APPROXIMATION THEORY FOR THE P-VERSION OF THE FINITE ELEMENT METHOD IN THREE DIMENSIONS IN THE FRAMEWROK OF JACOBI-WEIGHTED BESOV SPACES

Part I : Approximabilities of singular functions

Dedicated to Professor Ivo Babuška on the occasion of his 80-th birthday

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Abstract

This paper is the first in a series devoted to the approximation theory of the p-version of the finite element method in three dimensions. In this paper, we introduce the Jacobi-weighted Besov and Sobolev spaces in the three-dimensional setting and analyze the approximability of functions in the framework of these spaces. In particular, the Jacobi-weighted Besov and Sobolev spaces with three different weights are defined to precisely characterize the natures of the vertex-singularity, the edge singularity and vertex-edge singularity, and to explore their best approximabilities in terms of these spaces. In the forth coming Part II, we will apply the approximabilities of these singular functions to prove the optimal convergence of the p-version of the finite element method for elliptic problems in polyhedral domains, where the singularities of three different types occur and substantially govern the convergence of the finite element solutions.

Key words: p-version, finite element method, Jacobi-weighted Besov and Sobolev spaces, Jacobi projection, vertex singularity, edge singularity and vertex-edge singularity.

1. INTRODUCTION

Since the late 1970s, the p-version of the finite element method(FEM), which increases the degree of polynomials on a fixed mesh to obtain higher accuracy, has been widely used in engineering computations. There are

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several commercial and research codes based on the p and h-p versions of the finite element method, for example, MSC/PROBE (MacNeal Schwendler, CA, USA), Poly FEM(IBM, MA USA), MECHANICA (Rasna Corp., CA, USA), PHLEX (Computational Mechanics, TX, USA), STRESSCHECK (Engineering Software Research & Development, MO,USA), and STRIPE (Aeronautical Research Institute of Sweden).

In 1980 it was shown that the p-version of FEM in two dimensions converges at least as fast as the traditional h-version with quasi-uniform meshes, and that it converges twice as fast as the h-version of FEM if the solution has singularity of r^γ-type. Since then significant progress for the p-version in one and two dimensions has been made in the past two decades. The estimation of the upper bound of the approximation error in finite element solutions of the p-version in two dimensions were analyzed in [5, 6], and a detailed analysis of the p-version in one dimension is available in [10]. Very recently, the author and his collaborators have further developed the approximation theory of the p-version of finite element method and boundary element method(BEM) in the framework of the Jacobi-weighted Besov and Sobolev spaces [1, 2, 3, 4, 11, 12]. In this mathematical framework, the lower and upper bounds of approximation error in FEM solutions of the p-version and in BEM solutions of p- and h-p version for problems in polygonal domains were proved, and the optimal rate of convergence was mathematically established. The spectral method in the framework of the Jacobi-weighted Sobolev spaces has been studied and was successfully applied to singular differential equations [8, 13, 14].

In contrast to the p-version in one and two dimensions, the p-version of FEM in three dimensions is much less developed due to the complexity of three dimensional problems. Because of lacking of effective mathematical tools and theory to deal with the complexities of three dimensional singularities in the 1980's and 1990's, a few results and analysis are available in the literatures. The upper bounds in approximation error of the p-version in three dimensions was discussed for problems with singularities as a conjecture in [9] without proof, and analyzed in [15] for problems with smooth solutions belonging to $H^k(\Omega), k > 2$.

In this series of papers, we shall precisely characterize singularities and analyze the approximation to singular functions as well as smooth functions in $H^k(\Omega), k > 1$ in the framework of the Jacobi-weighted Besov and Sobolev spaces, and prove the optimal convergence of the p- version of FEM for problems on polyhedral domains. In the first paper of the series, we shall introduce the Jacobi-weighted Besov and Sobolev spaces in three dimensions and derive the approximation results for functions in these spaces, then verify that singular functions of different types, which arise from problems in polyhedral domains, belong to the corresponding Jacobi-weighted Besov spaces and prove their approximability by high-order polynomials. Since the approximation to functions in the Jacobi-weighted Besov and Sobolev spaces in one and two dimensions can be generalized to three dimensions without substantial difficulty and the approximability of singular functions follows from the general approximation properties for functions in the Jacobi-weighted Besov spaces and verification of the singularities in appropriate Jacobi-weighted Besov spaces, the crucial part of the paper is to prove that these singular functions belong to different Jacobi-weighted Besov spaces which are precisely designed according to the nature of these singular functions. It is well known that there are singularities of three different types in solutions of problems with piecewise analytic date and on polyhedral domains which severely govern the convergence of the FEM solution; namely vertex singularity, edge singularity and vertex-edge singularity. Since the vertex-edge singularity occurs in two directions and is anisotropic, the characterization of the vertex-edge singularity in the Jacobi-weighted Besov spaces is very different from those for the two dimensional setting [1, 2, 3, 4] and for the vertex singularity and the edge singularity, which reflects the major difficulty as well as significance of the paper. The main theorems of the paper are Theorem 5.2 and 5.3, i.e. $u(x) = \rho^\gamma \sin^\sigma\theta\,\chi(\rho)\,\Psi(\theta)\,\Phi(\phi) \in B^{s,\beta}_\kappa(Q)$ with $s = 2+2\min\{\sigma, \gamma+(1+\beta_3)/2\}+\beta_1+\beta_2$, the Jacobi weight $\beta = (\beta_1, \beta_2, \beta_3)$ with $\beta_i > -1$, arbitrary, and

$$\kappa = \begin{cases} 0 & \text{if } \sigma \neq \gamma + (1+\beta_3)/2, \\ 1/2 & \text{if } \sigma = \gamma + (1+\beta_3)/2. \end{cases}$$

where $Q = (-1,1)^3$ and (ρ, θ, ϕ) are the spherical coordinates with respect to the vertex $(-1,-1,-1)$ and the vertical line $L = \{x = (x_1, x_2, x_3) \mid x_1 = x_2 = -1, x_3 \in (-\infty, \infty)\}$, $\chi(\rho)$, $\Psi(\theta)$ and $\Phi(\phi)$ are the usual c^∞ cut-off functions. It follows immediately from the approximability of functions in the space $B^{s,\beta}_\kappa(Q)$ that

$$\|u - \psi\|_{L^2(Q)} \leq Cp^{-(2+2\min\{\sigma,\gamma+1/2\})}(1+\log p)^\kappa$$

and

$$\|u - \varphi\|_{H^1(R_0)} \leq Cp^{-2\min\{\sigma,\gamma+1/2\}}\,(1+\log p)^\kappa$$

with

$$\kappa = \begin{cases} 0 & \text{if } \sigma \neq \gamma + 1/2, \\ 1/2 & \text{if } \sigma = \gamma + 1/2. \end{cases}$$

where R_0 denotes a the conic subregion of Q which is the support of u, ψ and φ are the Jacobi projections of u on the space $\mathcal{P}_p(Q)$ of polynomials of degree $\leq p$ associated with the Legendre weight $\beta = (0,0,0)$ and the Chebyshev-Legendre weight $\beta = (-1/2, -1/2, 0)$, respectively. It is worth indicating that a logarithmic term appears in the error estimation if $\sigma = \gamma + 1/2$ although the function has no logarithmic singularity. This unique feature in three dimension is precisely explored by the Jacobi-weighted space $B^{s,\beta}_\kappa(Q)$ which is an interpolation space introduced by the modified K-method. The results of this paper and forth-coming ones will significantly improve the approximation theory of the p-version of FEM in three dimensions.

The scope of the paper is as follows. In Section 2 we introduce the Jacobi-weighted Besov spaces $B^{s,\beta}_{\nu}(Q)$ and Sobolev spaces $H^{s,\beta}_{\nu}(Q)$ with $Q=(-1,1)^3, s>0$ and integer $\nu\geq 0$, and derive error estimation of the Jacobi projections in the Jacobi-weighted Sobolev norms. In Section 3 we characterize the singularity and analyze the approximability for singular functions of $\rho^{\gamma}\log^{\nu}\rho$-type with $\gamma>0,\nu\geq 0$ in terms of the space $B^{s,\beta}_{\nu^*}(Q)$. The singularity and approximability of singular functions of $r^{\sigma}\log^{\mu} r$-type with $\sigma>0$ and $\mu\geq 0$ in terms of the space $B^{s,\beta}_{\mu^*}(Q)$ are analyzed in the next section. Section 5 focusses on the characterization of singularities and the best approximation in L^2- and H^1-norms for singular functions of the $\rho^{\gamma}\sin^{\sigma}\theta$-type and $\rho^{\gamma}\sin^{\sigma}\theta\,\log^{\nu}\rho\,\log^{\mu}\sin\theta$-type with $\gamma,\sigma>0$ and integers $\nu,\mu\geq 0$ in terms of the space $B^{s,\beta}_{\kappa}(Q)$. Some concluding remarks are given in the last section on the effectiveness of the Jacobi-weighted Sobolev and Besov spaces by comparing the error estimations of the h- and p-versions of FEM in terms of Besov and Sobolev spaces with and without the Jacobi weights.

2. JACOBI-WEIGHTED BESOV AND SOBOLEV SPACES

Let $Q=I^3=(-1,1)^3$, and let

$$w_{\alpha,\beta}(x)=\prod_{i=1}^{3}(1-x_i^2)^{\alpha_i+\beta_i} \tag{2.1}$$

be a weight function with integer $\alpha_i\geq 0$ and real number $\beta_i>-1$, which is refereed to as Jacobi weight. Obviously, the Jacobi polynomials and their derivatives are orthogonal with the weight $w_{\alpha,\beta}(x)$.

The Jacobi-weighted Sobolev space $H^{k,\beta}(Q)$ with integer k is defined as a closure of C^{∞} functions in the norm with the Jacobi weight

$$\|u\|^2_{H^{k,\beta}(Q)}=\sum_{|\alpha|=0}^{k}\int_Q |D^{\alpha}u|^2\, w_{\alpha,\beta}(x)\,dx \tag{2.2}$$

where $D^{\alpha}u=u_{x_1^{\alpha_1}x_2^{\alpha_2},x_3^{\alpha_3}}$, $\alpha=(\alpha_1,\alpha_2,\alpha_3), |\alpha|=\alpha_1+\alpha_2+\alpha_3$, and $\beta=(\beta_1,\beta_2,\beta_3)$. By $|u|_{H^{k,\beta}(Q)}$ we denote the semi-norm,

$$|u|^2_{H^{k,\beta}(Q)}=\sum_{|\alpha|=k}\int_Q |D^{\alpha}u|^2\, w_{\alpha,\beta}(x)\,dx.$$

Let $\mathcal{B}^{s,\beta}_{2,q}(Q)$ be the interpolation spaces defined by the K-method

$$\Big(H^{\ell,\beta}(Q),H^{k,\beta}(Q)\Big)_{\theta,q}$$

where $0<\theta<1, 1\le q\le\infty$, $s=(1-\theta)\ell+\theta k$, ℓ and k are integers, $\ell<k$, and

$$\|u\|_{\mathcal{B}^{s,\beta}_{2,q}(Q)} = \Big(\int_0^\infty t^{-q\theta}\,|K(t,u)|^q \frac{dt}{t}\Big)^{1/q},\ 1\le q<\infty \tag{2.3a}$$

$$\|u\|_{\mathcal{B}^{s,\beta}_{2,\infty}(Q)} = \sup_{t>0} t^{-\theta}\,K(t,u) \tag{2.3b}$$

where

$$K(t,u) = \inf_{u=v+w}\Big(\|v\|_{H^{\ell,\beta}(Q)} + t\|w\|_{H^{k,\beta}(Q)}\Big). \tag{2.4}$$

In particular, we are interested in the cases $q=2$ and $q=\infty$. We shall write for $s\ge 0$ and $q=2$

$$H^{s,\beta}(Q) = \mathcal{B}^{s,\beta}_{2,2}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,2}$$

with $0<\theta<1$ and $s=(1-\theta)\ell+\theta k$. This space is called the Jacobi-weighted Sobolev space with fractional order if s is not an integer. It has been proved that $\mathcal{B}^{s,\beta}_{2,2}(Q) = H^{m,\beta}(Q)$ if s is an integer m in two dimensions[1], it can be proved analogously in three dimensions.

For $q=\infty$, we shall write

$$B^{s,\beta}(Q) = \mathcal{B}^{s,\beta}_{2,\infty}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty}$$

which is refereed as the Jacobi-weighted Besov space. It is an exact interpolation space of θ-exponent according to [7].

