On the Moments of Traces of Matrices of Classical Groups

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Abstract

We consider random matrices, belonging to the groups $U(n)$, $O(n)$, $SO(n)$, and $Sp(n)$ and distributed according to the corresponding unit Haar measure. We prove that the moments of traces of powers of the matrices coincide with the moments of certain Gaussian random variables if the order of moments is low enough. Corresponding formulas, proved partly before by various methods, are obtained here in the framework of a unique method, reminiscent of the method of correlation equations of statistical mechanics. The equations are derived by using a version of the integration by parts.

1 Introduction

Consider the probability space, whose objects are $n \times n$ unitary matrices, and whose probability measure is unit the Haar measure on the group $U(n)$. Denote $\mathbf{E}\{...\}$ the expectation with respect to the measure. Let $a : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ be a function, defined on positive integers and assuming non-negative integer values such that only finitely many of them are different from zero. Given two such functions a and b, consider the moment

$$\mathbf{E}\left\{\prod_{l\geq 1}(\mathrm{Tr}U_n^l)^{a_l}\overline{(\mathrm{Tr}U_n^l)^{b_l}}\right\}. \tag{1.1}$$

Denote

$$\kappa(a) = \sum_{l\geq 1} l a_l < \infty. \tag{1.2}$$

It follows from the invariance of the Haar measure with respect to the transformation $U_n \to e^{i\varphi}U_n$, $\varphi \in [0, 2\pi)$ that (1.1) is zero if $\kappa(a) \neq \kappa(b)$. Hence, without loss of generality, we can confine ourselves to the moments, whose multi-indices a and b satisfy the condition: $\kappa(a) = \kappa(b)$. In this case we call

$$\kappa := \kappa(a) = \kappa(b) \tag{1.3}$$

the order of the corresponding moment, and we write

$$m_\kappa^{(n)}(a; b) = \mathbf{E}\left\{\prod_{l\geq 1}(\mathrm{Tr}U_n^l)^{a_l}\overline{(\mathrm{Tr}U_n^l)^{b_l}}\right\}. \tag{1.4}$$

In the recent paper [4] (see also [5, 8]) Diaconis and Evans proved the formulas:

$$m_\kappa^{(n)}(a;b) = \mu_\kappa(a;b),\ \kappa \le n, \tag{1.5}$$

where

$$\mu_\kappa(a;b) = \prod_{l\ge1} l^{a_l} a_l! \delta_{a_l,b_l}, \tag{1.6}$$

and

$$\mathbf{E}\left\{\mathrm{Tr}U_n^j\ \overline{\mathrm{Tr}U_n^k}\right\} = \min\{j,n\}\cdot\delta_{j,k}, \tag{1.7}$$

for all positive integers j.

Denote $\{X_l\}_{l\ge1}$ and $\{Y_l\}_{l\ge1}$ independent standard Gaussian random variables (i.e., of zero mean and of unit variance) and set $Z_l = (X_l + iY_l)/\sqrt{2}$. Then (1.4)–(1.6) are equivalent to [4]

$$\mathbf{E}\left\{\prod_{l\ge1}(\mathrm{Tr}U_n^l)^{a_l}\overline{(\mathrm{Tr}U_n^l)^{b_l}}\right\} = \mathbf{E}\left\{\prod_{l\ge1}(\sqrt{l}Z_l)^{a_l}\overline{(\sqrt{l}Z^l)^{b_l}}\right\},\ \kappa \le n. \tag{1.8}$$

Hence the mixed moments of traces of powers of matrices of $U(n)$ whose orders do not exceed n coincide with analogous moments of multiples of the standard Gaussian complex random variables. This is why this property is called in [9] the mock Gaussian property. Notice that a collection of random variables

$$\left\{(\mathrm{Tr}U_n^l)^{a_l}\overline{,\ (\mathrm{Tr}U_n^l)^{b_l}}\right\}_{l\ge1}$$

cannot be Gaussian for any two multi-indices a and b. A simple example is given by the pair

$$\left\{(\mathrm{Tr}U_n^l)^{a_l},\ \overline{(\mathrm{Tr}U_n^l)^{a_l}}\right\},\ l > n.$$

Indeed, according to formulas (2.21), (2.25), and (2.26) below, the characteristic function of the pair is not Gaussian.

Analogous result was obtained in [4, 5] also for the orthogonal group $O(n)$. Namely, let us view $O(n)$ as the probability space whose probability measure is the normalized to unity Haar measure of the group and denote again $\mathbf{E}\{...\}$ the expectation with respect to the measure. Given a multi-index $a : \mathbb{N}\setminus\{0\} \to \mathbb{N}$ with finitely many non-zero components, consider the moments

$$m_\kappa^{(n)}(a) := \mathbf{E}\left\{\prod_{l\ge1}(\mathrm{Tr}O_n^l)^{a_l}\right\}. \tag{1.9}$$

Then, according to [4, 5], we have

$$m_\kappa^{(n)}(a) = \mathbf{E}\left\{\prod_{l\ge1}\left(\sqrt{l}X_l + \eta_l\right)^{a_l}\right\}, \quad \eta_l = \left(1 + (-1)^l\right)/2, \tag{1.10}$$

where

$$\kappa \le n/2. \tag{1.11}$$

On the other hand, by using the explicit form of the matrices of $SO(2)$, it is easy to check that the collection of random variables

$$\big\{(\mathrm{Tr}O_n^l)^{a_l}\big\}_{l\geq 1}$$

is not Gaussian, at least for $n=2$.

Similarly, if

$$\widehat{m}_\kappa^{(n)}(a) := \mathbf{E}\left\{\prod_{l\geq 1}(\mathrm{Tr}S_n^l)^{a_l}\right\} \tag{1.12}$$

are the moments of symplectic matrices with respect to the unit Haar measure on $Sp(n)$, then we have [9]

$$\widehat{m}_\kappa^{(n)}(a) = \mathbf{E}\left\{\prod_{l\geq 1}\left(\sqrt{l}X_l - \eta_l\right)^{a_l}\right\},\ \kappa \leq n+1. \tag{1.13}$$

The proofs of (1.5)–(1.7), (1.10)–(1.11), and (1.13) in [4, 5] were based on the representation theory of the groups $U(n)$, $O(n)$, and $Sp(n)$, in particular, on the works [16] on the Brauer algebras of $O(n)$ and $Sp(n)$. Another proof of (1.5) was given in [11] (Appendix). It is based on certain identities for the Töplitz determinants [1]. Hughes and Rudnick [9] proved (1.10) for the group $SO(n)$ and $\kappa \leq n-1$ and (1.13), by using the combinatorics of the cumulant expansion, constructed by using the Weyl integration formulas for $SO(n)$ and $Sp(n)$.

