum_{x\in X(t)} \sum_{k=1}^{\infty} |x|(z_1^{x_1} z_2^{x_2})^k = \sum_{k=1}^{\infty} \sum_{x\in X(t)} |x| e^{-\alpha k\langle \tilde{x}, \delta(\tau(\tilde{x}))\rangle}, \tag{4.11}$$

where $\delta(t) := (\delta_1(t), \delta_2(t))$ and $\langle y, y'\rangle := y_1 y_1' + y_2 y_2'$ denotes the inner product of vectors $y = (y_1, y_2),\ y' = (y_1', y_2')$.

Let us now set

$$f(h) := \sum_{x\in X(t)} h|x| e^{-\alpha h\langle \tilde{x}, \delta(\tau(\tilde{x}))\rangle}, \tag{4.12}$$

so that, according to (4.11), we have

$$\mathbb{E}_z^\gamma[\ell_\Gamma(t)] = \sum_{k=1}^{\infty} \frac{f(k)}{k}. \tag{4.13}$$

Furthermore, note that

$$\begin{aligned} F(h) :=& \sum_{y_1=1}^{\infty} \sum_{0\le y_2\le t y_1} h|y| e^{-\alpha h\langle \tilde{y}, \delta(\tau(\tilde{y}))\rangle} \\ =& \sum_{m=1}^{\infty} \sum_{x\in X(t)} hm|x| e^{-\alpha hm\langle \tilde{x}, \delta(\tau(\tilde{x}))\rangle} \\ =& \sum_{m=1}^{\infty} f(hm). \end{aligned} \tag{4.14}$$

By the Möbius inversion formula (see Hardy and Wright 1960, Sect. 16.5, Theorem 270), the function $f(h)$ can be expressed as

$$f(h) = \sum_{m=1}^{\infty} \mu(m) F(hm), \tag{4.15}$$

where $\mu(m)$ is the *Möbius function* defined as follows: $\mu(1) = 1,\ \mu(m) = (-1)^q$ if m is a product of q different prime numbers, and $\mu(m) = 0$ if m is a multiple of the square of a prime number (see Hardy and Wright 1960, Sect. 16.3.)

For validity of the formula (4.15) it is sufficient that for all $h > 0$

$$\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} f(hkm) < \infty. \tag{4.16}$$

To check this condition, note that by (4.6), (4.12) and (4.14) we have, uniformly in h,

$$
\begin{aligned}
0 \le f(h) \le F(h) &\le h \sum_{y_1=1}^{\infty} \sum_{y_2=0}^{\infty} (y_1+y_2) e^{-\alpha h \delta_*(y_1+\rho y_2)} \\
&\le h \sum_{j=1}^{\infty} j^2 e^{-\alpha h \delta_* j/2} \\
&= h \, \frac{e^{-\alpha h \delta_*/2} + e^{-\alpha h \delta_*}}{(1-e^{-\alpha h \delta_*/2})^3} \\
&= O(1)\, \alpha^{-3} h^{-2},
\end{aligned} \tag{4.17}
$$

where we used that $y_1+\rho y_2 \ge (y_1+y_2)/2$ for all n large enough, since $\rho \to 1$ as $n \to \infty$. Therefore, $f(hkm) = O(1)\, \alpha^{-3}(hkm)^{-2}$, uniformly in k and m, hence the series (4.16) is convergent, as required.

Returning to the representation (4.13) and using that $n_1^{-1} = \rho\alpha^3$, from the formula (4.15) with $h = k$ we get

$$
\begin{aligned}
n_1^{-1}\, \mathbb{E}_z^{\gamma}[\ell_\Gamma(t)] &= \rho \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} m\mu(m)\, \frac{\alpha^3 F(km)}{km} \\
&= \rho \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} m\mu(m) \sum_{y_1=1}^{\infty} \sum_{0 \le y_2 \le t y_1} \alpha^3 |y| e^{-km\alpha\langle \tilde{y}, \delta(\tau(\tilde{y}))\rangle}.
\end{aligned} \tag{4.18}
$$

Taking into account the estimate (4.17) and using that $|\mu(m)| \le 1$, we see that the general term in the double sum over k, m in (4.18) admits a uniform bound of the form $O(1)\, k^{-3} m^{-2}$, which is a term of a convergent series. Therefore, we can apply Lebesgue's dominated convergence theorem to pass to the limit in (4.18) as $n \to \infty$.

In order to find this limit, note that the internal double series over y_1, y_2 in (4.18) is a Riemann sum for the integral

$$
\int_0^{\infty} \int_0^{t y_1} \sqrt{y_1^2 + y_2^2}\; e^{-km(y_1 \delta_1(y_2/y_1) + y_2 \delta_2(y_2/y_1))}\, dy_1 dy_2. \tag{4.19}
$$

Moreover, this sum does converge to the integral (4.19) as $\alpha \to 0$, since the integrand function in (4.19) is directly Riemann integrable, as follows from an estimation similar to (4.17).

The integral (4.19) can be easily evaluated by the substitution $y_1 = u$, $y_2 = us$. The Jacobian of this transformation equals

$$
\det \begin{pmatrix} 1 & 0 \\ s & u \end{pmatrix} = u,
$$

hence the integral (4.19) is reduced to

$$\begin{aligned}\int_0^\infty u^2\,du \int_0^t \sqrt{1+s^2}\, e^{-kmu(\delta_1(s)+s\delta_2(s))}\,ds \\ = \int_0^t \frac{\sqrt{1+s^2}}{\big(\delta_1(s)+s\delta_2(s)\big)^3}\,ds \int_0^\infty u_1^2 e^{-kmu_1}\,du_1 \\ = \frac{2}{(km)^3}\int_0^t \frac{\sqrt{1+s^2}}{\big(\delta_1(s)+s\delta_2(s)\big)^3}\,ds,\end{aligned}$$

where we used the obvious substitution $u_1 = u(\delta_1(s)+s\delta_2(s))$. Substituting this into (4.18) we obtain

$$\begin{aligned}\lim_{n\to\infty} n_1^{-1}\,\mathbb{E}_z^\gamma[\ell_\Gamma(t)] &= 2\sum_{k=1}^\infty \frac{1}{k^3}\sum_{m=1}^\infty \frac{\mu(m)}{m^2}\int_0^t \frac{\sqrt{1+s^2}}{\big(\delta_1(s)+s\delta_2(s)\big)^3}\,ds \\ &= \frac{2\zeta(3)}{\zeta(2)}\int_0^t \frac{\sqrt{1+s^2}}{\big(\delta_1(s)+s\delta_2(s)\big)^3}\,ds.\end{aligned}$$

Here the expression for the sum over m can be obtained using the Möbius inversion formula (4.15) with $f(h)=h^{-2}$ and $F(h)=\sum_{m=1}^\infty (hm)^{-2} = h^{-2}\zeta(2)$ (cf. Hardy and Wright 1960, Sect. 17.5, Theorem 287).

Therefore, recalling the condition (4.8) we obtain

$$\varkappa^3\int_0^t \frac{\sqrt{1+s^2}}{\big(\delta_1(s)+s\delta_2(s)\big)^3}\,ds = \ell_\gamma(t), \qquad 0\le t\le\infty. \tag{4.20}$$

Note that, according to the definitions (2.2), (2.3), we have $\ell_\gamma(t)\equiv 0$ for $t\in[0,t_0)$ and $\ell_\gamma(t)\equiv\ell_\gamma(\infty)$ for $t\in(t_1,\infty]$, while for $t\in(t_0,t_1)$ the derivative $\ell'_\gamma(t)$ is given by the formula (2.6). Differentiating the identity (4.20) with respect to t, we obtain equations (4.9), (4.10). □

Let us now check that the equation (4.10) has a suitable solution.

Proposition 4.2. *For $t\in[t_0,t_1]$ let us set*

$$\begin{aligned}\delta_1(t) &:= \varkappa\,\kappa_\gamma(t)^{1/3}\,\frac{c_\gamma\sqrt{1+t^2}}{c_\gamma+t}, \\ \delta_2(t) &:= \varkappa\,\kappa_\gamma(t)^{1/3}\,\frac{\sqrt{1+t^2}}{c_\gamma+t} \equiv \frac{\delta_1(t)}{c_\gamma},\end{aligned} \tag{4.21}$$

where $c_\gamma = g_\gamma(1)$ and the curvature κ_γ is given by (4.3). Then the functions $\delta_1(t)$, $\delta_2(t)$ satisfy the assumption (4.6) and the equation (4.10).