For the best approximation of the singular functions such as $\rho^\gamma\log^\nu\rho$, $\nu>0$ we need to introduce an interpolation space

$$B^{s,\beta}_\nu(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty,\nu}$$

with integer $\nu>0$ by a modified K-method,

$$\|u\|_{B^{s,\beta}_\nu(Q)} = \sup_{t>0}\frac{t^{-\theta}K(t,u)}{(1+|\log t|)^\nu} \tag{2.5}$$

Remark 2.1. The space $B^{s,\beta}_0(Q) = B^{s,\beta}(Q)$ is a standard exact interpolation space of θ-exponent, all important properties of exact interpolation spaces such as the reiteration theorem stands for $B^{s,\beta}(Q)$. It has been shown [1] that the space $B^{s,\beta}_\nu(Q)$ with $\nu>0$ is a uniform interpolation space, but not an exact one. Hence many important properties of exact interpolation spaces do not hold for the space $B^{s,\beta}_\nu(Q)$ with $\nu>0$, for instance, the reiteration theorem. Fortunately a partial reiteration theorem was proved which guarantees

$$\Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty,\nu} = \Big(H^{\ell',,\beta}(Q), H^{k',\beta}(Q)\Big)_{\theta',\infty,\nu}$$

as long as $(1-\theta)\ell + \theta k = (1-\theta')\ell' + \theta' k' = s$. Hence the space $B_\nu^{s,\beta}(Q)$ is well defined and does not depend on the individual values of ℓ and k, but their combination $(1-\theta)\ell + \theta k$.

For the definition and properties of exact interpolation spaces of exponent θ, we refer to [7]. For the partial reiteration theorem and various properties of uniform interpolation space $B_\nu^{s,\beta}(Q)$ with integer $\nu > 0$, we refer to [3].

Remark 2.2. For $\beta_1 = \beta_2 = \beta_3 = 0$, the spaces $B_\nu^{s,\beta}(Q)$ are referred to as Legendre-weighted Besov spaces, and for $\beta_1 = \beta_2 = -1/2, \beta_3 = 0$, they are referred to as Chebyshev-Legendre weighted Besov spaces, for $\beta_1 = \beta_2 = -1/2, \beta_3 > -1$, are referred as the Chebychev-weighted Besov spaces, etc.

We next study the approximation properties for functions in the Jacobi-weighted Sobolev spaces. Let $\mathcal{P}_p(Q)$ be set of all polynomials of (separate) degree $\leq p$. For $u \in H^{k,\beta}(Q), k \geq 0$, we have the Jacobi-Fourier expansion in $H^{0,\beta}(Q)$

$$u(x) = \sum_{i,j,k=0}^{\infty} C_{ijk}\, P_i(x_1, \beta_1)\, P_j(x_2, \beta_2)\, P_k(x_3, \beta_3)$$

where

$$P_n(x_i, \beta_i) = \frac{(1-x_i^2)^{-\beta}}{2^n\, n!} \frac{d^n\, (1-x_i^2)^{\beta+n}}{dx^n}$$

is the Jacobi polynomial of degree n in variable $x_i, 1 \leq i \leq 3$. Then

$$u_p(x) = \sum_{i,j,k=0}^{p} C_{ijk}\, P_i(x_1, \beta_1)\, P_j(x_2, \beta_2) P_k(x_3, \beta_3)$$

is the projection of $u(x)$ on $\mathcal{P}_p(Q)$.

Proposition 2.2 Let $u \in H^{k,\beta}(Q)$, and let $u_p(x)$ be the projection of $u(x)$ on $P_p(Q)$ in $H^{0,\beta}(Q)$. Then , $u_p(x)$ is the projection on $P_p(Q)$ in $H^{\ell,\beta}(Q)$ for all $0 \leq \ell \leq k$, and

$$|u_p|^2_{H^{\ell,\beta}(Q)} + |u - u_p|^2_{H^{\ell,\beta}(Q)} = |u|^2_{H^{\ell,\beta}(Q)}.$$

Proof The proposition was proved in [3] for two dimensions. The proof can be carried easily for one and three dimensions.

Due to the Proposition 2.2, u_p is refereed as the Jacobi projection, for which, we have the following approximation property.

Theorem 2.3 Let $u \in H^{k,\beta}(Q)$ with integer $k \geq 1$, $\beta_i > -1, i = 1,2$, and u_p be its $H^{0,\beta}(Q)$-projection onto $\mathcal{P}_p(Q)$. Then we have for integer $\ell \leq k \leq p+1$

$$|u - u_p|_{H^{\ell,\beta}(Q)} \leq C\, p^{-(k-\ell)}\, |u|_{H^{k,\beta}(Q)}. \tag{2.6}$$

Proof: The proof for one and two dimensions can be carried here for three dimensions, we will not give the details of the proof, instead refer to [1].

By a standard argument of interpolation spaces, we are able to generalize Theorem 2.3 to an approximation theorem for functions in the Jacobi-weighted Besov spaces $B^{s,\beta}(Q)$.

Theorem 2.4 Let $u \in B^{s,\beta}(Q)$, $s > 0$ with $\beta_i > -1, 1 \leq i \leq 3$, and let u_p be the Jacobi projection of u on $mathcalP_p(Q)$ with $p+1 \geq s$. Then for any real $\kappa \in [0,s)$ there holds

$$\|u - u_p\|_{H^{\kappa,\beta}(Q)} \leq C\, p^{-(s-\ell)}\, \|u\|_{B^{s,\beta}(Q)} \tag{2.7}$$

with constant C independent of p.

Theorem 2.5 Let $u \in B^{s,\beta}_{\nu}(Q)$, $s > 0, \nu > 0$ with $\beta_i > -1, 1 \leq i \leq 3$, and let u_p be the Jacobi projection of u on $P_p(Q)$ with $p+1 \geq s$. Then for any real $\kappa \in [0,s)$, there holds

$$\|u - u_p\|_{H^{\kappa,\beta}(Q)} \leq C\, p^{-(s-\ell)}\, (1 + \log p)^{\nu} \|u\|_{B^{s,\beta}_{\nu}(Q)} \tag{2.8}$$

with constant C independent of p.

The proof of Theorem 2.4 and 2.5 for integer κ can be found in [3], and a usual argument of interpolation spaces leads to the estimations for non-integer κ.

3. APPROXIMABILITY OF VERTEX-SINGULAR FUNCTIONS

Let $Q = (-1,1)^3$, and let (ρ,θ,ϕ) be the spherical coordinates with respect to the vertex $(-1,-1,-1)$ and the vertical line $L = \{x = (x_1,x_2,x_3) \mid x_1 = x_2 = -1, x_3 \in (-\infty,\infty)\}$ with $\rho = \{\sum_{i=1}^{3}(x_i+1)^2\}^{1/2}$, $\theta = \arctan \dfrac{r}{x_3+1}$ $= \arctan \dfrac{\{(x_1+1)^2 + (x_2+1)^2\}^{1/2}}{x_3+1} \in (0,\pi/2)$, and $\phi = \arctan \dfrac{x_2+1}{x_1+1} \in (0,\pi/2)$.

We now consider the singular functions with $\gamma > 0$

$$u(x) = \rho^{\gamma}\, \chi(\rho)\, \Phi(\theta,\phi) \tag{3.1}$$

and

$$v(x) = \rho^\gamma \log^\nu \rho\, \chi(\rho)\, \Phi(\theta,\phi) \tag{3.2}$$

with integer $\nu \geq 0$, where $\chi(\rho)$ and $\Phi(\theta,\phi)$ are C^∞ functions such that for $0 < \rho_0 < 1$

$$\chi(\rho) = 1 \quad \text{for } 0 < \rho < \rho_0/2, \quad \chi(\rho) = 0 \quad \text{for } \rho > \rho_0.$$

Hereafter, S_{κ_0} denotes a subset of the intersection of the unit sphere and Q such that the angles between the radial $A_1 - x$ and the x_i-axis is larger than κ_0. For $0 < \kappa_0 < \pi/4$, let

$$R_0 = R_{\rho_0,\kappa_0}\{x \in Q \mid 0 < \rho < \rho_0, (\theta,\phi) \in S_{\kappa_0}\}$$

as shown in Fig 3.1. Then there hold for $x \in R_0$

$$\begin{aligned} (2-\rho_0)(1+x_i) &\leq (1-x_i^2) \leq 2(1+x_i),\ 1 \leq i \leq 3, \\ \kappa_1 &\leq \frac{1+x_i}{1+x_j} \leq \kappa_2,\ 1 \leq i \leq 3 \end{aligned} \tag{3.3}$$

where $\kappa_2 = \max\{\tan\kappa_0, \cot\kappa_0\}$ and $\kappa_1 = 1/\kappa_2$.

The functions defined in (3.1) and (3.2) reflect a typical singularity, referred as the vertex singularity, which occurs in the solutions of problems on polyhedral domains and severely affect the convergence of the finite element solution. Therefore, finding the best approximation to these singular functions is essential for the error estimates of the finite element solutions for problems with such singularities. It is worth indicating that the vertex singularity is isotropic, hence the most appropriate Jacobi-weighted Besov and Sobolev spaces for their best approximation shall be isotropic as well.

3.1 SINGULAR FUNCTIONS OF ρ^γ-TYPE

Theorem 3.1 Let $u = \rho^\gamma\, \chi(\rho)\, \Phi(\theta,\phi)$ given in (3.1), and let $\beta = (\beta_1,\beta_2,\beta_3)$ with $\beta_i > -1, 1 \leq i \leq 3$, arbitrary. Then $u \in B^{s,\beta}(Q)$ and $u \in H^{s-\epsilon,\beta}(Q)$ with $s = 2\gamma + 3 + \sum_{i=1}^{3}\beta_i$ and $\epsilon > 0$, arbitrary.

Proof Let $u = u_1 + u_2$ with $u_1 = \chi_\delta(\rho)\, u$ and $u_2 = (1-\chi_\delta(\rho))u$ for $\delta \in (0,\rho_0)$. Then $u_1 \in H^{0,\beta}(Q)$, and

$$\|u_1\|^2_{H^{0,\beta}(Q)} \leq C\delta^{2\gamma+3+\sum_{i=1}^{3}\beta_i}. \tag{3.4}$$

It is easy to see that $u_2 \in H^{k,\beta}(Q)$, for any $k > 2\gamma + 3 + \sum_{i=1}^{3}\beta_i$, and

$$\|u_2\|^2_{H^{k,\beta}(Q)} \leq C\delta^{2\gamma+3-k+\sum_{i=1}^{3}\beta_i}. \tag{3.5}$$

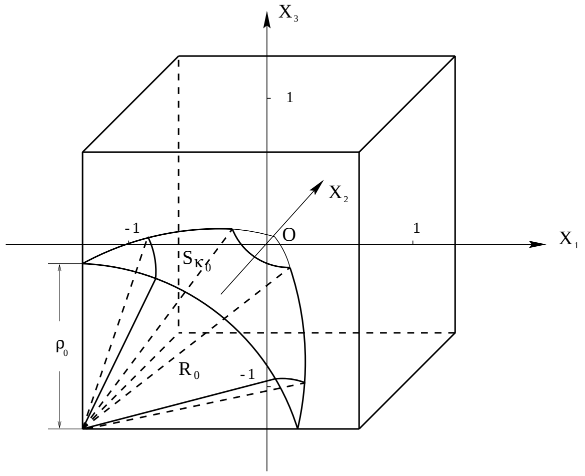

Fig. 3.1 Cubic Domain Q and sub region R_{r_0,κ_0}

Selecting $\delta = t^{\frac{2}{k}}$, we have for $t \in (0,1)$

$$\begin{aligned} K(t,u) &\leq C\delta^{\gamma+(3+\sum_{i=1}^{3}\beta_i)/2}(1+t\delta^{-k/2}) \\ &\leq C\delta^{\gamma+(3+\sum_{i=1}^{3}\beta_i)/2} \leq Ct^{\frac{2\gamma+3+\sum_{i=1}^{3}\beta_i}{k}} \end{aligned}$$

and for $t \geq 1$, there holds

$$K(t,u) \leq C\|u\|_{H^{0,\beta}(Q)}.$$

Letting $\theta = \dfrac{2\gamma+3+\sum_{i=1}^{3}\beta_i}{k}$, we have

$$\sup_{t>0} t^{-\theta}\, K(t,u) \leq C$$

which implies that $u \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,\infty} = B^{s,\beta}(Q)$ with $s = \theta k = 2\gamma + 3 + \sum_{i=1}^{3}\beta_i$.

If $\theta = \dfrac{2\gamma+3+\sum_{i=1}^{3}\beta_i - \epsilon}{k} = \dfrac{s-\epsilon}{k}$ with $\epsilon > 0$, arbitrary, then

$$\int_0^1 t^{-2\theta}|K(t,u)|^2\frac{dt}{t} \leq C\int_0^1 t^{-1+2\epsilon/k}dt \leq C.$$

which implies $u \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,2} = H^{s-\epsilon,\beta}(Q)$. □

The approximability of the singular function of ρ^{γ}-type is the consequence of Theorem 2.4 and Theorem 3.1.