We will prove (1.5) for $U(n)$, (1.10) for the both groups $O(n)$, and $SO(n)$ and for $\kappa \leq n-1$, and (1.13) by using a unique and elementary method, similar to the method of correlation equations of statistical mechanics. We also give proofs of (1.7) and of certain related formulas, by using standard means of the random matrix theory, in fact, the so called determinant form of the Weyl formula for the restriction of the Haar measure of $U(n)$ to the space of cental functions (see e.g. [13]). Similar formulas for other classical groups [12] yield analogs of (1.7), i.e. the variances of traces of matrices, for other classical groups (see e.g. [9], Section 2). For example, for the $Sp(n)$ we have

$$\mathbf{E}\{\mathrm{Tr}S_n^j\} = -\left\{\begin{array}{ll} \eta_j, & j\leq n,\\ 0, & j>n,\end{array}\right. \tag{1.14}$$

and

$$\begin{aligned}\mathbf{E}\Big\{\big(\mathrm{Tr}S_n^j - \mathbf{E}\{\mathrm{Tr}S_n^j\}\big)\big(\mathrm{Tr}S_n^k - \mathbf{E}\{\mathrm{Tr}S_n^k\}\big)\Big\} \qquad\qquad & \qquad (1.15)\\ = \delta_{j,k}\left\{\begin{array}{cc} j, & j\leq n/2,\\ j-1, & 1+n/2\leq j\leq n,\\ n, & j>n.\end{array}\right.\end{aligned}$$

Notice that to have (1.5), (1.10), and (1.13) in the full range ($\kappa \leq n$, $\kappa \leq n-1$ and $\kappa \leq n+1$ respectively) is important for comparison with results on the behavior of linear statistics of zeros of the Riemann ζ-function and the L-functions [10]. It is also of interest for the quantum chaos studies [8]. Certain questions related to the above moments were considered in [3, 2, 7, 14, 19, 18, 21].

2 Unitary Group

Our proof will be based on the following simple implication of the left invariance of Haar measure on $U(n)$.

Proposition 2.1. *Let $F : U(n) \times U(n) \to \mathbb{C}$ be a continuously differentiable function. Then for any $n \times n$ Hermitian matrix X we have*

$$\mathbf{E}\Big\{ F_1'(U_n, U_n^*) \cdot XU_n - F_2'(U_n, U_n^*) \cdot U_n^* X \Big\} = 0, \tag{2.1}$$

where U_n^* is the Hermitian conjugate of U_n, and F_1' and F_2' are derivatives of F with respect to its first and second argument correspondingly, i.e., linear applications in the vector space of $n \times n$ matrices.

The proposition follows from the fact that e^{-itX} is a unitary matrix for any real t and that $\mathbf{E}\left\{ F(e^{itX}U_n, U_n^* e^{-itX}) \right\}$ is independent of t because of the left invariance of the Haar measure of $U(n)$.

Lemma 2.2. *Denote T_l^p the operation, replacing the lth value a_l of a given multi-index a by $a_l + p$, $p \in \mathbb{Z}$: $(T_l^p a)_m = a_m + p\delta_{l,m}$. Let j be the left hand endpoint of the support of a. Then we have the following identities*

$$\begin{aligned}
m_\kappa^{(n)}(a;b) + \frac{1}{n}\Bigg((1-\delta_{j,1}) \sum_{l=1}^{j-1} m_\kappa^{(n)}(T_l T_{j-l} T_j^{-1} a; b) + j(a_j - 1) m_\kappa^{(n)}(T_j^{-2} T_{2j} a; b) \\
+ \sum_{l \geq j+1} l a_l m_\kappa^{(n)}(T_j^{-1} T_l^{-1} T_{j+l} a; b) \Bigg) \qquad (2.2) \\
= j b_j m_{\kappa-j}^{(n)}(T_j^{-1} a_k; T_j^{-1} b) + \frac{1}{n}\Bigg((1-\delta_{j,1}) \sum_{l=1}^{j-1} l b_l m_{\kappa-l}^{(n)}(T_{j-l} T_j^{-1} a; T_l^{-1} b) \\
+ \sum_{l \geq j+1} l b_l m_{\kappa-j}^{(n)}(T_j^{-1} a; T_{l-j}^1 T_l^{-1} b) \Bigg).
\end{aligned}$$

Proof. Choosing in (2.1) $X = zX^{(x,y)} + \overline{z}(X^{(x,y)})^T$, where z is an arbitrary complex number and

$$X^{(x,y)} = \{\delta_{px}\delta_{qy}\}_{p,q=1}^n, \tag{2.3}$$

we conclude that formula (2.1) is valid also in the case where X is replaced by $X^{(x,y)}$ (in fact, by any real matrix).

We apply (2.1) with $X = X^{(x,y)}$ to

$$F(U_n, U_n^*) = (U_n^j)_{x,y} \left(\mathrm{Tr}\, U_n^j\right)^{a_j - 1} \prod_{l \geq j+1} \left(\mathrm{Tr}\, U_n^l\right)^{a_l} \prod_{l \geq 1} \Big(\mathrm{Tr}\ (U_n^*)^l\Big)^{b_l}. \tag{2.4}$$

Taking into account the relations

$$\begin{aligned}\left((U_n^m)_{x,y}\right)' \cdot X^{(x,y)}U_n &= \sum_{i=0}^{m-1}\left(U_n^i X^{(x,y)} U_n^{m-i}\right)_{x,y} = \sum_{i=0}^{m-1}\left(U_n^i\right)_{x,x}\left(U_n^{m-i}\right)_{y,y} \\ &= \delta_{x,x}\left(U_n^m\right)_{y,y} + \sum_{i=1}^{m-1}\left(U_n^i\right)_{x,x}\left(U_n^{m-i}\right)_{y,y}, \\ (\mathrm{Tr}\,U_n^m)' &\cdot X^{(x,y)}U_n = m\mathrm{Tr}\,U_n^m X = m\left(U_n^m\right)_{y,x}, \\ (\mathrm{Tr}\,(U_n^*)^m)' &\cdot U_n^* X^{(x,y)} = m\left((U_n^*)^m\right)_{y,x},\end{aligned}$$