Proof. It is straightforward to verify that the equation (4.10) is satisfied. A lower bound of the form (4.6) follows from the assumption (4.2). $\square$

Remark 4.2. In the 'classical' case, where the curve $\gamma = \gamma_0$ is determined by the equation (2.12), it is easy to check that the curvature (4.1) is given by

$$\kappa_0(t) = \frac{c\,(1+t/c)^3}{2(1+t^2)^{3/2}}, \qquad 0 \le t \le \infty.$$

Hence, the expressions (4.21) take the form

$$\begin{aligned} \delta_1(t) &= \varkappa \left(\frac{c}{2}\right)^{1/3} \frac{1+t/c}{\sqrt{1+t^2}} \cdot \frac{c\sqrt{1+t^2}}{c+t} = \varkappa \left(\frac{c}{2}\right)^{1/3}, \\ \delta_2(t) &= \frac{\delta_1(t)}{c} = \varkappa \left(\frac{1}{2c^2}\right)^{1/3} \end{aligned}$$

(cf. Sinai 1994 and Bogachev and Zarbaliev 1999b).

5. Asymptotics of the expectation

In this section, we derive a few corollaries following from the choice of $z_1(x)$, $z_2(x)$ that we have made. Our first goal is to show that convergence in (4.8) is uniform in t.

Theorem 5.1. *Let the parameters $z_1(x)$, $z_2(x)$ $(x \in X)$ be chosen according to formulas* (4.4), *where the functions $\delta_1(t)$, $\delta_2(t)$ are given by the expressions* (4.9), (4.21). *Then convergence in* (4.8) *is uniform in $t \in [0, \infty]$, that is,*

$$\lim_{n\to\infty} \sup_{0\le t\le\infty} \left| n_1^{-1}\, \mathbb{E}_z^{\gamma}[\ell_\Gamma(t)] - \ell_\gamma(t) \right| = 0. \tag{5.1}$$

We will use the following simple criterion of uniform convergence (see Bogachev and Zarbaliev 2003).

Lemma 5.2. *Let us be given a sequence of functions $\{f_n\}$ defined on a finite interval $[a,b]$, such that for each n the function f_n is non-decreasing on $[a,b]$ and $f_n(t) \to f(t)$ as $n \to \infty$ for each $t \in [a,b]$, where f is a continuous function. Then $f_n(t) \to f(t)$ uniformly on $[a,b]$.*

Proof of Theorem 5.1. Note that for each n the function

$$f_n(t) := n_1^{-1}\, \mathbb{E}_z^{\gamma}[\ell_\Gamma(t)] = \frac{1}{n_1} \sum_{x\in X(t)} |x|\, \mathbb{E}_z^{\gamma}[\nu(x)]$$

is non-decreasing in t and that the limiting function $f(t) := \ell_\gamma(t)$ given by (2.3) is continuous on $[0,\infty]$. Hence, by Lemma 5.2 convergence (5.1) is uniform in t on every finite interval $[0,t^*]$. To complete the proof, it suffices to check that for any $\varepsilon>0$ and for large enough n, there exists $t^*<\infty$ such that for all $t\ge t^*$

$$n_1^{-1}\,\mathbb{E}_z^\gamma[\ell_\Gamma(\infty)-\ell_\Gamma(t)]\le\varepsilon. \tag{5.2}$$

Similarly to (4.11) and (4.17) we can write

$$\begin{aligned}\mathbb{E}_z^\gamma[\ell_\Gamma(\infty)-\ell_\Gamma(t)] &= \sum_{k=1}^\infty\sum_{x\notin X(t)}|x|\,e^{-\alpha k\langle\tilde{x},\delta(\tau(\tilde{x}))\rangle}\\ &\le\sum_{k=1}^\infty\sum_{x\notin X(t)}(x_1+x_2)\,e^{-\alpha k\delta_*(x_1+\rho x_2)/2}\\ &\le\sum_{k=1}^\infty\sum_{y_1=1}^\infty\sum_{y_2>ty_1}(y_1+y_2)\,e^{-\alpha k\delta_*(y_1+y_2)/2}.\end{aligned}\tag{5.3}$$

Note that the number of integer pairs (y_1,y_2) (with $y_1\ge1$, $y_2\ge0$) satisfying the conditions $y_1+y_2=j$ and $y_2>ty_1$ does not exceed $j/(t+1)$. Hence, again using the uniform estimate (4.17), we see that the right-hand side of (5.3) is bounded from above by

$$\sum_{k=1}^\infty\sum_{j=1}^\infty\frac{j^2}{t+1}\,e^{-\alpha k\delta_* j/2}=\frac{1}{t+1}\sum_{k=1}^\infty O(1)\,(\alpha k)^{-3}=O(1)\,\frac{1}{\alpha^3(t+1)}.$$

Recalling that $\alpha^3=1/(\rho n_1)\sim n_1^{-1}$, this implies the estimate (5.2) for all t large enough, and the proof is complete. $\square$

Denote by $\xi_1(t)$, $\xi_2(t)$ the coordinates of the right endpoint of $\Gamma(t)$:

$$\xi_i(t)=\sum_{x\in X(t)}x_i\nu(x)\qquad(i=1,2). \tag{5.4}$$

In particular, for $t=\infty$, the formulas (5.4) are reduced to (3.4) and yield the coordinates $\xi_i=\xi_i(\infty)$ of the right end of the entire polygon Γ.

It is natural to expect that with the parameters $z_1(x)$, $z_2(x)$ chosen in the previous section, expectation of the functions $n_1^{-1}\xi_1(t)$, $n_1^{-1}\xi_2(t)$ in the limit $n\to\infty$ should coincide, respectively, with the functions $u_\gamma(t)$, $g_\gamma(u_\gamma(t))$ associated with the given curve γ.

Theorem 5.3. *Suppose that the parameters $z_1(x)$, $z_2(x)$ ($x\in X$) are chosen according to formulas (4.4), where the functions $\delta_1(t)$, $\delta_2(t)$ are given by (4.9), (4.21). Then, uniformly in $t\in[0,\infty]$,*

$$\lim_{n\to\infty}n_1^{-1}\,\mathbb{E}_z^\gamma[\xi_1(t)]=u_\gamma(t),\qquad\lim_{n\to\infty}n_1^{-1}\,\mathbb{E}_z^\gamma[\xi_2(t)]=g_\gamma(u_\gamma(t)). \tag{5.5}$$

In particular, for $t = \infty$ this yields

$$\lim_{n\to\infty} n_1^{-1}\,\mathbb{E}_z^\gamma[\xi_1] = 1, \qquad \lim_{n\to\infty} n_1^{-1}\,\mathbb{E}_z^\gamma[\xi_2] = c_\gamma. \tag{5.6}$$

Proof. Like in the proof of Theorem 4.1, one can show that

$$n_1^{-1}\,\mathbb{E}_z^\gamma[\xi_1(t)] = \rho \sum_{k=1}^\infty \sum_{m=1}^\infty m\mu(m) \sum_{y_1=1}^\infty \sum_{y_2\le ty_1} \alpha^3 y_1 e^{-km\alpha\langle\tilde{y},\delta(\tau(\tilde{y}))\rangle}. \tag{5.7}$$

By the estimate (4.17) it follows that the general term of the double series (5.7) (over k, m) admits a uniform bound of the form $O(1)\,k^{-3}m^{-2}$, so Lebesgue's dominated convergence theorem applies. Assuming that $t_0 \le t \le t_1$ and passing to the limit similarly as in the proof of Theorem 4.1, we see that

$$\begin{aligned}
\lim_{n\to\infty} n_1^{-1}\,\mathbb{E}_z^\gamma[\xi_1(t)] &= \sum_{k=1}^\infty \sum_{m=1}^\infty m\mu(m) \int_0^\infty \int_0^{ty_1} y_1\, e^{-km\langle y,\delta(\tau(y))\rangle}\,dy_1 dy_2 \\
&= \sum_{k=1}^\infty \sum_{m=1}^\infty m\mu(m) \frac{2}{(km)^3} \int_{t_0}^t \frac{1}{\big(\delta_1(s)+s\delta_2(s)\big)^3}\,ds \\
&= 2\sum_{k=1}^\infty \frac{1}{k^3} \sum_{m=1}^\infty \frac{\mu(m)}{m^2} \int_{t_0}^t \frac{ds}{\varkappa^3 g_\gamma''(u_\gamma(s))} \\
&= \frac{2\zeta(3)}{\zeta(2)\varkappa^3} \int_0^{u_\gamma(t)} \frac{dg_\gamma'(u)}{g_\gamma''(u)} \\
&= u_\gamma(t).
\end{aligned} \tag{5.8}$$

