# SYMMETRY CLASSES OF DISORDERED FERMIONS

*by*

P. Heinzner, A. Huckleberry & M.R. Zirnbauer

***Abstract.*** **—** Building upon Dyson's fundamental 1962 article known in random-matrix theory as *the threefold way*, we classify disordered fermion systems with quadratic Hamiltonians by their unitary and antiunitary symmetries. Important physical examples are afforded by noninteracting quasiparticles in disordered metals and superconductors, and by relativistic fermions in random gauge field backgrounds.

The primary data of the classification are a Nambu space of fermionic field operators carrying a representation of some symmetry group. Our approach is to eliminate all of the unitary symmetries from the picture by transferring to an irreducible block of equivariant homomorphisms. After reduction, the block data specifying a linear space of symmetry-compatible Hamiltonians consist of a basic vector space $V$, a space of endomorphisms in $\mathrm{End}(V \oplus V^*)$, a bilinear form on $V \oplus V^*$ which is either symmetric or alternating, and one or two antiunitary symmetries that may mix $V$ with $V^*$. Every such set of block data is shown to determine an irreducible classical compact symmetric space. Conversely, every irreducible classical compact symmetric space occurs in this way.

This proves the correspondence between symmetry classes and symmetric spaces conjectured some time ago.

Keywords: disordered electron systems, random Dirac fermions, quantum chaos; representation theory, symmetric spaces

## 1. Introduction

In a famous and influential paper published in 1962 ("the threefold way: algebraic structure of symmetry groups and ensembles in quantum mechanics" [**D**]), Freeman J. Dyson classified matrix ensembles by a scheme that became fundamental to several areas of theoretical physics, including the statistical theory of complex many-body systems, mesoscopic physics, disordered electron systems, and the field of quantum chaos. Being set in the context of standard quantum mechanics, Dyson's classification asserted that "the most general matrix ensemble, defined with a symmetry group that may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs one of three known types." These three ensembles, or rather their underlying matrix spaces, are nowadays known as the Wigner-Dyson symmetry classes of orthogonal, unitary, and symplectic symmetry.

Over the last ten years, various matrix spaces beyond Dyson's threefold way have come to the fore in random-matrix physics and mathematics. On the physics side, such spaces arise in problems of disordered or chaotic fermions; among these are the Euclidean Dirac operator in a stochastic gauge field background [**VZ**], and quasiparticle excitations in disordered superconductors or metals in proximity to a superconductor [**A**]. In the mathematical research area of number theory, the study of statistical correlations in the values of the Riemann zeta function, and more generally of families of *L*-functions, has prompted some of the same extensions [**K**].

A brief account of why new structures emerge on the physics side is as follows. When Dirac first wrote down his famous equation in 1928, he assumed that he was writing an equation for the *wavefunction* of the electron. Later, because of the instability caused by negative-energy solutions, the Dirac equation was reinterpreted (via second quantization) as an equation for the *fermionic field operators* of a quantum field theory. A similar change of viewpoint is carried out in reverse in the Hartree-Fock-Bogoliubov mean-field description of quasiparticle excitations in superconductors. There, one starts from the equations of motion for linear superpositions of the electron creation and annihilation operators, and reinterprets them as a unitary quantum dynamics for what might be called the quasiparticle 'wavefunction'.

In both cases – the Dirac equation and the quasiparticle dynamics of a superconductor – there enters a structure not present in the standard quantum mechanics underlying Dyson's classification: the fermionic field operators are subject to a set of conditions known as the *canonical anticommutation relations*, and these are preserved by the quantum dynamics. Therefore, whenever second quantization is undone (assuming it *can* be undone) to return from field operators to wavefunctions, the wavefunction dynamics is required to preserve some extra structure. This puts a linear constraint on the allowed Hamiltonians. A good viewpoint to adopt is to attribute the extra invariant structure to the Hilbert space, thereby turning it into a Nambu space.