Theorem 3.2 For $u = \rho^\gamma \chi(\rho) \Phi(\theta,\phi)$ given in (3.1), there exists $\psi \in \mathcal{P}_p(Q)$ such that

$$\|u-\psi\|_{L^2(Q)} \le Cp^{-(2\gamma+3)} \|u\|_{B^{2\gamma+3,\beta}(Q)} \tag{3.6}$$

with $\beta = (0,0,0)$. Also, there exists $\varphi \in \mathcal{P}_p(Q)$ such that

$$\|u-\varphi\|_{H^1(R_0)} \le C\|u-\varphi\|_{H^{1,\beta}(Q)} \le Cp^{-(2\gamma+1)} \|u\|_{B^{2\gamma+2,\beta}(Q)} \tag{3.7}$$

with $\beta = (-1/3,-1/3,-1/3)$.

Proof By Theorem 3.1 $u \in B^{s,\beta}(Q)$ with $s = 2\gamma+3$ and $\beta = (0,0,0)$. Due to Theorem 2.3, the Jacobi projection ψ of u on $\mathcal{P}_p(Q)$ associated with the weight $\beta = (0,0,0)$ satisfies

$$\|u-\psi\|_{L^2(Q)} = \|u-\psi\|_{H^{0,\beta}(Q)} \le C\, p^{-(2\gamma+3)} \|u\|_{B^{2\gamma+3,\beta}(Q)}.$$

For $\beta = (-1/3,-1/3,-1/3)$, by Theorem 3.1, $u \in B^{s,\beta}(Q)$ with $s = 2\gamma+2$. Owing to Theorem 2.4, there holds for the Jacobi projection φ of u on $\mathcal{P}_p(Q)$, associated with the weight $\beta = (-1/3,-1/3,-1/3)$,

$$|u-\varphi|_{H^{\ell,\beta}(Q)} \le C\, p^{-(2\gamma+2-\ell)} \|u\|_{B^{2\gamma+2,\beta}(Q)}.$$

with $\ell = 0,1$. Note that

$$\|u-\varphi\|_{L^2(Q)} \le \|u-\varphi\|_{H^{0,\beta}(Q)} \le Cp^{-(2\gamma+2)} \|u\|_{B^{2\gamma+2,\beta}(Q)}. \tag{3.8}$$

Due to (3.3), for α with $|\alpha| = 1$ and for $x \in R_0$, there exist two constants C_1 and C_2 such that

$$C_1 \le \prod_{1\le i\le 3} (1-x_i^2)^{\alpha_i - 1/3} \le C_2. \tag{3.9}$$

Then, we have

$$\begin{aligned}
\int_{R_0} \Big|D^\alpha(u-\varphi)\Big|^2 dx \le\ & C \int_{R_0} \Big|D^\alpha(u-\varphi)\Big|^2 \prod_{1\le i\le 3} (1-x_i^2)^{\alpha_i-1/3}\, dx \\
\le\ & C \int_Q \Big|D^\alpha(u-\varphi)\Big|^2 \prod_{1\le i\le 3} (1-x_i^2)^{\alpha_i-1/3} dx \\
\le\ & Cp^{-2(2\gamma+1)} \|u\|^2_{B^{2\gamma+2,\beta}(Q)}
\end{aligned}$$

which together with (3.8) leads to (3.7). □

3.2 SINGULAR FUNCTIONS OF $\rho^\gamma \log^\nu \rho$-TYPE

It can be proved that the singular function $v(x) = \rho^\gamma \log^\nu \rho\, \chi(\rho)\, \Phi(\theta,\phi)$, given in (3.2), belongs to the space $B^{s-\epsilon,\beta}(Q)$ with $s = 2\gamma + 3 + \sum_{i=1}^3 \beta_i$ and $\epsilon > 0$, arbitrary. Consequently, the approximation error will lose a rate of $O(p^\epsilon)$. To avoid such a loss, the modified Jacobi-weighted Besov spaces will be the most appropriate spaces for the vertex-singular functions with logarithmic terms to describe the nature of singularity and to explore the best approximation.

Theorem 3.3 Let $v(x) = \rho^\gamma \log^\nu \rho\, \chi(\rho)\, \Phi(\theta,\phi)$, given in (3.2), and let $\beta = (\beta_1, \beta_2, \beta_3)$ with $\beta_i > -1, 1 \le i \le 3$, arbitrary. Then $v \in H^{s-\epsilon,\beta}(Q)$, and $v \in B^{s,\beta}_{\nu^*}(Q)$, with $s = 2\gamma + 3 + \sum_{i=1}^3 \beta_i$ and $\epsilon > 0$, arbitrary, and

$$\nu^* = \begin{cases} \max\{\nu - 1, 0\} & \text{if } \gamma \text{ is an integer,} \\ \nu & \text{if } \gamma \text{ is not an integer.} \end{cases} \tag{3.10}$$

Proof Let $v = v_1 + v_2$ with $v_1 = \chi_\delta(\rho)\, v$ and $v_2 = (1 - \chi_\delta(\rho))v$ with $\chi_\delta(\rho)$. Then $v_1 \in H^{0,\beta}(Q)$, and

$$\begin{aligned} \|v_1\|^2_{H^{0,\beta}(Q)} &= \int_Q |v|^2 \prod_{i=1}^3 (1 - x_i^2)^{\beta_i} dx \\ &\le C \int_0^\delta \rho^{2\gamma+2+\sum_{i=1}^3 \beta_i} |\log \rho|^{2\nu} d\rho \\ &\le C \delta^{2\gamma+3+\sum_{i=1}^3 \beta_i} |\log \delta|^{2\nu}. \end{aligned}$$

It is easy to see that $v_2 \in H^{k,\beta}(Q)$, for any $k > 2\gamma + 3 + \sum_{i=1}^3 \beta_i$, and there holds

$$\|v_2\|^2_{H^{k,\beta}(Q)} \le C \delta^{2\gamma+3-k+\sum_{i=1}^3 \beta_i} |\log \delta|^{2\nu}.$$

Selecting $\delta = t^{\frac{2}{k}}$, we have for $t \in (0,1)$

$$\begin{aligned} K(t,v) &\le C(\|v_1\|^2_{H^{0,\beta}(Q)} + t\|v_2\|^2_{H^{k,\beta}(Q)}) \\ &\le C\delta^{\gamma+(3+\sum_{i=1}^3 \beta_i)/2}(1 + t\delta^{-k/2})|\log \delta|^\nu \\ &\le C\delta^{\gamma+(3+\sum_{i=1}^3 \beta_i)/2}(1 + |\log t|)^\nu \end{aligned}$$

and for $t \ge 1$, there hold

$$K(t,v) \le C\|v\|_{H^{0,\beta}(Q)},$$

and

$$\sup_{t>1} \frac{t^{-\theta} K(t,v)}{(1 + |\log t|)^\nu} \le C\|v\|_{H^{0,\beta}(Q)}.$$

Letting $\theta = \dfrac{2\gamma + 3 + \sum_{i=1}^{3} \beta_i}{k}$, we have

$$\sup_{0<t<1} \frac{t^{-\theta} K(t,v)}{(1+|\log t|)^{\nu}} \le C$$

which implies that $v \in \left(H^{0,\beta}(Q), H^{k,\beta}(Q)\right)_{\theta,\infty,\nu} = B_{\nu}^{s,\beta}(Q)$ with $s = \theta k = 2\gamma + 3 + \sum_{i=1}^{3} \beta_i$.

Similarly arguing as in the proof of Theorem 3.1, and selecting $\theta = \dfrac{2\gamma + 3 + \sum_{i=1}^{3} \beta_i - \epsilon}{k} = \dfrac{s-\epsilon}{k}$ with $\epsilon > 0$, arbitrary, we have

$$\int_0^1 t^{-2\theta} |K(t,u)|^2 \frac{dt}{t} \le C \int_0^1 t^{-1+2\epsilon/k} (1+|\log t|)^{\nu} dt \le C,$$

which implies $u \in \left(H^{0,\beta}(Q), H^{k,\beta}(Q)\right)_{\theta,2} = H^{s-\epsilon,\beta}(Q)$.

If γ is an integer and the integer $\nu \ge 1$, we adopt a different composition of $v = v_1 + v_2$ for $\delta \in (0,1)$, namely,

$$v_1 = \rho^{\gamma} \left(\log^{\nu} \rho - \log^{\nu}(\rho+\delta)\right) \chi(\rho)\, \Phi(\theta,\phi)$$

and

$$v_2 = \rho^{\gamma} \log^{\nu}(\rho+\delta) \chi(\rho)\, \Phi(\theta,\phi).$$

Then $v_1 \in H^{0,\beta}(Q)$ and $v_2 \in H^{k,\beta}(Q)$ for any $k > 2\gamma + 3 + \sum_{i=1}^{3} \beta_i$. Using the arguments in [1, Theorem 3.9], we have

$$\|v_1\|^2_{H^{0,\beta}(Q)} \le C \delta^{2\gamma+3+\sum_{i=1}^{3}\beta_i} \, |\log \delta|^{2(\nu-1)} \tag{3.11}$$

and

$$\|v_2\|^2_{H^{k,\beta}(Q)} \le C \delta^{2\gamma-k+3+\sum_{i=1}^{3}\beta_i} \, |\log \delta|^{2(\nu-1)}. \tag{3.12}$$

(3.11) and (3.12) lead to

$$\begin{aligned} K(t,v) &\le C \delta^{\gamma+(3+\sum_{i=1}^{3}\beta_i)/2} (1 + t\delta^{-k/2}) |\log \delta|^{\nu-1} \\ &\le C \delta^{\gamma+(3+\sum_{i=1}^{3}\beta_i)/2} |\log \delta|^{\nu-1} \end{aligned}$$

and

$$\sup_{0<t<1} \frac{t^{-\theta} K(t,v)}{(1+|\log t|)^{\nu-1}} \le C$$

with $\delta = t^{\frac{2}{k}}$ and $\theta = \dfrac{2\gamma + 3 + \sum_{i=1}^{3} \beta_i}{k}$. This implies that $v \in B_{\nu-1}^{s,\beta}(Q)$ with $s = 2\gamma + 3 + \sum_{i=1}^{3} \beta_i$. $\square$

The precise characterization of singularity for the singular function of $\rho^\gamma \log^\nu \rho$-type given by Theorem 3.3 leads to the best approximation to the singular function of this type. The following theorem is a consequence of Theorem 2.5 and 3.3

Theorem 3.4 For $v = \rho^\gamma \log^\nu \rho\, \chi(\rho)\, \Phi(\theta,\phi)$ given in (3.2), there exists $\psi(x) \in \mathcal{P}_p(Q)$ such that

$$\|v-\psi\|_{L^2(Q)} \le Cp^{-(2\gamma+3)}\,(1+\log p)^{\nu^*}\,\|u\|_{B^{2\gamma+3,\beta}_{\nu^*}(Q)} \tag{3.13}$$

with $\beta = (0,0,0)$. Also, there exists $\varphi(x) \in \mathcal{P}_p(Q)$ such that

$$\|u-\varphi\|_{H^1(R_0)} \le Cp^{-(2\gamma+2)}\,(1+\log p)^{\nu^*}\,\|u\|_{B^{2\gamma+2,\beta}_{\nu^*}(Q)} \tag{3.14}$$

with $\beta = (-1/3,-1/3,-1/3)$. In both (3.13) and (3.14) ν^* is given in (3.10).

