Compressible Two-Phase Flows: Two-Pressure Models and Numerical Methods

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Abstract

A conservative hyperbolic model for compressible two-phase two-fluid model is studied and numerical methods for its approximate solution are proposed. The derivation of the governing equations of the model is based on the principles of extended thermodynamics. The field equations form a hyperbolic system of balance equations in conservative form, which guarantees the well-posedness of the initial-value problem (its solvability, at least locally in time). The system of governing equations consists of well-known conservation laws for the mixture mass, momentum, and energy, which are completed by the additional balance laws for the relative velocity of phases and for the volume concentration of one phase. The closure constitutive relation for the model is the equation of state for the mixture, which can be derived from known equations of state for each phase. The eigenstructure analysis of the one-dimensional case shows the existence of six real eigenvalues, four of which are connected with two speeds of sound in the pure phases, and two correspond to the mixture flow velocity. A corresponding, complete set of linearly independent eigenvectors is given explicitly and the nature of the associated characteristic fields is studied.

For the case of isentropic flow it is shown that in terms of the individual phase parameters the originally conservative system of governing equations can be transformed to the well-known non-conservative model of Baer-Nunziato-type. In this situation our model differs from the latter by the definition of interfacial pressure and by extra terms in the momentum equations related to lift forces.

Finite volume shock-capturing methods for solving numerically the governing equations are studied, test problems are proposed and numerical results are presented and discussed.

Key words: two-phase, two-pressure model, compressible flow, hyperbolic conservation laws, finite volume numerical methods

1 Introduction

The multiphase flow modelling is one of the most challenging fields of research in applied mathematics and computational fluid dynamics. But even in the case of two-phase compressible flow there is no final conventional form of the model and its governing equations. Certainly well mathematical properties of governing equations play a key role in the formulation of a model. At present it is practically conventional fact that the governing equations of compressible two-phase flow model must be hyperbolic. Ideally it would be very attractive if all governing equations of a model can be written in a conservative form, because it gives a straightforward way to define a discontinuous solution such a shock waves and contacts.

In this paper we discuss two-phase flow models, in which pressures of constituents of the mixture supposed to be different. Such type a model is called two-pressure two-phase

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model. The first two-pressure model was proposed in 1986 by Baer and Nunziato [1] and this paper was the basis for further modifications of the model and new approaches in the modelling of such a flows, see for example [2, 3, 4]. The governing equations in these models are based on the mass, momentum and energy conservation laws for each phase in which interfacial exchange terms are included. Note that these approaches lead to governing equations which are written in a non-conservative form, that is why a difficulties arise in a defining of a discontinuous solution [5, 6].

Here we consider a phenomenological conservative model derived by the principles of extended thermodynamics [7, 8, 9, 10] and proposed in [11, 12]. One-dimensional versions of this model are studied in [13, 14]. The governing equations are written in terms of the parameters of state for the mixture and taking into account a two-phase character of a flow. We study the mathematical properties of its one-dimensional version and give a full eigenstructure analysis. We also formulate a reduced version of equations in which thermal processes are ignored (isentropic model).

It interesting to note that although the conservative model is derived by the phenomenological principles and written in terms of the parameters of state for the mixture, there is a possibility to rewrite them in terms of parameters of state for individual phases and in the form which is similar to the Baer-Nunziato equations. In the paper we compare equations for conservative and Baer-Nunziato models for the isentropic case. One-dimensional conservative equations can be written in the Baer-Nunziato form, but there is a difference in the definition of interfacial pressure. For the multi-dimensional case, conservative equations written in the Baer-Nunziato form include additional differential source terms in the momentum equations which not appear in the original Baer-Nunziato equations and their modifications. Such terms in the momentum equations are called as lift forces [15].

A few numerical examples to illustrate the character of solutions of the model are given.

2 Conservative equations for two-phase compressible flow

We study the system of governing equations for two-phase two-fluid flow which has been proposed in [12]. These governing equations derived using the principles of extended thermodynamics [7, 8, 11]. The resulting system forms a system of partial differential equations written in a conservative form, which can be transformed to a symmetric hyperbolic system [12].

2.1 The multi-dimensional model

The three-dimensional system of evolution differential equations

$$\frac{\partial}{\partial t}(\rho\alpha) + \frac{\partial}{\partial x_k}(\rho u_k \alpha) = -\phi, \tag{1}$$

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_k}(\rho u_k) = 0, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho u_l) + \frac{\partial}{\partial x_k}(\rho u_l u_k + p\delta_{kl} + \rho w_l E_{w_k}) = 0, \tag{3}$$

$$\frac{\partial}{\partial t}(\rho c) + \frac{\partial}{\partial x_k}(\rho u_k c + \rho E_{w_k}) = 0, \tag{4}$$

$$\frac{\partial}{\partial t}w_k + \frac{\partial}{\partial x_k}(u_l w_l + E_c) = -(e_{klj}u_l\omega_j + \pi_k), \tag{5}$$

$$\frac{\partial}{\partial t}\Big(\rho(E + \frac{u_l u_l}{2})\Big) + \frac{\partial}{\partial x_k}\Big(\rho u_k(E + \frac{u_l u_l}{2}) + \Pi_k\Big) = 0. \tag{6}$$

should be completed by two additional compatibility relations which are discussed below. In the paper the conventional summation notation with respect to equal index is used.

The set of parameters characterizing the state of a mixture is: α - volume concentration of one of the phases (assume that it is the phase with prescribed number 1), c - the mass concentration of one of the phases (we also assume that it is the phase with number 1), ρ - the mass density of the mixture, u_l - the velocity of the mixture, w_l - the relative velocity of phases, p - the pressure of the mixture, E - the specific internal energy of the mixture (we also call it further Equation of state). Variables ϕ, ω_j, π_k are defined below, and δ_{kl} and e_{klj} are the unit tensor and unit pseudoscalar respectively. Π_k is the energy flux vector and is defined by the formula

$$\Pi_k = u_k p + \rho u_k w_l E_{w_l} + \rho E_c E_{w_l}. \tag{7}$$

The system consists of the balance equation for the volume concentration of one of the phases (1), the mass conservation equation for the mixture (2), total momentum conservation equation (3), mass concentration equation (4) for one of the phases, balance equation (5) for the relative velocity of phases, and energy conservation equation for the mixture (6).

Here we ignore many possible dissipative processes such as heat conductivity, viscous behavior of each phase, and others. We also ignore phase transition of the constituents of the mixture. The phase interaction includes the relaxation of phases pressures to a common value and an interfacial friction only. We also emphasize that the model is designed for processes in which the thermal non-equilibrium between the phases is small enough, that is why we take the mixture entropy only as the parameter of state characterizing the thermal behavior of a mixture. Nevertheless the range of processes for the modelling of which these equations can be applied is quite wide.

The internal energy E assumed to be a known function of the parameters of state for the mixture:

$$E = e(\alpha, \rho, c, S) + c(1-c)\frac{w_l w_l}{2}, \tag{8}$$

where e is the thermodynamic internal energy of the mixture, and S is the mixture entropy. The mixture pressure p is connected with the internal energy by the formula

$$p = \rho^2 E_\rho = \rho^2 \frac{\partial E}{\partial \rho} = \rho^2 \frac{\partial e(\alpha, \rho, c, S)}{\partial \rho}. \tag{9}$$

As it is noted above two processes of interphase exchange are taken in account in the model. First, the pressure relaxation is simulated by the source term ϕ in the equation (1):

$$\phi = \frac{\rho}{\tau} E_\alpha = \frac{\rho}{\tau} \frac{\partial E}{\partial \alpha}, \tag{10}$$

where τ is the pressure relaxation time which assumed to be a function of parameters of state. Second, the interfacial friction term π on the right hand side of equation (5) for the relative velocity:

$$\pi_k = \kappa E_{w_k} = \kappa \frac{\partial E}{\partial w_k} = \kappa c(1-c) w_k, \tag{11}$$

where κ is the coefficient of interfacial friction, which also can be a function of parameters of state.

The vector variable ω_j is not a parameter of state, but it is an auxiliary variable which is introduced to formulate the equation (5) in a conservative form. The introducing of auxiliary variables in systems of thermodynamically compatible conservative governing differential equations is caused by its specific structure and discussed in [11].

The vector ω_j is defined as the vorticity of the relative velocity vector

$$\omega_j = e_{jkl} \frac{\partial w_k}{\partial x_l}, \tag{12}$$

and it must satisfy an additional differential relation

$$\frac{\partial \omega_k}{\partial t} + \frac{\partial (u_l \omega_k - u_k \omega_l + e_{klj} \pi_j)}{\partial x_l} = 0. \tag{13}$$

This additional relation is a consequence of compatibility requirement for the system (1)-(6). To prove this it is necessary to apply the differential operator $e_{jkl}\partial/\partial x_l$ to equation (5):

$$\frac{\partial}{\partial t}\left(e_{jkl} \frac{\partial w_k}{\partial x_l} \right) + e_{jkl} \frac{\partial^2}{\partial x_k \partial x_l} (u_l w_l + E_c) = -e_{jkl} \frac{\partial}{\partial x_l} (e_{kmn} u_m \omega_n + \pi_k).$$

It is obvious that the second term in the left hand side of above formula is equal to 0 due to antisymmetric character of e_{jkl} with respect to subscripts k and l, namely, $e_{jkl} = -e_{jlk}$. Finally, using (12), this formula can be written in the form

$$\frac{\partial}{\partial t} \omega_j + \frac{\partial}{\partial x_l} (e_{jkl} e_{kmn} u_m \omega_n + e_{jkl} \pi_k) = 0,$$

which is equivalent to (13).

Emphasize again that the relative velocity vorticity ω_j is an auxiliary variable and its introducing is necessary to write the equation for relative velocity in a conservative form only. The system (1)-(6) can be rewritten in the form in which ω_j is neglected. To do this it is necessary to include to the system the following equation

$$\frac{\partial w_k}{\partial t} + u_l \frac{\partial w_k}{\partial x_l} + \frac{\partial E_c}{\partial x_k} + w_l \frac{\partial u_l}{\partial x_k} = -\pi_k \tag{14}$$

instead of equation (5). Equation (14) can be derived from the equation (5) by adding (12) multiplied by u_l. After solving the complete system (1)-(4),(14),(6) for variables $\alpha, \rho, u_l, c, w_k, S$ one can calculate ω_j using its definition (12), if it is necessary.

The system (1)-(6) is completely reasonable from the mathematical viewpoint. Firstly, all its equations are in a conservative form, that allows to define a discontinuous solution. Secondly, the simplified system, in which source term are neglected, can be written in a symmetric hyperbolic form (if the equation of state is a convex function). Finally, the system (1)-(6) is a thermodynamically compatible one, it means that the solution of the system admits an additional entropy balance law.

This entropy balance law has the following form:

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho u_k S}{\partial x_k} = Q = \frac{1}{E_S}\left(E_\alpha \phi + \rho E_{w_k} \pi_k\right). \tag{15}$$

The right hand side Q in the above equation is the entropy production and it is a non–negative quantity due to appropriate definitions (10) for ϕ and (11) for π_k:

$$Q = \frac{1}{E_S}\left(E_\alpha \phi + \rho E_{w_k} \pi_k\right) = \frac{\rho}{E_S}\left(\frac{1}{\tau} E_\alpha^2 + \kappa E_{w_k} E_{w_k}\right) \geq 0.$$

To derive this balance law for the entropy we can use the equivalent nonconservative form of the system (1)–(6):

$$\begin{aligned}
&\frac{d\rho}{dt} + \rho \frac{\partial u_k}{\partial x_k} = 0,\\
&\rho \frac{du_l}{dt} + \frac{\partial p}{\partial x_l} + \frac{\partial \rho w_l E_{w_k}}{\partial x_k} = 0,\\
&\rho \frac{dE}{dt} + p\frac{\partial u_k}{\partial x_k} + \rho w_i E_{w_k} \frac{\partial u_i}{\partial x_k} + \frac{\partial \rho E_c E_{w_k}}{\partial x_k} = 0,\\
&\rho \frac{d\alpha}{dt} = -\phi,\\
&\rho \frac{dc}{dt} + \frac{\partial \rho E_{w_k}}{\partial x_k} = 0,\\
&\frac{dw_k}{dt} + w_l \frac{\partial u_l}{\partial x_k} + \frac{\partial E_c}{\partial x_k} = -\pi_k.
\end{aligned}$$

Here $d/dt = \partial/\partial t + u_k \partial/\partial x_k$ is the material derivative.