It was conjectured some time ago [**A**] that extending Dyson's classification to the Nambu space setting, the relevant objects one is led to consider are large families of *symmetric spaces* of compact type. Past understanding of the systematic nature of the extended classification scheme relied on the mapping of disordered fermion problems to field theories with supersymmetric target spaces [**Z**] in combination with renormalization group ideas and the classification theory of Lie superalgebras.

An extensive review of the mathematics and physics of symmetric spaces, covering the wide range from the basic definitions to various random-matrix applications, has recently been given in [**C**]. That work, however, offers no answers to the question as to *why* symmetric spaces are relevant for symmetry classification, and under what assumptions the classification by symmetric spaces is complete.

In the present paper, we get to the bottom of the subject and, using a minimal set of tools from linear algebra, give a rigorous answer to the classification problem for disordered fermions. The rest of this introduction gives an overview over the mathematical model to be studied and a statement of the main result obtained.

We begin with a finite- or infinite-dimensional Hilbert space $V$ carrying a unitary representation of some compact Lie group $G_0$ – this is the group of unitary symmetries of the disordered fermion system. We emphasize that $G_0$ need not be connected; in fact, it might be just a finite group.

Let $W = V \oplus V^*$, called the Nambu space of fermionic field operators, be equipped with the induced $G_0$-representation. This means that $V$ is equipped with the given representation, and $g(f) := f \circ g^{-1}$ for $f \in V^*$, $g \in G_0$. Let $C : W \to W$ be the $\mathbb{C}$-antilinear involution determined by the Hermitian scalar product $\langle\, , \rangle_V$ on $V$. In physics this operator is called particle-hole conjugation. Another canonical structure on $W$ is the symmetric complex bilinear form $b : W \times W \to \mathbb{C}$ defined by

$$b(v_1 + f_1, v_2 + f_2) := f_1(v_2) + f_2(v_1) \; .$$

It encodes the canonical anticommutation relations for fermions, and is related to the unitary structure $\langle\, , \rangle$ of $W$ by $b(w_1, w_2) = \langle Cw_1, w_2 \rangle$ for all $w_1, w_2 \in W$.

It is assumed that $G_0$ is contained in a group $G$ – the total symmetry group of the fermion system – which is acting on $W$ by transformations that are either unitary or antiunitary. An element $g \in G$ either stabilizes $V$ or exchanges $V$ and $V^*$. In the latter case we say that $g \in G$ mixes, and in the former case we say that it is nonmixing.

The group $G$ is generated by $G_0$ and distinguished elements $g_T$ which act as antiunitary operators $T : W \to W$. These are referred to as distinguished 'time-reversal' symmetries, or $T$-symmetries for short. The squares of the $g_T$ lie in the center of the abstract group $G$; we therefore require that the antiunitary operators $T$ representing them satisfy $T^2 = \pm \mathrm{Id}$. The subgroup $G_0$ is defined as the set of all elements of $G$ which are represented as unitary, nonmixing operators on $W$.

If $T$ and $T_1$ are distinguished time-reversal operators, then $P := TT_1$ is a unitary symmetry. $P$ may be mixing or nonmixing. In the latter case, $P$ is in $G_0$. Therefore, modulo $G_0$, there exist at most two different $T$-symmetries. If there are exactly two such symmetries, we adopt the convention that $T$ is mixing and $T_1$ is nonmixing. Furthermore, it is assumed that $T$ and $T_1$ either commute or anticommute, i.e., $T_1 T = \pm T T_1$.

As explained throughout this article, all of these situations are well motivated by physical considerations and examples. We note that time-reversal symmetry (and all other $T$-symmetries) of the disordered fermion system may also be broken; in this case $T$ and $T_1$ are eliminated from the mathematical model and $G_0 = G$.

