THE ZEROS OF RANDOM POLYNOMIALS CLUSTER UNIFORMLY NEAR THE UNIT CIRCLE

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Abstract. Given a sequence of random polynomials, we show that, under some very general conditions, the roots tend to cluster near the unit circle, and their angles are uniformly distributed. In particular, we do not assume independence or equidistribution of the coefficients of the polynomial. We apply this result to various problems in both random and deterministic sequences of polynomials, including some problems in random matrix theory.

1. Introduction

We are interested in the asymptotics of the zeros of the random polynomial

$$P_N(Z) = \sum_{k=0}^{N} a_{N,k} Z^k$$

as $N \to \infty$. We will denote the zeros as $z_1, \dots, z_N$ rather than $z_1^{(N)}, \dots, z_N^{(N)}$ for simplicity.

Let

$$\nu_N(\rho) := \# \left\{ z_k \ : \ 1-\rho \le |z_k| \le \frac{1}{1-\rho} \right\}$$

denote the number of zeros of $P_N(Z)$ lying in the annulus bounded by $1-\rho$ and $\frac{1}{1-\rho}$, where $0 \le \rho \le 1$, and let

$$\nu_N(\theta, \phi) := \# \left\{ z_k \ : \ \theta \le \arg(z_k) < \phi \right\}$$

denote the number of zeros of $P_N(Z)$ whose argument lies between θ and ϕ, where $0 \le \theta < \phi \le 2\pi$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which the array $(a_{N,k})_{\substack{N \ge 1 \\ 0 \le k \le N}}$ is defined. The aim of this paper is to show that under some very general conditions on the distribution of the coefficients $a_{N,k}$, we have that

$$\lim_{N\to\infty} \frac{1}{N} \nu_N(\rho) = 1 \qquad (1)$$

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and

$$\lim_{N\to\infty} \frac{1}{N}\nu_N(\theta,\phi) = \frac{\phi-\theta}{2\pi}$$

either almost surely or in the p^{th} mean, according to the hypotheses we make. We say that the zeros cluster near the unit circle if (1) remains true when $\rho \to 0$ as $N \to \infty$. In many examples, a natural rescaling turns out to be $\rho = \alpha(N)/N$ (so clustering requires $\alpha(N) = o(N)$).

Almost all of our results will follow from the following:

Theorem 1. *Let (a_k) be a sequence of complex numbers which satisfy $a_0 \neq 0$ and $a_N \neq 0$. Denote the zeros of the polynomial*

$$P_N(Z) = \sum_{k=0}^{N} a_k Z^k$$

by z_i (for i from 1 to N), and for $0 \leq \rho \leq 1$ let

$$\nu_N(\rho) := \#\{z_k\ ,\ 1-\rho \leq |z_k| \leq 1/(1-\rho)\}$$

and for $0 \leq \theta < \phi < 2\pi$ let

$$\nu_N(\theta,\phi) := \#\{z_k,\ \theta \leq \arg(z_k) < \phi\}$$

Then, for $0 \leq \alpha(N) \leq N$

$$\left(1 - \frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)\right) \leq \frac{2}{\alpha(N)}\left(\log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|\right)$$

and there exists a constant C such that

$$\left|\frac{1}{N}\nu_N(\theta,\phi) - \frac{\phi-\theta}{2\pi}\right|^2 \leq \frac{C}{N}\left(\log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|\right)$$

Though the proof of this proposition is very simple, using only Jensen's formula and a result of Erdős and Turan [7], powerful results follow.

Note that the same function

$$F_N := \log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|$$

controls both the clustering of the zeros to the unit circle, and the uniformity in the distribution of their arguments.

Note further that this result holds for any $a_0,\dots,a_N$ subject to $a_0 a_N \neq 0$, and thus has consequences for non-random polynomials.

It is clear that if there exists a function $\alpha(N) = o(N)$ such that

$$F_N = o(\alpha(N)) \quad a.s. \tag{2}$$

then the zeros of the random polynomial

$$P(Z) = \sum_{n=0}^{N} a_n Z^n$$

satisfy

$$\lim_{N\to\infty} \frac{1}{N} \nu_N \left(\frac{\alpha(N)}{N} \right) = 1, \quad a.s.$$

and,

$$\lim_{N\to\infty} \frac{1}{N} \nu_N (\theta, \phi) = \frac{\phi - \theta}{2\pi}, \quad a.s.$$

(that is, the zeros cluster near the unit circle, and their arguments are uniformly distributed).

The bulk of this paper is concerned with finding conditions on the coefficients such that we may conclude that either $\mathbb{E}[F_N] = o(N)$ or that there exists a deterministic function $0 < \alpha(N) < N$ such that $F_N = o(\alpha(N))$ *a.s.* For example, in Theorem 8 we show that there exists an $\alpha(N)$ such that (2) holds if the a_k satisfy the following three conditions:

- There exists an $s > 0$ such that for all k, $\mu_k := \mathbb{E}\left[|a_k|^s\right] < \infty$.
- Furthermore, $\limsup_{k\to\infty} (\mu_k)^{1/k} = 1$.
- For some $0 < \delta \leq 1$ there exists $t > 0$ and a $q > 0$, such that for all N
$$\mathbb{E}\left[\frac{1}{|a_N|^t} \mathbb{1}_{\{|a_N| \leq \delta\}} \right] = O\left(N^q\right).$$

This can be interpreted as a generalization of a theorem of Shmerling and Hochberg [18] by removing the following requirements on the coefficients in the random polynomial: they are independent; they have a finite second moment; they have density functions.

Our results also enable us to deal with the general case of sequences of random polynomials (i.e. the coefficients of the polynomials are allowed to change with the degree). For example, consider the sequence of polynomials

$$P_N(Z) = \sum_{k=0}^{N} a_{N,k} Z^k$$

If there exists a positive function $\alpha(N) = o(N)$ such that

$$\lim_{N\to\infty} \frac{1}{\alpha(N)} \mathbb{E}\left[\log\left(\sum_{k=0}^{N} |a_{N,k}| \right) - \frac{1}{2} \log|a_{N,0}| - \frac{1}{2} \log|a_{N,N}| \right] = 0$$

then by Theorem 1 the zeros of this sequence of random polynomials cluster uniformly around the unit circle.

This more general case is dealt with in view of applications to characteristic polynomials of random unitary matrices. We recover for example the result of clustering of the zeros of the derivative of the characteristic polynomial of a random unitary matrix, as found in the work of Mezzadri, [16]. We shall return to the study of random matrix polynomials in a later paper.

The structure of this paper is as follows: In section 2 we review some of the relevant history of zeros of random polynomials, and describe which prior results which can be obtained as corollaries of our work. In section 3 we describe the basic estimates we need, and then in sections 4 and 5 we prove the main result, and use it to deduce clustering of zeros in many examples.

In Section 6 we study the random empirical measure

$$\mu_N = \frac{1}{N}\sum_{k=1}^{N}\delta_{z_k},$$

associated with the roots of random polynomials. We show that they converge in mean or almost-surely weakly to the Haar measure on the unit circle (i.e. the uniform measure on the unit circle).

2. Review of earlier work on random polynomials

Mark Kac [12] gave an explicit formula for the expectation of the number, $\nu_N(B)$, of zeros of

$$\sum_{n=0}^{N} a_n Z^n$$

in any Borel subset B of the real line, in the case where the variables $(a_n)_{n\geq 0}$ are real independent standard Gaussian. His results were expanded in various directions (for example, to the non-Gaussian case), but most of the work has focused on the real zeros (see [8], [6] and [17] for more details and references).

Almost fifty years later, L.A. Shepp and J. Vanderbei [17] extended the results of Kac to the case where B is any Borel subset of the complex plane. They noticed that as the degree of the polynomials N gets large, the zeros tend to cluster near the unit circle and are approximately uniformly distributed around the circle. I. Ibragimov and O. Zeitouni [10], using different techniques, have obtained similar results for i.i.d. coefficients in the domain of attraction of the stable law. They again observed the clustering of the zeros near the unit circle.