and the equality $U_n U_n^* = 1$, we obtain

$$\begin{aligned}\delta_{xx}\mathbf{E}\left\{(U_n^j)_{y,y}\alpha_-\beta\right\} &+ \sum_{i=1}^{j-1}\mathbf{E}\left\{\left(U_n^i\right)_{x,x}\left(U_n^{j-i}\right)_{y,y}\alpha_-\beta\right\} \\ &+ (a_j-1)j\mathbf{E}\left\{\left(U_n^j\right)_{x,y}\left(U_n^j\right)_{y,x}\left(\mathrm{Tr}\,U_n^j\right)^{a_j-2}\alpha_+\beta\right\} \\ &+ \sum_{l\geq j+1} a_l l\mathbf{E}\left\{\left(U_n^j\right)_{x,y}\left(U_n^l\right)_{y,x}\alpha(l)\beta\right\} \\ &- \sum_{l\geq 1} b_l l\mathbf{E}\left\{\left(U_n^j\right)_{x,y}\left((U_n^*)^l\right)_{y,x}\alpha_-\beta(l)\right\} = 0,\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}\alpha_- &= \left(\mathrm{Tr}\,U_n^j\right)^{a_j-1}\alpha_+,\ \alpha_+ = \prod_{l\geq j+1}\left(\mathrm{Tr}\,U_n^l\right)^{a_l},\ \beta = \prod_{l\geq 1}\left(\mathrm{Tr}\ (U_n^*)^l\right)^{b_l}, \\ \alpha(l) &= \left(\mathrm{Tr}\,U_n^j\right)^{a_j-1}\left(\mathrm{Tr}\,U_n^{j+1}\right)^{a_{j+1}}...\left(\mathrm{Tr}\,U_n^{l-1}\right)^{a_{l-1}}\left(\mathrm{Tr}\,U_n^l\right)^{a_l-1}\left(\mathrm{Tr}\,U_n^{l+}\right)^{a_{l+1}}..., \\ \beta(l) &= \left(\mathrm{Tr}\,U_n^*\right)^{b_1}...\left(\mathrm{Tr}\,(U^*)_n^{l-1}\right)^{b_{l-1}}\left(\mathrm{Tr}\,(U_n^*)^l\right)^{b_l-1}\left(\mathrm{Tr}\,(U_n^*)^{l+1}\right)^{b_{l+1}}....\end{aligned}$$

Since the moments (1.4) can be written as

$$\begin{aligned}&m_k^{(n)}(a;b) \\ &\quad = \sum_{x=1}^n \mathbf{E}\left\{(U_n^j)_{x,x}\left(\mathrm{Tr}\,U_n^j\right)^{a_j-1}\prod_{l\geq j+1}\left(\mathrm{Tr}\,U_n^l\right)^{a_l}\prod_{l\geq 1}\left(\mathrm{Tr}\ (U_n^*)^l\right)^{b_l}\right\},\end{aligned} \tag{2.6}$$

we apply to (2.5) the operation $n^{-1}\sum_{x,y=1}^n$ and we obtain, after regrouping terms, the assertion of the lemma. □

Remark. It will be important in what follows that the orders of all moments on the r.h.s. of (2.2) equal κ, while the orders of all moments on the l.h.s. are less then κ ($\kappa - 1$ at most).

Now we are ready to prove formulas (1.5)–(1.7) (Theorem 2.1 of [4]).

Theorem 2.3. *Let $m_k^{(n)}(a;b)$ be the moments of traces of powers of unitary matrices, defined in (1.2)–(1.4). Then we have formulas (1.5)–(1.7).*

Proof. We prove first (1.5). To this end we present the result of Lemma 2.2 in a more convenient form, reminiscent of that of the correlation equations (for instance, the Kirkwood-Salzburg equations) of statistical mechanics (see e.g. [17]).

Given a non-negative integer K, denote P_K the set of multi-indices such that

$$P_K = \left\{ a = \{a_l\}_{l\geq 1} : \sum_{l\geq 1} l a_l \leq K \right\}. \tag{2.7}$$

Consider the vector space $\mathcal{L}_K^{(U)}$ of collections of complex numbers, indexed by pairs (a, b) such that $\kappa(a) = \kappa(b)$ and $a, b \in P_K$, and call the integer κ of (1.3) the order of a component $v(a; b)$ of $v \in \mathcal{L}_K^{(U)}$. We define in $\mathcal{L}_K^{(U)}$ the uniform norm

$$||v||_U = \max_{a,b\in P_K} |v_\kappa(a; b)|. \tag{2.8}$$

Furthermore, we view the expression in the parentheses of the l.h.s. of (2.2), the first term of the r.h.s., and the expression in the parentheses of the r.h.s. of (2.2) as the results of action of certain linear operators on the vector $m_K^{(n)}$, whose components are the moments (1.4) of the orders $\kappa \leq K$. In other words, if j is the left hand endpoint of the support of a, then we set for $v \in \mathcal{L}_K^{(U)}$:

$$\begin{aligned}(A_U v)_\kappa(a; b) &= (1 - \delta_{j,1}) \sum_{l=1}^{j-1} v_\kappa(T_l T_{j-l} T_j^{-1} a; b) \\ &+ j(a_j - 1) v_\kappa(T_j^{-2} T_{2j} a; b) + \sum_{l\geq j+1} l a_l v_\kappa(T_j^{-1} T_l^{-1} T_{j+l} a; b),\end{aligned} \tag{2.9}$$

$$(B_U v)_\kappa(a; b) = j b_j v_{\kappa-j}(T_j^{-1} a; T_j^{-1} b), \tag{2.10}$$

$$\begin{aligned}(C_U v)_\kappa(a; b) &= (1 - \delta_{j,1}) \sum_{l=1}^{j-1} l b_l v_{\kappa-l}(T_{j-l} T_j^{-1} a; T_l^{-1} b) \\ &+ \sum_{l\geq j+1} l b_l v_{\kappa-j}(T_j^{-1} a; T_{l-j} T_j^{-1} b).\end{aligned} \tag{2.11}$$

With this notation we can rewrite (2.2) as the following equation in $\mathcal{L}_K^{(U)}$:

$$\left(I + n^{-1} A_U\right) m_K^{(n)} = B_U m_K^{(n)} + n^{-1} C_U m_K^{(n)}. \tag{2.12}$$

Besides, it is easy to see that the sequence (1.6)–(1.8) verifies the following recursion relation, valid for any $l \geq 1$:

$$\mu_\kappa(a; b) = l b_l \mu_{\kappa-l}(T_l^{-1} a; T_l^{-1} b). \tag{2.13}$$

By using (2.9)–(2.11) and (2.13), we can prove the following (see Appendix)

Lemma 2.4. *Let A_U, B_U and C_U be the linear operators, defined by (2.9)–(2.11). We have*

(i) $||A_U|| \leq (K - 1)$,

(ii) if μ_K is the vector of $\mathcal{L}_K$, whose components are given by (1.6) for all $\kappa \leq K$, then

$$B_U \mu_K = \mu_K, \; C_U \mu_K = A_U \mu_K.$$

Since $B_U m_K^{(n)}$ and $C_U m_K^{(n)}$ include the moments whose orders are strictly less than K, we can use (2.12) to find the moments of the order K, provided that the moments of lower orders are known. This suggests the use of the induction in K to prove formula (1.5).

Indeed, it is easy to check that for $K = 0, 1$ formula (1.5) holds: the equality $m_0^{(n)}(a;b) = 1$ is evident and the equality $m_1^{(n)}(a;b) = 1$ can already be deduced from (2.2) (it is also the normalization of the character of $U(n)$).