Proof By Theorem 3.3 $v \in B^{s,\beta}_{\nu^*}(Q)$ with $s = 2\gamma+3$ and $\beta = (0,0,0)$. Due to Theorem 2.3, the Jacobi projection ψ of u on $P_p(Q)$ associated with the weight $\beta = (0,0,0)$ satisfies

$$\|u-\psi\|_{L^2(Q)} = \|u-\psi\|_{H^{0,\beta}(Q)} \le C\,p^{-(2\gamma+3)}\,(1+\log p)^{\nu^*}\,\|u\|_{B^{2\gamma+3,\beta}_{\nu^*}(Q)}.$$

For $\beta = (-1/3,-1/3,-1/3)$, by Theorem 3.1, $v \in B^{s,\beta}_{\nu^*}(Q)$ with $s = 2\gamma+2$. Owing to Theorem 2.5, there holds for the Jacobi projection φ of u on $P_p(Q)$ associated with the weight $\beta = (-1/3,-1/3,-1/3)$,

$$|u-\varphi|_{H^{\ell,\beta}(Q)} \le C\,p^{-(2\gamma+2-\ell)}\,(1+\log p)^{\nu^*}\,\|u\|_{B^{2\gamma+2,\beta}_{\nu^*}(Q)}. \tag{3.15}$$

for $\ell = 0,1$. Due to (3.9) and Theorem 2.5, we have for $|\alpha| = 1$

$$\begin{aligned}\int_{R_0}\Big|D^\alpha(u-\varphi)\Big|^2 dx \;&\le C\int_Q \Big|D^\alpha(u-\varphi)\Big|^2 \prod_{1\le i\le 3}(1-x_i^2)^{\alpha_i-1/3}dx\\ &\le Cp^{-2(2\gamma+2)}\,(1+\log p)^{2\nu^*}\,\|u\|^2_{B^{2\gamma+2,\beta}_{\nu^*}(Q)}\end{aligned}$$

which together with (3.15) leads to (3.14). □

4. APPROXIMABILITY OF EDGE-SINGULAR FUNCTIONS

Let $Q = (-1,1)^3$, and let (r,ϕ,x_3) be the cylindrical coordinates with respect to the vertex (-1,-1,-1) and the vertical line $L = \{x = (x_1,x_2,x_3) \mid x_1 = x_2 = -1, x_3 \in (-\infty,\infty)\}$. Let $r = \{\sum_{i=1}^2 (x_i+1)^2\}^{1/2}$, and let $\phi = \arctan\dfrac{x_2+1}{x_1+1} \in (0,\pi/2)$.

We consider the singular function with $\sigma > 0$

$$u(x) = r^{\sigma}\,\chi(r)\,\Phi(\phi)\,\Psi(x_3) \tag{4.1}$$

and

$$v(x) = r^{\sigma}\,\log^{\mu} r\chi(r)\,\Phi(\phi)\,\Psi(x_3). \tag{4.2}$$

Here $\chi(r), \Psi(x_3)$ and $\Phi(\phi)$ are C^{∞} functions such that for $0 < r_0 < 1$,

$$\chi(r) = 1 \quad \text{for } 0 < r < r_0/2,\ \chi(r) = 0 \quad \text{for } r > r_0,$$

and for $0 < z_0 < 1/2$

$$\Psi(x_3) = 1 \quad \text{for } x_3 \in (-1 + 2z_0, 1 - 2z_0),\ \Psi(x_3) = 0 \quad \text{for } |x_3| \geq 1 - z_0.$$

Obviously, $u(x)$ and $v(x)$ have a support $R_{r_0,z_0} = \{x \in Q \mid 0 < r < r_0, |x_3| \leq 1 - z_0\} \subset Q$. For $0 < \phi_0 < \pi/4$, let

$$R_0 = R_{r_0,\phi_0,z_0}\{x \in Q \mid 0 < r < r_0, \phi_0 \leq \phi \leq \pi/2 - \phi_0, |x_3| \leq 1 - z_0\},$$

as shown in Fig. 4.1. Then there hold for $x \in R_0$

$$\begin{aligned} z_0(2 - z_0) &\leq (1 - x_3^2) \leq 1, \\ (2 - \rho_0)(1 + x_i) &\leq (1 - x_i^2) \leq 2(1 + x_i),\ 1 \leq i \leq 2, \\ \tan\phi_0 &\leq \frac{1 + x_2}{1 + x_1} \leq \cot\phi_0. \end{aligned} \tag{4.3}$$

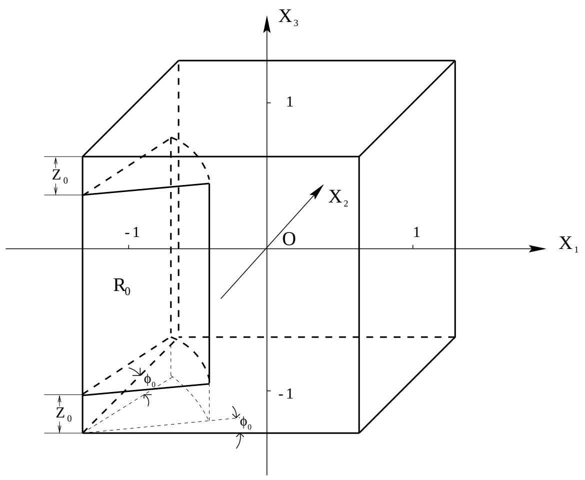

Fig. 4.1 Cubic Domain Q and sub region R_{r_0,ϕ_0,z_0}

The singular functions given in (4.1) and (4.2) reflect another typical singularity in the solutions of problems in polyhedral domains, and are referred as the edge singularity. The characterization of edge singularity in appropriate functional spaces is critical to its approximability and the convergence of the finite element solutions. Although the characterization and approximability of singular functions of $r^\sigma \ \log^\mu r$-type in $Q = (-1,1)^3$ are similar to those of vertex singular functions of $r^\gamma \ \log^\mu r$-type in two dimensions, it is worth pointing out that the edge singularity in three dimensions is anisotropic and the vertex singularity in two dimensions is isotropic. Therefore we will refer to [1] for the details of some arguments which are applicable to three dimensional setting, and emphasize the special features in three dimensions.

4.1 SINGULAR FUNCTIONS OF r^σ-TYPE

Theorem 4.1 Let $u(x) = r^\sigma\,\chi(r)\,\Phi(\phi)\,\Psi(x_3)$ given in (4.1), and let $\beta = (\beta_1, \beta_2, \beta_3)$ with $\beta_i > -1, 1 \le i \le 3$. Then $u \in B^{s,\beta}(Q)$ and $u \in H^{s-\epsilon,\beta}(Q)$ with $s = 2\sigma + 2 + \beta_1 + \beta_2$ and $\epsilon > 0$, arbitrary.

Proof Let $u = u_1 + u_2$ with $u_1 = \chi_\delta(r)\,u$ and $u_2 = (1 - \chi_\delta(r))u$ for $\delta \in (0, r_0)$. Note that

$$D^\alpha u_1 = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\Big(\chi_\delta \frac{\partial^{\alpha_3} u}{\partial x_3^{\alpha_3}}\Big)$$

and

$$D^\alpha u_2 = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\Big((1-\chi_\delta) \frac{\partial^{\alpha_3} u}{\partial x_3^{\alpha_3}}\Big).$$

Due to (4.3), the factor $(1 - x_3^2)^{\alpha_3+\beta_3}$ is bounded from above and below, and will not affect the regularity of singular function u. Therefore, the arguments for vertex singular functions in two dimensions [1] can be carried out here. Obviously, $u_1 \in H^{0,\beta}(Q)$, and

$$\|u_1\|^2_{H^{0,\beta}(Q)} \le C\delta^{2\sigma+2+\beta_1+\beta_2}. \tag{4.4}$$

It is easy to see that $u_2 \in H^{k,\beta}(Q)$, for any $k > 2\sigma + 2 + \beta_1 + \beta_2$, and

$$\|u_2\|^2_{H^{k,\beta}(Q)} \le C\delta^{2\sigma+2-k+\beta_1+\beta_2}. \tag{4.5}$$

For the details for derivation of (4.4) and (4.5) we refer to [1, Theorem 3.4].

Selecting $\delta = t^{\frac{2}{k}}$, we have for $t \in (0, 1)$

$$\begin{aligned} K(t,u) &\le C\delta^{\sigma+1+\beta_1/2+\beta_2/2}(1 + t\delta^{-k/2}) \\ &\le C\delta^{\sigma+1+\beta_1/2+\beta_2/2} \le Ct^{\frac{2\sigma+2+\beta_1+\beta_2}{k}} \end{aligned}$$

and for $t \ge 1$, there holds

$$K(t,u) \le C\|u\|_{H^{0,\beta}(Q)}.$$

Letting $\theta = \dfrac{2\sigma + 2 + \beta_1 + \beta_2}{k}$, we have

$$\sup_{t>0} t^{-\theta}\,K(t,u) \le C$$

which implies that $u \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,\infty} = B^{s,\beta}(Q)$ with $s = \theta k = 2\sigma + 2 + \beta_1 + \beta_2$.

If $\theta = \dfrac{2\sigma + 2 + \beta_1 + \beta_2 - \epsilon}{k} = \dfrac{s-\epsilon}{k}$ with $\epsilon > 0$, arbitrary, then

$$\int_0^1 t^{-2\theta}|K(t,u)|^2 \frac{dt}{t} \le C\int_0^1 t^{-1+2\epsilon/k}dt \le C.$$

which implies $u \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,2} = H^{s-\epsilon,\beta}(Q)$. $\square$

Theorem 4.1 and Theorem 2.4 lead to the best approximation of the singular function u.

Theorem 4.2 For $u(x) = r^\sigma\,\chi(r)\,\Phi(\phi)\,\Psi(x_3)$ given in (4.1), there exists $\psi(x) \in \mathcal{P}_p(Q)$ such that

$$\|u-\psi\|_{L^2(Q)} \le Cp^{-2(\sigma+1)}\ \|u\|_{B^{2\sigma+2,\beta}(Q)} \tag{4.6}$$

with $\beta_1 = \beta_2 = 0$ and $\beta_3 > -1$, arbitrary. Also, there exists $\varphi(x) \in \mathcal{P}_p(Q)$ such that

$$\|u-\varphi\|_{H^1(R_0)} \le C\|u-\varphi\|_{H^{1,\beta}(Q)} \le Cp^{-2\sigma}\ \|u\|_{B^{1+2\sigma,\beta}(Q)} \tag{4.7}$$

with $\beta_1 = \beta_2 = -1/2$ and $\beta_3 > -1$, arbitrary.

Proof Due to Theorem 4.1, $u \in B^{2\sigma+2,\beta}(Q)$ with $\beta_1 = \beta_2 = 0$ and $\beta_3 > -1$, arbitrary, which together with Theorem 2.4 leads to (4.6).

For $\beta_1 = \beta_2 = -1/2$ and $\beta_3 > -1$, arbitrary, $u \in B^{1+2\sigma,\beta}(Q)$. By Theorem 2.5 the Jacobi projection φ of u associated with the weight $\beta = (-1/2, -1/2, \beta_3)$

$$\|u-\varphi\|_{H^{\ell,\beta}(Q)} \le Cp^{-2\sigma-1+\ell}\ \|u\|_{B^{1+2\sigma,\beta}(Q)}$$

with $\ell = 0, 1$, which gives

$$p\|u-\varphi\|_{L^2(Q)} + \|\frac{\partial(u-\varphi)}{\partial x_3}\|_{L^2(Q)} \le Cp^{-2\sigma}\ \|u\|_{B^{1+2\sigma,\beta}(Q)}. \tag{4.8}$$

Due to (4.3), there holds for α with $\sum_{i=1}^2 \alpha_i = 1$ and for $x \in R_0$, there exist two constants C_1 and C_2 such that

$$C_1 \le (1-x_3^2)^{\alpha_i+\beta_3} \prod_{1\le i\le 2} (1-x_i^2)^{\alpha_i - 1/2} \le C_2.$$

which implies for $|\alpha| = 1$ with $\alpha_3 = 0$

$$\|D^\alpha(u-\varphi)\|_{L^2(R_0)} \le C\|u-\varphi\|_{H^{1,\beta}(Q)} \le Cp^{-2\sigma}\ \|u\|_{B^{1+2\sigma,\beta}(Q)}$$

This together with (4.8) leads to (4.7). $\square$

4.2 SINGULAR FUNCTIONS OF $r^\sigma \log^\mu r$-TYPE

For singularity with logarithmic terms we need to use the modified Jacobi-weighted Besov spaces for the best approximation.