However, this result about the clustering of the complex roots of random polynomials has already been observed by Šparo and Šur [19] in a general setting. They considered i.i.d. complex coefficients $(a_n)_{n\geq 0}$ such that

$$\mathbb{P}\left[a_k = 0\right] \neq 1, \quad k = 0, 1, \ldots, N$$

and,

$$\mathbb{E}\left[\log^+ |a_k|\right] < \infty, \quad k = 0, 1, \ldots, N$$

where $\log^+|a_k| = \max\{0, \log|a_k|\}$. They proved that

$$\lim_{N\to\infty} \frac{1}{N}\nu_N(\rho) = 1$$

and

$$\lim_{N\to\infty} \frac{1}{N}\nu_N(\theta,\phi) = \frac{\phi-\theta}{2\pi}$$

where the convergence holds in probability. Arnold [1] improved this result and proved that the convergence holds in fact almost surely and in the p^{th} mean if the moduli of a_k are equidistributed and $\mathbb{E}[|\log|a_k||] < \infty$ for $k = 0, 1, \ldots, N$. Recently, Shmerling and Hochberg [18] have shown that the condition on equidistribution can be dropped if $(a_n)_{n\geq 0}$ is a sequence of independent variables which have continuous densities f_n which are uniformly bounded in some neighborhood of the origin with finite means μ_n and standard deviations σ_n that satisfy the condition

$$\max\left\{\limsup_{n\to\infty} \sqrt[n]{|\mu_n|}\ ,\ \limsup_{n\to\infty} \sqrt[n]{|\sigma_n|}\right\} = 1,$$
$$\mathbb{P}[a_0 = 0] = 0$$

Finally, let us mention that the distribution of roots of random polynomials has also been investigated in physics, which among others, appear naturally in the context of quantum chaotic dynamics. Bogomolny *et. al.* [5] studied self-inversive polynomials, with $\overline{a_k} = a_{N-k}$, and a_k ($k = 0, 1, \ldots, \frac{N-1}{2}$) complex independent Gaussian variables with mean zero, and they proved that not only do the zeros cluster near the unit circle, but a finite proportion of them lie on it. This case is very interesting since it shows that at least in some special cases, we can drop the independence and equidistribution assumptions on the coefficients.

Theorems 6 and 7 of this paper includes and extends the above mentioned results on uniform clustering of zeros.

3. Basic estimates

In the first part of this paper, we will apply Jensen's formula repeatedly, so we recall it here (see [14], for example).

Lemma 2. *Let f be a holomorphic function in a neighborhood of the closed disc $\overline{D}_r = \{z \in \mathbb{C}\ ,\ |z| \leq r\}$, such that $f(0) \neq 0$. Let z_i be the zeros of f in $D_r = \{z \in \mathbb{C}\ ,\ |z| < r\}$, repeated according to their multiplicities, then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{\mathrm{i}\varphi}\right)\right| \,\mathrm{d}\varphi = \log|f(0)| + \sum_{z_i} \log\frac{r}{|z_i|} \tag{3}$$

We also use Jensen's inequality repeatedly, which states that if X is a positive random variable, such that $\mathbb{E}[\log X]$ exists, then

$$\mathbb{E}[\log X] \leq \log \mathbb{E}[X]$$

Before considering random polynomials, we will first state some fundamental results about zeros of deterministic polynomials.

For $N \geq 1$, let $(a_k)_{0 \leq k \leq N}$ be a sequence of complex numbers satisfying $a_0 a_N \neq 0$. From this sequence construct the polynomial

$$P_N(Z) = \sum_{k=0}^{N} a_k Z^k,$$

and denote its zeros by z_i (where i ranges from 1 to N). For $0 \leq \rho \leq 1$, we are interested in estimates for

$$\begin{aligned} \widetilde{\nu}_N(1-\rho) &= \# \{ z_j \ , \ |z_j| < 1-\rho \} \\ \overline{\nu}_N(1/(1-\rho)) &= \# \left\{ z_j \ , \ |z_j| > \frac{1}{1-\rho} \right\} \\ \nu_N(\rho) &= \# \left\{ z_j \ , \ 1-\rho \leq |z_j| \leq \frac{1}{1-\rho} \right\} \end{aligned}$$

which counts the number of zeros of the polynomial $P_N(Z)$ which lie respectively inside the open disc of radius $1-\rho$, outside the closed disc of radius $1/(1-\rho)$, and inside the closed annulus bounded by circles of radius $1-\rho$ and $1/(1-\rho)$.

Lemma 3. *For $N \geq 1$, let $(a_k)_{0 \leq k \leq N}$ be an sequence of complex numbers which satisfy $a_0 a_N \neq 0$. Then, for $0 < \rho < 1$*

$$\frac{1}{N} \widetilde{\nu}_N(1-\rho) \leq \frac{1}{N\rho} \left(\log \left(\sum_{k=0}^{N} |a_k| \right) - \log |a_0| \right), \tag{4}$$

$$\frac{1}{N} \overline{\nu}_N(1/(1-\rho)) \leq \frac{1}{N\rho} \left(\log \left(\sum_{k=0}^{n} |a_k| \right) - \log |a_N| \right) \tag{5}$$

and

$$\left(1 - \frac{1}{N} \nu_N(\rho) \right) \leq \frac{2}{N\rho} \left(\log \left(\sum_{k=0}^{N} |a_k| \right) - \frac{1}{2} \log |a_0| - \frac{1}{2} \log |a_N| \right) \tag{6}$$

Proof. An application of Jensen's formula, (3), with $r = 1$ yields

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| P_N(e^{\mathrm{i}\varphi}) \right| \, \mathrm{d}\varphi - \log |P_N(0)| = \sum_{|z_i|<1} \log \frac{1}{|z_i|}$$

where the sum on the right hand side is on zeros lying inside the open unit disk. We have the following minorization for this sum:

$$\begin{aligned} \sum_{|z_i|<1} \log \frac{1}{|z_i|} &\geq \sum_{|z_i|<1-\rho} \log \frac{1}{|z_i|} \\ &\geq \rho \widetilde{\nu}_N(1-\rho) \end{aligned}$$

since if $0 \leq \rho \leq 1$, then for all $|z_i| \leq 1-\rho$, $\log(1/|z_i|) \geq \rho$, and by definition there are $\widetilde{\nu}_N(1-\rho)$ such terms in the sum.

We also have the following trivial upper bound

$$\max_{\varphi\in[0,2\pi]} |P_N(e^{\mathrm{i}\varphi})| \leq \sum_{k=0}^{N} |a_k|,$$

and so

$$\begin{aligned} \rho\widetilde{\nu}_N\left(1-\rho\right) &\leq \frac{1}{2\pi}\int_0^{2\pi} \log|P_N(e^{\mathrm{i}\varphi})|\,\mathrm{d}\varphi - \log|a_0| \\ &\leq \log\left(\sum_{k=0}^{N}|a_k|\right) - \log|a_0| \end{aligned}$$

which gives equation (4).

To estimate the number of zeros lying outside the closed disc of radius $(1-\rho)^{-1}$, note that if z_0 is a zero of the polynomial $P_N\left(Z\right) = \sum_{k=0}^{N} a_k Z^k$, then $1/z_0$ is a zero of the polynomial $Q_N\left(Z\right) := Z^N P_N\left(\frac{1}{Z}\right) = a_N + a_{N-1}Z + \ldots + a_0 Z^N$. Therefore, the number of zeros of $P_N(Z)$ outside the closed disc of radius $1/(1-\rho)$ equals the number of zeros of $Q_N(Z)$ inside the open disc of radius $1-\rho$. Therefore, from (4) we get

$$\frac{1}{N}\overline{\nu}_N(1/(1-\rho)) \leq \frac{1}{N\rho}\left(\log\left(\sum_{k=0}^{N}|a_k|\right) - \log|a_N|\right)$$

which gives equation (5).

Since

$$N - \nu_N(\rho) = \widetilde{\nu}_N(1-\rho) + \overline{\nu}_N(1/(1-\rho))$$

we immediately get (6). □

To deal with the asymptotic distribution of the arguments of the zeros of random polynomials (that is, to show the angles are uniformly distributed) we use a result of Erdős and Turan [7]:

Lemma 4 (Erdős-Turan)**.** *Let $(a_k)_{0\leq k\leq N}$ be a sequence of complex numbers such that $a_0 a_N \neq 0$. For $0 \leq \theta < \phi \leq 2\pi$, let $\nu_N\left(\theta,\phi\right)$ denote the number of zeros of $P\left(Z\right) = \sum_{k=0}^{N} a_k Z^k$ which belong to the sector $\theta \leq \arg z < \phi$. Then*

$$\left|\frac{1}{N}\nu_N\left(\theta,\phi\right) - \frac{\phi-\theta}{2\pi}\right|^2 \leq \frac{C}{N}\left[\log\sum_{k=0}^{N}|a_k| - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|\right]$$

for some constant C

Remark. By considering the example $P_N(Z) = (z-1)^N$, we observe that $C > 1/\log 2$.