Assume now that $m_\kappa^{(n)}(a,b) = \mu_\kappa(a,b)$, $\forall \kappa \le K-1$. The r.h.s. of (2.10)–(2.11) contain the components of v, whose order does not exceed $K-1$. Hence we can write: $B_U m_K^{(n)} = B_U \mu_K$, $C_U m_K^{(n)} = C_U \mu_K$. These facts and the second assertion of the lemma allow us to replace (2.12) by the following relation

$$\left(I + n^{-1} A_U\right) m_K^{(n)} = \left(I + n^{-1} A_U\right) \mu_K. \tag{2.14}$$

Now the first assertion of the lemma implies that if $K \le n$, then the operator $I + n^{-1} A_U$ is invertible. Hence, for $K \le n$ (2.14) is equivalent to (1.5).

To prove (1.7) we first note that its part, corresponding to $j \le n$ is a particular case of (1.5). Hence we have to prove (1.7) for $j > n$. We will use the standard mean of the random matrix theory, so called determinant form of the joint probability density of eigenvalues of the random matrix U_n (the Circular Unitary Ensemble), widely used in the random matrix theory since the seminal paper by Dyson [6] (see also [13]). Namely, if $\lambda_\alpha = e^{i\theta_\alpha}$, $\theta_\alpha \in [0, 2\pi)$, $\alpha = 1, ..., n$ are eigenvalues of U_n, then their joint eigenvalue density with respect to the measure $d\theta_1 ... d\theta_n/(2\pi)^n$ on the n-dimensional torus is

$$p_n(\theta_1 ... \theta_n) = (n!)^{-1} |\Delta(\theta_1 ... \theta_n)|^2, \tag{2.15}$$

where

$$\Delta(\theta_1 ... \theta_n) = \prod_{1 \le \alpha < \beta \le n} (e^{i\theta_\alpha} - e^{i\theta_\beta}) = \det\{e^{i(j-1)\theta_\alpha}\}_{j,\alpha=1}^n. \tag{2.16}$$

Note, that (2.15)–(2.16) is, in fact, the Weyl integration formula for the restriction of the Haar measure on $U(n)$ to the space of central function [12, 22]. This implies the so-called determinant formula for the lth marginal density $p_{n,l}$ of p_n [6, 13]:

$$p_{n,l}(\theta_1 ... \theta_l) = \left[n(n-1)...(n-l+1)\right]^{-1} \det\{K_n(\theta_\alpha, \theta_\beta\}_{\alpha,\beta=1}^l, \tag{2.17}$$

where

$$K_n(\theta', \theta'') = \sum_{l=0}^{n-1} e^{il(\theta' - \theta'')}. \tag{2.18}$$

By using (2.17) for $l = 1, 2$, it is easy to prove (1.7). We will give below a bit different version of the above technique, to demonstrate corresponding simple and general mathematical mechanisms. Our presentation is rather close to those of [8, 19].

Given $j \ge 1$, consider the moments

$$\mathbf{E}\left\{ \left(\mathrm{Tr} U_n^j\right)^p \left(\overline{\mathrm{Tr} U_n^j}\right)^p \right\}, \; p = 0, 1, ... \tag{2.19}$$

and their generating function

$$F(t) = \sum_{p=0}^\infty \frac{t^{2p}}{(p!)^2} \mathbf{E}\left\{ \left(\mathrm{Tr} U_n^j\right)^p \left(\overline{\mathrm{Tr} U_n^j}\right)^p \right\}. \tag{2.20}$$

It follows from the invariance of the Haar measure with respect to the change $U_n \to e^{i\varphi}U_n$, $\varphi \in [0, 2\pi)$ that

$$\mathbf{E}\left\{\left(\mathrm{Tr}U_n^j\right)^p\left(\overline{\mathrm{Tr}U_n^j}\right)^q\right\} = \delta_{p,q}\mathbf{E}\left\{\left(\mathrm{Tr}U_n^j\right)^p\left(\overline{\mathrm{Tr}U_n^j}\right)^p\right\}.$$

Hence, $F(t)$ can be written as

$$\begin{aligned} F(t) &= \sum_{p,q=0}^{\infty}\frac{t^p t^q}{p!q!}\mathbf{E}\left\{\left(\mathrm{Tr}U_n^j\right)^p\left(\overline{\mathrm{Tr}U_n^j}\right)^q\right\} = \mathbf{E}\left\{\exp\left[t\mathrm{Tr}U_n^j + t\overline{\mathrm{Tr}U_n^j}\right]\right\} \\ &= \mathbf{E}\left\{\exp\left[\sum_{\alpha=1}^{n} te^{ij\theta_\alpha} + \sum_{\alpha=1}^{n} te^{-ij\theta_\alpha}\right]\right\} = \mathbf{E}\left\{\prod_{\alpha=1}^{n} g(\theta_\alpha)\right\}, \end{aligned} \tag{2.21}$$

where

$$g(\theta) = \exp\left[te^{ij\theta} + te^{-ij\theta}\right], \tag{2.22}$$

and in writing this formula we took into account the spectral theorem for unitary matrices, according to which $\mathrm{Tr}U_n^j = \sum_{\alpha=1}^n e^{ij\theta_\alpha}$.

Combining now (2.15)–(2.16), (2.21)–(2.22), and the Gram theorem (see e.g. [20]), we obtain that

$$F(t) = \det A_n, \tag{2.23}$$

where the entries A_{m_1,m_2} of the $n \times n$ matrix A_n are

$$A_{m_1,m_2} = \frac{1}{2\pi}\int_0^{2\pi} \exp\left[te^{ij\theta}\right]\exp\left[te^{-ij\theta}\right]e^{i(m_1-m_2)\theta}d\theta,\ 1 \le m_1, m_2 \le n. \tag{2.24}$$

Expanding the exponentials $\exp\left[te^{ij\theta}\right]$ and $\exp\left[-te^{ij\theta}\right]$ and taking into account that the difference $m_1 - m_2$ in the integrand of (2.24) varies between $-(n-1)$ and $(n-1)$ (see (2.15)–(2.16)), we find easily that if $j > n$, then $A_{m_1,m_2} = \delta_{m_1,m_2}f(t)$, where

$$f(t) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left[te^{ij\theta} + te^{-ij\theta}\right]d\theta = \sum_{p=0}^{\infty}\frac{t^{2p}}{(p!)^2} \tag{2.25}$$

is $I_0(2t)$, and I_0 is the modified Bessel function. This and (2.23) implies that

$$F(t) = (f(t))^n, \tag{2.26}$$

and in view of (2.20) we obtain (1.7) for $j > n$ just calculating the second derivative of (2.26) at $t = 0$. □

Remark 1. Formulas (2.25)–(2.26) allow us to find also other "binary" moments for $j > n$. For example

$$\mathbf{E}\left\{\left(\mathrm{Tr}U_n^j\right)^2\left(\overline{\mathrm{Tr}U_n^j}\right)^2\right\} = n(2n-1),\ j > n.$$

Remark 2. Formula (1.7) was obtained in [8] by a similar argument.