Given $W$ and the representation of $G$ on it, the object of interest is the real vector space $\mathcal{H}$ of $\mathbb{C}$-linear operators in $\mathrm{End}(W)$ that preserve the canonical structures $b$ and $\langle\, , \rangle$ of $W$ and commute with the $G$-action. Physically speaking, $\mathcal{H}$ is the space of 'good' Hamiltonians: the field operator dynamics generated by $H \in \mathcal{H}$ preserves both the canonical anticommutation relations and the probability in Nambu space, and is compatible with the prescribed symmetry group $G$.

When unitary symmetries are present, the space $\mathcal{H}$ decomposes by *blocks* associated with isomorphism classes of $G_0$-subrepresentations occurring in $W$. To formalize this, recall that two unitary representations $\rho_i : G_0 \to \mathrm{U}(V_i)$, $i = 1,2$, are equivalent if and only if there exists a unitary $\mathbb{C}$-linear isomorphism $\varphi : V_1 \to V_2$ so that $\rho_2(g)(\varphi(v)) = \varphi(\rho_1(g)(v))$ for all $v \in V_1$ and for all $g \in G_0$. Let $\hat{G}_0$ denote the space of equivalence classes of irreducible unitary representations of $G_0$. An element $\lambda \in \hat{G}_0$ is called an isomorphism class for short. By standard facts (recall that every representation of a compact group is completely reducible) the unitary $G_0$-representation on $V$ decomposes as an orthogonal sum over isomorphism classes:

$$V = \oplus_\lambda V_\lambda \, .$$

The subspaces $V_\lambda$ are called the $G_0$-isotypic components of $V$. Some of them may be zero. (Some of the isomorphism classes of $G_0$ may just not be realized in $V$.)

For simplicity suppose now that there is only one distinguished time-reversal symmetry $T$, and for any fixed $\lambda \in \hat{G}_0$ with $V_\lambda \neq 0$, consider the vector space $T(V_\lambda)$. If $T$ is nonmixing, i.e., $T : V \to V$, then $T(V_\lambda) \subset V$ must coincide with the isotypic component for the same or some other isomorphism class. (Since conjugation by $g_T$ is an automorphism of $G_0$, the decomposition into $G_0$-isotypic components is preserved by $T$.) If $T$ is mixing, i.e., $T : V \to V^*$, then $T(V_\lambda) = V^*_{\lambda'}$, still with some $\lambda' \in \hat{G}_0$.

Now define the block $B_\lambda$ to be the smallest $G$-invariant space containing $V_\lambda \oplus V^*_\lambda$. Note that if we are in the situation of nonmixing and $T(V_\lambda) \neq V_\lambda$, then

$$B_\lambda = \big(V_\lambda \oplus T(V_\lambda)\big) \oplus \big(V_\lambda \oplus T(V_\lambda)\big)^* \, .$$

On the other hand, if we are in the situation of mixing and $T(V_\lambda) \neq V^*_\lambda$, then

$$B_\lambda = \big(V_\lambda \oplus T(V^*_\lambda)\big) \oplus \big(V^*_\lambda \oplus T(V_\lambda)\big) \, .$$

The block $B_\lambda$ is halved if $T(V_\lambda) = V_\lambda$ resp. $T(V_\lambda) = V^*_\lambda$.

Note that if there are two distinguished $T$-symmetries, the above discussion is only slightly more complicated. In any case we now have the basic $G$-invariant blocks $B_\lambda$.

Because different blocks are built from representations of different isomorphism classes, the good Hamiltonians do not mix blocks. Thus every $H \in \mathcal{H}$ is a direct sum over blocks, and the structure analysis of $\mathcal{H}$ can be carried out for each block $B_\lambda$ separately. If $V_\lambda$ is infinite-dimensional, then to have good mathematical control we truncate to a finite-dimensional space $V_\lambda \subset V_\lambda$ and form the associated block $B_\lambda \subset W$. The truncation is done in such a way that $B_\lambda$ is a $G$-representation space and is Nambu.