Combining Lemmas 3 and 4 yields the following proposition:

Proposition 5. *Let $(a_k)_{0\leq k\leq N}$ be a sequence of complex numbers which satisfy $a_0 a_N \neq 0$. Denote the zeros of the polynomial*

$$P_N(Z) = \sum_{k=0}^{N} a_k Z^k$$

by z_i (for i from 1 to N), and for $0 \leq \rho \leq 1$ let

$$\nu_N(\rho) := \#\{z_i \;:\; 1-\rho \leq |z_i| \leq 1/(1-\rho)\}$$

and for $0 \leq \theta < \phi < 2\pi$ let

$$\nu_N(\theta,\phi) := \#\{z_i \;:\; \theta \leq \arg(z_i) < \phi\}$$

Then, for $0 \leq \alpha(N) \leq N$

$$\left(1 - \frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)\right) \leq \frac{2}{\alpha(N)}\left(\log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|\right) \tag{7}$$

and there exists a constant C such that

$$\left|\frac{1}{N}\nu_N(\theta,\phi) - \frac{\phi-\theta}{2\pi}\right|^2 \leq \frac{C}{N}\left(\log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|\right) \tag{8}$$

Remark. Note again that the same function

$$\log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|$$

controls both the clustering of the zeros near the unit circle, and the uniform distribution of the arguments of the zeros.

Remark. Note that for any complex coefficients a_k,

$$\log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N| \geq \log 2$$

Therefore, this method cannot detect when all the zeros are on the unit circle.

Remark. Note that if $a_k \mapsto \lambda a_k$ for some $\lambda \neq 0$, then the zeros of $P_N(Z)$ are unchanged, and

$$\log\left(\sum_{k=0}^{N}|\lambda a_k|\right) - \frac{1}{2}\log|\lambda a_0| - \frac{1}{2}\log|\lambda a_N| = \log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|$$

so, in some sense, this is a natural function to control the location of the zeros.

We are interested in the zeros of sequences of random polynomials. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which the array of random variables, $(a_{N,k})_{\substack{N\geq 1 \\ 0\leq k\leq N}}$, is defined. From this sequence we construct the random polynomial

$$P_N(Z) = \sum_{k=0}^{N} a_{N,k} Z^k.$$

We require no independence restriction on our random variables. We only assume that

$$\mathbb{P}[a_{N,0} = 0] = 0 \tag{9}$$

and

$$\mathbb{P}[a_{N,N} = 0] = 0, \tag{10}$$

for all N.

We recap the various types of convergence which we will see in this project: we say that X_N converges in probability to X if for all $\epsilon > 0$, $\mathbb{P}\{|X_N - X| > \epsilon\} \to 0$ as $N \to \infty$; we say that X_N converges in the p^{th} mean to X if $\mathbb{E}[|X_N - X|^p] \to 0$ as $N \to \infty$; we say that X_N converges almost surely to X if for all $\omega \in \Omega \setminus E$ (where E, called the exceptional set, is a measure zero subset of the measurable sets Ω), $\lim_{N\to\infty} X_N(\omega) = X(\omega)$. The fact that almost sure convergence for bounded variables implies the convergence in the p^{th} mean is a classical result in probability theory (see, for example, [11]). The fact that convergence in the mean square implies convergence in probability follows from Chebyshev's inequality.

4. Uniform clustering results for roots of random polynomials

Now we give several results for the uniform clustering of the zeros of random polynomials.

Theorem 6 (Main theorem)**.** *For $N \geq 1$, let $(a_{N,k})_{0\leq k\leq N}$ be an array of random complex numbers such that $\mathbb{P}[a_{N,0} = 0] = 0$ and $\mathbb{P}[a_{N,N} = 0] = 0$ for all N. Denote the zeros of the polynomial*

$$P_N(Z) = \sum_{k=0}^{N} a_{N,k} Z^k$$

by z_i, and for $0 \leq \rho \leq 1$, let

$$\nu_N(\rho) := \#\{z_i \ : \ 1-\rho \leq |z_i| \leq 1/(1-\rho)\}$$

and for $0 \leq \theta < \phi < 2\pi$, let

$$\nu_N(\theta, \phi) := \#\{z_i \ : \ \theta \leq \arg(z_i) < \phi\}$$

Let

$$F_N := \log\left(\sum_{k=0}^{N} |a_{N,k}|\right) - \frac{1}{2}\log|a_{N,0}| - \frac{1}{2}\log|a_{N,N}| \tag{11}$$

If

$$\mathbb{E}\left[F_N\right] = o(N) \qquad \text{as } N \to \infty \tag{12}$$

then there exists a positive function $\alpha(N)$ *satisfying* $\alpha(N) = o(N)$ *such that*

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\nu_N\left(\frac{\alpha\left(N\right)}{N}\right)\right] = 1$$

and

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\nu_N\left(\theta,\phi\right)\right] = \frac{\phi-\theta}{2\pi}$$

In fact the convergence also holds in probability and in the p^{th} *mean, for all positive* p*.*

Furthermore, if there exists a (deterministic) positive function $\alpha(N)$ *satisfying* $\alpha(N) \le N$ *for all* N*, such that*

$$F_N = o(\alpha(N)) \quad \text{almost surely} \tag{13}$$

then

$$\lim_{N\to\infty} \frac{1}{N}\nu_N\left(\frac{\alpha\left(N\right)}{N}\right) = 1, \quad a.s.$$

and

$$\lim_{N\to\infty} \frac{1}{N}\nu_N\left(\theta,\phi\right) = \frac{\phi-\theta}{2\pi}, \quad a.s.$$

Remark. It is clear that the only way for a sequence of polynomials not to have zeros which cluster uniformly to the unit circle is if there exists a constant $c > 0$ such that $\mathbb{E}\left[F_N\right] > cN$ for an infinite number of N.

Proof. The convergence in mean for $\nu_N(\alpha(N)/N)$ is a consequence of (7). We have

$$1 - \mathbb{E}\left[\frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)\right] \le \frac{2}{\alpha(N)}\,\mathbb{E}\left[F_N\right]$$

Therefore we see that the result follows for any positive function $\alpha(N)$ satisfying $\alpha(N) \le N$ for all N such that $\mathbb{E}\left[F_N\right]/\alpha(N) \to 0$, and such a function exists by assumption (12).

Similarly from (8) and (12) we have that

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{N}\nu_N\left(\theta,\phi\right) - \frac{\phi-\theta}{2\pi}\right|^2\right] &\le \frac{C}{N}\,\mathbb{E}\left[F_N\right] \\ &= o(1)\end{aligned}$$

Note that the mean square convergence implies convergence in the mean, as in the theorem, and also convergence in probability. Note further, that since the random variables are uniformly bounded ($0 \le \frac{1}{N}\nu_N\left(\theta,\phi\right) \le 1$), mean convergence implies convergence in the p^{th} mean for all positive p.

In the same way, the almost sure convergence of $\frac{1}{N}\nu_N(\alpha(N)/N)$ and $\frac{1}{N}\nu_N(\theta,\phi)$ follows immediately from (7) and (8), using (13). □

We shall now give some examples for which the hypotheses of Theorem 6 are satisfied.