Remark 3. If we confine ourselves to the case $j > n$ already in formula (2.21), then we can obtain (2.25)–(2.26) without (2.23)–(2.24). Indeed, consider a more general function

$$\begin{aligned} G(s_1,...,s_n,t_1,...,t_n) &= \mathbf{E}\left\{\exp\left[\sum_{\alpha=1}^{n} s_\alpha e^{ij_\alpha\theta_\alpha} + \sum_{\alpha=1}^{n} t_\alpha e^{-ij_\alpha\theta_\alpha}\right]\right\} \\ &= \mathbf{E}\left\{\prod_{\alpha=1}^{n} \exp\left[s_\alpha e^{ij_\alpha\theta_\alpha}\right]\exp\left[t_\alpha e^{-ij_\alpha\theta_\alpha}\right]\right\}. \end{aligned}$$

coinciding with (2.21) if $s_1 = ... = t_n = t$, $j_1 = ... = j_n = j$. Assume that all integers $j_1,...j_n$ are strictly bigger than n. Expanding every exponential in the second line of this formula and taking into account that the r.h.s. of (2.16) contains the exponentials $e^{il\theta_\alpha}$ with $|l| \le n-1$, we obtain easily that

$$G(s_1,...,s_n,t_1,...,t_n) = \prod_{\alpha=1}^{n} f(\sqrt{s_\alpha t_\alpha}). \tag{2.27}$$

Setting here $s_1 = ... = t_n = t$, we obtain in the r.h.s. the r.h.s. of (2.26). For $j_1 = ... = j_n = j > n$ we obtain the characteristic function of eigenvalues of U_n^j, and then (2.27) implies that eigenvalues of U_n^j are statistically independent if $j > n$. This interesting phenomenon was discussed in [15] in the a general context of compact Lee groups.

One more manifestation of the phenomenon is the closed form of the generating function of the family of moments

$$\mathbf{E}\left\{(\mathrm{Tr}U_n^{j_1})^{a_1}...\mathrm{Tr}U_n^{j_p})^{a_p}\overline{(\mathrm{Tr}U_n^{k_1})^{b_1}}...\overline{(\mathrm{Tr}U_n^{k_q})^{b_q}}\right\}$$

for any fixed $j_1,...j_p,k_1,...k_q$ that are all strictly bigger than n, and for $a_1,...a_p,b_1,...b_q$, varying over $\mathbb{N}$. In this case we consider a multi-variable analog of (2.21):

$$F(s_1,...,s_p,t_1,...,t_q) = \sum_{a_1=0}^{\infty}...\sum_{b_q=1}^{\infty}\prod_{\mu=1}^{p}\frac{s_\mu^{a_\mu}}{a_\mu!}(\mathrm{Tr}U_n^{j_\mu})^{a_\mu}\prod_{\nu=1}^{q}\frac{t_\nu^{b_\nu}}{b_\nu!}\overline{(\mathrm{Tr}U_n^{k_\nu})^{b_\nu}},$$

and we obtain (cf (2.26)–(2.25))

$$F(s_1,...s_p,t_1,...,t_q) = [f(s_1,...,s_p,t_1,...,t_q)]^n,$$

where

$$f(s_1,...,s_p,t_1,...,t_q) = \frac{1}{2\pi}\int_0^{2\pi}\exp\left[\sum_{\alpha=1}^{p} s_\alpha e^{ij_\alpha\theta} + \sum_{\beta=1}^{q} t_\beta e^{-ik_\beta\theta}\right]d\theta.$$

Remark 4. Since the proof of the formulas (1.5)–(1.6) for the mixed moments is based on the induction argument, it requires the both, equation (2.12) (or (2.2)) and the explicit form (1.5) (or (1.8)) of the moments for $\kappa \le n$. It is interesting to note in this connection that the explicit form can also be obtained from the equation (2.12) (or (2.2)) as the $n \to \infty$ limit of the moments. Indeed, by using (2.2), whose r.h.s. contains the moments of the order less than κ, we can prove by induction in κ that for any given K all the moments of the order less or equal K are uniformly bounded in n. Hence, the sequence $\{m_K^{(n)}\}_{n\ge K}$ of the vectors of the space $\mathcal{L}_K^{(U)}$ is bounded in n. Besides, it is easy to see that the norms of the operators B and C of (2.10) and (2.11) are bounded by K. Thus, by the compactness argument, the limit m_K of any converging subsequence of the sequence $\{m_K^{(n)}\}_{n\ge K}$ satisfies the equation $m_K = Bm_K$. In view of (2.10) and (2.13), this equation is equivalent to (1.5).

3 Orthogonal and Symplectic Groups

3.1 Orthogonal Groups

In this subsection we prove (1.10) for the groups $O(n)$ and $SO(n)$ and for $\kappa \le n-1$. As in the previous section we will use the following simple

Proposition 3.1. *Let $F : O(n) \to \mathbb{R}$ be a continuously differentiable function. Then, for any $n \times n$ real antisymmetric matrix X we have (cf. (2.1))*

$$\mathbf{E}\left\{F'(O_n) \cdot XO_n\right\} = 0, \tag{3.1}$$

where F' is the derivative of F.

The proposition follows from the fact that the expression $\mathbf{E}\left\{F(e^{tX}O_n)\right\}$ is independent of a real parameter t.

Remark. Since the matrices $e^{tX},\ t \in \mathbb{R}$ belong to $SO(n)$, formula (3.1) is also valid, if we replace $O(n)$ by $SO(n)$ and the unit Haar measure on $O(n)$ by the unit Haar measure on $SO(n)$. This implies that our result, whose derivation below is based on this formula, will be valid for the both groups: $O(n)$ and $SO(n)$.

Lemma 3.2. *Denote a_j the first from the left non-zero component of the multi-index $a = \{a_l\}_{l\ge 1}$. Then we have the following identity for the moments $m_\kappa^{(n)}(a)$ of (1.9):*

$$\begin{aligned} m_\kappa^{(n)}(a) + \frac{1}{n-1}\Bigg((1-\delta_{j,1})\sum_{l=1}^{j-1} m_\kappa^{(n)}(T_lT_{j-l}T_j^{-1}a) + j(a_j-1)m_\kappa^{(n)}(T_j^{-2}T_{2j}a) \\ + \sum_{l\ge j+1} la_l m_\kappa^{(n)}(T_j^{-1}T_l^{-1}T_{j+l}a)\Bigg) \\ = \frac{n}{n-1}\left(\eta_j m_{\kappa-j}^{(n)}(T_j^{-1}a) + j(a_j-1)m_{\kappa-2j}^{(n)}(T_j^{-2}a)\right) \\ + \frac{1}{n-1}\left(2(1-\delta_{j,1})\sum_{l<j/2} m_{\kappa-2l}^{(n)}(T_{j-2l}T_j^{-1}a) + \sum_{l\ge j+1} la_l m_{\kappa-2j}^{(n)}(T_j^{-1}T_{l-j}T_l^{-1}a)\right), \end{aligned} \tag{3.2}$$

where η_j is defined in (1.10).