Similarly, we obtain for $t_0 \le t \le t_1$

$$\begin{aligned}
\lim_{n\to\infty} n_2^{-1}\,\mathbb{E}_z^\gamma[\xi_2(t)] &= \sum_{k=1}^\infty \sum_{m=1}^\infty m\mu(m) \int_0^\infty \int_0^{ty_1} y_2\, e^{-km\langle y,\delta(\tau(y))\rangle}\,dy_1 dy_2 \\
&= \sum_{k=1}^\infty \sum_{m=1}^\infty m\mu(m) \frac{2}{(km)^3} \int_0^{ty_1} \frac{s}{\big(\delta_1(s)+s\delta_2(s)\big)^3}\,ds \\
&= \frac{2\zeta(3)}{\zeta(2)} \int_{t_0}^{u_\gamma(t)} \frac{s}{\varkappa^3 g_\gamma''(u_\gamma(s))}\,ds \\
&= \int_0^{u_\gamma(t)} \frac{g_\gamma'(u)\,dg_\gamma'(u)}{g_\gamma''(u)} \\
&= g_\gamma(u_\gamma(t)).
\end{aligned}$$

Uniform convergence in (5.5) can be proved similarly as in Theorem 5.1. $\square$

For the future applications, we need to estimate the rate of convergence in (5.6) with sufficient accuracy. To be able to do so, we have to require more smoothness of the function g_γ.

Assumption 5.1. In addition to Assumption 4.1, we now suppose that $g_\gamma \in C^3[0,1]$.

Theorem 5.4. *For $i = 1, 2$, one has $\mathbb{E}_z^\gamma[\xi_i] - n_i = O(n_1^{2/3})$ as $n \to \infty$.*

Proof. Consider the case $i = 1$. As shown in (5.7), we have

$$\begin{aligned}\mathbb{E}_z^\gamma[\xi_1] &= \sum_{k=1}^\infty \sum_{m=1}^\infty m\mu(m) \sum_{y_1=1}^\infty \sum_{y_2=0}^\infty y_1 e^{-km\alpha\langle \tilde{y}, \delta(\tau(\tilde{y}))\rangle} \\ &= \sum_{k=1}^\infty \sum_{m=1}^\infty \frac{\mu(m)}{k\alpha} F_1(km\alpha),\end{aligned}$$

where

$$F_1(h) := \sum_{y_1=1}^\infty \sum_{y_2=0}^\infty f_1(hy_1, hy_2), \qquad f_1(y_1, y_2) := y_1 e^{-\langle \tilde{y}, \delta(\tau(\tilde{y}))\rangle}. \tag{5.9}$$

Repeating the calculations as in (5.8), we note that

$$\int_0^\infty \int_0^\infty f_1(hy_1, hy_2)\, dy_1 dy_2 = \frac{2}{\rho h^2 \varkappa^3},$$

so that

$$\begin{aligned}&\sum_{k=1}^\infty \sum_{m=1}^\infty \frac{\mu(m)}{\alpha k} \left(\int_0^\infty \int_0^\infty f_1(hy_1, hy_2)\, dy_1 dy_2 \right)\bigg|_{h=\alpha k m} \\ &\qquad = \frac{2}{\rho \alpha^3 \varkappa^3} \sum_{k=1}^\infty \sum_{m=1}^\infty \frac{\mu(m)}{k^3 m^2} = \frac{1}{\rho \alpha^3} = n_1.\end{aligned} \tag{5.10}$$

Hence, we obtain the representation

$$\mathbb{E}_z^\gamma[\xi_1] - n_1 = \sum_{k=1}^\infty \sum_{m=1}^\infty \frac{\mu(m)}{\alpha k} \Delta_1(\alpha k m), \tag{5.11}$$

where

$$\Delta_1(h) := F_1(h) - \int_0^\infty \int_0^\infty f_1(hy_1, hy_2)\, dy_1 dy_2.$$

Using that $\delta_i(t) \ge \delta_* > 0$ and $\rho \le 1/2$, we have

$$F_1(h) \le \sum_{y_1=1}^\infty \sum_{y_2=0}^\infty h y_1 e^{-h(y_1+y_2)\delta_*/2} = \frac{h e^{-h\delta_*/2}}{(1 - e^{-h\delta_*/2})^3}.$$

Hence, $F_1(h) = O(h^{-2})$ as $h \to 0$ and $F_1(h) = O(h^{-\beta})$ for all $\beta > 0$ as $h \to +\infty$. Therefore, the function $F_1(h)$ is well defined for all $h > 0$ and its Mellin transform

$$M_1(s) := \int_0^\infty h^{s-1} F_1(h)\, dh \tag{5.12}$$

(see Titchmarsh 1986, § 1.5) is a regular function for $\operatorname{Re} s > 2$. From a two-dimensional version of the Muntz lemma (see Bogachev and Zarbaliev 2003), it then follows that the function $M_1(s)$ is in fact meromorphic in the semi-plane $\operatorname{Re} s > 1$ and has the single (simple) pole at point $s = 2$. Moreover, for all $1 < \operatorname{Re} s < 2$ the following formula is valid:

$$M_1(s) = \int_0^\infty h^{s-1} \Delta_1(h)\, dh. \tag{5.13}$$

The inversion formula for the Mellin transform (Widder 1946, Ch. VI, § 9, Theorem 9a) yields

$$\Delta_1(h) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h^{-s} M_1(s)\, ds, \qquad 1 < c < 2. \tag{5.14}$$

In order to make use of the formula (5.14), we need to find explicitly the analytic continuation of the function (5.12) to the strip $1 < \operatorname{Re} s < 2$. Let us use the Euler–Maclaurin summation formula (e.g., see Bhattacharya and Ranga Rao 1976, p. 264)

$$\sum_{y=0}^\infty f(y) = \int_0^\infty f(y)\, dy + \frac{1}{2} f(0) + \int_0^\infty B_1(y)\, f'(y)\, dy, \tag{5.15}$$

where $B_1(y) := y - [y] - 1/2$ and $[y]$ is the integer part of y. In view of Assumption 5.1 and equations (4.3), (4.21), we can apply this formula to the sum over y_2 in (5.9). Using the substitution $y_2 = t y_1/\rho$, we obtain

$$\begin{aligned} F_1(h) &= \sum_{y_1=1}^\infty h y_1 \int_0^\infty e^{-h\langle \tilde{y}, \delta(\tau(\tilde{y}))\rangle}\, dy_2 + \frac{1}{2} \sum_{y_1=1}^\infty h y_1 e^{-h y_1 \delta_1(0)} + O(1)\, \frac{e^{-\mathrm{const}\cdot h}}{h} \\ &= \frac{h}{\rho} \sum_{y_1=1}^\infty y_1^2 \int_0^\infty e^{-h y_1 \psi(t)}\, dt + O(1)\, \frac{e^{-\mathrm{const}\cdot h}}{h}, \end{aligned} \tag{5.16}$$

where

$$\psi(t) := \delta_1(t) + t\delta_2(t). \tag{5.17}$$

Keeping track of only the main term in (5.16) and writing dots for functions that are regular for $\operatorname{Re} s > 1$, the Mellin transform of $F_1(h)$ can be represented as follows:

$$\begin{aligned} M_1(s) &= \frac{1}{\rho} \int_0^\infty h^s \left(\sum_{y_1=1}^\infty y_1^2 \int_0^\infty e^{-h y_1 \psi(t)}\, dt \right) dh + \cdots \\ &= \frac{1}{\rho} \sum_{y_1=1}^\infty y_1^2 \int_0^\infty \left(\int_0^\infty h^s\, e^{-h y_1 \psi(t)}\, dh \right) dt + \cdots \\ &= \frac{1}{\rho} \sum_{y_1=1}^\infty \frac{1}{y_1^{s-1}} \int_0^\infty \frac{\Gamma(s+1)}{\psi(t)^{s+1}}\, dt + \cdots \\ &= \frac{1}{\rho}\, \zeta(s-1)\, \Gamma(s+1)\, \Psi(s) + \cdots \end{aligned} \tag{5.18}$$

where

$$\Psi(s) := \int_0^\infty \frac{1}{\psi(t)^{s+1}}\,dt.$$

Recalling (4.1) and (4.10), the function (5.17) is rewritten in the form

$$\psi(t) = \varkappa \kappa_\gamma^{1/3}\sqrt{1+t^2}, \qquad t_0 \le t \le t_1,$$

and Assumption 4.1 implies that the function $\Psi(s)$ is regular if $\operatorname{Re} s > 0$.