Corollary 6.1. *Let* $(a_{N,k})$ *be an array of random complex numbers which satisfy (9) and (10). Assume that* $\mathbb{E}[\log|a_{N,0}|] = o(N)$*, and* $\mathbb{E}[\log|a_{N,N}|] = o(N)$*, and that there exists a fixed* $s > 0$ *and a sequence* ε_N *tending to zero such that*

$$\sup_{0 \le k \le N} \mathbb{E}\left[|a_{N,k}|^s\right] \le \exp(\varepsilon_N N)$$

Then, there exists an $\alpha(N) = o(N)$ *such that* F_N*, defined in* (11)*, satisfies* $\mathbb{E}[F_N] = o(\alpha(N))$*, and so*

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)\right] = 1$$

and

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\frac{1}{N}\nu_N(\theta,\phi) - \frac{\phi-\theta}{2\pi}\right|\right] = 0$$

Proof. It is a consequence of Theorem 6 and the following chain of concavity inequalities:

$$\begin{aligned}
\mathbb{E}\left[\log\left(\sum_{k=0}^{N}|a_{N,k}|\right)\right] &= \frac{1}{s}\,\mathbb{E}\left[\log\left(\sum_{k=0}^{N}|a_{N,k}|\right)^s\right] \\
&\le \frac{1}{s}\,\mathbb{E}\left[\log\left(\sum_{k=0}^{N}|a_{N,k}|^s\right)\right] \\
&\le \frac{1}{s}\log\left(\sum_{k=0}^{N}\mathbb{E}\left[|a_{N,k}|^s\right]\right) \\
&\le \frac{1}{s}\log\left((N+1)\exp(\varepsilon_N N)\right) \\
&= \frac{1}{s}\left(\log(N+1) + N\varepsilon_N\right) = o(N)
\end{aligned}$$

since we assume $\varepsilon_N \to 0$ as $N \to \infty$. Therefore F_N, defined in (11), satisfies $F_N = o(N)$, and the result follows from Theorem 6. □

Remark. The Corollary shows that under some very general conditions (just some conditions on the size of the expected values of the modulus of the coefficients), without assuming any independence or equidistribution condition, the zeros of random polynomials tend to cluster uniformly near the unit circle. We can also remark that we do not assume that our coefficients must have density functions: they can be discrete-valued random variables.

Example. Let $a_{N,k}$ be a random variables distributed according to the Cauchy distribution with parameter $N(k+1)$. The first moment does not exist but some fractional moments do, and in particular we have for $0 \leq s < 1$

$$\begin{aligned}\mathbb{E}\left[|a_{N,k}|^s\right] &= \frac{N(k+1)}{\pi} \int_{-\infty}^{\infty} \frac{|x|^s}{x^2 + N^2(k+1)^2}\,\mathrm{d}x \\ &= \frac{1}{\pi} N^s (k+1)^s \Gamma(\frac{1}{2} + \frac{s}{2}) \Gamma(\frac{1}{2} - \frac{s}{2})\end{aligned}$$

Moreover,

$$\mathbb{E}\left[\log |a_{N,k}|\right] = \log(N(k+1))$$

Hence we can apply Corollary 6.1 and deduce that the zeros of the sequence of random polynomials with coefficients $(a_{N,k})_{\substack{N \geq 1 \\ 0 \leq k \leq N}}$ where $a_{N,k}$ are chosen from the Cauchy distribution with parameter $N(k+1)$ cluster uniformly around the unit circle.

Example. We can also interpret this result for sequences of deterministic polynomials, since then $\mathbb{E}\left[|a_{N,k}|\right] = |a_{N,k}|$. For example, for every sequence of polynomials with nonzero bounded integer coefficients, we have for all $\rho \in (0,1)$, $\lim_{N\to\infty} \frac{1}{N} \nu_N(\rho) = 1$ and similarly $\lim_{N\to\infty} \frac{1}{N} \nu_N(\theta, \phi) = \frac{\phi - \theta}{2\pi}$. Indeed, one can take $\rho = \alpha(N)/N$ for any sequence $\alpha(N) \leq N$ such that $\log N / \alpha(N) \to 0$.

Corollary 6.2. *Let (ε_N) be a sequence of positive real numbers, which satisfies $\lim_{N\to\infty} \varepsilon_N = 0$. Let $(a_{N,k})$ be an array of complex random variables such that for each N, $\exp(-\varepsilon_N N) \leq |a_{N,k}| \leq \exp(\varepsilon_N N)$ for all k.*

Then there exists a deterministic positive function $\alpha(N) = o(N)$ such that

$$\lim_{N\to\infty} \frac{1}{N} \nu_N \left(\frac{\alpha(N)}{N} \right) = 1, \quad a.s.$$

and

$$\frac{1}{N} \nu_N (\theta, \phi) \to \frac{\phi - \theta}{2\pi}, \quad a.s.$$

The convergence also holds in the p^{th} mean for all positive p.

Proof. With the hypotheses of the corollary, we have that

$$\begin{aligned}F_N &:= \log \left(\sum_{k=0}^{N} |a_{N,k}| \right) - \frac{1}{2} \log |a_{N,0}| - \frac{1}{2} \log |a_{N,N}| \\ &\leq \log((N+1)\exp(\varepsilon_N N)) - \log(\exp(-\varepsilon_N N)) \\ &\leq 2\varepsilon_N N + \log(N+1)\end{aligned}$$

and so for any positive function $\alpha(N)$ satisfying $\alpha(N) \leq N$ and $2\varepsilon_N N + \log N = o(\alpha(N))$ (for example, $\alpha(N) = \sqrt{\varepsilon_N} N + \log^2 N$), the result follows from the second half of Theorem 6. □

Example. Let $(a_{N,k})$, for fixed N, and $0 \leq k \leq N$, be discrete random variables taking values in $\{\pm 1, \ldots, \pm N\}$, not necessarily having the same distribution; then

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N}\nu_N\left(\frac{\log^{1+\gamma}(N)}{N}\right) &= 1, \quad a.s., \quad \forall\gamma > 0 \\ \frac{1}{N}\nu_N(\theta,\phi) &\to \frac{\phi-\theta}{2\pi}, \quad a.s. \end{aligned}$$

As a special case, we have the well known random polynomials of the form $\sum_{k=0}^{N} \mu_k Z^k$, with $\mu_k = \pm 1$, with probabilities p and $(1-p)$. Moreover, we have from the Markov inequality, the following rate for the convergence in probability:

$$\begin{aligned} \mathbb{P}\left[\left(1 - \frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)\right) > \varepsilon\right] &\leq \frac{1}{\varepsilon}\frac{C\log N}{\alpha(N)} \\ \mathbb{P}\left[\left|\frac{1}{N}\nu_N(\theta,\phi) - \frac{\phi-\theta}{2\pi}\right| > \varepsilon\right] &\leq \frac{1}{\varepsilon^2}\frac{C\log N}{N} \end{aligned}$$

where $\varepsilon > 0$.

4.1. **Self inversive polynomials.** The Theorem 6 also gives us an interesting result for self-inversive polynomials. These polynomials are of interest in physics (see [5]) and in random matrix theory (characteristic polynomials of random unitary matrices).

A polynomial $P(Z) = \sum_{k=0}^{N} a_k Z^k$ is said to be self-inversive if

$$\overline{a_N} P(Z) = a_0 Z^N \overline{P}(1/Z)$$

where $\overline{z}$ denotes the complex conjugate of z, and $\overline{P}(Z) = \overline{P(\overline{Z})}$. This implies

$$\overline{a_k} = \frac{\overline{a_0}}{a_N} a_{N-k}$$

for all k. One can see that the zeros of self-inversive polynomials lie either on the unit circle or are symmetric with respect to it, that is, if z is a zero, so is $1/\overline{z}$. So, with the notations of Theorem 3, we just have to check that $\frac{1}{N}\widetilde{\nu}_N\left(1 - \frac{\alpha(N)}{N}\right)$ tends to zero.

Corollary 6.3. *Let* $\left(P_N(Z) = \sum_{k=0}^{N} a_{Nk} Z^k\right)_{N=1}^{\infty}$ *be a sequence of random self inversive polynomials satisfying*

- $\mathbb{P}[a_{N,0} = 0] = 0$ *for all* N
- $\mathbb{E}[\log|a_{N,0}|] = o(N)$
- *There exists a fixed* $s > 0$ *and a positive sequence* ε_N *tending to zero as* $N \to \infty$ *such that for all* N, $\mathbb{E}[|a_{N,k}|^s] \leq \exp(\varepsilon_N N)$ *for all* k

then there exists a function $\alpha(N) \le N$ such that

$$\begin{aligned} \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)\right] &= 1, \\ \lim_{N\to\infty} \mathbb{E}\left[\left|\frac{1}{N}\nu_N(\theta,\phi) - \frac{\phi-\theta}{2\pi}\right|\right] &= 0 \end{aligned}$$

In fact the convergence holds in the p^{th} mean for any positive p.

Proof. It is a consequence of Corollary 6.1. □

Remark. Usually, one is interested in the case where $|a_{N,0}| = 1$; in this case there is only one condition to check: that for all N, $\mathbb{E}\left[|a_{N,k}|^s\right] \le \exp(\varepsilon_N N)$ for all k.

Remark. We can also prove results about almost sure convergence as in the general case.