The goal now is to compute the space of Hermitian operators on $B_\lambda$ which commute with the $G$-action and respect the canonical symmetric $\mathbb{C}$-bilinear form $b$ induced from that on $V \oplus V^*$ (such a space of operators realizes what is called a *symmetry class*).

For this, certain spaces of $G_0$-equivariant homomorphisms play an essential role, i.e., linear maps $S : V_1 \to V_2$ between $G_0$-representation spaces which satisfy

$$\rho_2(g) \circ S = S \circ \rho_1(g)$$

for all $g \in G_0$, where $\rho_i : G_0 \to \mathrm{U}(V_i)$, $i = 1,2$, are the respective representations. If it is clear which representations are at hand, we often simply write $g \circ S = S \circ g$ or $S = gSg^{-1}$. Thus we regard the space $\mathrm{Hom}_{G_0}(V_1,V_2)$ of equivariant homomorphisms as the space of $G_0$-fixed vectors in the space $\mathrm{Hom}(V_1,V_2)$ of all linear maps. If $V_1 = V_2 = V$, then these spaces are denoted by $\mathrm{End}_{G_0}(V)$ and $\mathrm{End}(V)$ respectively.

Roughly speaking, there are two steps for computing the relevant spaces of Hermitian operators. First, the block $B_\lambda$ is replaced by an analogous block $H_\lambda$ of $G_0$-equivariant homomorphisms from a fixed representation space $R_\lambda$ of isomorphism class $\lambda$ and/or its dual $R^*_\lambda$ to $B_\lambda$. The space $H_\lambda$ carries a canonical form (called either $s$ or $a$) which is induced from $b$. As the notation indicates, although the original bilinear form on $B_\lambda$ is symmetric, this induced form is either symmetric or alternating.

Change of parity occurs in the most interesting case when there is a nontrivial $G_0$-isomorphism $\psi : R_\lambda \to R^*_\lambda$. In that case there exists a bilinear form $F_\psi : R_\lambda \times R_\lambda \to \mathbb{C}$ defined by $F_\psi(r,t) = \psi(r)(t)$, which is either symmetric or alternating. In a certain sense the form $b$ is a product of $F_\psi$ and a canonical form on $H_\lambda$. Thus, if $F_\psi$ is alternating, then the canonical form on $H_\lambda$ must also be alternating.

After transferring to the space $H_\lambda$, in addition to the canonical bilinear form $s$ or $a$ we have a unitary structure and conjugation by one or two distinguished time-reversal symmetries. Such a symmetry $T$ may be mixing or not, and both $T^2 = \mathrm{Id}$ and $T^2 = -\mathrm{Id}$ are possible. The second main step of our work is to understand these various cases, each of which is directly related to a classical symmetric space of compact type. Such are given by a classical Lie algebra $\mathfrak{g}$ which is either $\mathfrak{su}_n$, $\mathfrak{usp}_{2n}$, or $\mathfrak{so}_n(\mathbb{R})$.

In the notation of symmetric spaces we have the following situation. Let $\mathfrak{g}$ be the Lie algebra of *antihermitian* endomorphisms of $H_\lambda$ which are isometries (in the sense of Lie algebra elements) of the induced complex bilinear form $b = s$ or $b = a$. This is of compact type, because it is the intersection of the Lie algebra of the unitary group of $H_\lambda$ and the complex Lie algebra of the group of isometries of $b$. Conjugation by the antiunitary mapping $T$ defines an involution $\theta : \mathfrak{g} \to \mathfrak{g}$.