Using the formula

$$dE = \frac{\partial E}{\partial \alpha} d\alpha + \frac{\partial E}{\partial \rho} d\rho + \frac{\partial E}{\partial c} dc + \frac{\partial E}{\partial w_k} dw_k + \frac{\partial E}{\partial S} dS$$

we find

$$\frac{dS}{dt} = \frac{1}{E_S}\left(\frac{dE}{dt} - E_\alpha \frac{d\alpha}{dt} + E_\rho \frac{d\rho}{dt} + E_c \frac{dc}{dt} + E_{w_k} \frac{dw_k}{dt}\right)$$

Now using equations of latter nonconservative system one can derive the required entropy balance law (15).

Sometimes it is convenient to use the complete system of governing differential equations (1)-(6) in which the energy conservation law (6) replaced by the entropy balance equations (15). The energy conservation law is a must if we study discontinuous solutions.

2.2 Reformulation of the model

Note that in the previous consideration we used parameters of state for the mixture, such as mixture mass density, volume and mass fractions of the phases, mixture velocity and relative velocity. Now we present a different formulation of the system of governing equations with the use of individual mass density and velocity for each phase. The two phases of the mixture are characterized by the volume concentrations α_1, α_2 and mass concentrations c_1, c_2 with constraints

$$\alpha_1 + \alpha_2 = 1, \quad c_1 + c_2 = 1.$$

The definition of individual phase parameters of state by the mixture parameters are as follows:

$$\rho_1 = \frac{c_1\rho}{\alpha_1} = \frac{c\rho}{\alpha}, \quad \rho_2 = \frac{c_2\rho}{\alpha_2} = \frac{(1-c)\rho}{(1-\alpha)}, \tag{16}$$

$$(u_1)_k = u_k + c_2 w_k = u_k + (1-c)w_k, \quad (u_2)_k = u_k - c_1 w_k = u_k - c w_k, \tag{17}$$

where ρ_i is the mass density of i-th phase, $(u_i)_k$ is the k-th component of velocity vector of i-th phase, c_i is the mass concentration of i-th phase, and α_i is the volume concentration of i-th phase.

The definition of mixture parameters by the individual parameters follows from (16) and (17)

$$\rho = \alpha_1\rho_1 + \alpha_2\rho_2, \quad u_k = c_1(u_1)_k + c_2(u_2)_k = \alpha_1\rho_1(u_1)_k + \alpha_2\rho_2(u_2)_k. \tag{18}$$

Now we formulate the main assumption which allows to rewrite equations in terms of parameters characterizing each phase. It concerns the thermodynamic internal energy of the mixture e. We suppose that it can be derived from the two known equations of state of each phase as follows:

$$e(\alpha, \rho, c, S) = c_1 e_1(\rho_1, S) + c_2 e_2(\rho_2, S) = c e_1\left(\frac{c\rho}{\alpha}, S\right) + (1-c) e_2\left(\frac{(1-c)\rho}{(1-\alpha)}, S\right). \tag{19}$$

Here e_i is the internal specific energy of i-th phase. Note that we assume that the entropy S is the common entropy for the mixture. It is clear that in limiting cases $c_1 = 1, c_2 = 0$ and $c_1 = 0, c_2 = 1$ we have $e = e_1(\rho_1, S)$ and $e = e_2(\rho_2, S)$ respectively.

Now using formula (19) for internal energy, definition (16), and identities for differentials derived from (16)

$$d\rho_1 = -\frac{c\rho}{\alpha^2}d\alpha + \frac{\rho}{\alpha}dc + \frac{c}{\alpha}d\rho, \quad d\rho_2 = \frac{(1-c)\rho}{(1-\alpha)^2}d\alpha - \frac{\rho}{1-\alpha}dc + \frac{1-c}{1-\alpha}d\rho,$$

we obtain the following formulae for derivatives of the equation of state

$$\frac{\partial e}{\partial \alpha} = -\frac{\rho_1^2}{\rho}\frac{\partial e_1}{\partial \rho_1} + \frac{\rho_2^2}{\rho}\frac{\partial e_2}{\partial \rho_2} = \frac{p_2 - p_1}{\rho}, \tag{20}$$

$$\frac{\partial e}{\partial c} = e_1 + \frac{\rho c}{\alpha}\frac{\partial e_1}{\partial \rho_1} - e_2 - \frac{\rho(1-c)}{(1-\alpha)}\frac{\partial e_2}{\partial \rho_2} = e_1 + \frac{p_1}{\rho_1} - e_2 - \frac{p_2}{\rho_2}, \tag{21}$$

$$\frac{\partial e}{\partial \rho} = \frac{\alpha_1}{\rho^2}\rho_1^2\frac{\partial e_1}{\partial \rho_1} + \frac{\alpha_2}{\rho^2}\rho_2^2\frac{\partial e_2}{\partial \rho_2} = \frac{\alpha p_1 + (1-\alpha)p_2}{\rho^2}, \tag{22}$$

$$\frac{\partial e}{\partial S} = c\frac{\partial e_1}{\partial S} + (1-c)\frac{\partial e_2}{\partial S}. \tag{23}$$

Note that from (22) the formula for the mixture pressure p as an average of phases pressures p_1 and p_2 follows as

$$p = \alpha p_1 + (1-\alpha)p_2. \tag{24}$$

Now using formulae (18), (20), (21), (24) the system (1)-(6) supplemented by the steady compatibility relation (12) can be written in an equivalent form

$$\begin{aligned}
&\frac{\partial}{\partial t}(\alpha_1\rho_1 + \alpha_2\rho_2) + \frac{\partial}{\partial x_k}(\alpha_1\rho_1(u_1)_k + \alpha_2\rho_2(u_2)_k) = 0,\\
&\frac{\partial}{\partial t}(\alpha_1\rho_1(u_1)_l + \alpha_2\rho_2(u_2)_l) + \frac{\partial}{\partial x_k}(\alpha_1\rho_1(u_1)_l(u_1)_k + \alpha_2\rho_2(u_2)_l(u_2)_k + p\delta_{lk}) = 0,\\
&\frac{\partial}{\partial t}((\alpha_1\rho_1 + \alpha_2\rho_2)\alpha) + \frac{\partial}{\partial x_k}((\alpha_1\rho_1(u_1)_k + \alpha_2\rho_2(u_2)_k)\alpha) = -\phi,\\
&\frac{\partial}{\partial t}(\alpha_1\rho_1) + \frac{\partial}{\partial x_k}(\alpha_1\rho_1(u_1)_k) = 0,\\
&\frac{\partial}{\partial t}((u_1)_k - (u_2)_k) + \frac{\partial}{\partial x_k}\left(\frac{(u_1)_l(u_1)_l}{2} - \frac{(u_2)_l(u_2)_l}{2} + e_1 + \frac{p_1}{\rho_1} - e_2 - \frac{p_2}{\rho_2}\right) = -\Gamma_k,\\
&\frac{\partial}{\partial t}\left(\alpha_1\rho_1 e_1 + \alpha_2\rho_2 e_2 + \frac{(u_1)_l(u_1)_l}{2} + \frac{(u_2)_l(u_2)_l}{2}\right) +\\
&\frac{\partial}{\partial x_k}\left(\alpha_1\rho_1(u_1)_k\left(e_1 + \frac{p_1}{\rho_1} + \frac{(u_1)_l(u_1)_l}{2}\right) + \alpha_2\rho_2(u_2)_k\left(e_2 + \frac{p_2}{\rho_2} + \frac{(u_2)_l(u_2)_l}{2}\right)\right) = 0,\\
&\frac{\partial((u_1)_k - (u_2)_k)}{\partial x_j} - \frac{\partial((u_1)_j - (u_2)_j)}{\partial x_k} = -e_{kjl}\omega_l,
\end{aligned} \tag{25}$$

where the source terms are

$$\phi = \frac{1}{\tau}(p_2 - p_1), \quad \Gamma_k = e_{klj}u_l\omega_j + \pi_k, \quad \pi_k = \kappa c(1-c)w_k.$$

The entropy balance equation is transformed to

$$\frac{\partial}{\partial t}((\alpha_1\rho_1 + \alpha_2\rho_2)S) + \frac{\partial}{\partial x_k}((\alpha_1\rho_1(u_1)_k + \alpha_2\rho_2(u_2)_k)S) = Q,$$

where the entropy production is

$$Q = \frac{1}{e_S}\left(\frac{(p_1-p_2)^2}{\rho\tau} + \rho\kappa c^2(1-c)^2((u_1)_k - (u_2)_k)((u_1)_k - (u_2)_k)\right) \geq 0.$$

Further we will see that the choice of individual parameters of a state as primitive variables is more convenient for the eigenstructure analysis and gives a possibility to derive an explicit formulae for eigenvalues and eigenvectors.

3 Analysis of the model

In this section we formulate one-dimensional version of the system for two-phase flow described in the previous section. We shall study two different kind of the system using two different sets of primitive variables, one of them is mixture parameters of state and another one is based on the individual parameters of phases.

3.1 One-dimensional equations

Assume that the mixture flows along $x = x_1$ axis and hence the mixture velocity u_k and the relative velocity w_k have only one component each, $u = u_1$ and $w = w_1$ respectively. Then the system (1)-(6) can be written in the form

$$\partial_t U + \partial_x F(U) = S(U), \tag{26}$$

where U, $F(U)$, and $S(U)$ are vectors of conserved variables, fluxes and source terms respectively, which are defined by

$$U = \begin{pmatrix} \rho \\ \rho\alpha \\ \rho u \\ \rho c \\ w \\ \rho(E + \frac{u^2}{2}) \end{pmatrix}, \tag{27}$$

$$F(U) = \begin{pmatrix} \rho u \\ \rho u\alpha \\ \rho u^2 + p + \rho w E_w \\ \rho uc + \rho E_w \\ uw + E_c \\ \rho u(E + \frac{u^2}{2}) + pu + \rho uw E_w + \rho E_c E_w \end{pmatrix}, \tag{28}$$

$$S(U) = \begin{pmatrix} 0 \\ -\frac{\rho}{\tau} E_\alpha \\ 0 \\ 0 \\ -\kappa c(1-c)w \\ 0 \end{pmatrix}. \tag{29}$$

The system is closed by the equation of state for the mixture

$$E(\alpha, \rho, c, S, w) = e(\alpha, \rho, c, S) + c(1-c)\frac{w^2}{2}, \tag{30}$$

where e is defined by (19). The pressure and derivatives of E with respect to w, c, α are

$$p = \rho^2 E_\rho = \rho^2 e_\rho, \quad E_w = c(1-c)w, \quad E_c = e_c + (1-2c)\frac{w^2}{2}, \quad E_\alpha = e_\alpha.$$

3.2 Primitive-variable formulation

The system (26)-(29) can be rewritten in a quasilinear form using the vector of primitive variables

$$W = (\rho, \alpha, u, c, w, S)^T \tag{31}$$

as follows

$$\partial_t W + A(W)\partial_x W = Q(W), \tag{32}$$

where the matrix $A(W)$ is

$$\begin{pmatrix} u & 0 & \rho & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 0 & 0 \\ \frac{1}{\rho}(p_\rho + c(1-c)w^2) & \frac{p_\alpha}{\rho} & u & \frac{1}{\rho}(p_c + \rho(1-2c)w^2) & 2c(1-c)w & \frac{p_S}{\rho} \\ c(1-c)\frac{w}{\rho} & 0 & 0 & u + (1-2c)w & c(1-c) & 0 \\ e_{c\rho} & e_{\alpha c} & w & e_{cc} - w^2 & u + (1-2c)w & e_{cS} \\ 0 & 0 & 0 & 0 & 0 & u \end{pmatrix},$$

and $Q(W) = U_W^{-1} S(U(W))$.

The approximate analysis of eigenvalues of the matrix A has been done in [12]. It is shown that there are six real roots of the equation for eigenvalues

$$\det[A(W) - \lambda I] = 0, \tag{33}$$

which can be calculated explicitly in the case of zero relative velocity $w = 0$. Unfortunately in general if $w \neq 0$ only approximate formula, assuming w a small value, for eigenvalues and eigenvectors has been derived in [12].