4.2. **The derivative of the characteristic polynomial.** The characteristic polynomial of a random unitary matrix was introduced by Keating and Snaith [13] as a model to understand statistical properties of the Riemann zeta function, $\zeta(s)$. We can apply the methods developed in this paper to study the location of the zeros of the derivative of the characteristic polynomial, first considered by Mezzadri [16] in order to model the horizontal distribution of the zeros of $\zeta'(s)$. Having a good understanding of the location of the zeros of $\zeta'(s)$ is important, because if there are no zeros to the left of the vertical line $\mathfrak{Re}(s) = 1/2$, then the Riemann Hypothesis would be true.

Denote the characteristic polynomial of an $N \times N$ unitary matrix M by

$$\begin{aligned} \Lambda_M(Z) &= \det(M - ZI) \\ &= \sum_{k=0}^{N} (-1)^k Sc_{N-k}(M) Z^k, \end{aligned}$$

where Sc_j denotes the j^{th} secular coefficient of the matrix M. Since all the zeros of $\Lambda_M(Z)$ lie on the unit circle, it follows that $\Lambda_M(Z)$ is self-inversive. The derivative is given by

$$\Lambda'_M(Z) = \sum_{k=0}^{N-1} (-1)^{k+1}(k+1) Sc_{N-k-1}(M) Z^k.$$

We will use the following fact about secular coefficients averaged over Haar measure, due to Haake *et. al.* [9]:

$$\mathbb{E}\left[Sc_j(M)\overline{Sc_k(M)}\right] = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

and so

$$\mathbb{E}\left[\sum_{k=0}^{N-1}(k+1)^2\,|Sc_{N-1-k}(M)|^2\right] = \frac{1}{6}N(N+1)(2N+1)$$

Furthermore, $Sc_{N-1}(M) = \det M\overline{\operatorname{Tr} M}$, and so

$$\mathbb{E}\left[\log|Sc_{N-1}(M)|\right] = \mathbb{E}\left[\log|\operatorname{Tr} U|\right] \le \frac{1}{2}\log\mathbb{E}\left[|\operatorname{Tr} U|^2\right] = \frac{1}{2}\log 2$$

Therefore, Theorem 6.1 allows us to deduce that if $\alpha(N)$ tends to infinity faster than $\log N$, then

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)\right] = 1$$

By completely different methods, which are special to $\Lambda'_M(Z)$, Mezzadri [16] has previously shown that the zeros cluster (in fact his results give an asymptotic expansion for the rate of clustering).

5. Classical Random Polynomials

Let us now consider the special, but very important, case of the classical random polynomials as mentioned in the first section, that is

$$P_N(Z) = \sum_{k=0}^{N} a_k Z^k \tag{14}$$

These polynomials have been extensively studied (see, for example, [2] or [8] for a complete account). The uniform clustering of the zeros have often been noticed in some special cases of i.i.d. coefficients, as in [10], [12], [17] for example (but these papers are concerned with the density distribution of the zeros as is mentioned in section 2), and it has been proved in more general cases by Arnold [1] in the case of equidistributed coefficients, and by Shmerling and Hochberg [18] in the case of independent and non equidistributed coefficients. We shall now see that we can recover and improve the results in [1] and [18].

The results of the previous section take a simpler form in the special case of random polynomials of the form (14). The conditions (9) and (10) become

$$\mathbb{P}[a_N = 0] = 0,\ \text{ for all } N \ge 0 \tag{15}$$

We will restate Theorem 6 for this classical case.

Theorem 7. *Let $(a_k)_{k\ge 0}$ be a sequence of complex random variables which satisfy (15). Let*

$$F_N := \log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|$$

If

$$\mathbb{E}\left[F_N\right] = o(N) \qquad \text{as } N \to \infty$$

then there exists a positive function $\alpha(N)$ *satisfying* $\alpha(N) = o(N)$ *such that*

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)\right] = 1$$

and

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\nu_N\left(\theta,\phi\right)\right] = \frac{\phi-\theta}{2\pi}$$

In particular, the convergence also holds in probability and in the p^{th} *mean, for all positive* p*, since* $0 \le \frac{1}{N}\nu_N\left(\alpha(N)/N\right) \le 1$ *and* $0 \le \frac{1}{N}\nu_N\left(\theta,\phi\right) \le 1$*.*

Furthermore, if there exists a (deterministic) positive function $\alpha(N)$ *satisfying* $\alpha(N) \le N$ *for all* N*, such that*

$$F_N = o(\alpha(N)) \qquad \text{almost surely}$$

then

$$\lim_{N\to\infty} \frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right) = 1, \qquad a.s.$$

and

$$\lim_{N\to\infty} \frac{1}{N}\nu_N\left(\theta,\phi\right) = \frac{\phi-\theta}{2\pi}, \qquad a.s.$$

Remark. Again, we can observe that the result holds for the special case $\nu_N\left(\rho\right)$, for fixed $\rho \in (0,1)$. One simply takes $\alpha(N) = \rho N$.

Theorem 7 shows that under some very general conditions (assuming neither independence nor equidistribution) we have a uniform clustering of the zeros of random polynomials near the unit circle.

Example. We shall now present two examples to show that our results are not completely sharp. Denote by $\widetilde{\nu}_N(r)$ the number of zeros in the disc of radius r centered at 0. Using classical results about random polynomials with coefficients (a_n) which are i.i.d. standard Gaussian ([8], [17]), it can be shown that as $N \to \infty$

$$\mathbb{E}\left[\frac{1}{N}\widetilde{\nu}_N\left(1-\frac{\alpha(N)}{N}\right)\right] \sim \begin{cases} \frac{1}{2\alpha(N)} & \text{if } \alpha(N) \to \infty \\ \frac{1}{2\alpha(N)} - \frac{1}{\exp(2\alpha(N))-1} & \text{if } \alpha\left(N\right) \to \alpha \neq 0 \\ 1/2 & \text{if } \alpha(N) \to 0 \end{cases}$$

If the zeros are to cluster, then we must have $\mathbb{E}\left[\frac{1}{N}\widetilde{\nu}_N\left(1-\frac{\alpha(N)}{N}\right)\right] \to 0$. Hence in this case we must have $\alpha(N) \to \infty$. However, there exists a constant $c > 0$ such that $\mathbb{E}[F_N] > c\log N$, and we can only deduce clustering from our results when $\alpha(N)/\log N \to \infty$. This is not surprising since our results are presented in great generality, and if one knows specific information about the distribution of the a_k it is plausible that specialized techniques would give more information about clustering.

Our second example concerns polynomials which have all their roots on the unit circle, for example $Z^N - 1$. Since for *any* polynomial, $F_N \geq \log 2$, our results can only deduce clustering when $\alpha(N) \to \infty$, despite the fact that in this case it holds true for any $\alpha(N) \geq 0$.

We will now show some cases where Theorem 7 allows us to deduce almost sure convergence of the zeros to the unit circle.

Theorem 8. *Let $(a_n)_{n\geq 0}$ be a sequence of complex random variables. Assume that there exists some $s \in (0,1]$ such that*

$$\forall k \quad \mu_k := \mathbb{E}\left[|a_k|^s\right] < \infty$$

and for some $0 < \delta \leq 1$ there exists $t > 0$, such that for all N

$$\forall N \quad \xi_N := \mathbb{E}\left[\frac{1}{|a_N|^t} \mathbb{1}_{\{|a_N| \leq \delta\}}\right] = O\left(N^q\right) \tag{16}$$

for some $q > 0$. Assume further that:

$$\limsup_{k\to\infty} \left(\mu_k\right)^{1/k} = 1$$

or, equivalently, there exists a sequence (ε_N) tending to zero such that

$$\sum_{k=0}^{N} \mu_k = \exp(N\varepsilon_N).$$

Then for any deterministic positive sequence $\alpha(N)$ satisfying $\alpha(N) = o(N)$ and $\frac{\alpha(N)}{N\varepsilon_N + \log N} \to \infty$,

$$\lim_{N\to\infty} \frac{1}{N} \nu_N\left(\frac{\alpha(N)}{N}\right) = 1, \quad a.s.$$

and

$$\lim_{N\to\infty} \frac{1}{N} \nu_N\left(\theta, \phi\right) = \frac{\phi - \theta}{2\pi}, \quad a.s.$$

In fact the convergence also holds in the p^{th} mean for every positive p.