Furthermore, it is well known that the gamma-function $\Gamma(s)$ is analytic for $\operatorname{Re} s > 0$, whereas the zeta-function $\zeta(s)$ has the single pole at point $s = 1$ (see Titchmarsh 1939, § 4.41, 4.43). It follows that the right-hand side of (5.18) is regular in the strip $1 < \operatorname{Re} < 2$ and hence provides the required analytic continuation of the function $M_1(s)$ originally defined by (5.12).

Setting $h = \alpha k m$ and returning to formulas (5.11) and (5.14), we get for $1 < c < 2$

$$\begin{aligned}\mathbb{E}_z^\gamma[\xi_1] - n_1 &= \sum_{k=1}^\infty \sum_{m=1}^\infty \frac{\mu(m)}{\alpha k}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{M_1(s)}{(km\alpha)^s}\,ds \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{M_1(s)\,\zeta(s+1)}{\alpha^{s+1}\,\zeta(s)}\,ds.\end{aligned} \tag{5.19}$$

Using that $\zeta(s) \ne 0$ for $\operatorname{Re} s \ge 1$, we can transform the contour of integration $\operatorname{Re} s = c$ in (5.19) to the union of a small semi-circle $s = 1 + r e^{it}$ $(-\pi/2 \le t \le \pi/2)$ and two vertical lines, $s = 1 \pm it$ $(t \ge r)$. Furthermore, studying the resolution (5.18), one can show that $M_1(1 \pm it) = O(|t|^{-2})$ as $t \to \infty$. As a result, the right-hand side of (5.19) is bounded by $O(\alpha^{-2})$. This proves Theorem 5.4 for $i = 1$.

The case $i = 2$ is considered along the same lines, by setting

$$F_2(h) := \sum_{y_1=1}^\infty \sum_{y_2=0}^\infty f_2(hy_1, hy_2), \qquad f_2(y_1, y_2) := y_2 e^{-\langle \tilde{y}, \delta(\tau(\tilde{y}))\rangle}.$$

A crucial point is then to check that

$$\int_0^\infty \int_0^\infty f_2(hy_1, hy_2)\,dy_1 dy_2 = \frac{2c_\gamma}{\rho^2 h^2 \varkappa^3},$$

which implies the identity

$$\begin{aligned}&\sum_{k=1}^\infty \sum_{m=1}^\infty \frac{\mu(m)}{\alpha k}\int_0^\infty \left(\int_0^\infty f_2(hy_1, hy_2)\,dy_1 dy_2\right)\Bigg|_{h=\alpha k m} \\ &\qquad = \frac{2c_\gamma}{\rho^2 \alpha^3 \varkappa^3} \sum_{k=1}^\infty \sum_{m=1}^\infty \frac{\mu(m)}{k^3 m^2} = \frac{c_\gamma}{\rho^2 \alpha^3} = \frac{c_n n_1}{c_n} = n_2.\end{aligned} \tag{5.20}$$

The latter calculation, along with (5.10), explains the specific choice of the parameter α made in (4.5). $\square$

6. Asymptotics of the second order moments

For the random variable $\nu(x)$ with geometric distribution (3.3), its variance is given by

$$\mathbb{V}\mathrm{ar}_z^\gamma[\nu(x)] = \frac{z_1^{x_1} z_2^{x_2}}{(1 - z_1^{x_1} z_2^{x_2})^2} = \sum_{k=1}^\infty k(z_1^{x_1} z_2^{x_2})^k.$$

Using that the random variables $\nu(x)$ are independent for different $x \in X$, from (3.4) we have

$$\sigma_i^2 := \mathbb{V}\mathrm{ar}_z^\gamma[\xi_i] = \sum_{x\in X} x_i^2\, \mathbb{V}\mathrm{ar}_z^\gamma[\nu(x)] = \sum_{k=1}^\infty \sum_{x\in X} k x_i^2 (z_1^{x_1} z_2^{x_2})^k. \tag{6.1}$$

Similarly, the covariance is given by

$$\sigma_{12} := \mathbb{C}\mathrm{ov}_z^\gamma(\xi_1, \xi_2) = \sum_{x\in X} x_1 x_2\, \mathbb{V}\mathrm{ar}_z^\gamma[\nu(x)] = \sum_{k=1}^\infty \sum_{x\in X} k x_1 x_2 (z_1^{x_1} z_2^{x_2})^k. \tag{6.2}$$

Theorem 6.1. *Suppose that the parameters $z_1(x)$, $z_2(x)$ $(x \in X)$ are chosen subject to formulas* (4.4), *with $\delta_1(t)$, $\delta_2(t)$ defined in* (4.9), (4.21). *Then, as $n \to \infty$,*

$$\begin{gathered} \sigma_1^2 \sim \frac{3n_1^{4/3}}{\varkappa} \int_0^1 \frac{du}{g_\gamma''(u)^{1/3}}\,, \qquad \sigma_2^2 \sim \frac{3n_1^{4/3}}{\varkappa} \int_0^1 \frac{g_\gamma'(u)^2 du}{g_\gamma''(u)^{1/3}}\,, \\ \sigma_{12} \sim \frac{3n_1^{4/3}}{\varkappa} \int_0^1 \frac{g_\gamma'(u)\, du}{g_\gamma''(u)^{1/3}}\,. \end{gathered} \tag{6.3}$$

Proof. Consider the case $i = 1$. Substituting (3.4) into (6.1) and using the Möbius inversion formula (see (4.15)), we obtain

$$\begin{aligned} \sigma_1^2 &= \sum_{k=1}^\infty \sum_{x\in X} k x_1^2 e^{-k\alpha\langle\tilde{x},\delta(\tau(\tilde{x}))\rangle} \\ &= \sum_{k=1}^\infty \sum_{m=1}^\infty k m^2 \mu(m) \sum_{y_1=1}^\infty \sum_{y_2=0}^\infty y_1^2 e^{-km\alpha\langle\tilde{y},\delta(\tau(\tilde{y}))\rangle}. \end{aligned} \tag{6.4}$$

Arguing as in the proof of Theorem 4.1, we obtain

$$\begin{aligned} \lim_{n\to\infty} \alpha^4 \sum_{y_1=1}^\infty \sum_{y_2=0}^\infty y_1^2 e^{-km\alpha\langle\tilde{y},\delta(\tau(\tilde{y}))\rangle} &= \int_0^\infty \!\! \int_0^\infty y_2^2 e^{-km(y_1\delta_1(y_2/y_1)+y_2\delta_2(y_2/y_1))}\, dy_1 dy_2 \\ &= \int_0^\infty u^3\, du \int_0^\infty e^{-kmu(\delta_1(t)+t\delta_2(t))}\, dt \\ &= \frac{6}{(km)^4} \int_0^\infty \frac{1}{(\delta_1(t)+t\delta_2(t))^4}\, dt\,. \end{aligned}$$

Returning to (6.4) and using (4.9), (4.10), we get

$$\begin{aligned}\lim_{n\to\infty}\alpha^4\sigma_1^2 &= 6\sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{m=1}^{\infty}\frac{\mu(m)}{m^2}\int_0^{\infty}\frac{1}{(\delta_1(t)+t\delta_2(t))^4}\,dt\\ &= \frac{6\zeta(3)}{\zeta(2)}\int_{t_0}^{t_1}\frac{1}{\varkappa^4 g_\gamma''(u_\gamma(t))^{4/3}}\,dt\\ &= \frac{3}{\varkappa}\int_0^1\frac{1}{g_\gamma''(u)^{1/3}}\,du,\end{aligned}$$

and the first formula in (6.3) follows, since $\alpha=(\rho n_1)^{-1/3}$ and $\rho\to 1$ as $n\to\infty$.