The good Hamiltonians (restricted to the reduced block $H_\lambda$) are the *Hermitian* operators $h \in \mathrm{i}\mathfrak{g}$ such that at the level of group action the one-parameter groups $\mathrm{e}^{-\mathrm{i}th}$ satisfy $T\mathrm{e}^{-\mathrm{i}th} = \mathrm{e}^{+\mathrm{i}th}T$, i.e., $\mathrm{i}h \in \mathfrak{g}$ must anticommute with $T$. Equivalently, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition of $\mathfrak{g}$ into $\theta$-eigenspaces, the space of operators which is to be computed is the $(-1)$-eigenspace $\mathfrak{p}$. The space of good Hamiltonians restricted to $H_\lambda$ then is $\mathrm{i}\mathfrak{p}$. Since the appropriate action of the Lie group $\mathsf{K}$ (with Lie algebra $\mathfrak{k}$) on this space is just conjugation, one identifies $\mathrm{i}\mathfrak{p}$ with the tangent space $\mathfrak{g}/\mathfrak{k}$ of an associated symmetric space $\mathsf{G}/\mathsf{K}$ of compact type.

It should be underlined that there is more than one symmetric space associated to a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We are most interested in the one consisting of the physical time-evolution operators $\mathrm{e}^{-\mathrm{i}th}$; if $\mathsf{G}$ (not to be confused with the symmetry group $G$) is the semisimple and simply connected Lie group with Lie algebra $\mathfrak{g}$, this is given as the image of the compact symmetric space $\mathsf{G}/\mathsf{K}$ under the Cartan embedding into $\mathsf{G}$ defined by $g\mathsf{K} \mapsto g\theta(g)^{-1}$, where $\theta : \mathsf{G} \to \mathsf{G}$ is the induced group involution.

The following mathematical result is a conseqence of the detailed classification work in Sects. 3, 4 and 5.

***Theorem 1.1**. — The symmetric spaces which occur under these assumptions are irreducible classical symmetric spaces $\mathfrak{g}/\mathfrak{k}$ of compact type. Conversely, every irreducible classical symmetric space of compact type occurs in this way.*

We emphasize that here the notion *symmetric space* is applied flexibly in the sense that depending on the circumstances it could mean either the infinitesimal model $\mathfrak{g}/\mathfrak{k}$ or the Cartan-embedded compact symmetric space $\mathsf{G}/\mathsf{K}$.

Let us mention that direct sums $\mathfrak{g}/\mathfrak{k} \oplus \mathfrak{g}/\mathfrak{k}$ may occur in examples where the original situation is irreducible, e.g., when the initial block $V \oplus V^*$ is invariant under two distinguished time-reversal symmetries. But the main object of interest would seem to be the irreducible classical symmetric space of compact type.

Theorem 1.1 settles the question of symmetry classes in disordered fermion systems; in fact every physics example is handled by one of the situations above.

The paper is organized as follows. In Sect. 2, starting from physical considerations we motivate and develop the model that serves as the basis for subsequent mathematical work. Sect. 3 proves a number of results which are used to eliminate the group of unitary symmetries $G_0$. The main work of classification is given in Sect. 4 and Sect. 5. In Sect. 4 we handle the case where there is at most one distinguished time-reversal operator present, and in Sect. 5 the case where there are two. There are numerous situations that must be considered, and in each case we precisely describe the symmetric space which occurs.

Various examples taken from the physics literature are listed in Sect. 6, illustrating the general classification theory.

## 2. Disordered fermions with symmetries

'Fermions' is the physics name for the elementary particles which all matter is made of. The goal of present article is to establish a symmetry classification of Hamiltonians which are *quadratic* in the fermion creation and annihilation operators. To motivate this restriction note that at the fundamental level, any Hamiltonian for fermions is of Dirac type; thus it is always quadratic in the fermion operators, albeit with time-dependent coefficients that are themselves operators. At the nonrelativistic or effective level, quadratic Hamiltonians arise in the Hartree-Fock mean-field approximation for metals and the Hartree-Fock-Bogoliubov approximation for superconductors. By the

Landau-Fermi liquid principle, such mean-field or noninteracting Hamiltonians give an adequate description of physical reality at very low temperatures.