It turns out that if to rewrite the system using another set of primitive variables then the eigenstructure analysis can be done explicitly. Let us take the following vector of primitive variables

$$V = (v_1, v_2, v_3, v_4, v_5, v_6)^T = (\alpha, S, \rho_1, u_1, \rho_2, u_2)^T \tag{34}$$

which is connected with $W = (\rho, \alpha, u, c, w, S)^T$ by formulae (16) and (17) in which the one-dimensional character of flow is taken in account:

$$u_1 = u + c_2 w = u + (1-c)w, \quad u_2 = u - c_1 w = u - cw.$$

In order to study the eigenstructure of the one-dimensional equations one can pass to the new set of variables in the equation (32) directly, using the Jacobian matrix of variables transformation. But it is quite complicated procedure and it is better to use an equivalent one-dimensional version of governing equations system (25) which is as follows

$$\begin{aligned}
&\frac{\partial}{\partial t}(\alpha_1\rho_1 + \alpha_2\rho_2) + \frac{\partial}{\partial x}(\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2) = 0, \\
&\frac{\partial}{\partial t}(\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2) + \frac{\partial}{\partial x}(\alpha_1\rho_1 u_1^2 + \alpha_2\rho_2 u_2^2 + \alpha_1 p_1 + \alpha_2 p_2) = 0, \\
&\frac{\partial}{\partial t}((\alpha_1\rho_1 + \alpha_2\rho_2)\alpha) + \frac{\partial}{\partial x}((\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2)\alpha) = -\phi, \\
&\frac{\partial}{\partial t}(\alpha_1\rho_1) + \frac{\partial}{\partial x}(\alpha_1\rho_1 u_1) = 0, \\
&\frac{\partial}{\partial t}(u_1 - u_2) + \frac{\partial}{\partial x}\left(\frac{u_1^2}{2} - \frac{u_2^2}{2} + e_1 + \frac{p_1}{\rho_1} - e_2 - \frac{p_2}{\rho_2}\right) = -\pi, \\
&\frac{\partial}{\partial t}\left(\alpha_1\rho_1 e_1 + \alpha_2\rho_2 e_2 + \frac{u_1^2}{2} + \frac{u_2^2}{2}\right) + \\
&\frac{\partial}{\partial x}\left(\alpha_1\rho_1 u_1\left(e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2}\right) + \alpha_2\rho_2 u_2\left(e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2}\right)\right) = 0,
\end{aligned} \tag{35}$$

where the source terms are

$$\phi = \frac{1}{\tau}(p_2 - p_1), \quad \pi = \kappa c(1-c)(u_1 - u_2).$$

The entropy balance law in the one-dimensional case becomes

$$\frac{\partial}{\partial t}((\alpha_1\rho_1 + \alpha_2\rho_2)S) + \frac{\partial}{\partial x}((\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2)S) = Q,$$

where the entropy production is

$$Q = \frac{1}{e_S}\left(\frac{(p_1 - p_2)^2}{\rho\tau} + \rho\kappa c^2(1-c)^2(u_1 - u_2)^2\right) \geq 0.$$

Further transformation of equations in order to simplify them and write them in a quasilinear form leads us to the following system:

$$\begin{aligned}
&\frac{\partial\alpha}{\partial t} + u\frac{\partial\alpha}{\partial x} = -\frac{\phi}{\rho},\\
&\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = \frac{Q}{\rho},\\
&\frac{\partial\rho_1}{\partial t} + u_1\frac{\partial\rho_1}{\partial x} + \rho_1\frac{\partial u_1}{\partial x} + \frac{\alpha_2}{\alpha_1}\frac{\rho_1\rho_2}{\rho}(u_1 - u_2)\frac{\partial\alpha}{\partial x} = \frac{\rho_1\phi}{\alpha_1\rho},\\
&\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + \frac{1}{\rho_1}\frac{\partial p_1}{\partial x} + \frac{p_1 - p_2}{\rho}\frac{\partial\alpha}{\partial x} + \frac{\alpha_2\rho_2}{\rho}(T_1 - T_2)\frac{\partial S}{\partial x} = -\frac{\alpha_2\rho_2}{\rho}\pi,\\
&\frac{\partial\rho_2}{\partial t} + u_2\frac{\partial\rho_2}{\partial x} + \rho_2\frac{\partial u_2}{\partial x} + \frac{\alpha_1}{\alpha_2}\frac{\rho_1\rho_2}{\rho}(u_1 - u_2)\frac{\partial\alpha}{\partial x} = -\frac{\rho_2\phi}{\alpha_2\rho},\\
&\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + \frac{1}{\rho_2}\frac{\partial p_2}{\partial x} + \frac{p_1 - p_2}{\rho}\frac{\partial\alpha}{\partial x} - \frac{\alpha_1\rho_1}{\rho}(T_1 - T_2)\frac{\partial S}{\partial x} = \frac{\alpha_1\rho_1}{\rho}\pi.
\end{aligned}$$

Here $\rho = \alpha_1\rho_1 + \alpha_2\rho_2$ and $T_i = \partial e_i/\partial S$. Certainly we assume $\alpha_i \neq 0$ and $\alpha_i \neq 1$

Now it is not difficult to rewrite the latter system in quasilinear form in terms of the set of primitive variables (34)

$$\partial_t V + B(V)\partial_x V = Z(V), \tag{36}$$

where the matrix $B(V)$ is

$$\begin{pmatrix}
u & 0 & 0 & 0 & 0 & 0\\
0 & u & 0 & 0 & 0 & 0\\
a_1 & 0 & u_1 & \rho_1 & 0 & 0\\
b & d_1 & \frac{C_1^2}{\rho_1} & u_1 & 0 & 0\\
a_2 & 0 & 0 & 0 & u_2 & \rho_2\\
b & d_2 & 0 & 0 & \frac{C_2^2}{\rho_2} & u_2
\end{pmatrix}.$$

Here

$$\begin{aligned}
a_1 = \frac{\alpha_2}{\alpha_1}\frac{\rho_1\rho_2}{\rho}(u_1 - u_2), \quad a_2 = \frac{\alpha_1}{\alpha_2}\frac{\rho_1\rho_2}{\rho}(u_1 - u_2), \quad b = \frac{p_1 - p_2}{\rho},\\
d_1 = \frac{1}{\rho_1}\frac{\partial p_1}{\partial S} + \frac{\alpha_2\rho_2}{\rho}(T_1 - T_2), \quad d_2 = \frac{1}{\rho_2}\frac{\partial p_2}{\partial S} - \frac{\alpha_1\rho_1}{\rho}(T_1 - T_2),\\
C_1^2 = \frac{\partial p_1}{\partial\rho_1}, \quad C_2^2 = \frac{\partial p_2}{\partial\rho_2}.
\end{aligned}$$

3.3 Eigenstructure and characteristic fields

Due to the structure of the matrix B the equation for eigenvalues

$$\det[B - \lambda I] = 0$$

takes the form

$$(\lambda - u)^2[(\lambda - u_1)^2 - C_1^2][(\lambda - u_2)^2 - C_2^2] = 0. \tag{37}$$

It is clear now that the system (36) has six real eigenvalues. Assuming $C_1 \geq C_2$ we can order them in the following manner

$$\lambda_1 = u_1 - C_1, \lambda_2 = u_2 - C_2, \lambda_3 = \lambda_4 = u, \lambda_5 = u_2 + C_2, \lambda_6 = u_1 + C_1. \tag{38}$$

We see that eigenvalues $\lambda_{3,4}$ are the mixture velocity. The functions $C_1 = \sqrt{\partial p_1/\partial \rho_1}$ and $C_2 = \sqrt{\partial p_2/\partial \rho_2}$ coincide with the speeds of sounds in the phases with numbers 1 and 2 respectively. Hence eigenvalues $\lambda_{1,6}$ correspond to sound waves propagating in the first phase moving with the velocity u_1, and $\lambda_{2,5}$ correspond to the similar sound wave in the second phase.

As is noted (in the beginning of the second section) the presented equations of two-phase flow can be transformed to a symmetric hyperbolic system, that is why the set of linearly independent eigenvectors for matrix B exists. Now we derive an explicit formulae for eigenvectors.

The right eigenvectors corresponding to the set of eigenvalues $\lambda_i, (i = 1, ..., 6)$ can be chosen by the following way:

$$R^{(1)} = (0, 0, \rho_1, -C_1, 0, 0)^T,$$

$$R^{(2)} = (0, 0, 0, 0, \rho_2, -C_2)^T,$$

$$R^{(3)} = (z_1 z_2, 0, \mu_1 z_2, \nu_1 z_2, \mu_2 z_1, \nu_2 z_1)^T,$$

$$R^{(4)} = (0, z_1 z_2, \rho_1 d_1 z_2, -(u_1 - u) d_1 z_2, \rho_2 d_2 z_1, -(u_2 - u) d_2 z_1)^T,$$

$$R^{(5)} = (0, 0, 0, 0, \rho_2, C_2)^T,$$

$$R^{(6)} = (0, 0, \rho_1, C_1, 0, 0)^T,$$

where

$$z_1 = (u_1 - u)^2 - C_1^2, \quad z_2 = (u_2 - u)^2 - C_2^2,$$

$$\mu_1 = -(u_1 - u)a_1 + \rho_1 b, \quad \nu_1 = \frac{C_1^2}{\rho_1} a_1 - (u_1 - u)b,$$

$$\mu_2 = -(u_2 - u)a_2 + \rho_2 b, \quad \nu_2 = \frac{C_2^2}{\rho_2} a_2 - (u_2 - u)b.$$

So, we have the set of linearly independent eigenvectors, although the multiple eigenvalues $\lambda_3 = \lambda_4$ exist, and in principle there is a possibility for some eigenvalues corresponding to sound waves to be equal for some values of flow parameters.

Now we study characteristic fields properties. Denoting by

$$\nabla_V \lambda = \left(\frac{\partial \lambda}{\partial v_1}, ..., \frac{\partial \lambda}{\partial v_6} \right)$$

one can prove that for the eigenvalues $\lambda_1, \lambda_2, \lambda_5, \lambda_6$ the property

$$\nabla_V \lambda_i \cdot R^{(i)} \neq 0$$

holds. Actually the value $\nabla_V \lambda_i \cdot R^{(i)}$ computed for the mentioned eigenvalue-eigenvector pairs is equal to one of four values

$$\pm \frac{\partial(\rho_i C_i)}{\partial \rho_i} \quad (i = 1, 2),$$

which are not equal to 0 for admissible equations of state. Let us consider, for example, the first eigenvalue-eigenvector pair $(\lambda_1, R^{(1)})$. It is clear that

$$\frac{\partial \lambda_1}{\partial \alpha} = 0, \ \frac{\partial \lambda_1}{\partial S} = 0, \ \frac{\partial \lambda_1}{\partial \rho_1} = -\frac{\partial C_1}{\partial \rho}, \ \frac{\partial \lambda_1}{\partial u_1} = 1, \ \frac{\partial \lambda_1}{\partial \rho_2} = 0, \ \frac{\partial \lambda_1}{\partial u_2} = 0.$$

Now using the expression for $R^{(1)}$ we have

$$\nabla_V \lambda_1 \cdot R^{(1)} = -\rho_1 \frac{\partial C_1}{\partial \rho_1} - C_1 = -\frac{\partial(\rho_1 C_1)}{\partial \rho_1} \neq 0.$$

The property of genuine nonlinearity is valid for the eigenvalue $\lambda_3 = u$ which corresponds to the equation for the void fraction α, but there exist locally exceptional cases, namely if $u_1 - u_2 = 0$, then

$$\nabla_V \lambda_3 \cdot R^{(3)} = 0.$$

Actually by definition $\lambda_3 = u = (\alpha_1 \rho_1 u_1 + \alpha_2 \rho_2 u_2)/(\alpha_1 \rho_1 + \alpha_2 \rho_2)$. Hence

$$\frac{\partial \lambda_3}{\partial \alpha} = \frac{\rho_1 \rho_2}{\rho^2}(u_1 - u_2), \frac{\partial \lambda_3}{\partial S} = 0, \frac{\partial \lambda_3}{\partial \rho_1} = \frac{\alpha_1 \alpha_2 \rho_2}{\rho^2}(u_1 - u_2), \frac{\partial \lambda_3}{\partial u_1} = \frac{\alpha_1 \rho_1}{\rho},$$

$$\frac{\partial \lambda_3}{\partial \rho_2} = -\frac{\alpha_1 \alpha_2 \rho_1}{\rho^2}(u_1 - u_2), \frac{\partial \lambda_3}{\partial u_2} = \frac{\alpha_2 \rho_2}{\rho}.$$

Now using the expression for $R^{(3)}$ we have

$$\nabla_V \lambda_3 \cdot R^{(3)} = \frac{(u_1 - u_2)}{\rho^2}(\rho_1 \rho_2 z_1 z_2 + \alpha_1 \alpha_2 \rho_2 \mu_1 z_2 + \alpha_1 \rho_1 \rho \nu_1 z_2 - \alpha_1 \alpha_2 \rho_1 \mu_2 z_1 + \alpha_2 \rho_2 \rho \nu_2 z_1),$$

and one can prove that latter expression is not equal to 0 if $u_1 - u_2 \neq 0$.