Proof. We write (1.9) as

$$m_k^{(n)}(a) = \sum_{x=1}^n \mathbf{E}\left\{(O_n^j)_{x,x}\left(\mathrm{Tr}\, O_n^j\right)^{a_j-1}\prod_{l\ge j+1}\left(\mathrm{Tr}\, O_n^l\right)^{a_l}\right\}, \tag{3.3}$$

and use Proposition 3.1 with

$$F(O_n) = (O_n^j)_{x,y}\left(\mathrm{Tr}\, O_n^j\right)^{a_j-1}\prod_{l\ge j+1}\left(\mathrm{Tr}\, O_n^l\right)^{a_l}, \tag{3.4}$$

and with

$$X = Y^{(x,y)} := \{\delta_{px}\delta_{qy} - \delta_{py}\delta_{qx}\}_{p,q=1}^n. \tag{3.5}$$

Taking into account the relations

$$\begin{aligned}
\left((O_n^m)_{x,y}\right)' \cdot Y^{(x,y)} O_n &= \sum_{i=0}^{m-1} \left(O_n^i Y^{(x,y)} O_n^{m-i}\right)_{x,y} \\
&= \sum_{i=0}^{m-1} \left(\left(O_n^i\right)_{x,x} \left(O_n^{m-i}\right)_{y,y} - \left(O_n^i\right)_{x,y} \left(O_n^{m-i}\right)_{x,y}\right) \\
&= \delta_{x,x} \left(O_n^m\right)_{y,y} + \sum_{i=1}^{m-1} \left(O_n^i\right)_{x,x} \left(O_n^{m-i}\right)_{y,y} - \delta_{x,y} \left(O_n^m\right)_{x,y} \\
&- \sum_{i=1}^{m-1} \left(O_n^i\right)_{x,y} \left(\left(O_n^{m-i}\right)^T\right)_{y,x}, \\
\left(\mathrm{Tr}\, O_n^m\right)' \cdot Y^{(x,y)} O_n &= m \mathrm{Tr}\, O_n^m Y^{(x,y)} = m \left(O_n^m\right)_{y,x} - m \left(O_n^m\right)_{x,y} \\
&= m \left(O_n^m\right)_{y,x} - m \left(\left(O_n^m\right)^T\right)_{y,x},
\end{aligned}$$

and also the equality $O_n O_n^T = 1$, where O_n^T is the transposed matrix, we obtain

$$\begin{aligned}
\delta_{xx} \mathbf{E}\left\{(O_n^j)_{y,y} \alpha_-\right\} - \delta_{xy} \mathbf{E}\left\{(O_n^j)_{x,y} \alpha_-\right\} + \sum_{i=1}^{j-1} \mathbf{E}\left\{\left(O_n^i\right)_{x,x} \left(O_n^{j-i}\right)_{y,y} \alpha_-\right\} \qquad (3.6) \\
- \sum_{i=1}^{j-1} \mathbf{E}\left\{\left(O_n^i\right)_{x,y} \left(O_n^{j-i}\right)_{y,x}^T \alpha_-\right\} \\
+ (a_j - 1) j \mathbf{E}\left\{\left(O_n^j\right)_{x,y} \left(\left(O_n^j\right)_{y,x} - \left(O_n^j\right)_{y,x}^T\right) \left(\mathrm{Tr}\, O_n^j\right)^{a_j - 2} \alpha_+\right\} \\
+ \sum_{l \geq j+1} a_l l \mathbf{E}\left\{\left(O_n^j\right)_{x,y} \left(\left(O_n^l\right)_{y,x} - \left(O_n^l\right)_{y,x}^T\right) \alpha(l)\right\} = 0,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_- &= \left(\mathrm{Tr}\, O_n^j\right)^{a_j - 1} \prod_{l \geq j+1} \left(\mathrm{Tr}\, O_n^l\right)^{a_l}, \ \alpha_+ = \prod_{l \geq j+1} \left(\mathrm{Tr}\, O_n^l\right)^{a_l}, \\
\alpha(l) &= \left(\mathrm{Tr}\, O_n^j\right)^{a_j - 1} \left(\mathrm{Tr}\, O_n^{j+1}\right)^{a_{j+1}} \ldots \left(\mathrm{Tr}\, O_n^{l-1}\right)^{a_{l-1}} \left(\mathrm{Tr}\, O_n^l\right)^{a_l - 1} \prod_{m \geq l+1} \left(\mathrm{Tr}\, O_n^m\right)^{a_m}.
\end{aligned}$$

Applying to (3.6) the operation $n^{-1} \sum_{x,y=1}^n$, regrouping terms and using (3.3), we obtain (3.2). □

Theorem 3.3. *Let $m_\kappa^{(n)}(a)$ be the moments (1.9) of traces of powers of orthogonal matrices, belonging to $O(n)$ or $SO(n)$. Then we have formulas (1.10) for all $\kappa \leq n-1$.*

Proof. As in the previous section we consider the vector space $\mathcal{L}_K^{(O)}$ of collections of real numbers, indexed by the multi-index $a = \{a_l\}_{l \geq 1}$ of the set P_K of (2.7), and we call the integer κ of (1.3) the order of a component $v(a)$ of $v \in \mathcal{L}_K^{(O)}$, if the index a of the component satisfies (1.3). We define in $\mathcal{L}_K^{(O)}$ the uniform norm:

$$||v||_O = \max_{a \in P_K} |v(a)|. \qquad (3.7)$$

Furthermore, we define the linear operators A_O, B_O, and C_O as follows. If a_j is the first from the left non-zero component of $a \in P_K$, then

$$\begin{aligned}(A_O v)_\kappa(a) &= (1-\delta_{j,1})\sum_{l=1}^{j-1} v_\kappa(T_l T_{j-l} T_j^{-1} a) \\ &+ j(a_j-1)v_\kappa(T_j^{-2}T_{2j}a) + \sum_{l\geq j+1} l a_l v_\kappa(T_j^{-1}T_l^{-1}T_{j+l}a),\end{aligned} \tag{3.8}$$

$$(B_O v)_\kappa(a) = \eta_j v_{\kappa-j}(T_j^{-1}a) + j(a_j-1)v_{\kappa-2j}(T_j^{-2}a), \tag{3.9}$$

$$\begin{aligned}(C_O v)_\kappa(a) &= 2(1-\delta_{j,1})\sum_{l<j/2} v_{\kappa-2l}(T_{j-2l}T_j^{-1}a) \\ &+ \eta_j v_{\kappa-j}(T_j^{-1}a) + j(a_j-1)v_{\kappa-2j}(T_j^{-2}a) \\ &+ \sum_{l\geq j+1} l a_l v_{\kappa-2j}(T_j^{-1}T_{l-j}T_l^{-1}a).\end{aligned} \tag{3.10}$$

With this notation we can rewrite (3.2) as (cf (2.12))

$$\left(I+(n-1)^{-1}A_O\right) m_K^{(n)} = B_O m_K^{(n)} + (n-1)^{-1} C_O m_K^{(n)}. \tag{3.11}$$

Besides, if we denote $\mu_\kappa(a)$ the r.h.s. of (1.10), then the integration by parts yields the relation (2.13)

$$\mu_\kappa(a) = \eta_l \mu_{\kappa-l}(T_l^{-1}a) + l(a_l-1)\mu_{\kappa-2l}(T_l^{-2}a), \tag{3.12}$$

valid for any $l \geq 1$ and $a_l \geq 1$.