Similar considerations in the case $i=2$ yield

$$\begin{aligned}\lim_{n\to\infty}\alpha^4\sigma_2^2 &= \frac{6\zeta(3)}{\zeta(2)}\int_{t_0}^{t_1}\frac{t^2}{\varkappa^4 g_\gamma''(u_\gamma(t))^{4/3}}\,dt\\ &= \frac{3}{\varkappa}\int_0^1\frac{g_\gamma'(u)^2}{g_\gamma''(u)^{1/3}}\,du.\end{aligned}$$

Finally, applying similar arguments to the representation (6.2) we get

$$\begin{aligned}\lim_{n\to\infty}\alpha^4\sigma_{12} &= \frac{6\zeta(3)}{\zeta(2)}\int_{t_0}^{t_1}\frac{t}{\varkappa^4 g_\gamma''(u_\gamma(t))^{4/3}}\,dt\\ &= \frac{3}{\varkappa}\int_0^1\frac{g_\gamma'(u)}{g_\gamma''(u)^{1/3}}\,du.\end{aligned}$$

The proof Theorem 6.1 is complete. $\square$

7. Local limit theorem

Recall that the random variables $\{\nu(x)\}_{x\in X}$, with respect to the distribution $\mathbb{Q}_z^\gamma$, are independent and have geometric distribution with parameter $z_1^{x_1}z_2^{x_2}$. In particular,

$$\mathbb{E}_z^\gamma[\nu(x)]=\frac{z_1^{x_1}z_2^{x_2}}{1-z_1^{x_1}z_2^{x_2}}\,,\qquad \mathbb{V}\mathrm{ar}_z^\gamma[\nu(x)]=\frac{z_1^{x_1}z_2^{x_2}}{(1-z_1^{x_1}z_2^{x_2})^2}\,. \tag{7.1}$$

Moreover, the characteristic function of $\nu(x)$ is given by

$$f_{\nu(x)}(t)=\frac{1-z_1^{x_1}z_2^{x_2}}{1-z_1^{x_1}z_2^{x_2}e^{it}}\,, \tag{7.2}$$

and hence the characteristic function of the vector $\xi=\sum_{x\in X}x\nu(x)$ is given by

$$f_\xi(\lambda)=\prod_{x\in X}f_{\nu(x)}(\langle x,\lambda\rangle)=\prod_{x\in X}\frac{1-z_1^{x_1}z_2^{x_2}}{1-z_1^{x_1}z_2^{x_2}\,e^{i\langle x,\lambda\rangle}}\,. \tag{7.3}$$

Let $f^*_{\nu(x)}(t)$ be the characteristic function of the centered random variable $\nu^0(x) := \nu(x) - \mathbb{E}^\gamma_z[\nu(x)]$. Let us set $a_z := \mathbb{E}^\gamma_z[\xi]$, then the characteristic function of the vector $\xi - a_z$ is given by

$$f^*_\xi(\lambda) = \mathbb{E}^\gamma_z\bigl[e^{i\langle\lambda,\,\xi - a_z\rangle}\bigr] = \prod_{x\in X} f^*_{\nu(x)}(\langle\lambda, x\rangle). \tag{7.4}$$

Let K_z be the covariance matrix of ξ,

$$K_z = \mathbb{C}\mathrm{ov}^\gamma_z(\xi,\xi) = \begin{pmatrix} \mathbb{V}\mathrm{ar}^\gamma_z[\xi_1] & \mathbb{C}\mathrm{ov}^\gamma_z(\xi_1,\xi_2) \\ \mathbb{C}\mathrm{ov}^\gamma_z(\xi_1,\xi_2) & \mathbb{V}\mathrm{ar}^\gamma_z[\xi_2] \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}. \tag{7.5}$$

Furthermore, let V_z be a symmetric positive definite matrix such that

$$V_z^2 = K_z^{-1}, \qquad K_z = V_z^{-2} \tag{7.6}$$

(the inverse matrix K_z^{-1} does exist since K_z is non-degenerate by Theorem 6.1).

Let $\phi(y)$ denote the probability density function of the standard two-dimensional normal distribution (with zero mean and identity covariance matrix), that is,

$$\phi(y) = \frac{1}{2\pi}\, e^{-|y|^2/2}, \qquad y \in \mathbb{R}^2. \tag{7.7}$$

Let us also denote by $\phi_z(y)$ the probability density function of the normal distribution with mean a_z and covariance matrix K_z,

$$\phi_z(y) := (\det K_z)^{-1/2}\phi((y - a_z)V_z) \tag{7.8}$$

We are now in a position to state a local limit theorem for the asymptotics of the probability $\mathbb{Q}^\gamma_z\{\xi = n\}$ as $n \to \infty$. This theorem will play a crucial role in order to pass, using equation (3.5), from the unconditional distribution $\mathbb{Q}^\gamma_z$ to the conditional one, $\mathbb{P}^\gamma_n$. Let us point out that the random variables ξ_1, ξ_2 are given by equations (3.4), so we deal here with a two-dimensional local limit theorem for independent, non-identically distributed summands.

Theorem 7.1. *Suppose that the parameters $z_1(x)$, $z_2(x)$ ($x \in X$) are chosen subject to formulas* (4.4), *with $\delta_1(t)$, $\delta_2(t)$ defined in* (4.21). *Let $m = (m_1, m_2)$ be a two-dimensional vector with non-negative integer components, and set $y_{m,n} := (m - a_z)V_z$. Then, uniformly in m,*

$$\mathbb{Q}^\gamma_z\{\xi = m\} - \phi_z(m) = O(n_1^{-5/3}). \tag{7.9}$$

For the proof of this theorem, we will need some (mostly well known) auxiliary facts about the matrix norm (for details, see Bogachev and Zarbaliev 2003).

Lemma 7.2. *Let A and B be symmetric matrices such that $A = B^2$. Then*

$$\|A\| = \|B\|^2.$$

Lemma 7.3. *If A is a symmetric non-degenerate matrix of order 2, then*

$$\|A^{-1}\| = \frac{\|A\|}{|\det A|}.$$

Lemma 7.4. *Let $A = (a_{ij})$ be a matrix of size $d \times d$. Then*

$$\|A\|^2 \le \sum_{i,j=1}^{d} a_{ij}^2.$$

Lemma 7.5. *Let $A = (a_{ij})$ be a matrix of size $d \times d$ with non-negative entries. Then*

$$\|A\|^2 \ge \frac{1}{d} \sum_{j=1}^{d} a_{jj}^2.$$

The next lemmas contain some useful analytic estimates. In what follows, the notation $a_n \asymp b_n$ means that

$$0 < \liminf_{n\to\infty} \frac{a_n}{b_n} \le \limsup_{n\to\infty} \frac{a_n}{b_n} < \infty.$$

Lemma 7.6. *For all $\lambda \in \mathbb{R}^2$,*

$$|f_\xi^*(\lambda)| \le e^{-J_\alpha(\lambda)}, \tag{7.10}$$

where

$$J_\alpha(\lambda) := \sum_{x\in X} e^{-\alpha\langle\delta,\, x\rangle} \mathbf{1}_{\{\cos\langle\lambda, x\rangle \le 0\}}. \tag{7.11}$$

Proof. We have

$$|f_\xi^*(\lambda)| = |f_\xi(\lambda)| = \prod_{x\in X} |f_{\nu(x)}(\langle\lambda,\, x\rangle)|.$$

From (7.2) it is seen that

$$|f_{\nu(x)}(t)| = \frac{1 - z^x}{|1 - z^x e^{it}|}.$$

Note that

$$|1 - z^x e^{it}|^2 = (1 - z^x e^{it})(1 - z^x e^{-it}) = 1 - 2z^x \cos t + z^{2x} \ge 1,$$

if $\cos t \le 0$. Therefore,

$$\begin{aligned} |f_\xi(\lambda)| &\le \exp\Biggl\{\sum_{x\in X} \ln |f_{\nu(x)}(\langle\lambda,\, x\rangle)| \cdot \mathbf{1}_{\{\cos\langle\lambda, x\rangle \le 0\}}\Biggr\} \\ &\le \exp\Biggl\{\sum_{x\in X} \ln(1 - z^x) \cdot \mathbf{1}_{\{\cos\langle\lambda, x\rangle \le 0\}}\Biggr\} \\ &\le \exp\Biggl\{-\sum_{x\in X} z^x\, \mathbf{1}_{\{\cos\langle\lambda, x\rangle \le 0\}}\Biggr\}. \end{aligned}$$

Recalling expressions (4.4) we obtain (7.10). □

Lemma 7.7. *As $n \to \infty$,*

$$\det K_z \sim \left(\frac{3}{\varkappa}\right)^2 \left(\int_0^1 \frac{du}{g_\gamma''(u)^{1/3}} \int_0^1 \frac{g_\gamma'(u)^2\,du}{g_\gamma''(u)^{1/3}} - \left(\int_0^1 \frac{g_\gamma'(u)\,du}{g_\gamma''(u)^{1/3}}\right)^2\right) n_1^{8/3}. \tag{7.12}$$

Proof. From the definition (7.5) we have $\det K_z = \sigma_1^2\sigma_2^2 - \sigma_{12}^2$, so (7.12) follows from Theorem 6.1. □

Lemma 7.8. *The norm of the matrix K_z is of order $n_1^{4/3}$,*

$$\|K_z\| \asymp n_1^{4/3} \qquad (n \to \infty).$$

Proof. Using Lemma 7.4 and Theorem 6.1 we get

$$\|K_z\|^2 \le \sigma_1^4 + 2\sigma_{12}^2 + \sigma_2^4 = O(n_1^{8/3}). \tag{7.13}$$