As for the characteristic field connected with the entropy equation eigenvalue $\lambda_4 = u$, one can prove that

$$\nabla_V \lambda_4 \cdot R^{(4)} \equiv 0.$$

To do this we have to use only the fact that $\nabla_V \lambda_3 = \nabla_V \lambda_4$ and the expression for $R^{(4)}$.

So we have formulated and studied the one-dimensional conservative system of governing equations for two-phase two-pressure compressible flow which allows to develop a mathematically correct theory for simple waves such as shock, centred and contact waves. In the next section we compare a reduced isentropic conservative model with one of the most popular two-pressure nonconservative model.

4 Comparison of models

In this paper we deal with conservative model of two-phase compressible flow with two velocities and two pressures. In recent years the most established approach for modelling two-phase two-pressure compressible flows has been proposed in [1]. The approach consists of representation two-phase flow as two separate continua coupled by momentum and energy exchange. The resulting system of governing equations is hyperbolic but it is not in a conservative form. Several modifications of this approach [2],[4],[3],[**?**] has been done. Models based on this approach we call Baer-Nunziato (BN) -type models. In this section we compare the governing equations discussed in previous sections with the governing equations of the BN-type model. As was noted in the Section 2 our conservative model is designed for processes in which the thermal state of phases is almost in equilibrium state, while BN-type models can take into account significant thermal non-equilibrium. That is why it is reasonable to compare the reduced conservative and BN models in which thermal processes are ignored and the phases behavior is isentropic.

In this section we will show that the one-dimensional versions of the models are similar, that is to say, they can be written by the selfsame way formally. But there is a difference in the definition of interfacial pressure and certainly this difference can give different results in the modelling of some phenomena. The models differ much more in the multidimensional case: the conservative model contain lift forces [15] in the individual phase momentum equations which arise intrinsically due to the structure of conservative governing equations. The BN-type models do not contain terms corresponding to lift forces.

The general BN-type model consists of partial differential balance equations for mass, momentum, and energy for each of two phases completed by the evolution equation for the volume fraction. Further we consider a simplified BN model in which thermal processes are ignored and hence the energy balance equations for each phase can be neglected. The reduced isentropic model is governed by the five equations system

$$
\begin{aligned}
&\frac{\partial \alpha_1}{\partial t} + (u_I)_k \frac{\partial \alpha_1}{\partial x_k} = \mu(p_1 - p_2),\\
&\frac{\partial \alpha_1\rho_1}{\partial t} + \frac{\partial \alpha_1\rho_1(u_1)_k}{\partial x_k} = 0,\\
&\frac{\partial \alpha_1\rho_1(u_1)_i}{\partial t} + \frac{\partial \alpha_1\rho_1(u_1)_i(u_1)_k}{\partial x_k} + \frac{\partial \alpha_1 p_1}{\partial x_i} = p_I \frac{\partial \alpha_1}{\partial x_i} + \lambda((u_2)_i - (u_1)_i),\\
&\frac{\partial \alpha_2\rho_2}{\partial t} + \frac{\partial \alpha_2\rho_2(u_2)_k}{\partial x_k} = 0,\\
&\frac{\partial \alpha_2\rho_2(u_2)_i}{\partial t} + \frac{\partial \alpha_2\rho_2(u_2)_i(u_2)_k}{\partial x_k} + \frac{\partial \alpha_2 p_2}{\partial x_i} = p_I \frac{\partial \alpha_2}{\partial x_i} - \lambda((u_2)_i - (u_1)_i).
\end{aligned}
\tag{39}
$$

Here, as in the previous sections, α_i are the volume concentrations of phases ($\alpha_1+\alpha_2 = 1$), ρ_i are the mass densities, $(u_i)_k$ are the k-th component of the velocity of i-th phase, $|u_i|^2 = (u_i)_1^2 + (u_i)_2^2 + (u_i)_3^2$, p_i is the pressure of i-th phase. The quantities $(u_I)_k$ and p_I are called interfacial velocity and interfacial pressure respectively. They are determined by the same way as average velocity and average pressure for the conservative model in previous sections:

$$
(u_I)_k = (u)_k = \frac{\alpha_1\rho_1(u_1)_k + \alpha_2\rho_2(u_2)_k}{\alpha_1\rho_1 + \alpha_2\rho_2}, \quad p_I = p = \alpha_1 p_1 + \alpha_2 p_2.
$$

Here we assume that for each phase its pressure is a known function of its density

$$p_i = p_i(\rho_i), \quad (i = 1, 2).$$

It is interesting to compare also the one-dimensional versions of the conservative and BN-type models. Assuming that the mixture flows in the $x = x_1$ direction and velocities u_1 and u_2 only exist, the one-dimensional version of BN equations is as follows

$$\begin{aligned}
&\frac{\partial \alpha_1}{\partial t} + u_I \frac{\partial \alpha_1}{\partial x} = \mu(p_1 - p_2),\\
&\frac{\partial \alpha_1\rho_1}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1}{\partial x} = 0,\\
&\frac{\partial \alpha_1\rho_1 u_1}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1^2}{\partial x} + \frac{\partial \alpha_1 p_1}{\partial x} = p_I \frac{\partial \alpha_1}{\partial x} + \lambda(u_2 - u_1),\\
&\frac{\partial \alpha_2\rho_2}{\partial t} + \frac{\partial \alpha_2\rho_2 u_2}{\partial x} = 0,\\
&\frac{\partial \alpha_2\rho_2 u_2}{\partial t} + \frac{\partial \alpha_2\rho_2 u_2^2}{\partial x_k} + \frac{\partial \alpha_2 p_2}{\partial x} = p_I \frac{\partial \alpha_2}{\partial x} - \lambda(u_2 - u_1).
\end{aligned} \tag{40}$$

First of all we compare one-dimensional versions of conservative model from Section 3 and the BN-type equations (40). Here we also consider a reduced conservative model in which thermal processes are neglected. Such a modification of the system (35) is as follows

$$\begin{aligned}
&\frac{\partial}{\partial t}(\alpha_1\rho_1 + \alpha_2\rho_2) + \frac{\partial}{\partial x}(\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2) = 0,\\
&\frac{\partial}{\partial t}(\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2) + \frac{\partial}{\partial x}(\alpha_1\rho_1 u_1^2 + \alpha_2\rho_2 u_2^2 + \alpha_1 p_1 + \alpha_2 p_2) = 0,\\
&\frac{\partial}{\partial t}((\alpha_1\rho_1 + \alpha_2\rho_2)\alpha) + \frac{\partial}{\partial x}((\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2)\alpha) = -\phi,\\
&\frac{\partial}{\partial t}(\alpha_1\rho_1) + \frac{\partial}{\partial x}(\alpha_1\rho_1 u_1) = 0,\\
&\frac{\partial}{\partial t}(u_1 - u_2) + \frac{\partial}{\partial x}\left(\frac{u_1^2}{2} - \frac{u_2^2}{2} + e_1 + \frac{p_1}{\rho_1} - e_2 - \frac{p_2}{\rho_2}\right) = -\pi.
\end{aligned} \tag{41}$$

It is obvious that using the first and third equations of the system (41) one can derive the evolution equation for the volume fraction $\alpha = \alpha_1$ in the form

$$\frac{\partial \alpha}{\partial t} + u\frac{\partial \alpha}{\partial x} = -\phi, \tag{42}$$

It is also clear that from the first and fourth equations of the system (41) the mass conservation equation for the second phase follows

$$\frac{\partial}{\partial t}(\alpha_2\rho_2) + \frac{\partial}{\partial x}(\alpha_2\rho_2 u_2) = 0. \tag{43}$$

And finally, after some cumbersome transformation, using the second (total momentum) and fifth (relative velocity) equations of the system (41), one can derive the momentum balance equations for each phase:

$$\frac{\partial \alpha_1\rho_1 u_1}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1^2}{\partial x} + \frac{\partial \alpha_1 p_1}{\partial x} = \frac{\alpha_2\rho_2 p_1 + \alpha_1\rho_1 p_2}{\rho}\frac{\partial \alpha_1}{\partial x} - \frac{\alpha_1\alpha_2\rho_1\rho_2}{\rho}\pi, \tag{44}$$

$$\frac{\partial \alpha_2\rho_2 u_2}{\partial t} + \frac{\partial \alpha_2\rho_2 u_2^2}{\partial x_k} + \frac{\partial \alpha_2 p_2}{\partial x} = \frac{\alpha_2\rho_2 p_1 + \alpha_1\rho_1 p_2}{\rho}\frac{\partial \alpha_2}{\partial x} + \frac{\alpha_1\alpha_2\rho_1\rho_2}{\rho}\pi. \tag{45}$$

So, combining equations (42)-(45) and the fourth equation of the system (41) we obtain the complete one-dimensional system for isentropic two-phase flow

$$\begin{aligned}
&\frac{\partial \alpha}{\partial t} + u_I \frac{\partial \alpha}{\partial x} = -\phi, \\
&\frac{\partial}{\partial t}(\alpha_1 \rho_1) + \frac{\partial}{\partial x}(\alpha_1 \rho_1 u_1) = 0, \\
&\frac{\partial}{\partial t}(\alpha_2 \rho_2) + \frac{\partial}{\partial x}(\alpha_2 \rho_2 u_2) = 0, \\
&\frac{\partial \alpha_1 \rho_1 u_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 u_1^2}{\partial x} + \frac{\partial \alpha_1 p_1}{\partial x} = \hat{p}_I \frac{\partial \alpha_1}{\partial x} - \frac{\alpha_1 \alpha_2 \rho_1 \rho_2}{\rho} \pi, \\
&\frac{\partial \alpha_2 \rho_2 u_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 u_2^2}{\partial x_k} + \frac{\partial \alpha_2 p_2}{\partial x} = \hat{p}_I \frac{\partial \alpha_2}{\partial x} + \frac{\alpha_1 \alpha_2 \rho_1 \rho_2}{\rho} \pi.
\end{aligned} \tag{46}$$

Recalling the definition of the source terms (see Section 3)

$$\phi = \frac{1}{\tau}(p_2 - p_1), \quad \pi = \kappa \frac{\alpha_1 \alpha_2 \rho_1 \rho_2}{\rho^2}(u_1 - u_2),$$

we conclude that the latter system is similar to the one-dimensional version of BN model (40) if to denote

$$\mu = -\frac{1}{\tau}, \quad \lambda = -\kappa \frac{(\alpha_1 \alpha_2 \rho_1 \rho_2)^2}{\rho^3}.$$

The definition of interfacial velocity is the same for both models

$$u_I = u = \frac{\alpha_1 \rho_1 u_1 + \alpha_2 \rho_2 u_2}{\alpha_1 \rho_1 + \alpha_2 \rho_2},$$

but the definition of interfacial pressure is different. In the BN-type model we have $p_I = \alpha_1 p_1 + \alpha_2 p_2$ while in the conservative model

$$\hat{p}_I = \frac{\alpha_2 \rho_2 p_1 + \alpha_1 \rho_1 p_2}{\alpha_1 \rho_1 + \alpha_2 \rho_2}.$$

Such a difference in definition of internal pressure in the BN model and conservative model can lead to a different behavior of solutions for concrete problems. Although in the case of very fast pressure relaxation the solutions could be close because in this case the phases pressures must be very close to a common uniform value.

The difference between two models becomes more significant in the multidimensional case. Now we will not present the complete comparison of governing equations for both models. We note only that the mass conservation equations for each phase and relaxation equation for the volume fraction remain identical for both models. But in the phase momentum balance equations derived from the present conservative model an additional terms connected with phase vorticities arise.

Actually, from the total momentum equation (second equation in (25)) and the relative velocity equation (fifth equation in (25)) for the mixture, after cumbersome transformations the equations for phases momentum follow

$$\frac{\partial \alpha_1 \rho_1 (u_1)_i}{\partial t} + \frac{\partial \alpha_1 \rho_1 (u_1)_i (u_1)_k}{\partial x_k} + \frac{\partial \alpha_1 p_1}{\partial x_i} = \hat{p}_I \frac{\partial \alpha_1}{\partial x_i} + F_i - \frac{\alpha_1 \alpha_2 \rho_1 \rho_2}{\rho} \pi, \tag{47}$$

$$\frac{\partial \alpha_2 \rho_2 (u_2)_i}{\partial t} + \frac{\partial \alpha_2 \rho_2 (u_2)_i (u_2)_k}{\partial x_k} + \frac{\partial \alpha_2 p_2}{\partial x_i} = \hat{p}_I \frac{\partial \alpha_2}{\partial x_i} - F_i + \frac{\alpha_1 \alpha_2 \rho_1 \rho_2}{\rho} \pi. \tag{48}$$

We can see that in equations (47),(48) an additional extra terms F_i arise which are not presented in the BN-type equations (39). These terms are defined as follows

$$F_i = \rho c_1 c_2((u_1)_k - (u_2)_k)\left(c_1\left(\frac{\partial(u_2)_i}{\partial x_k} - \frac{\partial(u_2)_k}{\partial x_i}\right) + c_2\left(\frac{\partial(u_1)_i}{\partial x_k} - \frac{\partial(u_1)_k}{\partial x_i}\right)\right).$$

Here $c_i = (\alpha_i\rho_i)/\rho$, $i = 1, 2$ are the phase mass fractions.