By using (3.8)–(3.12), we can prove an analog of Lemma 2.4 (see Lemma A.1of the Appendix). Thus, the rest of the proof of (1.10) for $\kappa \leq n-1$ coincides with that of the unitary case, taking into account that $m_0^{(n)}(a) = 1$ and $m_1^{(n)}(a) = 0$. The first equality is evident, and the second follows already from (3.2) (it is implied also by the orthogonality of the characters of the groups $O(n)$ and $SO(n)$ to a constant).

Remark. It can also be shown that the explicit form (3.12) of the moments (1.1) can be obtained by passing to the limit $n \to \infty$ in (3.11), analogously to the unitary case.

3.2 Symplectic Group

Here we will prove (1.13). Recall that $Sp(n)$ is the subgroup of $U(n)$ for an even order

$$n = 2\nu,$$

consisting of unitary matrices such that $J = UJU^T$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_\nu.$$

These matrices can be viewed as $\nu \times \nu$ matrices, whose entries are 2×2 blocks. We will index the blocks by pairs of the Latin characters, running each from 1 to ν and the entries of a block by pairs of the Greek characters, assuming values ± 1, so that

$$S_n = \{S_{\alpha x,\beta y}\}_{x,y=1;\alpha,\beta=\pm 1}^{\nu}.$$

In particular we have: $J_{\alpha,x;\beta,y} = -\alpha\delta_{\alpha,-\beta}\delta_{x,y}$. If a $n \times n$ matrix X is such that $S_n = e^{itX}$ belongs to $Sp(n)$ for some real t, then X is Hermitian and its blocks have the form

$$\begin{pmatrix} a_{x,y} & b_{x,y} \\ \overline{b}_{x,y} & -\overline{a}_{x,y} \end{pmatrix},$$

where $a_{x,y}$ and $b_{x,y}$ are complex numbers ($a_{x,y}$ are real for the diagonal blocks $x = y$).

For any matrix of this form we have an analog of Propositions 2.1 and 3.1:

$$\mathbf{E}\left\{F'(S_n) \cdot XS_n\right\} = 0. \tag{3.13}$$

The above form of X implies that it suffices to use (3.13) with the "basis" matrices (cf (2.3) and (3.5))

$$\begin{aligned} X^{(\xi,x;\eta,y)} &= \left(X^{(x,y)} \otimes q^{(\xi,\xi)} - X^{(y,x)} \otimes q^{(-\xi,-\xi)}\right)\delta_{\xi,\eta} \\ &+ \left(X^{(x,y)} + X^{(y,x)}\right) \otimes q^{(\xi,-\xi)}\delta_{\xi,-\eta}, \end{aligned}$$

where

$$X^{(x,y)} = \{\delta_{xp}\delta_{yq}\}_{p,q=1}^{\nu}, \quad q^{(\xi,\eta)} = \{\delta_{\xi\alpha}\delta_{\eta\beta}\}_{\alpha,\beta=\pm},$$

and with (cf (2.4) and (3.4))

$$F(S_n) = \left(S_n^j\right)_{\xi,x;\eta,y}\left(\mathrm{Tr}\, S_n^j\right)^{a_j-1}\prod_{l\geq j+1}\left(\mathrm{Tr}\, S_n^l\right)^{a_l}.$$

Now, by using the scheme of proofs of Lemmas 2.2 and 3.2 we obtain the identity

$$\left(I + (n+1)^{-1}A_S\right)\widehat{m}_K^{(n)} = B_S\widehat{m}_K^{(n)} + (n+1)^{-1}C_S\widehat{m}_K^{(n)}, \tag{3.14}$$

where (cf (2.9)–(2.11) and (3.8)–(3.10))

$$(A_S v)_\kappa(a) = (A_O v)_\kappa(a), \tag{3.15}$$

$$(B_S v)_\kappa(a) = -\eta_j v_{\kappa-j}(T_j^{-1}a) + j(a_j - 1)v_{\kappa-2j}(T_j^{-2}a), \tag{3.16}$$

$$\begin{aligned} (C_S v)_\kappa(a) &= -2(1-\delta_{j,1})\sum_{l<j/2} v_{\kappa-2l}(T_{j-2l}T_j^{-1}a) \\ &+ \eta_j v_{\kappa-j}(T_j^{-1}a) - j(a_j-1)v_{\kappa-2j}(T_j^{-2}a)\big) \\ &+ \sum_{l\geq j+1} l a_l v_{\kappa-2j}(T_j^{-1}T_{l-j}T_l^{-1}a). \end{aligned} \tag{3.17}$$

Besides, if we denote $\hat{\mu}_\kappa(a)$ the r.h.s. of (1.13), then the integration by parts yields the relation (cf (2.13) and (3.12))

$$\hat{\mu}_\kappa(a) = -\eta_l\hat{\mu}_{\kappa-l}(T_l^{-1}a) + l(a_l - 1)\hat{\mu}_{\kappa-2l}(T_l^{-2}a), \tag{3.18}$$

valid for any $l \geq 1$ and $a_l \geq 1$.

By using (3.15)–(3.18), we can prove analogs of Lemmas 2.4 and A.1 of the Appendix. Thus, the rest of the proof of (1.13) is the same as its analogs in the unitary and/or the orthogonal case.

Remark. It can also be shown that the r.h.s. of (1.13) can be obtained by passing to the limit $n \to \infty$ in (3.14), analogously to the unitary and the orthogonal cases.

Appendix

Proof of Lemma 2.4. Let $v \in \mathcal{L}_K^{(U)}$ be a vector of the unit norm, where $\mathcal{L}_K^{(U)}$ is defined by (2.7)–(2.8). Then we have for its components:

$$|v(a;b)| \leq 1,\ \forall a, b \in P_K.$$

Then (2.9) yields

$$\begin{aligned} |(A_O v)_\kappa(a;b)| &\leq \left((1-\delta_{j,1}) \sum_{l=1}^{j-1} |v_\kappa(T_l T_{j-l} T_j^{-1} a; b)| + j(a_j - 1)|v_\kappa(T_j^{-2} T_{2j} a; b)| \right. \\ &+ \left. \sum_{l=j+1}^{k} l a_l |v_\kappa(T_j^{-1} T_l^{-1} T_{j+l} a; b)| \right) \\ &\leq \left((j-1) + j(a_j - 1) + \sum_{l \geq j+1} l a_l \right) \leq \left(\sum_{l \geq j} l a_l - 1 \right) \leq K - 1. \end{aligned}$$

This implies the first assertion of the lemma.