On the other hand, by Lemma 7.5 and Theorem 6.1

$$\begin{aligned}\|K_n\|^2 &\ge \frac{1}{2}\,(\sigma_1^4 + \sigma_2^4) \ge \sigma_1^2\sigma_2^2\\ &\sim \left(\frac{3}{\varkappa}\right)^2 n_1^{8/3} \int_0^1 \frac{du}{g_\gamma''(u)^{1/3}} \cdot \int_0^1 \frac{g_\gamma'(u)^2\,du}{g_\gamma''(u)^{1/3}}.\end{aligned}$$

and the lower bound follows. □

Lemma 7.9. *As $n \to \infty$, we have $\|V_z\| \asymp n_1^{-2/3}$.*

Proof. Since $V_z^2 = K_z^{-1}$, Lemma 7.2 implies $\|V_z\|^2 = \|K_z^{-1}\|$. Furthermore, Lemma 7.3 yields

$$\|K_z^{-1}\| = \frac{\|K_z\|}{\det K_z},$$

and it remains to apply Lemmas 7.7 and 7.8 to complete the proof. □

For the proof of Theorem 7.1, we will need to estimate the so-called *Lyapunov quotient* defined as

$$L_z := \|K_z^{-1}\|^{3/2} \sum_{x\in X} |x|^3\, \mathbb{E}_z\, |\nu^0(x)|^3. \tag{7.14}$$

By Lemma 7.3, this expression can be rewritten as follows

$$L_z = \left(\frac{\|K_z\|}{\det K_z}\right)^{3/2} \sum_{x\in X} |x|^3\, \mathbb{E}_z\, |\nu^0(x)|^3. \tag{7.15}$$

Lemma 7.10. *Suppose that the parameters z_1 and z_2 are chosen subject to* (4.4), (4.21). *Then the Lyapunov quotient* (7.14) *is of order $n_1^{-1/3}$, that is,*

$$L_z \asymp n_1^{-1/3}. \tag{7.16}$$

Proof. According to Lemma 7.7 and 7.4, we have

$$\det K_z \asymp n_1^{8/3}, \qquad \|K_z\| = O(n_1^{4/3}). \tag{7.17}$$

Note that by Jensen's inequality

$$|x|^3 = (x_1^2 + x_2^2)^{3/2} \le \sqrt{2}\,(x_1^3 + x_2^3),$$

and similarly to the proof of Theorem 6.1 one obtains

$$\begin{aligned}\sum_{x\in X} |x|^3\, \mathbb{E}_z\, |\nu^0(x)|^3 &= O(1) \sum_{x\in X} (x_1^3 + x_2^3)\, \mathbb{E}_z[\nu(x)^3] \\ &= O(\alpha_1^{-5}) = O(n_1^{5/3}).\end{aligned} \tag{7.18}$$

Substituting estimates (7.17) and (7.18) into (7.15), we get the upper bound for L_z:

$$L_z = O\big(n_1^{-4/3}\big)^{3/2} n_1^{5/3} = O(n_1^{-1/3}).$$

To obtain the lower bound, note that by Lemma 7.5 and Theorem 6.1

$$\|K_z\| \ge \frac{1}{\sqrt{2}}\left(\sigma_1^4 + \sigma_2^4\right)^{1/2} \ge (\sigma_1^2\sigma_2^2)^{1/2} \sim \frac{2}{(\delta_1\delta_2)^{1/2}}\, n_1^{4/3}$$

(cf. (7.13)). Furthermore, using the Lyapunov inequality we have

$$\mathbb{E}_z^\gamma\, |\nu^0(x)|^3 \ge \big(\mathbb{E}_z^\gamma\, |\nu^0(x)|^2\big)^{3/2} = \big(\mathbb{V}\mathrm{ar}_z^\gamma[\nu(x)]\big)^{3/2},$$

so that

$$\sum_{x\in X} |x|^3\, \mathbb{E}_z^\gamma\, |\nu^0(x)|^3 \ge \sum_{x\in X} x_1^3 \big(\mathbb{V}\mathrm{ar}_z^\gamma[\nu(x)]\big)^{3/2}.$$

Arguing similarly as in the proof of Theorem 6.1, one can show that, as $n \to \infty$, the last sum is asymptotically equivalent (up to a positive constant) to α^{-5}, which is of order of $n_1^{5/3}$ (cf. (7.18)).

To conclude, the lower bounds for all the terms involved in the expression (7.15) prove to be consistent with the upper bounds, and therefore the lemma is proved. □

The proof of the next lemma can be found in Bogachev and Zarbaliev (2003).

Lemma 7.11. *Let $\lambda = (\lambda_1, \lambda_2)$ be such that $|\lambda| \le L_z^{-1}$. Then*

$$\left| f_\xi^*(\lambda V_z) - e^{-|\lambda|^2/2} \right| \le 16 L_z |\lambda|^3 e^{-|\lambda|^2/3}. \tag{7.19}$$

We can now proceed to the proof of Theorem 7.1.

Proof of Theorem 7.1. By the Fourier inversion formula, we can write

$$\begin{aligned}\mathbb{Q}_z\{\xi=m\}&=\frac{1}{(2\pi)^2}\int_T e^{-i\langle\lambda,m\rangle}f_\xi(\lambda)\,d\lambda\\&=\frac{1}{(2\pi)^2}\int_T e^{-i\langle\lambda,m-a_z\rangle}f_\xi^*(\lambda)\,d\lambda,\end{aligned}\tag{7.20}$$

where $T:=\{\lambda=(\lambda_1,\lambda_2):|\lambda_i|\le\pi,\ i=1,2\}$. On the other hand, the characteristic function corresponding to the probability density $\phi_z(y)$ is given by

$$f_z(\lambda)=e^{ia_z\lambda-|\lambda V_z^{-1}|^2/2},\qquad \lambda\in\mathbb{R}^2.$$

Hence, by the inversion formula we obtain

$$\phi_z(m)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-i\langle\lambda,m-a_z\rangle-|\lambda V_z^{-1}|^2/2}\,d\lambda.\tag{7.21}$$

Note that if $|\lambda V_z^{-1}|\le L_z^{-1}$ then, according to Lemmas 7.9 and 7.10,

$$|\lambda|\le|\lambda V_z^{-1}|\cdot\|V_z\|\le L_z^{-1}\|V_z\|=O(n_1^{-1/3})=o(1),\tag{7.22}$$

which implies that $\lambda\in T$. Using this observation, from (7.20) and (7.21) we get

$$\bigl|\mathbb{Q}_n\{\xi=m\}-\phi_z(m)\bigr|\le I_1+I_2+I_3,\tag{7.23}$$

where

$$\begin{aligned}I_1&:=\frac{1}{(2\pi)^2}\int_{|\lambda V_z^{-1}|\le L_z^{-1}}\bigl|f_\xi^*(\lambda)-e^{-|\lambda V_z^{-1}|^2/2}\bigr|\,d\lambda,\\I_2&:=\frac{1}{(2\pi)^2}\int_{|\lambda V_z^{-1}|>L_z^{-1}}e^{-|\lambda V_z^{-1}|^2/2}\,d\lambda,\\I_3&:=\frac{1}{(2\pi)^2}\int_{T\cap\{|\lambda V_z^{-1}|>L_z^{-1}\}}|f_\xi^*(\lambda)|\,d\lambda.\end{aligned}$$

By the substitution $\lambda=\tilde\lambda V_z$, the integral I_1 is reduced to

$$\begin{aligned}I_1&=\frac{(\det K_z)^{-1/2}}{(2\pi)^2}\int_{|\tilde\lambda|\le L_z^{-1}}\bigl|f_\xi^*(\tilde\lambda V_z)-e^{-|\tilde\lambda|^2/2}\bigr|\,d\tilde\lambda\\&=O(n_1^{-4/3})\,L_z\int_{\mathbb{R}^2}|\tilde\lambda|^3e^{-|\tilde\lambda|^2/6}\,d\tilde\lambda=O(n_1^{-5/3}),\end{aligned}\tag{7.24}$$

on account of Lemmas 7.7, 7.10 and 7.11.