Theorem 4.3 Let $v(x) = r^\sigma\ \log^\mu r\ \chi(r)\ \Phi(\phi)\ \Psi(x_3)$ given in (4.2), and let $\beta = (\beta_1, \beta_2, \beta_3)$ with $\beta_i > -1, 1 \le i \le 3$. Then $v \in B^{s,\beta}_{\mu^*}(Q)$, and $v \in H^{s-\epsilon,\beta}(Q)$ with $s = 2\sigma + 2 + \beta_1 + \beta_2$ and $\epsilon > 0$, arbitrary and

$$\mu^* = \begin{cases} \max\{\mu-1, 0\} & \text{if } \gamma \text{ is an integer,} \\ \mu & \text{if } \gamma \text{ is not an integer.} \end{cases} \tag{4.9.}$$

Proof Let $v = v_1 + v_2$ be the decomposition same as one in the proof of Theorem 4.1, i.e. $v_1 = \chi_\delta(r)\, v$ and $v_2 = (1-\chi_\delta(r))v$ for $\delta \in (0, r_0)$ Then $v_1 \in H^{0,\beta}(Q)$, and

$$\begin{aligned}\|v_1\|^2_{H^{0,\beta}(Q)} &= \int_Q |v|^2 \prod_{i=1}^{3}(1-x_i^2)^{\beta_i} dx \\ &\leq C \int_0^\delta r^{2\sigma+1+\beta_1+\beta_2} |\log r|^{2\nu} dr \\ &\leq C\delta^{2\sigma+2+\beta_1+\beta_2} |\log \delta|^{2\nu}.\end{aligned}$$

Also, $v_2 \in H^{k,\beta}(Q)$, and there holds for any $k > 2\sigma + 2 + \beta_1 + \beta_2$

$$\|v_2\|^2_{H^{k,\beta}(Q)} \leq C\delta^{2\sigma+2-k+\beta_1+\beta_2} |\log \delta|^{2\mu}.$$

Selecting $\delta = t^{\frac{2}{k}}$, we have for $t \in (0, 1)$

$$\begin{aligned}K(t, v) &\leq C(\|v_1\|_{H^{0,\beta}(Q)} + t\|v_2\|_{H^{k,\beta}(Q)}) \\ &\leq C\delta^{\sigma+1+\beta_1/2+\beta_2/2}(1 + t\delta^{-k/2})|\log \delta|^\mu \\ &\leq C\delta^{\sigma+1+\beta_1/2+\beta_2/2}(1 + |\log t|)^\mu,\end{aligned}$$

and for $t \geq 1$, there holds

$$K(t, v) \leq C\|v\|_{H^{0,\beta}(Q)}.$$

Letting $\theta = \dfrac{2\sigma + 2 + \beta_1 + \beta_2}{k}$, we have

$$\sup_{t>0} \frac{t^{-\theta} K(t, v)}{(1 + |\log t|)^\mu} \leq C,$$

which implies that $v \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,\infty,\nu} = B^{s,\beta}_\nu(Q)$ with $s = \theta k = 2\sigma + 2 + \beta_1 + \beta_2$.

Similarly arguing as in the proof of Theorem 4.1, and selecting $\theta = \dfrac{2\sigma + 2 + \beta_1 + \beta_2 - \epsilon}{k} = \dfrac{s - \epsilon}{k}$ with $\epsilon > 0$, arbitrary, we have

$$\int_0^1 t^{-2\theta} |K(t, u)|^2 \frac{dt}{t} \leq C \int_0^1 t^{-1+2\epsilon/k}(1 + |\log t|)^\mu dt \leq C,$$

which implies $u \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,2} = H^{s-\epsilon,\beta}(Q)$.

If σ is an integer and the integer $\mu \geq 1$, we adopt a different decomposition of $v = v_1 + v_2$, namely, for $\delta \in (0, 1)$

$$v_1 = r^\sigma\, (\log^\nu r - \log^\mu (r + \delta))\, \chi(r)\, \Phi(\phi)\, \Psi(x_3)$$

and

$$v_2 = r^\sigma \log^\mu(r+\delta)\chi(r)\,\Phi(\phi)\,\Psi(x_3).$$

Then $v_1 \in H^{0,\beta}(Q)$ and $v_2 \in H^{k,\beta}(Q)$ for any $k > 2\sigma + 2 + \sum_{i=1}^2 \beta_i$. By [1, Theorem 3.3], we have

$$\|v_1\|^2_{H^{0,\beta}(Q)} \le C\delta^{2\sigma+2+\beta_1+\beta_2}\,|\log\delta|^{2(\mu-1)} \tag{4.10}$$

and

$$\|v_2\|^2_{H^{k,\beta}(Q)} \le C\delta^{2\sigma-k+2+\beta_1+\beta_2}\,|\log\delta|^{2(\mu-1)}. \tag{4.11}$$

(4.10) and (4.11) lead to

$$\begin{aligned} K(t,v) &\le C\delta^{\sigma+1+\beta_1/2+\beta_2/2}(1+t\delta^{-k/2})|\log\delta|^{\mu-1} \\ &\le C\delta^{\sigma+1+\beta_1/2+\beta_2/2}|\log\delta|^{\mu-1} \end{aligned}$$

and

$$\sup_{0<t<1} \frac{t^{-\theta}K(t,v)}{(1+|\log t|)^{\mu-1}} \le C$$

with $\delta = t^{\frac{2}{k}}$ and $\theta = \dfrac{2\sigma+2+\beta_1+\beta_2}{k}$. This implies that $v \in B^{s,\beta}_{\mu-1}(Q)$ with $s = 2\sigma + 2 + \beta_1 + \beta_2$. □

Theorem 4.3 gives a precise characterization of the singular function of $r^\gamma \log^\mu r$-type, which avoids a loss of $O(p^\epsilon)$ in the approximation error.

Theorem 4.4 For $v(x) = r^\sigma \log^\mu r\ \chi(r)\,\Phi(\phi)\,\Psi(x_3)$ given in (4.2), there exists $\psi(x) \in \mathcal{P}_p(Q)$ such that

$$\|v-\psi\|_{L^2(Q)} \le Cp^{-(2\sigma+2)}(1+\log p)^{\mu*}\ \|v\|_{B^{2\sigma+2,\beta}_{\mu^*}(Q)} \tag{4.12}$$

with $\beta_1 = \beta_2 = 0$ and $\beta_3 > -1$, arbitrary. Also, there exists $\varphi(x) \in \mathcal{P}_p(Q)$ such that

$$\|v-\varphi\|_{H^1(R_0)} \le C\|v-\varphi\|_{H^{1,\beta}(Q)} \le Cp^{-2\sigma}(1+\log p)^{\mu^*}\ \|v\|_{B^{1+2\sigma,\beta}_{\mu^*}(Q)} \tag{4.13}$$

with $\beta_1 = \beta_2 = -1/2$ and $\beta_3 > -1$, arbitrary. In both (4.12) and (4.13) μ^* is given in (4.9).

Proof The approximability of the singular function v is the consequence of Theorem 2.5 and Theorem 4.3. We will not elaborate details of the proof, which are similar to those for Theorem 3.4 and Theorem 4.2. □

5. APPROXIMABILITY OF VERTEX-EDGE SINGULAR FUNCTIONS

Let $Q = (-1, 1)^3$, and let (ρ, θ, ϕ) be the spherical coordinates with respect to the vertex $(-1, -1, -1)$ and the vertical line $L = \{x = (x_1, x_2, x_3) \mid x_1 = x_2 = -1, x_3 \in (-\infty, \infty)\}$ as in Section 3.

We now consider the singular functions with real $\gamma, \sigma > 0$ and integers $\nu, \mu \geq 0$,

$$u(x) = \rho^\gamma \sin^\sigma \theta \, \chi(\rho) \, \Psi(\theta) \, \Phi(\phi) \tag{5.1}$$

and

$$v(x) = \rho^\gamma \log^\nu \rho \sin^\sigma \theta \log^\mu \sin \theta \, \chi(\rho) \, \Psi(\theta) \, \Phi(\phi) \tag{5.2}$$

where $\rho = \{(x_1 + 1)^2 + (x_2 + 1)^2 + (x_3 + 1)^2\}^{1/2}, \chi(\rho)$ and $\Phi(\phi)$ are C^∞ cut-off functions defined in Section 3 and 4 with $0 < \rho_0 < 1$, respectively, and $\Psi(\theta)$ is a C^∞ function such that for $\theta_0 \in (0, \pi/2)$

$$\Psi(\theta) = 1 \quad \text{for } 0 \leq \theta \leq \theta_0/2, \quad \Psi(\theta) = 0 \quad \text{for } \theta \geq \theta_0.$$

Obviously, u has a support $R_{\rho_0,\theta_0} = \{x \in Q \mid 0 < \rho < \rho_0, \theta \in (0, \theta_0)\} \subset Q$. For $0 < \phi_0 < \pi/4$, let

$$R_0 = R_{\rho_0,\theta_0,\phi_0} = \{x \in Q \mid 0 < \rho < \rho_0, \theta \in (0, \theta_0), \phi \in (\phi_0, \pi/2 - \phi_0)\}$$

as shown in Fig. 5.1. Then there hold for $x \in R_0$

$$\begin{aligned} (2 - \rho_0)(1 + x_i) &\leq (1 - x_i^2) \leq 2(1 + x_i), \; 1 \leq i \leq 3, \\ \frac{1 + x_3}{1 + x_i} &\geq \cot \theta_0, 1 \leq i \leq 2, \\ \tan \phi_0 &\leq \frac{1 + x_2}{1 + x_1} \leq \cot \phi_0. \end{aligned} \tag{5.3}$$

The singularity of the functions given in (5.1) and (5.2) is the well-known vertex-edge singularity for problems on polyhedral domains, which reflect the major difficulties in characterization of the singularity and analysis of the approximability. They combine the vertex and edge singularities, and are anisotropic. The combination of two types of singularities make the analysis totally different from those for the two dimensional setting and for the vertex-singularity and the edge-singularity. The designing the Jacobi-weighted Besov spaces and proving the regularities in these spaces for the best approximation are extremely difficult and elegant.

5.1 SINGULAR FUNCTIONS OF $\rho^\gamma \sin^\sigma \phi$-TYPE

Lemma 5.1 Let $u(x) = \rho^\gamma \sin^\sigma \theta \, \chi(\rho) \, \Psi(\theta) \, \Phi(\phi)$ given in (5.1). Then $u \in H^{s,\beta}(Q)$ with $\beta_i > -1, 1 \leq i \leq 3$ for $s < 2 + 2\min\{\gamma + (1+\beta_3)/2, \sigma\} + \beta_1 + \beta_2$.

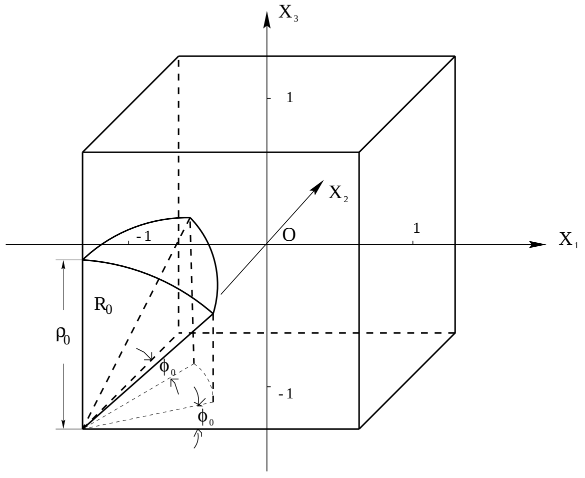

Fig. 5.1 Cubic Domain Q and sub region $R_{\rho_0,\phi_0,\theta_0}$

Proof Note that

$$|D^\alpha u| \le C\rho^{\gamma-|\alpha|}|\sin\theta|^{\sigma-\alpha_1-\alpha_2} \tag{5.4}$$

which implies that for $|\alpha| < 2+2\min\{\gamma+(1+\beta_3)/2,\sigma\}+\beta_1+\beta_2$

$$\begin{aligned} &\int_Q |D^\alpha u|^2 \prod_{i=1}^3 (1-x_i^2)^{\alpha_i+\beta_i} dx \\ \le\ & C\int_{R_0} \rho^{2\gamma-|\alpha|+2+\sum_{i=1}^3\beta_1}|\sin\theta|^{2\sigma-\alpha_1-\alpha_2+1+\sum_{i=1}^2\beta_i} d\rho d\theta d\phi < \infty. \end{aligned}$$

This proves the lemma for integer $s=k$. By a typical argument of interpolation spaces we are able to prove the lemma for non-integer s in general. $\Box$

Theorem 5.2 Let $u(x)=\rho^\gamma \sin^\sigma\theta\,\chi(\rho)\,\Psi(\theta)\,\Phi(\phi)$ given in (5.1), and let $\beta=(\beta_1,\beta_2,\beta_3)$ with $\beta_i > -1, 1\le i\le 3$. Then $u\in H^{s-\epsilon,\beta}(Q)$ and $u\in B^{s,\beta}_\kappa(Q)$ with $s=2+2\min\{\sigma,\gamma+(1+\beta_3)/2\}+\beta_1+\beta_2$, $\epsilon>0$, arbitrary, and

$$\kappa = \begin{cases} 0 & \text{if } \sigma \ne \gamma+(1+\beta_3)/2, \\ 1/2 & \text{if } \sigma = \gamma+(1+\beta_3)/2. \end{cases} \tag{5.5}$$

Proof Since $r=\rho\sin\theta=\{(1+x_1)^2+(1+x_2)^2\}^{1/2}$ we write

$$u(x)=\rho^{\gamma-\sigma}\,r^\sigma\,\chi(\rho)\,\Phi(\phi)\,\Psi(\theta),$$

and the estimate (5.4) can be written as

$$|D^\alpha u(x)| \le C\rho^{\gamma-|\alpha|}|\sin\theta|^{\sigma-\alpha_1-\alpha_2} \le C(1+x_3)^{\gamma-\sigma-\alpha_3}\,r^{\sigma-\alpha_1-\alpha_2}. \tag{5.6}$$

By $\phi_\delta(r)$ we denote a C^∞ function such that $\varphi_\delta(r) = 1$ for $r < \delta$ and $\varphi_\delta(r) = 0$ for $r > 2\delta$ with $0 < \delta < \rho_0/2$. Let $u_1 = \varphi_\delta(r)u$ and $u_2 = (1-\phi_\delta(r))u$. Then $u_1 \in H^{0,\beta}(Q)$ due to Lemma 5.1, and $u_2 \in H^{k,\beta}(Q)$ for any $k > 2 + 2\max\{\gamma + (1+\beta_3)/2, \sigma\} + \beta_1 + \beta_2$.