Proof. Note that (16) implies $\mathbb{P}\{|a_k| = 0\} = 0$. Therefore, from Theorem 7 it is sufficient to prove that for the choice of $\alpha(N) = o(N)$ given in the theorem,

$$\frac{1}{\alpha(N)}\left(\log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|\right) \to 0 \quad a.s.$$

For $0 < \delta \leq 1$ we have

$$\log 2 \leq \log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N| \leq$$

$$\frac{1}{s}\log\left(1+\sum_{k=0}^{N}|a_k|^s\right) + \frac{1}{2t}\left(\log\frac{1}{|a_0|}\right)\mathbb{1}_{\{|a_0|\leq\delta\}} + \frac{1}{2t}\left(\log\frac{1}{|a_N|}\right)\mathbb{1}_{\{|a_N|\leq\delta\}}$$
$$+\log\frac{1}{\delta}$$

so since $\alpha(N) \to \infty$, it is sufficient to show that

$$\frac{1}{\alpha(N)}\log\left(1+\sum_{k=0}^{N}|a_k|^s\right) = 0 \quad a.s$$

and

$$\frac{1}{\alpha(N)}\left(\log\frac{1}{|a_N|^t}\right)\mathbb{1}_{\{|a_N|\leq\delta\}} = 0 \quad a.s.$$

We are first going to prove that $\lim_{N\to\infty}\frac{1}{\alpha(N)}\log\left(1+\sum_{k=0}^{N}|a_k|\right) = 0$, $a.s.$ for our sequence $\alpha(N)$.

Consider first the case when $\sum_{k=0}^{\infty}\mu_k$ is finite. By the monotone convergence theorem, the sum $\sum_{k=0}^{N}|a_k|^s$ converges almost surely as $N \to \infty$ to an integrable random variable X. Therefore, since $\alpha(N)$ tends to infinity as $N \to \infty$, we see that

$$\lim_{N\to\infty}\frac{1}{\alpha(N)}\frac{1}{s}\log\left(1+\sum_{k=0}^{N}|a_k|^s\right) = 0 \quad a.s.$$

We can thus assume that $\sum_{k=0}^{\infty}\mu_k = \infty$. Given $\varepsilon > 0$, take $\beta > 0$ such that $\log(1+\beta) \leq \varepsilon/3$. As $\limsup_{k\to\infty}(\mu_k)^{\frac{1}{k}} = 1$, and $\mu_k < \infty$ for all k, there exists a constant $C = C(\beta)$ such that for all k we have $\mu_k \leq C(1+\beta)^k$. Hence, for N sufficiently large,

$$0 \leq \log\left(\sum_{k=0}^{N}\mu_k\right) \leq \log C + (N+1)\log(1+\beta) - \log(\beta)$$

Thus,

$$0 \leq \frac{1}{N+1}\log\left(\sum_{k=0}^{N}\mu_k\right) \leq \frac{1}{N+1}\log C + \log(1+\beta) - \frac{1}{N+1}\log(\beta)$$

There exists N' such that for $N \geq N'$,

$$\begin{aligned}\frac{1}{N+1}\log C &\leq \varepsilon/3\\ \frac{1}{N+1}|\log\beta| &\leq \varepsilon/3\end{aligned}$$

Hence, for all $\varepsilon > 0$, we found $N_0 = \max(N', k_0)$, such that for all $N \geq N_0$ we have $\frac{1}{N+1} \log \left(\sum_{k=0}^{N} \mu_k \right) \leq \varepsilon$, which implies $\log \left(\sum_{k=0}^{N} \mu_k \right) = o(N)$. We can thus write for $N \geq 0$:

$$\log \left(\sum_{k=0}^{N} \mu_k \right) = \varepsilon_N N$$

with $\varepsilon_N \to 0$ and $\varepsilon_N N \to \infty$.

Since $(N+1)^2/(k+1)^2 \geq 1$ for all $0 \leq k \leq N$, we have

$$\begin{aligned} &\log \left(1 + \sum_{k=0}^{N} |a_k|^s \right) \\ &\qquad \leq \log \left(1 + (N+1)^2 \exp(\varepsilon_N N) \sum_{k=0}^{N} \frac{|a_k|^s \exp(-\varepsilon_N N)}{(k+1)^2} \right) \\ &\qquad\qquad \leq 2 \log(N+1) + \varepsilon_N N + \log \left(1 + \sum_{k=0}^{N} \frac{|a_k|^s \exp(-\varepsilon_N N)}{(k+1)^2} \right) \end{aligned}$$

Now, as

$$\sum_{k=0}^{N} \mu_k = \exp(\varepsilon_N N),$$

we have

$$\sum_{k=0}^{\infty} \mathbb{E} \left[\frac{|a_k|^s \exp(-\varepsilon_N N)}{(k+1)^2} \right] \leq \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} < \infty$$

We deduce from the monotone convergence theorem that

$$\sum_{k=0}^{N} \frac{|a_k|^s \exp(-\varepsilon_N N)}{(k+1)^2}$$

converges almost surely to an integrable random variable. Hence, taking $\alpha(N)$ to be any positive function such that

$$\frac{\alpha(N)}{\varepsilon_N N + \log N} \to \infty$$

we have

$$\lim_{N \to \infty} \frac{1}{\alpha(N)} \log \left(1 + \sum_{k=0}^{N} |a_k|^s \right) = 0, \quad a.s.$$

Now, let us show that for the same sequence $\alpha(N)$, we have

$$\frac{1}{\alpha(N)} \left(\log \frac{1}{|a_N|^t} \right) \mathbb{1}_{\{|a_N| \leq \delta\}} = 0, \quad a.s.$$

From (16) we have

$$0 \leq \log\left(\frac{1}{|a_N|^t}\right)\mathbb{1}_{\{|a_N|\leq\delta\}} \leq \log\left(1+\frac{1}{|a_N|^t}\mathbb{1}_{\{|a_N|\leq\delta\}}\right)$$
$$\leq (q+2)\log(N+1) + \log\left(1+\frac{1}{(N+1)^{q+2}|a_N|^t}\mathbb{1}_{\{|a_N|\leq\delta\}}\right)$$

From the Markov inequality, we have, for any $\varepsilon > 0$:

$$\mathbb{P}\left[\left(\frac{1}{(N+1)^{q+2}|a_N|^t}\mathbb{1}_{\{|a_N|\leq\delta\}}\right) > \varepsilon\right] \leq \frac{1}{\varepsilon}\frac{\xi_N}{(N+1)^{q+2}}$$

As $\xi_N = O(N^q)$, for N large enough,

$$\mathbb{P}\left[\left(\frac{1}{(N+1)^{q+2}|a_N|^t}\mathbb{1}_{\{|a_N|\leq\delta\}}\right) > \varepsilon\right] \leq \frac{1}{\varepsilon}\frac{C}{(N+1)^2}$$

for some positive constant C. Hence by the Borel-Cantelli lemma,

$$\frac{1}{(N+1)^{q+2}|a_N|^t}\mathbb{1}_{\{|a_N|\leq\delta\}} \to 0 \quad a.s.$$

We can conclude that if $\alpha(N)$ goes to infinity faster than $\log N$ (which our choice of $\alpha(N)$ does), then

$$\lim_{N\to\infty}\frac{1}{\alpha(N)}\left(\log\left(\frac{1}{|a_N|^t}\right)\mathbb{1}_{\{|a_N|\leq\delta\}}\right) = 0, \quad a.s.$$

and the theorem follows. □

Corollary 8.1. *Let $(a_n)_{n\geq 0}$ be a sequence of complex random variables such that the moduli $(|a_n|)$ are from p different probability distributions on the positive real line, say $(F_j(dx))_{1\leq j\leq p}$. Assume that there exists some $s > 0$ such that*

$$\int_0^\infty x^s F_j(dx) < \infty$$

and that there exists some $0 < \delta \leq 1$ such that there exists some $t > 0$ such that

$$\int_0^\delta x^{-t} F_j(dx) < \infty$$

for any $\delta \in (0,1]$. Then for any deterministic positive sequence $\alpha(N) = o(N)$ such that $\alpha(N)/\log N \to \infty$,

$$\lim_{N\to\infty}\frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right) = 1, \quad a.s.$$

and

$$\lim_{N\to\infty}\frac{1}{N}\nu_N(\theta,\phi) = \frac{\phi-\theta}{2\pi}, \quad a.s.$$

In fact the convergence also holds in the p^{th} mean for every positive p.