To estimate I_2, we again use the substitution $\lambda=\tilde\lambda V_z$ and pass to polar coordinates to get

$$I_2=\frac{1}{(2\pi)^2}\int_{|\lambda|>L_z^{-1}}e^{-|\lambda|^2/2}\,d\lambda=\frac{1}{2\pi}\int_{L_z^{-1}}^\infty\rho\,e^{-\rho^2/2}\,d\rho=(2\pi)^{-1}e^{-L_z^{-2}/2},\tag{7.25}$$

which is $o(n_1^{-\beta})$ for any $\beta > 0$, as follows from Lemma 7.10.

For I_3 we obtain, using Lemma 7.6,

$$I_3 = O(1) \int_{T\cap\{|\lambda V_z^{-1}|>L_z^{-1}\}} e^{-J_\alpha(\lambda)}\, d\lambda\,. \tag{7.26}$$

The condition $|\lambda V_z^{-1}| > L_z^{-1}$ implies that $|\lambda| > \sqrt{2}\, c\alpha$ and hence $\max\{|\lambda_1|, |\lambda_2|\} > c\alpha$, where $c > 0$ is a suitable (small enough) constant. Indeed, assuming the contrary, from (4.4) and Lemmas 7.2, 7.8 and 7.10 it would follow that

$$1 < L_z|\lambda V_z^{-1}| \le \sqrt{2}\, c\alpha\, L_z\|K_z\|^{1/2} = O(c),$$

which leads to a contradiction if we let $c \downarrow 0$. Hence, the estimate (7.26) is reduced to

$$I_3 = O(1) \left(\int_{|\lambda_1|>c\alpha} + \int_{|\lambda_2|>c\alpha} \right) e^{-J_\alpha(\lambda)}\, d\lambda, \tag{7.27}$$

where $J_\alpha(\lambda)$ is given by (7.11).

Note that, by Assumption 4.1 and formulas (4.21), the functions $\delta_1(t)$, $\delta_2(t)$ are bounded above, $\sup_t \delta_i(t) \le \delta^* < \infty$. Hence, we obtain the estimate

$$J_\alpha(\lambda) \ge \sum_{x\in X} e^{-\alpha\delta^*(x_1+x_2)} \mathbf{1}_{\{\cos\langle x,\lambda\rangle\le 0\}}. \tag{7.28}$$

In estimation of the first integral in (7.27), we may assume without loss of generality that $\lambda_1 > 0$ (the other cases are considered similarly). Keeping in the sum (7.28) only pairs of the form $x = (x_1, 1)$, we get

$$J_\alpha(\lambda) \ge \sum_{x_1=1}^{\infty} e^{-\alpha\delta^*(x_1+1)} \mathbf{1}_{\{\cos(\lambda_1 x_1+\lambda_2)\le 0\}}. \tag{7.29}$$

It is easy to see that $\cos(\lambda_1 x_1 + \lambda_2) \le 0$ for all $k \ge 0$ such that

$$2\pi k + \frac{\pi}{2} \le \lambda_1 x_1 + \lambda_2 \le 2\pi k + \frac{3\pi}{2}. \tag{7.30}$$

In turn, these inequalities are valid for all integers x_1 such that $x_* < x_1 \le x^*$, where

$$\begin{aligned} x^* = x^*(k) &:= \left[\frac{2\pi k}{\lambda_1} + \frac{3\pi}{2\lambda_1} - \frac{\lambda_2}{\lambda_1} \right], \\ x_* = x_*(k) &:= \left[\frac{2\pi k}{\lambda_1} + \frac{\pi}{2\lambda_1} - \frac{\lambda_2}{\lambda_1} \right] \end{aligned} \tag{7.31}$$

(the square brackets denote the integer part of a number). Hence, (7.29) yields

$$J_\alpha(\lambda) \ge \sum_{k=0}^{\infty} \sum_{x_1=x_*+1}^{x^*} e^{-\alpha\delta^*(x_1+1)} \ge e^{-\alpha\delta^*} \sum_{k=0}^{\infty} (x^* - x_*)\, e^{-\alpha\delta^* x^*}. \tag{7.32}$$

Using the elementary inequality $[a+b] \ge [a]+[b]$ and also that $[b] \ge b/2$ for $b \ge 1$, from (7.31) we get

$$x^* - x_* \ge \left[\frac{\pi}{\lambda_1}\right] \ge \frac{\pi}{2\lambda_1},$$

and (7.32) amounts to

$$J_\alpha(\lambda) \ge \frac{\pi}{2\lambda_1}\, e^{-\alpha\delta^*} \sum_{k=0}^{\infty} e^{-\alpha\delta^* x^*}. \tag{7.33}$$

Recalling that $\lambda_2 \ge -\pi$ and $\lambda_1 > c\alpha$, we have

$$\alpha x^* \le \frac{\alpha}{\lambda_1}\left(2\pi k + \frac{3\pi}{2} - \lambda_2\right) \le \frac{2\pi\alpha k}{\lambda_1} + \frac{5\pi}{2c}.$$

Substitute this into (7.33) to obtain

$$J_\alpha(\lambda) \ge \frac{C}{\lambda_1}\, e^{-\alpha\delta^*} \sum_{k=0}^{\infty} e^{-2\pi k\alpha\delta^*/\lambda_1} = \frac{Ce^{-\alpha\delta^*}}{\lambda_1(1-e^{-2\pi\alpha\delta^*/\lambda_1})} \ge \frac{Ce^{-\alpha\delta^*}}{2\pi\alpha\delta^*} \asymp n_1^{1/3},$$

since $\alpha \asymp n_1^{1/3}$. As a result, the first integral on the right-hand side of (7.27) is bounded from above by $O(1)\exp(-\mathrm{const}\cdot n_1^{1/3}) = o(n_1^{-5/3})$, in accord with the statement of the theorem.

The second integral in (7.27) where $|\lambda_2| > c\alpha$, is estimated similarly by reducing summation in (7.11) to that over $x = (1, x_2)$ only (cf. (7.29)). Therefore, $I_3 = o(n_1^{-5/3})$, and this completes the proof of the theorem. □

Corollary 7.12. *The probability $\mathbb{Q}_z^\gamma\{\xi = n\}$ satisfies the following asymptotic bound:*

$$\liminf_{n\to\infty} n_1^{4/3}\, \mathbb{Q}_z^\gamma\{\xi = n\} \ge 0. \tag{7.34}$$

Proof. In Theorem 7.1, let us take m to be $n = (n_1, n_2)$, then $y_{n,n} = (n - a_z)V_z$. According to Theorem 5.4, $a_z = n + O(n_1^{2/3})$. Together with Lemma 7.9 this implies

$$|y_{n,n}| \le |n - a_z| \cdot \|V_z\| = O(1) \qquad (n \to \infty).$$

Hence, from (7.8) and (7.12) we get

$$\liminf_{n\to\infty} n_1^{4/3}\phi_z(y_{n,n}) = \frac{1}{2\pi} \liminf_{n\to\infty} n_1^{4/3}(\det K_z)^{-1/2} e^{-|y_{n,n}|^2/2} \ge \mathrm{const} > 0. \tag{7.35}$$

On the other hand, (7.9) and (7.35) imply

$$\begin{aligned}\liminf_{n\to\infty} n_1^{4/3}\, \mathbb{Q}_z^\gamma\{\xi = n\} &= \liminf_{n\to\infty} \left(n_1^{4/3}\phi_z(y_{n,n}) + O(n_1^{-1/3})\right)\\ &= \liminf_{n\to\infty} n_1^{4/3}\phi_z(y_{n,n}) > 0,\end{aligned}$$

and (7.34) is proved. □

8. Law of Large Numbers

Our next result states that with respect to the distribution $\mathbb{Q}_z^\gamma$, the given curve γ is the limit of polygons Γ under the scaling with the coefficient n_1^{-1}.