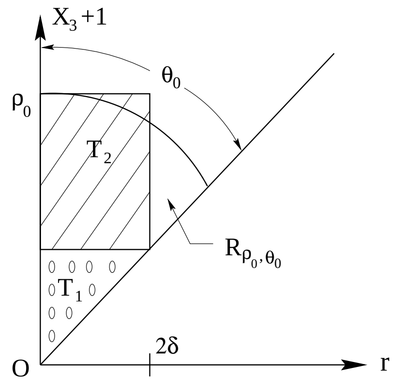

Fig. 5.2, Regions T_1, T_2 and R_{ρ_0,θ_0}

Let R_{ρ_0,θ_0} be the projection of $R_{\rho_0,\theta_0,\phi_0}$ on the $r - x_3$ plane,

$$R_{\rho_0,\theta_0} = \{(r, x_3) \mid r\cot\theta_0 \le (1 + x_3) \le (\rho_0^2 - r^2)^{1/2}, 0 \le r \le \rho_0 \sin\theta_0\},$$

and by T_1 and T_2 we denote the triangular and rectangular regions in the $r - x_3$ plane, respectively,

$$T_1 = \{(r, x_3) \mid \ r\cot\theta_0 \le 1 + x_3 \le 2\delta\cot\theta_0, 0 \le r \le 2\delta\}$$

and

$$T_2 = \{(r, x_3) \mid \ 2\delta\cot\theta_0 \le 1 + x_3 \le \rho_0, 0 \le r \le 2\delta\}$$

as shown in Fig. 5.2. Obviously, Supp.$u_1 \subset T_1 \cup T_2$.

Due to (5.6) there holds

$$\begin{aligned} \|u_1\|^2_{H^{0,\beta}(Q)} &= \int_{R_{\rho_0,\theta_0,\phi_0},} |\varphi_\delta u|^2 \rho^{\sum_{i=1}^3 \beta_i} |\sin\theta|^{\beta_1+\beta_2} dx \\ &\le C \int_{R_{\rho_0,\theta_0}} |\varphi_\delta u|^2 (1+x_3)^{\beta_3}\, r^{1+\beta_1+\beta_2} dr dx_3 \qquad (5.7) \\ &\le C \int_{T_1\cup T_2} (1+x_3)^{2(\gamma-\sigma)+\beta_3}\, r^{2\sigma+1+\beta_1+\beta_2} dr dx_3. \end{aligned}$$

Letting $\tilde{x}_3 = x_3 + 1$, we have by a simple calculation

$$
\begin{aligned}
&\int_{T_1} (1+x_3)^{2(\gamma-\sigma)+\beta_3}\, r^{2\sigma+1+\beta_1+\beta_2} dr dx_3 \\
&\le C \int_0^{2\delta\cot\theta_0} \tilde{x}_3^{2(\gamma-\sigma)+\beta_3}\, d\tilde{x}_3 \int_0^{\tilde{x}_3 tan\theta_0} r^{2\sigma+1+\sum_{i=1}^2 \beta_i} dr \\
&\le C \int_0^{2\delta\cot\theta_0} \tilde{x}_3^{2\gamma+2+\sum_{i=1}^3 \beta_i}\, d\tilde{x}_3 \\
&\le C\delta^{2\gamma+\sum_{i=1}^3 \beta_i+3}.
\end{aligned}
\tag{5.8}
$$

We also have for $\sigma \neq \gamma + (1+\beta_3)/2$

$$
\begin{aligned}
&\int_{T_2} (1+x_3)^{2(\gamma-\sigma)+\beta_3}\, r^{2\sigma+1+\beta_1+\beta_2} dr dx_3 \\
&\le C \int_0^{2\delta} r^{2\sigma+1+\beta_1+\beta_2} dr \int_{2\delta\cot\theta_0}^{\rho_0} \tilde{x}_3^{2(\gamma-\sigma)+\beta_3}\, d\tilde{x}_3 \\
&\le C(1+\delta^{2(\gamma-\sigma)+1+\beta_3})\delta^{2\sigma+2+\beta_1+\beta_2} \\
&\le C(\delta^{2\gamma+3+\sum_{i=1}^3 \beta_i} + \delta^{2\sigma+2+\beta_1+\beta_2})
\end{aligned}
\tag{5.9}
$$

and for $\sigma = \gamma + (1+\beta_3)/2$

$$
\begin{aligned}
&\int_{T_2} (1+x_3)^{2(\gamma-\sigma)+\beta_3}\, r^{2\sigma+1+\beta_1+\beta_2} dr dx_3 \\
&\le C \int_0^{2\delta} r^{2\sigma+1+\beta_1+\beta_2} dr \int_{2\delta\cot\theta_0-1}^{\rho_0-1} (1+x_3)^{2(\gamma-\sigma)+\beta_3}\, dx_3 \\
&\le C(1+|\log\delta|)\delta^{2\sigma+2+\beta_1+\beta_2},
\end{aligned}
\tag{5.10}
$$

which together with (5.7)-(5.10) yields

$$
\|u_1\|^2_{H^{0,\beta}(Q)} \le C(1+|\log\delta|)^{2\kappa}\delta^{2+2\min\{\gamma+(1+\beta_3)/2,\sigma\}+\beta_1+\beta_2} \tag{5.11}
$$

with κ given in (5.5).

We next estimate $\|u_2\|_{H^{k,\beta}(Q)}$. Note that

$$
\frac{\partial^k u_2}{\partial x_1^k} = (1-\varphi_\delta)\frac{\partial^k u}{\partial x_1^k} - \sum_{l=0}^{k-1} \binom{k}{l} \frac{\partial^l u}{\partial x_1^l}\frac{\partial^{k-l}\varphi_\delta}{\partial x_1^{k-l}},
$$

and for $0 \le l \le k$

$$
\Big|\frac{\partial^{k-l}\varphi_\delta}{\partial x_1^{k-l}}\Big| \le C\delta^{-(k-l)}.
$$

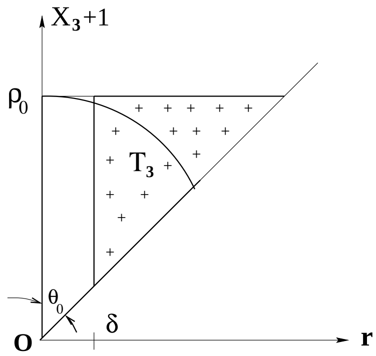

Fig. 5.3 Region T_3

Let

$$T_3 = \{(r, x_3) \mid \; r\cot\theta_0 \le 1 + x_3 \le \rho_0, \delta \le r \le \rho_0 \tan\theta_0\}$$

as shown in Fig. 5.3. Obviously, Supp. $(1-\varphi_\delta)\dfrac{\partial^k u}{\partial x_1^k}$ and Supp. $\dfrac{\partial^{k-l}\varphi_\delta}{\partial x_1^{k-l}}$ are contained in T_3 for $0 \le l < k$. . It is seen that

$$\begin{aligned}
&\int_Q |\frac{\partial^k u_2}{\partial x_1^k}|^2 (1-x_1^2)^{k+\beta_1} \prod_{i=2}^{3} (1-x_i^2)^{\beta_i} dx \\
\le\;\; & C\int_{R_0} \Big(\Big|\frac{\partial^k u}{\partial x_1^k}\Big|^2 |1-\varphi_\delta|^2 + \sum_{l=0}^{k-1} \Big|\frac{\partial^l u}{\partial x_1^l}\Big|^2 \Big|\frac{\partial^{k-l}\varphi_\delta}{\partial x_1^{k-l}}\Big|^2 \rho^{k+\sum_{i=1}^3 \beta_i} |\sin\theta|^{k+\beta_1+\beta_2} \Big) dx.
\end{aligned}$$

Due to (5.6) there hold

$$\begin{aligned}
&\int_{R_0} \Big|\frac{\partial^k u}{\partial x_1^k}\Big|^2 |\varphi_\delta|^2 \rho^{k+\sum_{i=1}^3 \beta_i} |\sin\theta|^{k+\beta_1+\beta_2} dx \\
&\le C\int_{T_3} (1+x_3)^{2(\gamma-\sigma)-k+\beta_3}\, r^{2\sigma+1-k+\beta_1+\beta_2} dr dx_3 \\
&\le C\int_{\delta\cot\theta_0}^{\rho_0} \tilde{x}_3^{2(\gamma-\sigma)+\beta_3}\, d\tilde{x}_3 \int_{\delta}^{\tilde{x}_3 \tan\theta_0} r^{2\sigma+1-k+\beta_1+\beta_2} dr \qquad (5.12) \\
&\le C\delta^{2\sigma+2-k+\beta_1+\beta_2} \int_{\delta\cot\theta_0}^{\rho_0} \tilde{x}_3^{2(\gamma-\sigma)+\beta_3} d\tilde{x}_3 \\
&\le C(1+|\log\delta|)^{2\kappa} \delta^{2\gamma+3-k+\sum_{i=1}^3 \beta_i},
\end{aligned}$$

and for $l < k$

(5.13)

$$\int_{R_0} \Big|\frac{\partial^l u}{\partial x_1^l}\Big|^2 \Big|\frac{\partial^{k-l}\varphi_\delta}{\partial x_1^{k-l}}\Big|^2 \rho^{k+\sum_{i=1}^3 \beta_i} |\sin\theta|^{k+\beta_1+\beta_2} dx$$

$$\leq C\delta^{-2(k-l)} \int_{\delta\cot\theta_0}^{2\delta\cot\theta_0} \tilde{x}_3^{2(\gamma-\sigma)+\beta_3}\, d\tilde{x}_3 \int_{\delta}^{\tilde{x}_3\tan\theta_0} r^{2(\sigma-l)+1+k+\beta_1+\beta_2} dr$$

$$\leq C\delta^{2(\sigma+1)+\beta_1+\beta_2-k} \int_{\delta\cot\theta_0}^{2\delta\cot\theta_0} \tilde{x}_3^{2(\gamma-\sigma)+\beta_3} d\tilde{x}_3$$

$$\leq C(1+|\log\delta|)^{2\kappa}\delta^{2\gamma+\sum_{i=1}^3\beta_i+3-k}.$$

A combination of (5.12) and (5.13) leads to

$$\int_Q \Big|\frac{\partial^k u_2}{\partial x_1^k}\Big|^2 (1-x_1^2)^{k+\beta_1} \prod_{i=1}^{2}(1-x_i^2)^{\beta_i} dx \leq C|\log\delta|^{2\kappa}\delta^{2\gamma+\sum_{i=1}^3\beta_i+3-k}.$$

The estimate on $\dfrac{\partial^k u_2}{\partial x_3^k}$ can be carried similarly. Due to (5.6), there hold

$$\Big|\frac{\partial^k u_2}{\partial x_3^k}\Big| = \Big|\varphi_\delta \frac{\partial^k u}{\partial x_3^k}\Big| \leq C(1+x_3)^{\gamma-\sigma-k}\, r^\sigma$$

and

$$\int_Q \Big|\frac{\partial^k u_2}{\partial x_3^k}\Big|^2 \prod_{i=1}^{2}(1-x_i^2)^{\beta_i}(1-x_3^2)^{k+\beta_3} dx$$

$$\leq C \int_{T_3} \Big|\frac{\partial^k u}{\partial x_3^k}\Big|^2 \rho^{k+\sum_{i=1}^3\beta_i} |\sin\theta|^{\beta_1+\beta_2} dx$$

$$\leq C \int_{\delta}^{\rho_0\tan\theta_0} r^{2\sigma+1+\beta_1+\beta_2} dr \int_{r\cot\theta_0}^{\rho_0} \tilde{x}_3^{2(\gamma-\sigma)-k+\beta_3}\, d\tilde{x}_3$$

$$\leq C \int_{\delta}^{\rho_0\tan\theta_0} r^{2\gamma+2+\sum_{i=1}^3\beta_i-k} dr$$

$$\leq C \quad \delta^{2\gamma+3-k+\sum_{i=1}^3\beta_i}.$$

We can treat all terms of $D^\alpha u_2$ with $|\alpha| \leq k$ in similar way, which gives for $k > 2\max\{\sigma, \gamma+1/2+\beta_3\} + 2 + \beta_1 + \beta_2$