Proof. This is immediate from Theorem 8. □

Corollary 8.2. *Let $(a_n)_{n\geq 0}$ be a sequence of complex random variables such that the moduli $(|a_n|)$ have densities which are uniformly bounded in a neighborhood of the origin. Assume that there exists some $s\in(0,1]$ such that*

$$\begin{aligned}\forall N, \quad \mu_N &\equiv \mathbb{E}\left[|a_N|^s\right] < \infty\\ \limsup_{k\to\infty} (\mu_k)^{\frac{1}{k}} &= 1\end{aligned}$$

Then there exists a deterministic sequence $\alpha(N) = o(N)$ such that

$$\lim_{N\to\infty} \frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right) = 1, \quad a.s.$$

and

$$\lim_{N\to\infty} \frac{1}{N}\nu_N(\theta,\phi) = \frac{\phi-\theta}{2\pi}, \quad a.s.$$

In fact the convergence also holds in the p^{th} mean for every positive p.

Proof. It suffices to notice that in this special case, $\sup_N \xi_N \leq C$ for some positive constant C. □

Example. Let $P_N(Z) = \sum_{k=0}^N a_k Z^k$, with a_k being distributed on $\mathbb{R}_+$ with Cauchy distribution with parameter $k^{-\sigma}$, $\sigma > 0$. This distribution has density

$$\frac{2}{\pi k^\sigma}\frac{1}{x^2+k^{-2\sigma}}$$

on the positive real line. The conditions of Theorem 8 are satisfied since $\mu_k := \mathbb{E}\left[a_k^{1/2}\right] \leq \frac{C}{k^\sigma}$ and $\xi_N := \mathbb{E}\left[\frac{1}{a_N^{1/2}}\mathbb{1}_{\{|a_N|\leq 1\}}\right] \leq Ck^\sigma$. Therefore, if $\alpha(N) = o(N)$ is such that $\alpha(N)/\log N \to \infty$, then

$$\lim_{N\to\infty} \frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right) = 1, \quad a.s.$$

and

$$\lim_{N\to\infty} \frac{1}{N}\nu_N(\theta,\phi) = \frac{\phi-\theta}{2\pi}, \quad a.s.$$

Again, the convergence also holds in the p^{th} mean for every positive p.

We can still weaken the hypotheses and still have mean convergence.

Proposition 9. *Let $(a_n)_{n\geq 0}$ be a sequence of complex random variables. Assume that there exists some $s\in(0,1]$ such that*

$$\begin{aligned}\forall N, \quad \mu_N &\equiv \mathbb{E}\left[|a_N|^s\right] < \infty\\ \limsup_{k\to\infty} (\mu_k)^{\frac{1}{k}} &= 1\end{aligned}$$

and some $t > 0$, such that

$$\forall N, \quad \xi_N \equiv \mathbb{E}\left[\frac{1}{|a_N|^t}\mathbb{1}_{\{|a_N|\leq\delta\}}\right] < \infty$$

for any $\delta \in (0,1]$, and

$$\log(1+\xi_N) = o(N)$$

Then:

$$\lim_{N\to\infty} \mathbb{E}\left[\left(1 - \frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)\right)^p\right] = 0, \quad \forall p > 0$$

for some sequence $\alpha(N) = o(N)$, $0 < \alpha(N) < N$ and

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\frac{1}{N}\nu_N(\theta,\phi) - \frac{\phi-\theta}{2\pi}\right|^p\right] = 0, \quad \forall p > 0$$

Proof. We first go through the same arguments as previously for the mean convergence and then conclude to the p^{th} mean convergence because of the boundedness of $\frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)$ and $\frac{1}{N}\nu_N(\theta,\phi)$. □

Again, as in the previous section, we can specialize our results to the special case of deterministic coefficients.

Proposition 10. *Let (ε_n) be a sequence of positive real numbers tending to zero, and let $(a_n)_{n\geq 0}$ be a sequence of complex numbers such that for all $n \geq 1$*

$$\exp(-\varepsilon_n n) \leq |a_n| \leq \exp(+\varepsilon_n n)$$

Then there exists a positive function $\alpha(N) = o(N)$ such that zeros of the polynomial $\sum_{k=0}^N a_k Z^k$ satisfy

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right) &= 1 \\ \lim_{N\to\infty} \frac{1}{N}\nu_N(\theta,\phi) &= \frac{\phi-\theta}{2\pi}, \end{aligned}$$

6. Convergence of the empirical measure

In this section, we study the convergence of the random empirical measure associated with the zeros of a random polynomial. We use elementary results about convergence of probability measures that can be found in textbooks such as [15], [3].

Let $\left(P_N(Z) = \sum_{k=0}^N a_{N,k} Z^k\right)_{N=1}^{\infty}$ be a sequence of random polynomials, such that $\mathbb{P}\{a_{N,0} = 0\} = 0$ and $\mathbb{P}\{a_{N,N} = 0\} = 0$. Let $(z_k)_{1\leq k\leq N}$ denote the zeros of $P_N(Z)$. Let

$$\mu_N \equiv \frac{1}{N}\sum_{k=1}^N \delta_{z_k}$$

denote the empirical (random) probability measure associated with the zeros on $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. For every continuous and bounded function,

$$\int f \, \mathrm{d}\mu_N \equiv \langle f, \mu_N \rangle = \frac{1}{N} \sum_{k=1}^{N} f(z_k)$$

Recall that a sequence of probability measures (λ_n) is said to converge weakly to a probability measure λ if for all bounded and continuous functions f,

$$\int f \, \mathrm{d}\lambda_N \to \int f \, \mathrm{d}\lambda$$

or with our notations:

$$\langle f, \lambda_N \rangle \to \langle f, \lambda \rangle \tag{17}$$

As $\mathbb{C}^*$, endowed with the metric $d(z_1, z_2) \equiv |z_1 - z_2| + \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$, is a locally compact polish space, we can in fact take the space $C_K(\mathbb{C}^*)$ of continuous functions with compact support as space of test functions in (17). Following Bilu [4] we call a function $f : \mathbb{C}^* \to \mathbb{C}$ standard, if

$$f\left(re^{\mathrm{i}\varphi}\right) = g(r) \exp(\mathrm{i}p\varphi)$$

where $g : \mathbb{R}_+^* \to \mathbb{C}$ is continuous and compactly supported, and $p \in \mathbb{Z}$.

Lemma 11. *The linear space, generated by the standard functions, is dense in the space of all compactly supported functions $\mathbb{C}^* \to \mathbb{C}$ (with the sup-norm).*

Proof. See [4]. □

Corollary 11.1. *Let (λ_n) be a sequence of probability measures on $\mathbb{C}^*$, and λ one more probability measure on $\mathbb{C}^*$. Assume that*

$$\langle f, \lambda_N \rangle \to \langle f, \lambda \rangle, \; N \to \infty$$

for any standard function. Then (λ_n) converges weakly to λ.

Proposition 12. *Let $\left(P_N(Z) = \sum_{k=0}^{N} a_{Nk} Z^k\right)_{N=1}^{\infty}$ be a sequence of random polynomials.*

(1) *If*

$$\lim_{N\to\infty} \frac{1}{N} \nu_N \left(\frac{\alpha(N)}{N}\right) = 1, \quad a.s.$$

and

$$\frac{1}{N} \nu_N(\theta, \phi) \to \frac{\phi - \theta}{2\pi} a.s.$$

then the sequence of random measures (μ_N) converges almost surely weakly to the Haar measure on the unit circle, that is to say for all bounded continuous functions $f : \mathbb{C}^* \to \mathbb{C}$, we have:

$$\langle f, \mu_N \rangle \to \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathrm{i}\varphi}) \, \mathrm{d}\varphi, \quad a.s$$

(2) *If*

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\nu_N\left(\frac{\alpha(N)}{N}\right)\right]=1$$

and

$$\lim_{N\to\infty}\mathbb{E}\left[\left|\frac{1}{N}\nu_N(\theta,\phi)-\frac{\phi-\theta}{2\pi}\right|\right]=0$$

then the sequence of measures (μ_N) *converges in mean weakly to the Haar measure on the unit circle, that is to say for all continuous functions* $f:\mathbb{C}^*\to\mathbb{C}$ *, we have:*

$$\lim_{N\to\infty}\mathbb{E}\left[\left|\langle f,\mu_N\rangle-\frac{1}{2\pi}\int_0^{2\pi}f(e^{\mathrm{i}\varphi})\,\mathrm{d}\varphi\right|\right]=0$$

In fact the convergence holds in the p^{th} *mean, for all positive* p.