In the present section, starting from a physical framework, we develop the appropriate model that will serve as the basis for the mathematical work done later on. Please be advised that *disorder*, though advertised in the title of the section and the title of paper, will play no explicit role here. Nevertheless, disorder (and/or chaos) are the indispensable agents that *must be present* in order to remove specific and nongeneric features from the physical system and make a classification by basic symmetries meaningful. In other words, what we carry out in this paper is the first step of a two-step program. This first step is to identify in the total space of Hamiltonians some linear subspaces that are relevant (in Dyson's sense) from a symmetry perspective. The second step is to put probability measures on these spaces and work out the disorder averages and statistical correlation functions of interest. It is this latter step that ultimately justifies the first one and thus determines the name of the game.

**2.1. The Nambu space model for fermions. —** The starting point for our considerations is the formalism of second quantization. Its relevant aspects will now be reviewed so as to introduce the key physical notions as well as the proper mathematical language.

Let $i = 1, 2, \ldots$ label an orthonormal set of quantum states for a single fermion. Second quantizing the many-fermion system means to associate with each $i$ a pair of operators $c_i^\dagger$ and $c_i$, which are called fermion creation and annihilation operators, respectively. These operators are subject to the *canonical anticommutation relations*

$$\begin{aligned} &c_i^\dagger c_j^\dagger + c_j^\dagger c_i^\dagger = 0 \,, \\ &c_i c_j + c_j c_i = 0 \,, \\ &c_i^\dagger c_j + c_j c_i^\dagger = \delta_{ij} \,, \end{aligned}$$

for all $i, j$. They act in a Fock space, i.e., in a vector space with a distinguished vector, called the 'vacuum', which is annihilated by all of the operators $c_i$ $(i = 1, 2, \ldots)$. Applying $n$ creation operators to the vacuum one gets a state vector for $n$ fermions. A *field operator* $\psi$ is a linear combination of creation and annihilation operators,

$$\psi = \sum_i \left( v_i c_i^\dagger + f_i c_i \right) \,,$$

with complex coefficients $v_i$ and $f_i$.

To put this in mathematical terms, let $V$ be the complex Hilbert space of single-fermion states. (We do not worry here about complications due to the dimension of $V$ being infinite. Later rigorous work will be carried out in the finite-dimensional setting.) Fock space then is the exterior algebra $\wedge V$, with the vacuum being the one-dimensional subspace of constants. Creating a single fermion amounts to exterior multiplication by a vector $v \in V$ and is denoted by $\varepsilon(v)$. To annihilate a fermion, one contracts with an element $f$ of the dual space $V^*$; this operation on $\wedge V$ is written

$\iota(f)$. In that framework the canonical anticommutation relations read

$$\begin{aligned}\varepsilon(v)\varepsilon(\tilde{v})+\varepsilon(\tilde{v})\varepsilon(v)&=0\ ,\\ \iota(f)\iota(\tilde{f})+\iota(\tilde{f})\iota(f)&=0\ ,\\ \iota(f)\varepsilon(v)+\varepsilon(v)\iota(f)&=f(v)\ .\end{aligned}$$

They can be viewed as the defining relations of an associative algebra generated by the vector space $W:=V\oplus V^*$ which is isomorphic to the space of field operators $\psi$.

This algebra, called the Clifford algebra $C(W)$, comes with a natural grading by the degree:

$$C(W)=\mathbb{C}\oplus C^1(W)\oplus C^2(W)\oplus\dots\ ,$$

where $C^1(W)\cong W$. In the sequel, we shall focus on the components $C^1(W)$ and $C^2(W)$. Since we only consider Hamiltonians that are quadratic in the creation and annihilation operators, we will be able to reduce the second-quantized formulation to standard single-particle quantum mechanics, albeit on the doubled space $W$ carrying some extra structure. $W$ is sometimes referred to as *Nambu space* in physics.