(5.14)
$$\|u_2\|^2_{H^{k,\beta}(Q)} \leq C(1+|\log\delta|)^{2\kappa}\delta^{2\gamma+\sum_{i=1}^3\beta_i+3-k}.$$

Therefore, we have by (5.11) and (5.14)

$$\begin{aligned} K(t,u) &= \inf_{u=v+w}\{\|v\|_{H^{0,\beta}(Q)} + t\,\|w\|_{H^{k,\beta}(Q)}\} \\ &\le C(\|u_1\|_{H^{0,\beta}(Q)} + t\,\|u_2\|_{H^{k,\beta}(Q)}) \\ &\le C(1+|\log\delta|)^{\kappa}\delta^{1+\min\{\gamma+(1+\beta_3)/2,\sigma\}+\beta_1/2+\beta_2/2}(1+t\,\delta^{-k/2}). \end{aligned}$$

Selecting $\delta = t^{2/k}$, we have for $0 < t < 1$

$$K(t,u) \le C(1+|\log t|)^{\kappa} t^{\frac{2+2\min\{\gamma+(1+\beta_3)/2,\sigma\}+\beta_1+\beta_2}{k}}.$$

For $t >\geq 1$, it always holds

$$K(t,u) \le C\|u_1\|_{H^{0,\beta}(Q)}.$$

Choosing $\theta = \dfrac{2+2\min\{\gamma+(1+\beta_3)/2,\sigma\}+\beta_1+\beta_2}{k}$, we have

$$\sup_{t>0}\frac{t^{-\theta}\,K(t,u)}{(1+|\log t|)^{\kappa}} \le C$$

which implies that $u \in \left(H^{0,\beta}(Q), H^{k,\beta}(Q)\right)_{\theta,\infty,\kappa} = B^{s,\beta}_{\kappa}(Q)$ with $s = \theta k = 2\min\{\gamma+(1+\beta_3)/2,\sigma\}+\beta_1+\beta_2$ and κ given in (5.10).

Selecting $\theta = \dfrac{2+2\min\{\gamma+(1+\beta_3)/2,\sigma\}+\beta_1+\beta_2-\epsilon}{k} = \dfrac{s-\epsilon}{k}$ with $\epsilon > 0$, arbitrary, gives for either $\sigma = \gamma+(1+\beta_3)/2$ or $\sigma \ne \gamma+(1+\beta_3)/2$

$$\int_0^{\infty} t^{-2\theta}|K(t,u)|^2\frac{dt}{t} \le C$$

which implies $u \in \left(H^{0,\beta}(Q), H^{k,\beta}(Q)\right)_{\theta,2} = H^{s-\epsilon,\beta}(Q)$. □

A combination of Theorem 5.2 and Theorem 2.4-2.5 leads to the approximability of the singular function of $\rho^{\gamma}\sin^{\sigma}\phi$-type.

Theorem 5.3 There exists $\psi(x) \in P_p(Q)$ such that for $\beta = (0,0,0)$ and $s = 2+2\min\{\sigma,\gamma+1/2\}$

$$\|u-\psi\|_{L^2(Q)} \le Cp^{-(2+2\min\{\sigma,\gamma+1/2\})}\,\|u\|_{B^{s,\beta}(Q)}. \tag{5.15}$$

if $\sigma \ne \gamma+1/2$, and

$$\|u-\psi\|_{L^2(Q)} \le Cp^{-(2+2\min\{\sigma,\gamma+1/2\})}(1+\log p)^{1/2}\,\|u\|_{B^{s,\beta}_{1/2}(Q)}. \tag{5.16}$$

Also, there exists $\varphi(x) \in \mathcal{P}_p(Q)$ such that for $\beta = (-1/2,-1/2,0)$ and $s = 1+2\min\{\sigma,\gamma+1/2\}$

$$\|u-\varphi\|_{H^1(R_0)} \le Cp^{-2\min\{\sigma,\gamma+1/2\}}\,\|u\|_{B^{s,\beta}(Q)} \tag{5.17}$$

if $\sigma \neq \gamma + 1/2$, and

$$\|u - \varphi\|_{H^1(R_0)} \leq Cp^{-2\sigma}(1 + \log p)^{1/2} \, \|u\|_{B^{s,\beta}_{1/2}(Q)} \tag{5.18}$$

if $\sigma = \gamma + 1/2$.

Proof For $\beta = (0,0,0)$, Theorem 5.2 indicates that $u \in B^{s,\beta}(Q)$ if $\sigma \neq \gamma + 1/2$ and $u \in B^{s,\beta}_{1/2}(Q)$ if $\sigma = \gamma + 1/2$ with $s = 2 + 2\min\{\sigma, \gamma + 1/2\}$, which together with Theorem 2.4-2.5 leads to (5.14) and (5.15) immediately.

Also for $\beta = (-1/2, -1/2, 0)$, Theorem 5.2 tells that $u \in B^{s,\beta}(Q)$ if $\sigma \neq \gamma + 1/2$ and $u \in B^{s,\beta}_{1/2}(Q)$ if $\sigma = \gamma + 1/2$ with $s = 1 + 2\min\{\sigma, \gamma + 1/2\}$. Due to Theorem 2.4-2.5, there exists $\varphi(x) \in \mathcal{P}_p(Q)$ such that for $\ell = 0, 1$

$$|u - \varphi|_{H^{\ell,\beta}(Q)} \leq Cp^{-(2\min\{\sigma,\gamma+1/2\}+1-\ell)} \, \|u\|_{B^{s,\beta}(Q)} \tag{5.19}$$

if $\sigma \neq \gamma + 1/2$, and

$$|u - \varphi|_{H^{\ell,\beta}(Q)} \leq Cp^{-(2\min\{\sigma,\gamma+1/2\}+1-\ell)}(1 + \log p)^{1/2} \, \|u\|_{B^{s,\beta}_{1/2}(Q)} \tag{5.20}$$

if $\sigma = \gamma + 1/2$. Note that

$$|u - \varphi|_{L^2(Q)} \leq C|u - \varphi|_{H^{0,\beta}(Q)}. \tag{5.21}$$

Due to (5.3), there holds for $x \in R_0 = R_{\rho_0,\theta_0,\phi_0}$ and $|\alpha| = 1$

$$C_1 \leq (1 + x_1)^{\alpha_1 - 1/2}(1 + x_2)^{\alpha_2 - 1/2}(1 + x_3)^{\alpha_3} \leq C_2 \tag{5.22}$$

where two positive constants C_1 and C_2 are independent of x. This implies that for $|\alpha| = 1$

$$\begin{aligned}
&\int_{R_0} \Big|D^\alpha(u - \varphi)\Big|^2 dx \\
&\leq \; C \int_{R_0} \Big|D^\alpha(u - \varphi)\Big|^2 \prod_{i=1}^{2}(1 + x_i)^{\alpha_i - 1/2}(1 + x_3)^{\alpha_3} \, dx \\
&\leq \; C \int_{R_0} \Big|D^\alpha(u - \varphi)\Big|^2 \prod_{i=1}^{2}(1 - x_i^2)^{\alpha_i - 1/2}(1 - x_3^2)^{\alpha_3} dx \\
&\leq \; C|u - \varphi|^2_{H^{1,\beta}(Q)}
\end{aligned}$$

which together with (5.19)-(5.21) leads to (5.17) and (5.18), and complete the proof. □

5.2 SINGULAR FUNCTIONS OF $\rho^\gamma \log^\nu \rho \sin^\sigma \theta \log^\mu \sin\theta$-TYPE

Since the function given in (5.2) can be written as

$$\begin{aligned} v(x) = &\ \rho^{\gamma-\sigma} r^\sigma \log^\nu \rho\, (\log\rho - \log r)^\mu \chi(\rho)\, \Psi(\theta)\, \Phi(\phi) \\ = &\ \rho^{\gamma-\sigma} r^\sigma \log^\nu \rho\, \chi(\rho)\, \Psi(\theta)\, \Phi(\phi) \textstyle\sum_{l=0}^{\mu} \binom{\mu}{l} (-1)^{\mu-l} \log^l \rho \log^{\mu-l} r \end{aligned}$$

We need to analyze the functions of this type

$$\begin{aligned} w(x) = &\ \rho^{\gamma-\sigma} r^\sigma \log^{\nu+l} \rho \log^{\mu-l} r\, \chi(\rho)\, \Psi(\theta)\, \Phi(\phi) \\ = &\ \rho^{\gamma-\sigma} r^\sigma \log^{\nu'} \rho \log^{\mu'} r\, \chi(\rho)\, \Psi(\theta)\, \Phi(\phi). \end{aligned}$$

with $\nu', \mu' \geq 0$.

Theorem 5.4 Let $\beta = (\beta_1, \beta_2, \beta_3)$ with $\beta_i > -1, 1 \leq i \leq 3$. Then $w \in H^{s-\epsilon,\beta}(Q)$ and $w(x) \in B^{s,\beta}_{\kappa'}(Q)$ with $s = 2+2\min\{\gamma+(1+\beta_3)/2, \sigma\}+\beta_1+\beta_2$, $\epsilon > 0$, arbitrary, and

$$\kappa' = \begin{cases} \mu' & \text{if } \sigma < \gamma + (1+\beta_3)/2, \\ \mu' + \nu' + 1/2 & \text{if } \sigma = \gamma + (1+\beta_3)/2, \\ \mu' + \nu' & \text{if } \sigma > \gamma + (1+\beta_3)/2. \end{cases} \tag{5.22}$$

Proof We decompose the function into $w = w_1 + w_2$ with $w_1 = \varphi_\delta(r) u$ and $w_2 = (1 - \varphi_\delta(r)) u$, where $\varphi_\delta(r)$ is a C^∞ function defined as previously. It is easy to verify that $w_1 \in H^{0,\beta}(Q)$ and $w_2 \in H^{k,\beta}(Q)$ for any $k > 2 + 2\max\{\gamma + (1+\beta_3)/2, \sigma\} + \beta_1 + \beta_2$.

Let $R_{\rho_0,\theta_0}, T_i, 1 \leq i \leq 3$ be the regions defined as in previous section and shown in Fig. 5.2-5.3. There holds
(5.24)

$$\|w_1\|^2_{H^{0,\beta}(Q)} = \int_{R_{\rho_0,\theta_0,\phi_0,}} |\varphi_\delta w|^2 \rho^{\sum_{i=1}^3 \beta_i} |\sin\theta|^{\beta_1+\beta_2} dx$$

$$\leq C \int_{R_{\rho_0,\theta_0}} |\varphi_\delta u|^2 (1+x_3)^{\beta_3}\, r^{1+\beta_1+\beta_2} dr dx_3$$

$$\leq C \int_{T_1 \cup T_2} (1+x_3)^{2(\gamma-\sigma)+\beta_3} \log^{2\nu'}(1+x_3)\, r^{2\sigma+1+\beta_1+\beta_2} \log^{2\mu'} r dr dx_3.$$

Letting $\tilde{x}_3 = x_3 + 1$, as analogue to the estimates (5.7), we have
(5.25)

$$\int_{T_1} (1+x_3)^{2(\gamma-\sigma)+\beta_3} \log^{2\nu'}(1+x_3)\, r^{2\sigma+1+\beta_1+\beta_2} \log^{2\mu'} r dr dx_3$$

$$\leq C \int_0^{2\delta\cot\theta_0} \tilde{x}_3^{2(\gamma-\sigma)+\beta_3} \log^{2\nu'} \tilde{x}_3\, d\tilde{x}_3 \int_0^{\tilde{x}_3 \tan\theta_0} r^{2\sigma+1+\beta_1+\beta_2} \log^{2\mu'} r dr$$

$$\leq C \int_0^{2\delta\cot\theta_0} \tilde{x}_3^{2\gamma+2+\sum_{i=1}^3 \beta_i} \log^{2(\nu'+\mu')} \tilde{x}_3\, d\tilde{x}_3$$
$$\leq C\delta^{2\gamma+\sum_{i=1}^3 \beta_i+3}(1+|\log\delta|)^{2(\nu'+\mu')}.$$