Proof. By corollary 11.1, we need only prove this result for standard functions.

We will first prove part (1), the almost sure convergence case. Let $f(z)$ be a standard function, that is $f\left(re^{\mathrm{i}\varphi}\right)=g(r)\,e^{\mathrm{i}p\varphi}$ where $g:\mathbb{R}_+\to\mathbb{C}$ and $p\in\mathbb{Z}$. We must distinguish between two cases when $p=0$ and when $p\neq 0$.

When $p=0$, we have to prove that $\langle f,\mu_N\rangle\to g(1)$ almost surely. As g is continuous, $\forall\varepsilon>0$, there exists $\delta>0$, such that for all $r\in[1-\delta,1/(1-\delta)]$, $|g(r)-g(1)|<\varepsilon$. Denoting $\mathcal{D}=[1-\delta,1/(1-\delta)]$ and $\|g\|_\infty=\sup_{r>0}|g(r)|$, we then have

$$\begin{aligned}|\langle f,\mu_N\rangle-g(1)| &= \frac{1}{N}\left|\sum_{|z_k|\in\mathcal{D}}(g(|z_k|)-g(1))+\sum_{|z_k|\notin\mathcal{D}}(g(|z_k|)-g(1))\right|\\ &\leq \frac{1}{N}\nu_N(1-\delta)\,\varepsilon+2\|g\|_\infty\left(1-\frac{1}{N}\nu_N(1-\delta)\right)\end{aligned}$$

The result then follows from the assumption that $\lim_{N\to\infty}\frac{1}{N}\nu_N(1-\delta)=1$, *a.s.*

Now consider the case when $p\neq 0$. As $\frac{1}{2\pi}\int g(1)\exp(\mathrm{i}p\varphi)\,\mathrm{d}\varphi=0$ we must show

$$\lim_{N\to\infty}|\langle f,\mu_N\rangle|=0\quad a.s.$$

For this, we notice that:

$$\begin{aligned}\left|\frac{1}{N}\sum_{k=1}^N f(z_k)\right| &= \frac{1}{N}\left|\sum_{k=1}^N g(|z_k|)\exp(\mathrm{i}p\vartheta_k)\right|\\ &\leq \frac{|g(1)|}{N}\left|\sum_{k=1}^N\exp(\mathrm{i}p\vartheta_k)\right|+\frac{1}{N}\sum_{k=1}^N|g(|z_k|)-g(1)|\end{aligned}$$

where ϑ_k is the argument of z_k in $[0, 2\pi)$. As $\frac{1}{N}\nu_N(\theta, \phi) \to \frac{\phi-\theta}{2\pi}$ *a.s.*, we can apply Weyl's theorem for uniformly distributed sequence of real numbers to deduce that

$$\lim_{N\to\infty} \frac{1}{N}\left|\sum_{k=1}^{N} \exp(\mathrm{i}p\vartheta_k)\right| = 0, \ a.s.$$

We have already shown that $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} |g(|z_k|) - g(1)| = 0$, *a.s.*, and this completes the proof of part (1).

Part (2) of the theorem concerns the case of mean convergence. As before, we let $f\left(re^{\mathrm{i}\varphi}\right) = g(r)\exp(\mathrm{i}p\varphi)$ be a standard function, and again we must distinguish between $p = 0$ and $p \neq 0$. The proof in the case $p = 0$ does not change. Indeed, we still have

$$|\langle f, \mu_N\rangle - g(1)| \leq \frac{\nu_N(1-\delta)}{N}\varepsilon + 2\frac{N - \nu_N(1-\delta)}{N}\|g\|_\infty$$

leading to

$$\begin{aligned}\mathbb{E}\left[|\langle f, \mu_N\rangle - g(1)|\right] &\leq \mathbb{E}\left[\frac{1}{N}\nu_N(1-\delta)\right]\varepsilon + 2\,\mathbb{E}\left[1 - \frac{1}{N}\nu_N(1-\delta)\right]\|g\|_\infty \\ &\to 0 \text{ as } N \to \infty\end{aligned}$$

For the case $p \neq 0$, we still have

$$\left|\frac{1}{N}\sum_{k=1}^{N} f(z_k)\right| \leq \frac{|g(1)|}{N}\left|\sum_{k=1}^{N}\exp(\mathrm{i}p\vartheta_k)\right| + \frac{1}{N}\sum_{k=1}^{N}|g(|z_k|) - g(1)|$$

Hence

$$\begin{aligned}&\mathbb{E}\left[\left|\frac{1}{N}\sum_{k=1}^{N} f(z_k)\right|\right] \\ &\qquad\leq \mathbb{E}\left[\frac{|g(1)|}{N}\left|\sum_{k=1}^{N}\exp(\mathrm{i}p\vartheta_k)\right|\right] + \frac{1}{N}\,\mathbb{E}\left[\sum_{k=1}^{N}|g(|z_k|) - g(1)|\right]\end{aligned}$$

Again, the case $p = 0$ shows that

$$\lim_{N\to\infty}\frac{1}{N}\,\mathbb{E}\left[\sum_{k=1}^{N}|g(|z_k|) - g(1)|\right] = 0$$

and so to complete the proof, we just have to show that

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\left|\sum_{k=1}^{N}\exp(\mathrm{i}p\vartheta_k)\right|\right] = 0$$

which follows from the following lemma. □

Lemma 13. *If*

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\frac{1}{N}\nu_N(\theta,\phi) - \frac{\phi-\theta}{2\pi}\right|\right] = 0$$

and if $f(z) = g(\arg z)$ *where* g *is a continuous function defined on the torus* $\mathbb{R}/2\pi\mathbb{Z}$*, we have*

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\langle f, \mu_N\rangle - \frac{1}{2\pi}\int_0^{2\pi} f(e^{\mathrm{i}\varphi})\,\mathrm{d}\varphi\right|\right] = 0 \tag{18}$$

Proof. We can assume that f is real valued; otherwise we would consider the real and imaginary parts. In the special case when $f(z) = \mathbb{1}_{[\theta,\phi)}(\arg z)$, (18) is exactly our assumption: $\lim_{N\to\infty} \mathbb{E}\left[\left|\frac{1}{N}\nu_N(\theta,\phi) - \frac{\phi-\theta}{2\pi}\right|\right] = 0$. It is easy to see that (18) holds for finite linear combination of such functions, and hence for step functions. Now, if g is continuous, for any $\varepsilon > 0$, there exist two step functions g_1 and g_2 such that $g_1 \le g \le g_2$, and

$$\frac{1}{2\pi}\int_0^{2\pi} (g_2(\varphi) - g_1(\varphi))\,\mathrm{d}\varphi \le \varepsilon$$

For simplicity, let $\overline{g} := \frac{1}{2\pi}\int_0^{2\pi} g(\varphi)\,\mathrm{d}\varphi$. Letting $f(z) = g(\arg z)$, $f_1(z) = g_1(\arg z)$ and $f_2(z) = g_2(\arg z)$ we then have

$$\langle f_1,\mu_N\rangle - \overline{g_1} - (\overline{g} - \overline{g_1}) \le \langle f,\mu_N\rangle - \overline{g} \le \langle f_2,\mu_N\rangle - \overline{g_2} - (\overline{g} - \overline{g_2})$$

Hence:

$$|\langle f,\mu_N\rangle - \overline{g}| \le |\langle f_2,\mu_N\rangle - \overline{g_2}| + |\langle f_1,\mu_N\rangle - \overline{g_1}| + (\overline{g} - \overline{g_1}) + (\overline{g_2} - \overline{g})$$

and

$$\mathbb{E}\,|\langle f,\mu_N\rangle - \overline{g}| \le \mathbb{E}\,|\langle f_2,\mu_N\rangle - \overline{g_2}| + \mathbb{E}\,|\langle f_1,\mu_N\rangle - \overline{g_1}| + 2\varepsilon$$

The lemma follows from the fact that $\lim_{N\to\infty}\mathbb{E}\left[|\langle f_j,\mu_N\rangle - \overline{g_j}|\right] = 0$, for $j = 1,2$ by the assumption of the lemma. □

Corollary 13.1. *In each case of convergence of the proposition, we have in fact:*

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\langle f, \mu_N\rangle - \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})\,\mathrm{d}\varphi\right|^p\right] = 0$$

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