Terms F_i describe the forces arising for the flow with nonzero relative velocity and caused by the phases vorticities. Such type of force is called as lift force, see for example [15].

So there is a difference between governing equations for isentropic processes in the BN-type model and conservative model. The distinctions appear in the phases momentum balance equations. First distinction is in the definition of the interfacial pressure. The second one appears in multidimensional case only, namely, the lift forces are presented in the conservative model, but they are not presented in the BN-type model.

5 Numerical methods

Here we consider finite volume schemes for solving numerically the two-phase flow equations studied in previous sections of this paper. We restrict the presentation to systems of the form

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{S}(\mathbf{Q}) \ , \tag{49}$$

where $\mathbf{Q}$ is the vector of conserved variables, $\mathbf{F}(\mathbf{Q})$ is the vector of fluxes and $\mathbf{S}(\mathbf{Q})$ is the vector of sources, assumed to be a *non-differential* term, e.g. algebraic.

Finite volume schemes for solving (49) numerically are constructed by first integrating (49) over a control volume $V \equiv [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]$ of dimensions

$$\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \ , \ \ \Delta t = t^{n+1} - t^n \ . \tag{50}$$

The result is the exact relation

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}\left[\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}\right] + \Delta t \mathbf{S}_i \ , \tag{51}$$

where

$$\left.\begin{aligned} \mathbf{Q}_i^n &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}(x, t^n)\, dx \ , \\ \mathbf{F}_{i+1/2} &= \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{F}(\mathbf{Q}(x_{i+1/2}, \tau))\, d\tau \ , \\ \mathbf{S}_i &= \frac{1}{\Delta t}\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{S}(\mathbf{Q}(x, \tau))\, dx\, d\tau \ . \end{aligned}\right\} \tag{52}$$

Conservative numerical schemes may be constructed from the exact relation (51)-(52) by defining suitable approximations to (52) leading to numerical fluxes and numerical sources, still denoted by the symbols $\mathbf{F}_{i+1/2}$ and $\mathbf{S}_i$ respectively.

Here we consider centred, as distinct from upwind, schemes. The most well-known centred schemes is that given by the Lax-Friedrichs numerical flux, namely

$$\mathbf{F}^{LF}_{i+1/2} = \frac{1}{2}\left[\mathbf{F}(\mathbf{Q}^n_i) + \mathbf{F}(\mathbf{Q}^n_{i+1})\right] - \frac{\Delta x}{\Delta t}\left[\mathbf{Q}^n_{i+1} - \mathbf{Q}^n_i\right] \ . \tag{53}$$

The conservative scheme (51) along with the numerical flux (53) gives the Lax-Friedrichs scheme, which is first-order accurate, mononote and stable to CFL unity, where CFL is the maximum Courant, or CFL, number. This scheme is very simple but is the most diffusive of all stable three-point schemes and thus is not recommended for practical use.

An improvement is obtained by using the FORCE flux [16]- [18]

$$\left.\begin{aligned} \mathbf{F}^{FORCE}_{i+1/2} &= \tfrac{1}{4}[\mathbf{F}(\mathbf{Q}^n_i) + 2\mathbf{F}(\mathbf{Q}^{n+\frac{1}{2}}_{i+\frac{1}{2}}) + \mathbf{F}(\mathbf{Q}^n_{i+1}) - \tfrac{\Delta x}{\Delta t}(\mathbf{Q}^n_{i+1} - \mathbf{Q}^n_i)], \\ \mathbf{Q}^{n+\frac{1}{2}}_{i+\frac{1}{2}} &= \tfrac{1}{2}[\mathbf{Q}^n_i + \mathbf{Q}^n_{i+1}] - \tfrac{1}{2}\tfrac{\Delta t}{\Delta x}[\mathbf{F}(\mathbf{Q}^n_{i+1}) - \mathbf{F}(\mathbf{Q}^n_i)]. \end{aligned}\right\}$$

This numerical flux leads to a conservative scheme that is first-order accurate, monotone and more accurate than that of Lax-Friedrichs. Recently, the scheme has been proved to be convergent, for a 2×2 system of non-linear conservation laws [19].

A second-order non-oscillatory (TVD) extension of the FORCE flux is the SLIC scheme [18], which is based on MUSCL linear reconstructions in each cell i, leading to two boundary extrapolated values

$$\mathbf{Q}^L_i = \mathbf{Q}^n_i - \frac{1}{2}\hat{\Delta}_i \ , \quad \mathbf{Q}^R_i = \mathbf{Q}^n_i + \frac{1}{2}\hat{\Delta}_i \ ,$$

where $\hat{\Delta}_i = \phi\Delta_i$ is a *TVD limited* slope (actually a vector difference), ϕ is a slope limiter function and Δ_i is a slope of the form

$$\Delta_i = \omega\Delta_{i-\frac{1}{2}} + (1-\omega)\Delta_{i+\frac{1}{2}}, \quad \Delta_{i+\frac{1}{2}} = Q^n_{i+1} - Q^n_i$$

with ω a parameter in the real interval $[0, 1]$. One usually takes $\omega = 0$.

In the scheme, the boundary extrapolated values are then evolved in time by half a time step as follows:

$$\hat{\mathbf{Q}}^L_i = \mathbf{Q}^L_i - \frac{1}{2}\frac{\Delta t}{\Delta x}[\mathbf{F}(\mathbf{Q}^R_i) - \mathbf{F}(\mathbf{Q}^L_i)] \ , \ \ \hat{\mathbf{Q}}^R_i = \mathbf{Q}^R_i - \frac{1}{2}\frac{\Delta t}{\Delta x}[\mathbf{F}(\mathbf{Q}^R_i) - \mathbf{F}(\mathbf{Q}^L_i)] \ .$$

In the final step of the scheme we evaluate the FORCE flux $\mathbf{F}^{FORCE}_{i+1/2}(\mathbf{A}, \mathbf{B})$ with arguments $\mathbf{A} = \hat{\mathbf{Q}}^R_i$ and $\mathbf{B} = \hat{\mathbf{Q}}^L_{i+1}$.

The resulting scheme is the second-order non-oscillatory SLIC scheme, which is simple and easily applicable to complex non-linear systems, such as those studied in this paper. Full details SLIC scheme are found in [17]. Regarding the source terms, there are various approaches that can be used. For details see Chapter 15 of [17].

6 Numerical Examples

In this section we study two test problems numerically using the reduced isentropic version of the conservative model for two-phase flow. The governing equations are taken

in the form (41) neglecting the interfacial interactions, namely pressure relaxation and interfacial friction. Obviously such a simplified system will not accurately describe phenomena involving strong shock waves but will still be useful for a valuable range of problems. Moreover, for this model it is easy to analyse the mathematical character of the system and to test new idea regarding numerical methods for multiphase flow problems. For the source terms we assume $\phi = 0$ and $\pi = 0$.

Each of the test problems represents a Riemann problem for an air/water mixture. Suppose that index 1 denotes the air parameters and index 2 denotes the water parameters. The closing constitutive relationships which are needed to close the model are the isentropic equations of state relating pressures to densities. For air, the isentropic perfect gas law is

$$p_1 = A_1 \left(\frac{\rho_1}{\rho_1^0} \right)^{\gamma_1}, \tag{54}$$

with constants $\rho_1^0 = 1kg/m^3, A_1 = 10^5 Pa, \gamma = 1.4$.

For water we take the Tait's equation of state

$$p_2 = A_2 \left(\frac{\rho_2}{\rho_2^0} \right)^{\gamma_2} - A_3, \tag{55}$$

with $\rho_2^0 = 10^3 kg/m^3, A_2 = 8.5 \times 10^8 Pa, A_3 = 8.4999 \times 10^8 Pa, \gamma_2 = 2.8$.

For the initial conditions in both test problems we assume that the mixture has constant volume fractions of air and water of 0.9 and 0.1 respectively and that the pressure of air and water in the initial data are equal and it is a given value. The values of phase densities in the initial data can be evaluated by formulae (54) and (55).

For both test problems the computational domain is the real interval $[0, 1]$. In each test the initial data defines a Riemann problem with a left section $[0, 1/2)$ and a right section $[1/2, 1]$.

6.1 Test 1: Symmetric four-rarefaction problem

This test problem has initial data:

Left section: $p = 10^9 Pa, u_1 = u_2 = -1.8 \times 10^3 m/s$,
Right section: $p = 10^9 Pa, u_1 = u_2 = 1.8 \times 10^3 m/s$.

Figs. 1 to 4 show computed results at time $0.11 \times 10^{-5} s$, using various meshes, for liquid and gas densities, liquid and gas velocities, mixture density and pressure, mixture velocity and relative velocity. The CFL coefficient used in the calculations was CFL= 0.9.

Fig. 1 shows a reference numerical solution computed with a fine mesh of 4000 cells and the best of the numerical methods presented in this paper, namely SLIC, which is a second-order TVD method. The structure of the solution contains four symmetric rarefaction waves, two for each phase. The solution also contains a middle region in which the density of both phases is very low. In fact this is the main feature that motivates the choice of this test problem for assessing numerical methods. For single-phase flow, it is a well documented fact that the computation of low density flows is indeed very challenging; it is known that all linearized Riemann solvers, for example, will lead to the computation of negative densities [20].

The four-rarefaction structure becomes more complex when we examine the mixture quantities. Due to the superposition of the two waves for each of the two phases we end up with a structure that contains, apparently, six waves. See for example the mixture

density and the relative velocity. The computed results of Fig. 1 look very satisfactory in that all the expected features of the solution are well resolved. We note, however, that there is a small spurious overshoot in the solution; this is more evident in the relative velocity profile.

Figs. 2 to 4 show numerical results for a coarse mesh of 100 cells (thin line) compared with the reference solution (thick line), for three schemes, namely Lax-Friedrichs, FORCE and SLIC. It is obvious that the Lax-Friedrichs scheme is too inaccurate to be used in practice. As judging from the results of these figures, the best scheme is the second-order TVD method, SLIC.

6.2 Test 2: Symmetric four-shock problem

This test problem describes the collision of a compressed mixture with discontinuous initial data:

$$\textit{Left section}\text{: } p = 10^8 Pa, u_1 = u_2 = 3.0 \times 10^3 m/s,$$
$$\textit{Right section}\text{: } p = 10^8 Pa, u_1 = u_2 = -3.0 \times 10^3 m/s.$$

Figs. 5 to 8 show results at time $0.22 \times 10^{-5} s$, for various meshes, of liquid and gas densities, liquid and gas velocities, mixture density and pressure, mixture velocity and relative velocity. The CFL coefficient used in the calculations was CFL= 0.9.

Fig. 5 shows a reference numerical solution computed with a fine mesh of 4000 cells and the best of the numerical methods presented in this paper, namely SLIC, which is a second-order TVD method.

The structure of the solution contains four symmetric shock waves, two for each phase. The point in proposing this test problem is to assess the ability of numerical schemes to resolve shock waves in general, and shock waves in two phase flow in particular. One of the difficulties of numerical methods when computing shock waves is robustness, the code may *crush*. Then one would like to resolve the shock with high resolution (a couple of cells across the wave) and without spurious oscillations in the vicinity of the discontinuity. The second-order TVD scheme SLIC appears to comply quite satisfactorily with these requirements.

The results of Fig. 5 look very satisfactory in that all the expected main features of the solution are well resolved. We do observe, however, some small spurious oscillations in the vicinity of the shock waves. It is known that the TVD (Total Variation Diminishing) constraint does not theoretically apply to non-linear systems, and therefore the TVD method will not strictly avoid all spurious oscillations.

Figs. 6 to 8 show numerical results for a coarse mesh of 100 cells (thin line) compared with the reference solution (thick line), for three schemes, namely Lax-Friedrichs, FORCE and SLIC. As judging from the results of figures 6 to 8, the best scheme is the second-order TVD method, SLIC.

Finally, in Fig. 9 we show a numerical convergence test for the SLIC scheme as applied to the four-shock test problem. To this end we select a single quantity, namely the mixture density. Numerical results (thin line) for sequence of four meshes: 100, 200, 400, 800 are compared with the reference solution of Fig. 5. We observe that the second and third meshes do already give very satisfactory results.