Analogously to (5.8)-(5.10) we have for $\sigma \neq \gamma + (1+\beta_3)/2$,
(5.26)

$$\int_{T_2} (1+x_3)^{2(\gamma-\sigma)+\beta_3} \log^{2\nu'}(1+x_3)\, r^{2\sigma+1+\beta_1+\beta_2} \log^{2\mu'} r dr dx_3$$

$$\leq C \int_0^{2\delta} r^{2\sigma+1+\beta_1+\beta_2} \log^{2\mu'} r dr \int_{2\delta\cot\theta_0}^{\rho_0} \tilde{x}_3^{2(\gamma-\sigma)+\beta_3} \log^{2\nu'} \tilde{x}_3\, d\tilde{x}_3$$

$$\leq C(1+\delta^{2(\gamma-\sigma)+1+\beta_3}|\log\delta|^{2\nu'})\delta^{2\sigma+2+\beta_1+\beta_2}|\log\delta|^{2\mu'}$$
$$\leq C(\delta^{2\gamma+3+\sum_{i=1}^3 \beta_i}|\log\delta|^{2(\nu'+\mu')} + \delta^{2\sigma+2+\beta_1+\beta_2}|\log\delta|^{2\mu'})$$

and for $\sigma = \gamma + (1+\beta_3)/2$,

$$\int_{T_2} (1+x_3)^{2(\gamma-\sigma)+\beta_3} \log^{2\nu'}(1+x_3)\, r^{2\sigma+1+\beta_1+\beta_2} \log^{2\mu'} r dr dx_3$$

$$\leq C \int_0^{2\delta} r^{2\sigma+1+\beta_1+\beta_2} \log^{2\mu'} r dr \int_{2\delta\cot\theta_0}^{\rho_0} \tilde{x}_3^{-1} \log^{2\nu'} \tilde{x}_3\, d\tilde{x}_3 \tag{5.27}$$

$$\leq C(1+|\log\delta|)^{2\nu'+1})\delta^{2\sigma+2+\beta_1+\beta_2}|\log\delta|^{2\mu'}$$
$$\leq C|\log\delta|^{2(\nu'+\mu')+1}\, \delta^{2\sigma+2+\beta_1+\beta_2}.$$

Combining (5.25)-(5.27) yields

$$\|w_1\|^2_{H^{0,\beta}(Q)} \leq C|\log\delta|^{2\kappa'}\delta^{2+2\min\{\sigma,\gamma+(1+\beta_3)/2\}+\beta_1+\beta_2}. \tag{5.28}$$

Similarly we have the estimate on $\|w_2\|^2_{H^{k,\beta}(Q)}$,

$$\|w_2\|^2_{H^{k,\beta}(Q)} \leq C|\log\delta|^{2\kappa'}\delta^{2+2\min\{\sigma,\gamma+(1+\beta_3)/2\}+\beta_1+\beta_2-k} \tag{5.29}$$

It follows from (5.28) and (5.29) that

$$K(t,w) \leq C|\log\delta|^{\kappa'}\delta^{1+\min\{\sigma,\gamma+(1+\beta_3)/2\}+\beta_1/2+\beta_2/2}\,(1+t\delta^{-k/2})$$

Selecting $\delta = t^{2/k}$ and $\theta = \dfrac{2 + 2\min\{\gamma + (1+\beta_3)/2, \sigma\} + \beta_1 + \beta_2}{k}$, we have for $0 < t < 1$

$$\frac{t^{-\theta} K(t,u)}{(1 + |\log t|)^{\kappa'}} \le C$$

which implies the desired characterization of the singularity of the function $w(x)$ in the spaces $B^{s,\beta}_{\kappa'}(Q)$ with $s = 2 + 2\min\{\gamma + (1+\beta_3)/2, \sigma\} + \beta_1 + \beta_2$ and κ' given in (5.22).

Selecting $\theta = \dfrac{2 + 2\min\{\gamma + (1+\beta_3)/2, \sigma\} + \beta_1 + \beta_2 - \epsilon}{k} = \dfrac{s-\epsilon}{k}$ with $\epsilon > 0$, arbitrary, we have

$$\int_0^\infty t^{-2\theta} |K(t,u)|^2 \frac{dt}{t} \le C$$

which implies $u \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,2} = H^{s-\epsilon,\beta}(Q)$. □

The following theorem on the characterization of singularity of the function $v(x)$ is a corollary of Theorem 5.4.

Theorem 5.5 Let $v(x)$ be given in (5.2), and let $\beta_i > -1, 1 \le i \le 3$. Then $w \in H^{s-\epsilon,\beta}(Q)$ and $v(x) \in B^{s,\beta}_{\kappa}(Q)$ with $s = 2 + 2\min\{\gamma + (1+\beta_3)/2, \sigma\} + \beta_1 + \beta_2$,$\epsilon > 0$, arbitrary, and

$$\kappa = \begin{cases} \mu & \text{if } \sigma < \gamma + (1+\beta_3)/2, \\ \mu + \nu + 1/2 & \text{if } \sigma = \gamma + (1+\beta_3)/2, \\ \mu + \nu & \text{if } \sigma > \gamma + (1+\beta_3)/2. \end{cases} \tag{5.30}$$

Characterization of singularity of the function $v(x)$ by Theorem 5.5 and the approximation property described in Theorem 2.5 give the approximability of $v(x)$.

Theorem 5.6 Let $v(x)$ be given in(5.2). Then there exists $\psi(x) \in \mathcal{P}_p(Q)$ such that for $\beta = (0,0,0)$ and $s = 2 + 2\min\{\sigma, \gamma + 1/2\}$

$$\|v - \psi\|_{L^2(Q)} \le C p^{-(2+2\min\{\sigma,\gamma+1/2\})} (1 + \log p)^{\kappa} \, \|u\|_{B^{s,\beta}_{\kappa}(Q)}. \tag{5.31}$$

Also, there exists $\varphi(x) \in \mathcal{P}_p(Q)$ such that for $\beta = (-1/2, -1/2, 0)$ and $s = 1 + 2\min\{\sigma, \gamma + 1/2\}$

$$\|v - \varphi\|_{H^1(R_0)} \le C p^{-2\min\{\sigma,\gamma+1/2\}} (1 + \log p)^{\kappa} \, \|u\|_{B^{s,\beta}_{\kappa}(Q)}. \tag{5.32}$$

κ in (5.31) and (5.32) is given in (5.30).

Proof By Theorem 5.5 $v(x) \in B^{2\min\{\gamma+1/2,\sigma\}+2,\beta}_{\kappa}(Q)$ with κ specified by (5.30), in particular, for $\beta = (0,0,0)$ and $\beta = (-1/2,-1/2,0)$.

Applying Theorem 2.5 with $\beta = (0,0,0)$ leads to (5.31) immediately. Applying Theorem 2.5 with $\beta = (-1/2,-1/2,0)$ and arguing as in the proof of Theorem 5.3 we can easily obtain (5.32). □

Remark κ given in (5.30) reduces to (5.6) if $\nu = \mu = 0$. κ depends on ν and μ, but also on the relation between γ and σ. When $\sigma = \gamma + (1+\beta_3)/2$, $v(x) \in B^{s,\beta}_{\kappa}(Q)$ with κ increased by an extra value of $1/2$. Consequently, an extra loss of a factor $(1+\log p)^{1/2}$ happens in the error estimate (5.31) and (5.32), which was mentioned in [16] for the p-version of BEM. Whether the extra value of $1/2$ can be removed or not is an open question for further investigation. Fortunately, the extra value of $1/2$ appears in κ not in s.

6. CONCLUDING REMARKS

The singularities of singular functions in three dimensions and their approximabilities have been analyzed in the framework of the Jacobi-weighted Besov and Sobolev spaces. To precisely characterizing the singularities and investigate the approximabilities for singular functions of three different types, Jacobi-weighted Besov and Sobolev spaces associated with three different Jacobi weights are elegantly designed. The most difficult as well as most significant work is the characterization of the functions with the singularity of $\rho^\gamma \log^\nu \rho \sin^\sigma \theta \log^\mu \sin\theta$-type in the Besov space $B^{s,\beta}_{\kappa}(Q)$ with κ given in (5.30). The singularity of this type is anisotropic and totally different from the singularity in two dimension. The key for success is the decomposition of the singular function with a cut-off function $\varphi_\delta(r)$, instead of $\varphi_\delta(\rho)$ and $\varphi_\delta(\theta)$ although the singularity appears in ρ and θ. After having tried various decompositions we are convinced that only this decomposition can lead to our desired results. For the best approximation of these singular functions we select different weights, namely, $\beta = (-1/3,-1/3,-1/3), \beta = (-1/2,-1/2,\beta_3), \beta = (-1/2,-1/2,0)$, respectively. We are also convinced that only this selection can give us the best error estimation in L^2- and H^1-norms. Once the weights are properly selected the approximation results follows in natural way. Our approach for error estimation for singular functions are different from usual approach, namely we do not analyze directly approximation of singular functions, but verify that they belong to certain Jacobi-weighted Besov spaces.

Although the treatments for singular functions in three dimensions are quite different from those in one and two dimensions and much more difficult, it is worth indicating that the structures of Jacobi-weighted spaces are basically the same. The difference lies only in the selection of Jacobi weights and the way to prove that singular function belong to the Jacobi-weighted spaces. Hence the mathematical framework of the Jacobi-weighted Besov and Sobolev spaces is robust and uniform for problems in one, two and three dimensions.

Table 6.1. The value of k and s in Sobolev, Besov and Jacobi-weighted Besov spaces for functions of $\rho^\gamma, r^\sigma, \rho^\gamma \sin^\sigma \theta$-type

Space	$H^k(Q)$	$H^s(Q)$	$B^s(Q)$	$H^{k,\beta}(Q)$	$B^{s,\beta}(Q)$
ρ^γ	$3/2+[\gamma]$	$3/2+\gamma-\epsilon$	$3/2+\gamma$	$2+2\gamma-\epsilon$	$2+2\gamma$
r^σ	$1+[\sigma]$	$1+\sigma-\epsilon$	$1+\sigma$	$1+2\sigma-\epsilon$	$1+2\sigma$
$\rho^\gamma \sin^\sigma \theta$	$1+[\lambda]$	$1+\lambda-\epsilon$	$1+\lambda$	$1+2\lambda-\epsilon$	$1+2\lambda$

Table 6.2. Accuracy of approximation in H^1-norm to singular functions of $\rho^\gamma, r^\sigma, \rho^\gamma \sin^\sigma \theta$-type by the h- and p-version based on Sobolev, Besov and Jacobi-weighted Besov spaces

	h version		p version		
Space	$H^s(Q)$	$B^s(Q)$	$H^s(Q)$	$B^s(Q)$	$B^{s,\beta}(Q)$
ρ^γ	$h^{1/2+\gamma-\epsilon}$	$h^{1/2+\gamma+1/2}$	$p^{-(1/2+\gamma-\epsilon)}$	$p^{-(\gamma+1/2)}$	$p^{-(2\gamma+1)}$
r^σ	$h^{\sigma-\epsilon}$	h^σ	$p^{-(\sigma-\epsilon)}$	$p^{-\sigma}$	$p^{-2\sigma}$
$\rho^\gamma \sin^\sigma \theta$	$h^{\lambda-\epsilon}$	h^λ	$p^{-(\lambda-\epsilon)}$	$p^{-\lambda}$	$p^{-2\lambda}$

In Table 6.1 and Table 6.2 $\lambda = \min\{\gamma + 1/2, \sigma\}, \sigma \neq \gamma + 1/2$, $\beta = (-1/3, -1/3, -1/3), \beta = (-1/2, -1/2, \beta_3), \beta = (-1/2, -1/2, 0)$ for ρ^γ, r^σ and $\rho^\gamma \sin^\sigma \theta$, respectively.

The singular functions with singularities of three different types are typical and appear in the solution of problems with piecewise analytical date on polyhedral domains, which govern the convergence of the finite element solutions of the h-, p- and h-p version(associated with quasi-uniform meshes). The function spaces used for characterizing the singularities depends on the nature of singularities as well as the type of the finite element methods. Thus the selection of function spaces is crucial to the best approximation for the finite element solutions. The Table 6.1 and 6.2 tell us how the functional spaces used for characterization of singularities and error analysis affect the estimation of approximation error measured in H^1-norm. Hence we can conclude that the Jacobi-weighted Besov is the best theoretical tool for analyzing approximation of functions by the p- and h-p version (associated with quasi-uniform meshes) of the finite element method. Meanwhile, it can be shown that it has no substantial impact on the error estimation for the classical h-version of the finite element method.

Finally, the framework we set up in three dimensions can be used for the spectral and the boundary element methods, and the analysis and results parallel to those for the finite element can be established for the spectral and the boundary element methods without substantial difficulties.

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