Theorem 8.1. *Let the parametric functions $z_1(x)$, $z_2(x)$ ($x \in X$) be chosen according to formulas* (4.4), *with $\delta_1(t)$, $\delta_2(t)$ defined in* (4.21). *Then for any $\varepsilon > 0$*

$$\lim_{n\to\infty} \mathbb{Q}_z^\gamma \left\{ d_1(n_1^{-1}\Gamma, \gamma) \le \varepsilon \right\} = 1.$$

Proof. In view of Theorem 5.1, we only need to check that for each $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{Q}_z^\gamma \left\{ \sup_{0\le t\le\infty} \left| \frac{1}{n_1}\ell_\Gamma(t) - \frac{1}{n_1}\,\mathbb{E}_z^\gamma[\ell_\Gamma(t)] \right| > \varepsilon \right\} = 0. \tag{8.1}$$

Note that the random process

$$\tilde{\ell}_\Gamma(t) := \ell_\Gamma(t) - \mathbb{E}_z^\gamma[\ell_\Gamma(t)] \qquad (0 \le t \le \infty) \tag{8.2}$$

has independent increments and zero mean, hence it is a martingale with respect to the natural filtration $\mathcal{F}_t := \sigma\{\nu(x),\, x \in X(t),\, t \in [0,\infty]\}$. From the definition of $\ell_\Gamma(t)$ (see (4.7)), it is also clear that $\tilde{\ell}_\Gamma(t)$ is a cadlag process, that is, its paths are everywhere right-continuous and have left limits. Therefore, applying Kolmogorov–Doob's submartingale inequality (see Yeh 1995, Corollary 2.1) we obtain

$$\begin{aligned}
\mathbb{Q}_z^\gamma \left\{ \sup_{0\le t\le\infty} \left|\tilde{\ell}_\Gamma(t)\right| > n_1\varepsilon \right\} &\le \frac{1}{(n_1\varepsilon)^2} \sup_{0\le t\le\infty} \mathbb{V}\mathrm{ar}_z^\gamma[\ell_\Gamma(t)] \\
&\le \frac{1}{n_1^2\varepsilon^2}\, \mathbb{V}\mathrm{ar}_z^\gamma[\ell_\Gamma] \\
&\le \frac{1}{n_1^2\varepsilon^2} \left(\mathbb{V}\mathrm{ar}_z^\gamma[\xi_1] + \mathbb{V}\mathrm{ar}_z^\gamma[\xi_2] \right) \\
&= O(1)\, n_1^{-2/3} \to 0,
\end{aligned}$$

according to Theorem 6.1. Hence, (8.1) is proved. □

Let us now prove a conditional version of a Law of Large Numbers, which states that polygons $\Gamma \in \mathcal{L}_n$ converge, as $n \to \infty$, to the curve γ under the measure $\mathbb{P}_n^\gamma$.

Theorem 8.2. *For all $\varepsilon > 0$*

$$\lim_{n\to\infty} \mathbb{P}_n^\gamma \left\{ d_1(n_1^{-1}\Gamma, \gamma) \le \varepsilon \right\} = 1. \tag{8.3}$$

Proof. Similarly to the proof of Theorem 8.1, it suffices to show that for each $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}_n^\gamma \left\{ \sup_{0\le t\le\infty} \left| \frac{1}{n_1} \tilde{\ell}_\Gamma(t) \right| > \varepsilon \right\} = 0,$$

where the random process $\tilde{\ell}_\Gamma(t)$ is defined in (8.2). According to formula (3.5),

$$\begin{aligned} \mathbb{P}_n^\gamma \left\{ \sup_{0\le t\le\infty} \left|\tilde{\ell}_\Gamma(t)\right| > \varepsilon n_1 \right\} &= \frac{\mathbb{Q}_z^\gamma \left\{ \sup_{0\le t\le\infty} \left|\tilde{\ell}_\Gamma(t)\right| > \varepsilon n_1,\ \Gamma \in \mathcal{L}_n \right\}}{\mathbb{Q}_z^\gamma\{\Gamma \in \mathcal{L}_n\}} \\ &\le \frac{\mathbb{Q}_z^\gamma \left\{ \sup_{0\le t\le\infty} \left|\tilde{\ell}_\Gamma(t)\right| > \varepsilon n_1 \right\}}{\mathbb{Q}_z^\gamma\{\xi = n\}}. \end{aligned} \tag{8.4}$$

As was mentioned in the proof of Theorem 8.1, the random process $(\tilde{\ell}_\Gamma(t), \mathcal{F}_t)_{t\in[0,\infty]}$ is a cadlag martingale, hence by Kolmogorov–Doob's submartingale inequality we get

$$\begin{aligned} \mathbb{Q}_z^\gamma \left\{ \sup_{0\le t\le\infty} \left|\tilde{\ell}_\Gamma(t)\right| > n_1\varepsilon \right\} &\le \frac{1}{(n_1\varepsilon)^6} \sup_{0\le t\le\infty} \mathbb{E}_z^\gamma \left|\tilde{\ell}_\Gamma(t)\right|^6 \\ &\le \frac{1}{n_1^6\varepsilon^6}\, \mathbb{E}_z^\gamma \left|\ell_\Gamma - \mathbb{E}_z^\gamma[\ell_\Gamma]\right|^6 \\ &= \frac{1}{n_1^6\varepsilon^6}\, \mathbb{E}_z^\gamma \left| \sum_{x\in X} |x| \big(\nu(x) - \mathbb{E}_z^\gamma[\nu(x)]\big) \right|^6. \end{aligned} \tag{8.5}$$

The expectation on the right-hand part of (8.5) can be estimated using the following lemma (see Bogachev and Zarbaliev 2003).

Lemma 8.3. *For each $k \in \mathbb{N}$ and $i = 1, 2$*

$$\mathbb{E}_z\big(\xi_i - \mathbb{E}_z[\xi_i]\big)^{2k} = O(n_1^{4k/3}) \qquad (n \to \infty).$$

Applying this lemma with $k = 3$, from (8.5) we obtain

$$\mathbb{Q}_z^\gamma \left\{ \sup_{0\le t\le\infty} \left|\tilde{\ell}_\Gamma(t)\right| > n_1\varepsilon \right\} = O(n_1^{-2}). \tag{8.6}$$

On the other hand, by Corollary 7.12 the denominator of the fraction (8.4) decays no faster than at order $n^{-4/3}$. Combining this with the estimate (8.6), we conclude that the right-hand part of (8.4) can be written as $O(n_1^{-2/3})$, which tends to zero as $n \to \infty$. Therefore, Theorem 8.2 is proved. ∎

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References

[1] Bárány, I. (1995). The limit shape of convex lattice polygons. *Discrete Comput. Geom.*, **13**, 279–295.

[2] Bingham, N. H., Goldie, C. M. and Teugels, J. L. (1989). *Regular Variation.* Paperback edition (with additions). Cambridge Univ. Press, Cambridge.

[3] Bhattacharya, R.N. and Ranga Rao, R. (1976). *Normal Approximation and Asymptotic Expansions.* Wiley, New York.

[4] Bogachev, L.V. and Zarbaliev, S.M. (1999a). Approximation of convex functions by random polygonal lines. *Math. Doklady*, **59**, 46–49.

[5] Bogachev, L.V. and Zarbaliev, S.M. (1999b). Limit theorems for a class of random convex polygons. *Russian Math. Surveys*, **54**, 155–156.

[6] Bogachev, L.V. and Zarbaliev, S.M. (2003). Limit theorems for random convex polygons. Research Report STAT 03/01, Department of Statistics, University of Leeds. Available at `www.maths.leeds.ac.uk/~bogachev/Papers/`

[7] Hardy, G.H. and Wright, E.M. (1960). *An Introduction to the Theory of Numbers.* 4th ed., Oxford University Press, Oxford.

[8] Khinchin, A.Y. (1960). *Mathematical Foundations of Quantum Statistics.* Graylock Press, Albany, N.Y.

[9] Sinai, Ya.G. (1994). A probabilistic approach to the analysis of the statistics of convex polygonal lines. *Funct. Anal. Appl.*, **28**, 108–113.

[10] Titchmarsh, E.C. (1939). *The Theory of Functions.* 2nd ed., Oxford University Press, Oxford.

[11] Titchmarsh, E.C. (1986). *Introduction to the Theory of Fourier Integrals.* 3rd ed., Chelsea Publishing Co., New York.

[12] Vershik, A.M. (1994). The limit form of convex integral polygons and related problems. *Funct. Anal. Appl.*, **28**, 13–20.

[13] Vershik, A.M. (1996). Statistical mechanics of combinatorial partitions, and their limit configurations. *Funct. Anal. Appl.*, **30**, 90–105.

[14] Vershik, A.M. (1997). Limit distribution of the energy of a quantum ideal gas from the viewpoint of the theory of partitions of natural numbers. *Russian Math. Surveys*, **52**, 379–386.

[15] Vershik, A.M., Freiman, G.A. and Yakubovich, Yu.V. (1999). A local limit theorem for random partitions of natural numbers. *Theory Probab. Appl.*, **44**, 453–468.

[16] Widder, D.V. (1946). *The Laplace Transform.* Princeton University Press, Princeton.

[17] Yeh, J. (1995). *Martingales and Stochastic Analysis.* World Scientific, Singapore.

[18] Zarbaliev, S. (1998). Certain two-dimensional local limit theorems with applications to the statistics of convex polygons. In *Abstracts of Communications, 7th Vilnius Conference on Probab. Theory and Math. Statistics, August 12–27, 1998.* Vilnius, pp. 461–462.