7 Conclusions

The governing equations of a conservative two-phase two-pressure model based on extended thermodynamics principles are studied. The system is hyperbolic and is written in conservation-law form. In the one-dimensional case the system has six equations, for which the full eigenstructure analysis is performed, namely, explicit formulae for eigenvalues and eigenvectors are derived and the corresponding characteristic fields are studied.

The reduced isentropic conservative system is formulated and compared with the corresponding isentropic Baer-Nunziato-type model. It is shown that the one-dimensional version of the present conservative system is similar to the Baer-Nunziato-type model except for the definition of interfacial pressure, which is included into the differential source terms in the momentum equations. Moreover, it turns out that for the multidimensional case the lift forces included in the momentum equations of the present model do not appear in the Baer-Nunziato-type model.

Conservative shock-capturing of the centred type are then proposed to solve the governing equations. Test problems are proposed and numerical results are shown and discussed.

Acknowledgements

The paper is based on research partially performed during the participation of the authors in the programme *Nonlinear Hyperbolic Waves in Phase Dynamics and Astrophysics* at the Isaac Newton Institute for Mathematical Sciences, University of Cambridge. The second author acknowledges the support provided by the Isaac Newton Institute for Mathematical Sciences, University of Cambridge, UK, as co-organizer of the six-months programme on *Nonlinear Hyperbolic Waves in Phase Dynamics and Astrophysics*, January to July 2003, and the associated EPSRC senior visiting fellowship, grant GR N09276. The first author acknowledges the Italian Ministry of Education and Research (MIUR) for the financial support provided during his stay as visiting professor at the Department of Civil and Environmental Engineering, University of Trento.

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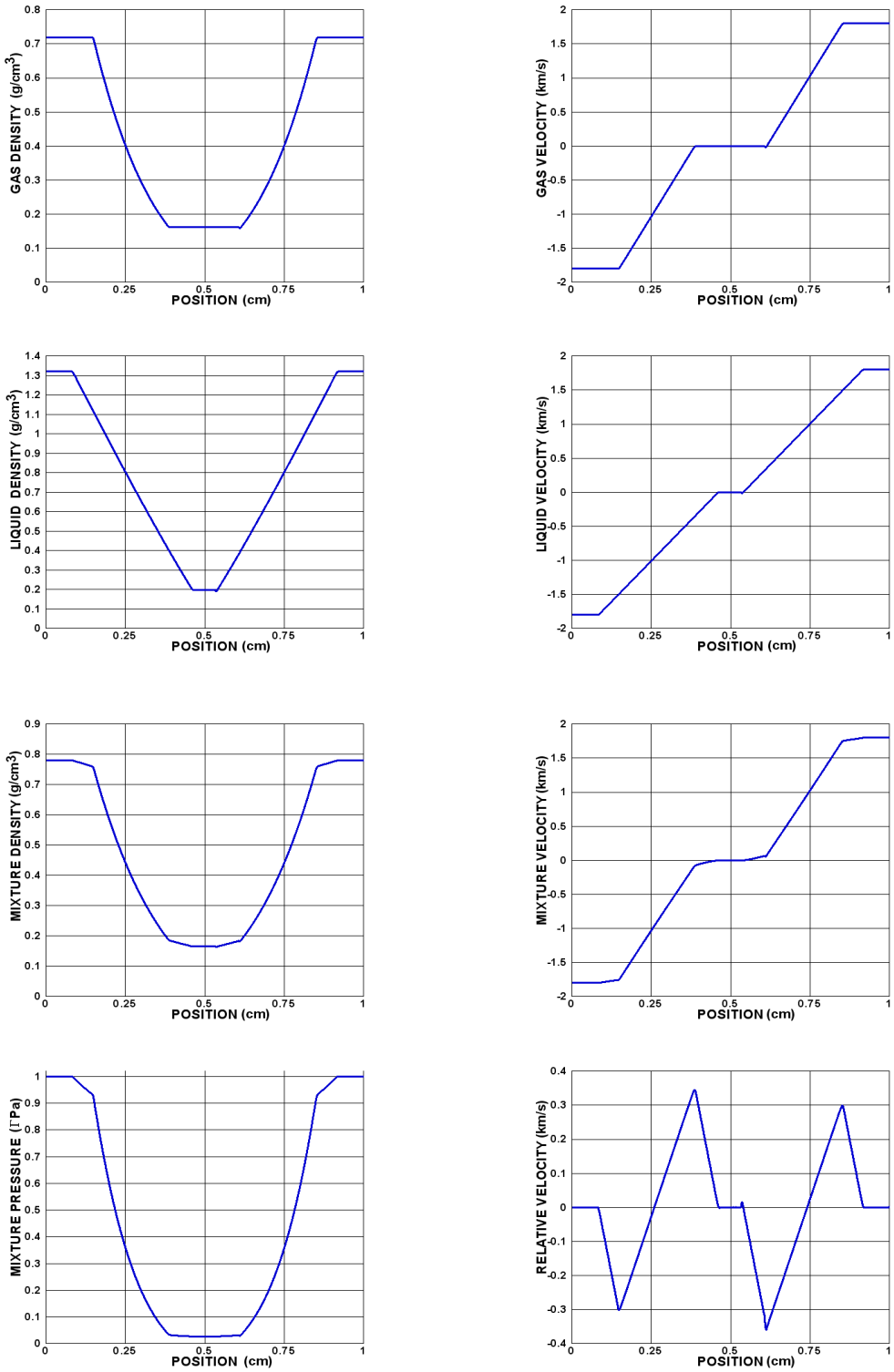

Figure 1: Test 1. Numerical results for 4-rarefaction waves Riemann Problem for 4000 mesh cells, TVD method, SUPERBEE limiter.

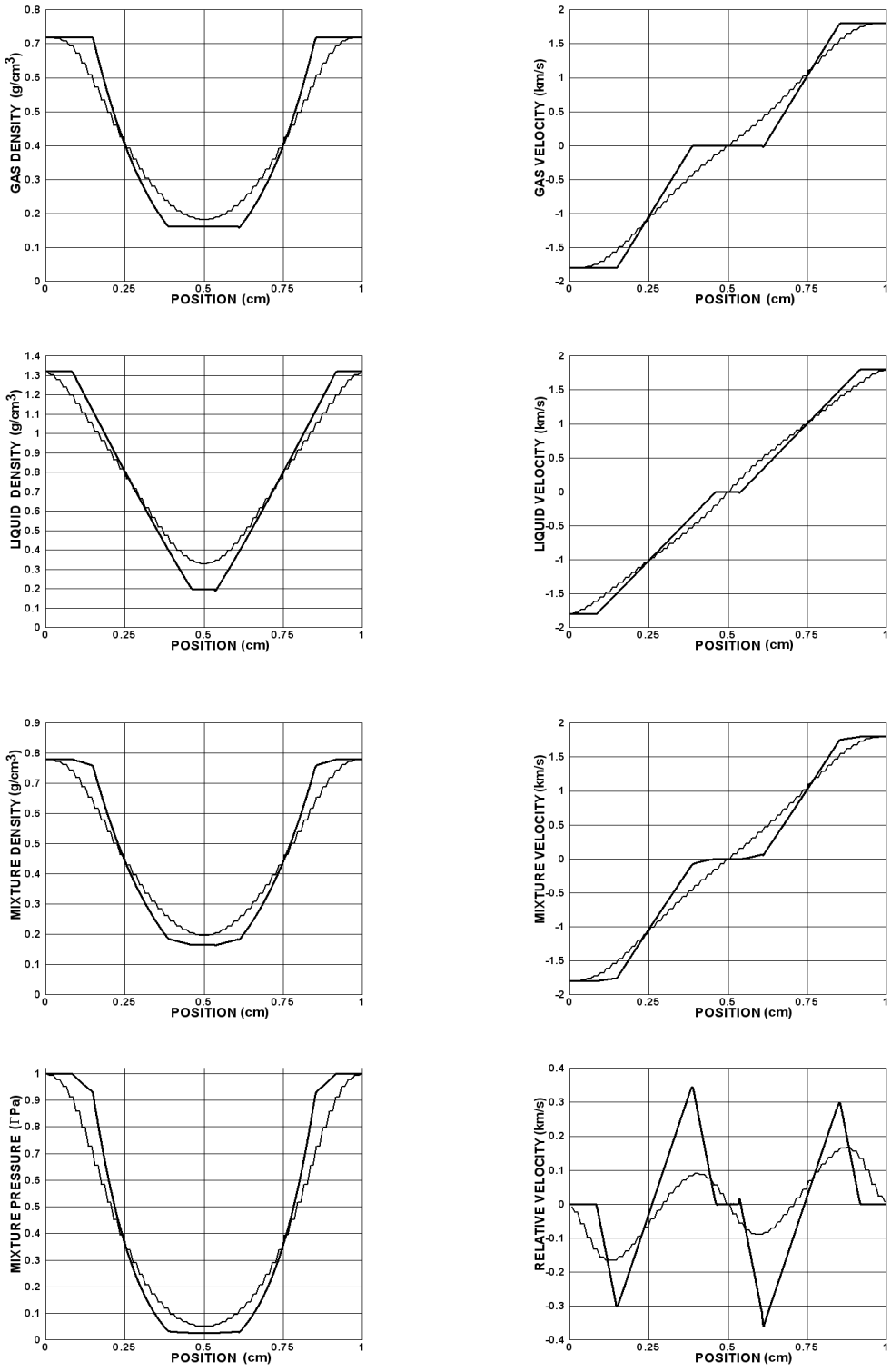

Figure 2: Test 1. Four-rarefaction test problem. Reference solution (thick line) and numerical solution (thin line) obtained by the Lax–Friedrichs method with mesh of 100 cells.

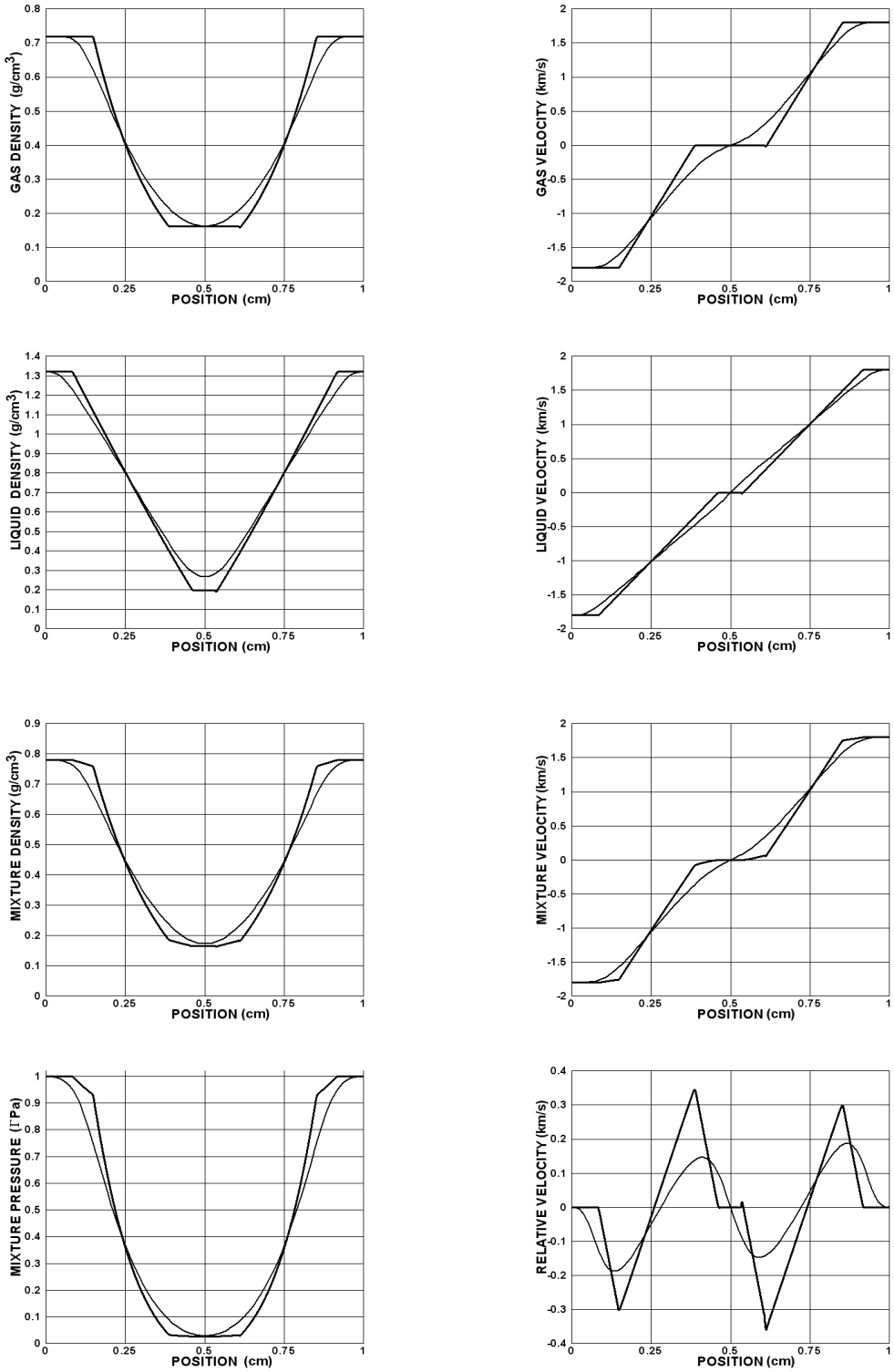

Figure 3: Test 1. Four-rarefaction test problem. Reference solution (thick line) and numerical solution (thin line) obtained by the FORCE method with mesh of 100 cells.