To prove the second assertion we denote $(A_1\mu)_\kappa(a;b)$ and $(C_1\mu)_\kappa(a;b)$ the first terms on the r.h.s of (2.9) and (2.11) respectively with μ as v. We have then, in view of (3.12),

$$\begin{aligned} (C_1\mu)_\kappa(a;b) : &= (1-\delta_{j,1}) \sum_{l=1}^{j-1} l b_l \mu_{\kappa-l}(T_{j-l} T_j^{-1} a; T_l^{-1} b) \\ &= (1-\delta_{j,1}) \sum_{l=1}^{j-1} \mu_\kappa(T_l T_{j-l} T_j^{-1} a; b) = (A_1 \mu_\kappa)(a;b). \end{aligned}$$

Likewise, denote $(A_2\mu)_\kappa(a;b)$ the sum of the second and the third terms on the r.h.s. of (2.9) and $(C_2\mu)_\kappa(a;b)$ the second term on the r.h.s. of (2.11) with μ as v. We obtain from (1.8) and (2.13):

$$\begin{aligned} (C_2\mu)_k(a;b) : &= \sum_{l=j+1}^{k} l b_l \mu_{\kappa-j}(T_j^{-1} a; T_{l-j} T_l^{-1} b) \\ &= \sum_{l=1}^{k-j} \mu_{\kappa+l}(T_j^{-1} T_{j+l} a; T_l b) = \sum_{l=1}^{k-j} l(b_l+1) \mu_\kappa(T_l^{-1} T_j^{-1} T_{j+l} a; b). \end{aligned}$$

In addition, formula (1.8), its "diagonality" in (a,b) in particular, implies

$$l(b_l+1)\mu_\kappa(T_l^{-1} T_j^{-1} T_{j+l} a; b) = \begin{cases} 0, & l < j, \\ j(a_j - 1)\mu_\kappa(T_j^{-2} T_{2j} a; b), & l = j, \\ l a_l \mu_\kappa(T_l^{-1} T_j^{-1} T_{j+l} a; b), & l \leq j+1. \end{cases}$$

The last formulas and the expression

$$(A_2\mu)_\kappa(a;b) := j(a_j - 1)\mu_\kappa(T_j^{-2} T_{2j} a; b) + \sum_{l=j+1}^{k} l a_l \mu_\kappa(T_j^{-1} T_l^{-1} T_{j+l} a; b)$$

yield evidently the equality $(A_2\mu)_\kappa(a;b) = (C_2\mu)_\kappa(a;b)$.

The equality $(B\mu)_\kappa(a;b) = \mu_\kappa(a;b)$ is, in fact, (2.13). $\square$

Lemma A.1. *Let A_O, B_O, and C_O be the operators (3.8)–(3.10), acting in the space $\mathcal{L}_K^{(O)}$, defined by (3.7). Then*

(i) $||A_O|| \leq (K-1)$;

(ii) *if μ_K is the vector of $\mathcal{L}_K^{(O)}$, whose components are given by the l.h.s. of (1.10), then*

$$B_O\mu_K = \mu_K,\ A_O\mu_K = C_O\mu_K.$$

Proof. The first assertion of the lemma can be proved by using the same argument as that in the proof of Lemma 2.4.

The proof of the second assertion of the lemma is also similar to the respective proof in Lemma 2.4, being based now on the explicit form (3.8)–(3.10) of the operators A_O, B_O, and C_O of (3.8)–(3.10), and on (1.10) and (3.12), in particular on the relations:

$$\mu_\kappa(a) = \left\{ \begin{array}{cc} \eta_l\mu_{\kappa-l}(T_l^{-1}a), & a_l = 1, \\ (l+\eta_l)\mu_{\kappa-2l}(T_l^{-2}a), & a_l = 2. \end{array} \right. \tag{A.1}$$

Denote $(A_1\mu)_\kappa(a)$ the sum of the two first terms on the r.h.s of (3.8) with μ as v, and $(C_1\mu)_\kappa(a)$ the analogous sum on the r.h.s of (3.10) with μ as v. We have then in view of (A.1):

$$\begin{aligned}(A_1\mu_\kappa)(a) &= \left(2(1-\delta_{j,1})\sum_{l<j/2}\eta_l\eta_{j-l} + \eta_j \left\{ \begin{array}{cc} 0 & j \text{ is odd,} \\ j/2+\eta_{j/2} & j \text{ is even;} \end{array}\right.\right)\mu_{\kappa-j}(T_j^{-1}a) \\ &= \mu_{\kappa-j}(T_j^{-1}a)\left\{ \begin{array}{cc} 0, & j \text{ is odd,} \\ j-1, & j \text{ is even;} \end{array}\right.\end{aligned}$$

and

$$\begin{aligned}(C_1\mu)_\kappa(a) &= 2(1-\delta_{j,1})\sum_{l<j/2}\eta_{j-2l}\mu_{\kappa-j}(T_j^{-1}a) + \eta_j \\ &= \mu_{\kappa-j}(T_j^{-1}a) \times \left\{ \begin{array}{cc} 0, & j \text{ is odd,} \\ j-1, & j \text{ is even.} \end{array}\right.\end{aligned}$$

Thus $(A_1\mu)_\kappa(a) = (C_1\mu)_\kappa(a)$.

Denote now $(A_2\mu)_\kappa(a)$ the sum of the second and of the third term on the r.h.s. of (3.8) μ as v, and $(C_2\mu)_\kappa(a)$ the analogous terms in (3.10). Let k be the right endpoint of the support of the multi-index a. There are several cases, depending on the relative values of k and j of the endpoints of the support of a. We analyze, for instance, the case, where $k \geq 3j$ (other cases can be treated analogously). By using again (1.10) or (3.12), we obtain after some algebra

$$\begin{aligned}(A_2\mu)_\kappa(a) &= (A'\mu)_\kappa(a) + (D\mu)_\kappa(a), \\ (C_2\mu)_\kappa(a) &= (C'\mu)_\kappa(a) + (D\mu)_\kappa(a),\end{aligned}$$

where

$$
\begin{aligned}
(D\mu)_\kappa(a) &= \mu_{\kappa+2j}(T_{2j}a) + \sum_{l=j+1}^{k-j} \mu_{\kappa+2l}(T_j^{-1}T_lT_{l+j}a) - \sum_{l=j}^{k-2j} \mu_{\kappa-l}(T_j^{-1}T_{l+j}a),\\
(A'\mu)_\kappa(a) &= -\sum_{l=k-j+1}^{k} \eta_l\eta_{j+l}\mu_{\kappa-j}(T_j^{-1}a), \quad (C'\mu)_\kappa(a) = -\sum_{l=1}^{j} \eta_l\eta_{j+l}\mu_{\kappa-j}(T_j^{-1}a).
\end{aligned}
$$

Now it is easy to find that the sums in $(A'\mu_K)_\kappa(a)$ and $(C'\mu_K)_\kappa(a)$ coincide (being equal zero if j is odd and $j/2$ if j is even). □

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