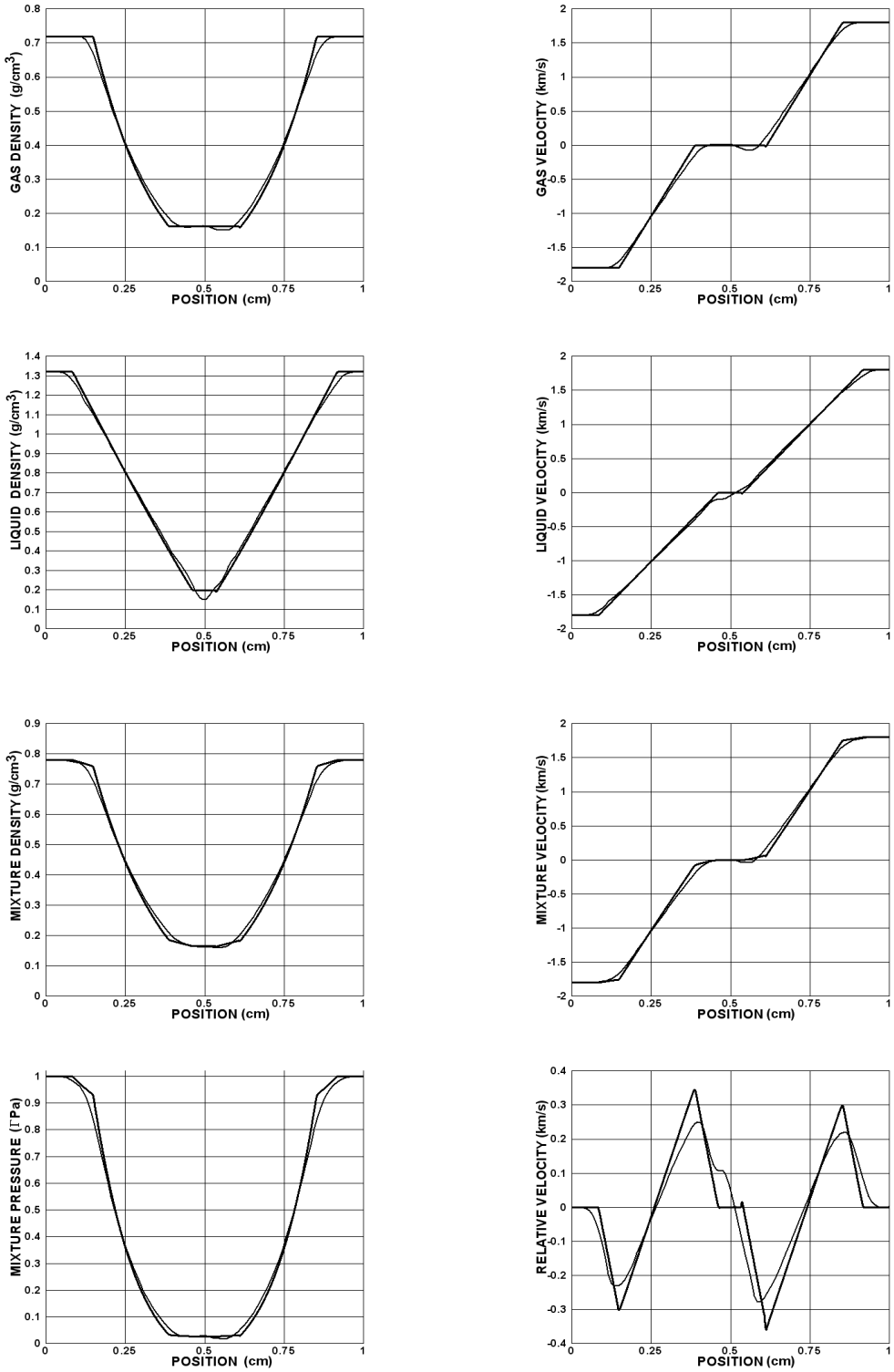

Figure 4: Test 1. Four-rarefaction test problem. Reference solution (thick line) and numerical solution (thin line) obtained by the SLIC method with mesh of 100 cells.

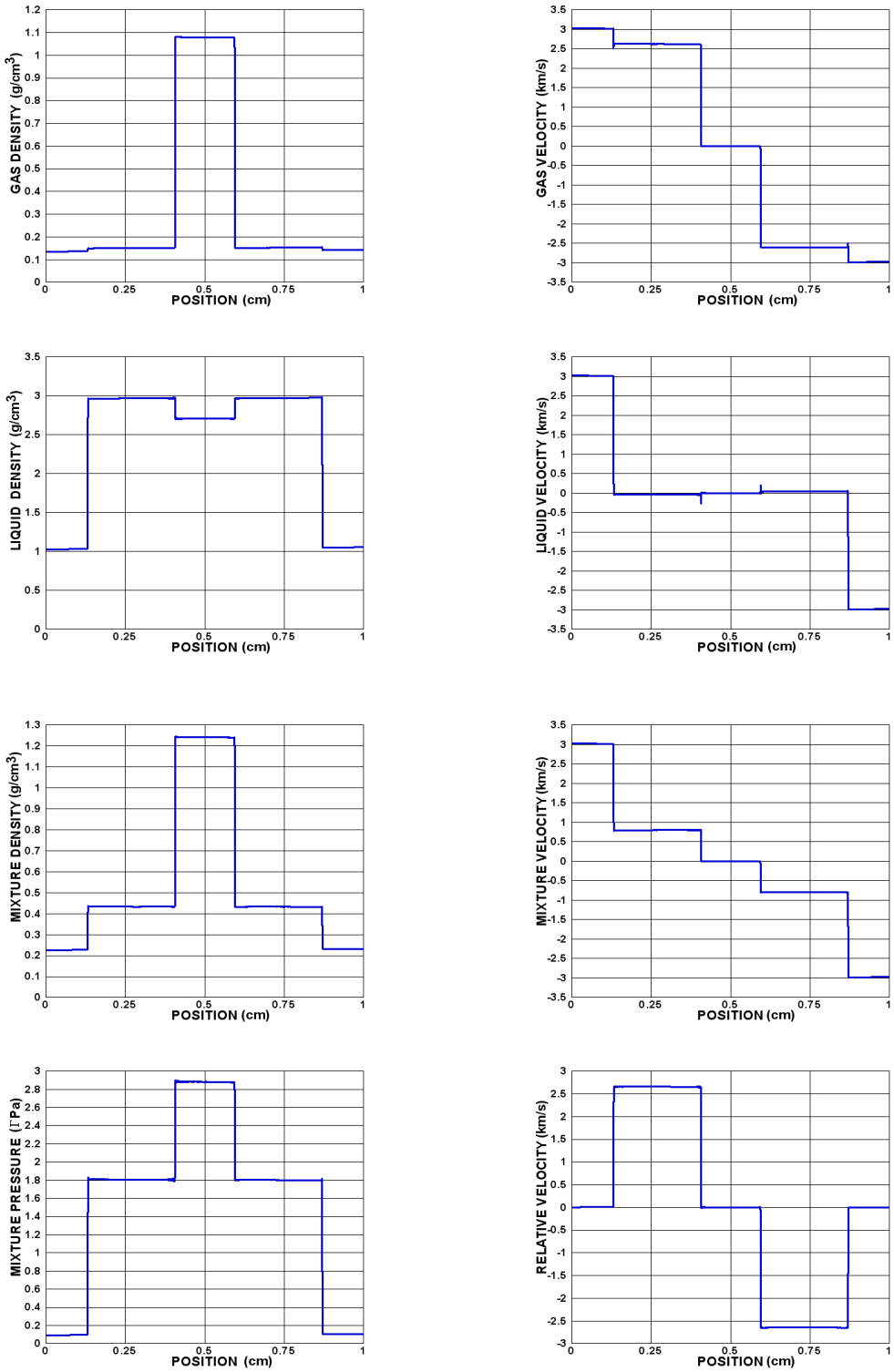

Figure 5: Test 2. Four-shock test problem. Reference solution using SLIC, SUPERBEE limiter and mesh of 4000 cells.

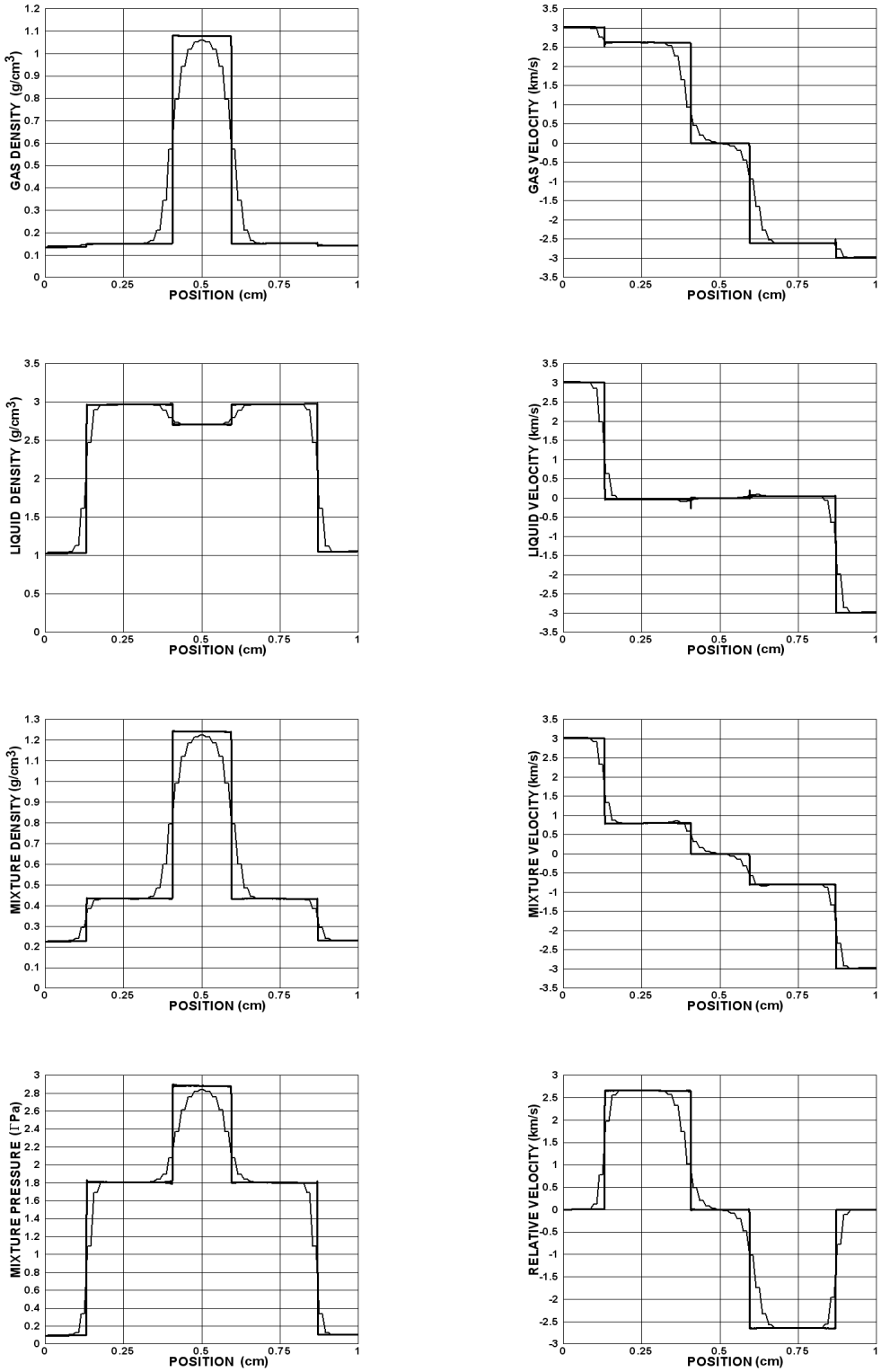

Figure 6: Test 2. Four-shock test problem. Reference solution (thick line) and numerical solution (thin line) obtained by the Lax–Friedrichs method for 100 mesh cells.

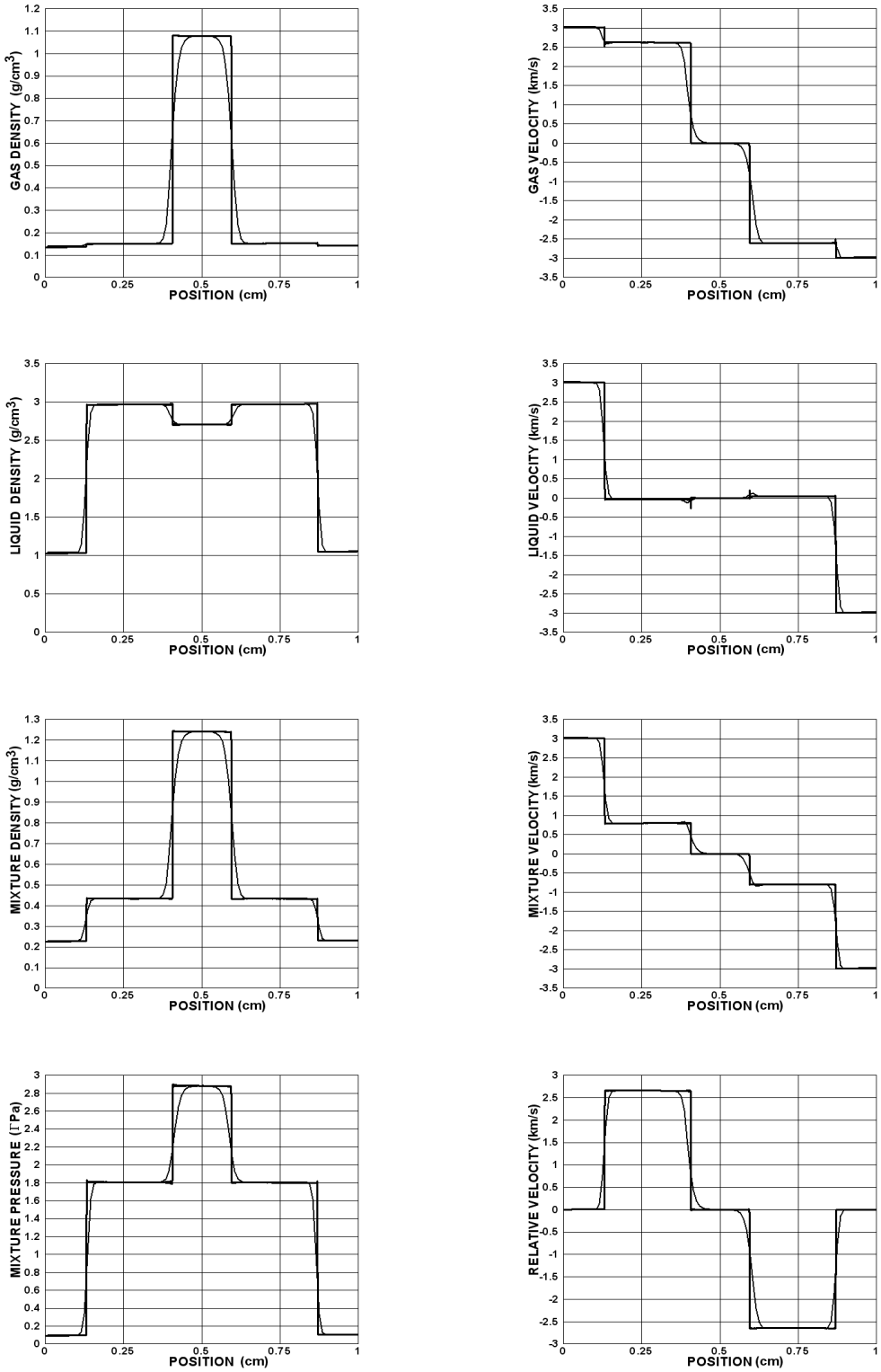

Figure 7: Test 2. Four-shock test problem. Reference solution (thick line) and numerical solution (thin line) obtained by the FORCE method for 100 mesh cells.

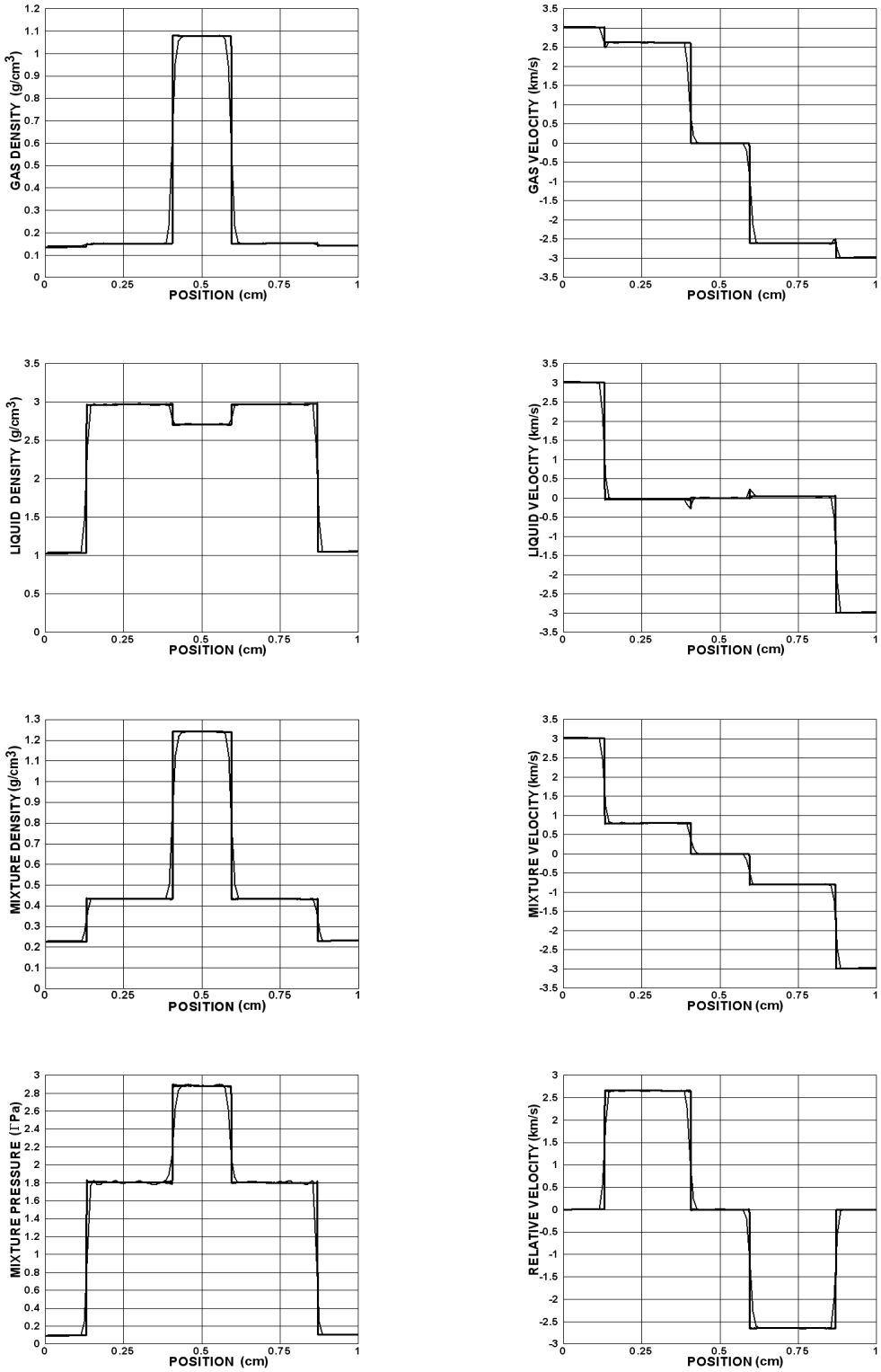

Figure 8: Test 2. Four-shock test problem. Reference solution (thick line) and numerical solution (thin line) obtained by the SLIC method for 100 mesh cells.

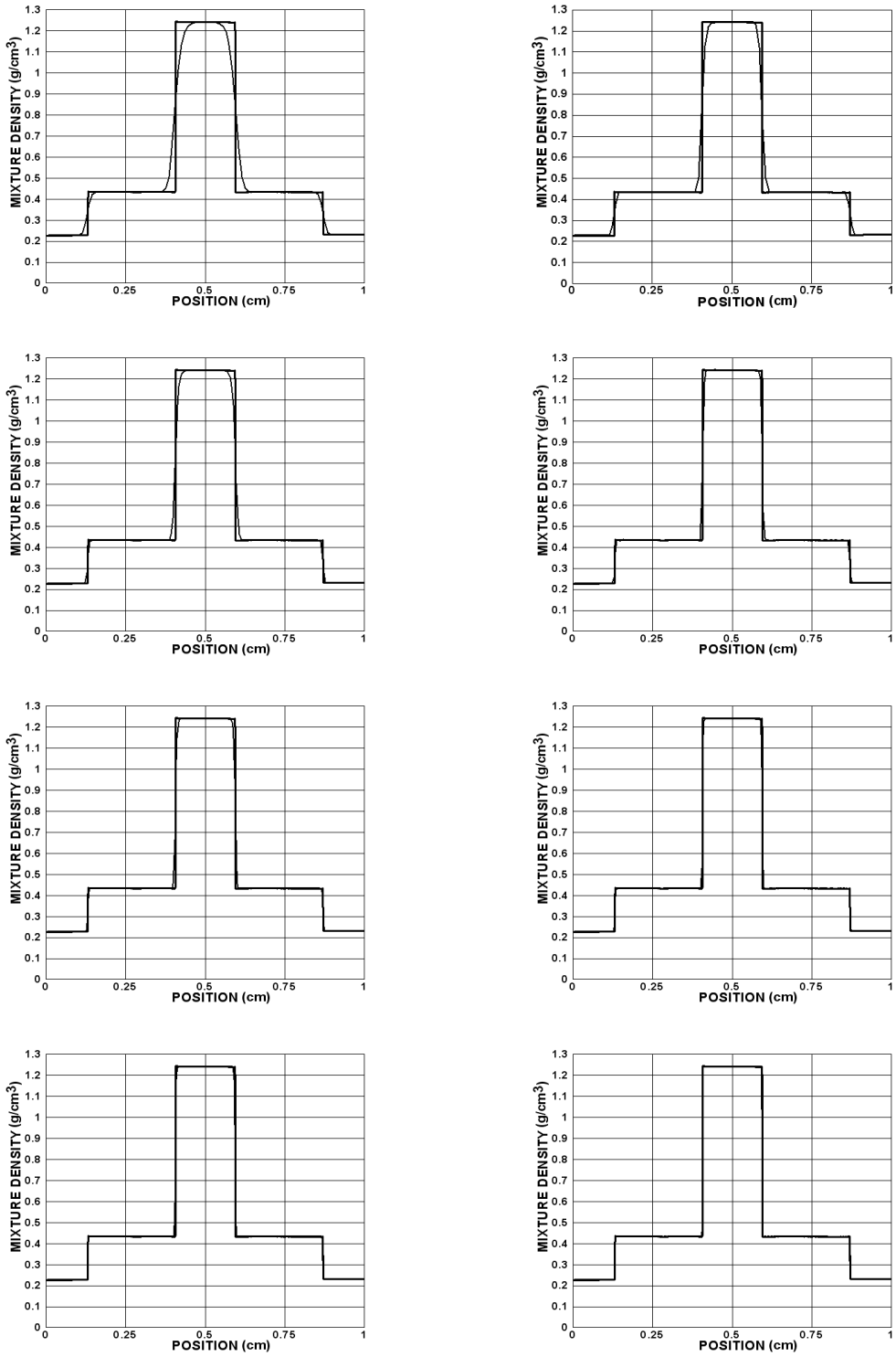

Figure 9: Test 2. Numerical convergence of mixture density for 4-shock waves Riemann Problem. Reference solution (thick line) and numerical solution (thin line) obtained by the FORCE (left column) and SLIC (right column) methods with meshes of 100, 200, 400, and 800 cells.

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