On $W$ there exists a canonical symmetric complex bilinear form $b$ defined by

$$b(v+f,\tilde{v}+\tilde{f})=\tilde{f}(v)+f(\tilde{v})=\textstyle\sum_i(\tilde{f}_i v_i+f_i\tilde{v}_i)\ .$$

The significance of this bilinear form in the present context lies in the fact that it encodes on $W$ the canonical anticommutation relations (CAR) obeyed by the Clifford algebra generators in $C^1(W)$. Indeed, we can view a field operator $\psi=\sum_i(v_i c_i^\dagger+f_i c_i)$ either as a vector $\psi=v+f\in V\oplus V^*$, or equivalently as an operator $\psi=\varepsilon(v)+\iota(f)\in C^1(W)$. Adopting the operator perspective, we get from CAR that

$$\psi\tilde{\psi}+\tilde{\psi}\psi=\tilde{f}(v)+f(\tilde{v})=\textstyle\sum_i\left(\tilde{f}_i v_i+f_i\tilde{v}_i\right)\ .$$

Switching to the vector perspective we have the same answer from $b(\psi,\tilde{\psi})$. Thus

$$\psi\tilde{\psi}+\tilde{\psi}\psi=b(\psi,\tilde{\psi})\ .$$

***Definition 2.1***. — *In the Nambu space model for fermions one identifies the space $C^1(W)$ of field operators $\psi=\varepsilon(v)+\iota(f)$ with the complex vector space $W=V\oplus V^*$ equipped with its canonical unitary structure $\langle\ ,\ \rangle$ and canonical symmetric complex bilinear form b.*

***Remark***. — Having already expounded the physical origin of the symmetric bilinear form, let us now specify the canonical unitary structure of $W$. The complex vector space $V$, being isomorphic to the Hilbert space of single-particle states, comes with a Hermitian scalar product (or unitary structure) $\langle\ ,\ \rangle_V$. Given $\langle\ ,\ \rangle_V$ define a $\mathbb{C}$-antilinear bijection $C:V\to V^*$ by

$$Cv=\langle v,\cdot\rangle_V\ ,$$

and extend this to an antilinear transformation $C : W \to W$ by the requirement $C^2 = \mathrm{Id}$. Thus $C|_{V^*} = (C|_V)^{-1}$. The operator $C$ is called *particle-hole conjugation* in physics. Using $C$, transfer the unitary structure from $V$ to $V^*$ in the natural way:

$$\langle f, \tilde{f} \rangle_{V^*} := \overline{\langle Cf, C\tilde{f} \rangle}_V = \langle C\tilde{f}, Cf \rangle_V \ .$$

The canonical unitary structure of $W$ is then given by

$$\langle v + f, \tilde{v} + \tilde{f} \rangle = \langle v, \tilde{v} \rangle_V + \langle f, \tilde{f} \rangle_{V^*} = \sum_i \left( \bar{v}_i \tilde{v}_i + \bar{f}_i \tilde{f}_i \right) \ .$$

Thus $\langle \, , \rangle$ is the orthogonal sum of the Hermitian scalar products on $V$ and $V^*$.

***Proposition 2.2***. — *The canonical unitary structure and symmetric complex bilinear form of $W$ are related by*

$$\langle \psi, \tilde{\psi} \rangle = b(C\psi, \tilde{\psi}) \ .$$

*Proof*. — Given an orthonormal basis $c_1^\dagger, c_1, c_2^\dagger, c_2, \ldots$ this is immediate from

$$C \sum_i (v_i c_i^\dagger + f_i c_i) = \sum_i (\bar{v}_i c_i + \bar{f}_i c_i^\dagger)$$

and the expressions for $\langle \, , \rangle$ and $b$ in components. □

Returning to the physics way of telling the story, consider the most general Hamiltonian $H$ which is quadratic in the single-fermion creation and annihilation operators:

$$H = \tfrac{1}{2} \sum_{ij} A_{ij} \left( c_i^\dagger c_j - c_j c_i^\dagger \right) + \tfrac{1}{2} \sum_{ij} \left( B_{ij} c_i^\dagger c_j^\dagger + \bar{B}_{ij} c_j c_i \right) \ .$$

The Hamiltonians $H$ act on the field operators $\psi$ by the commutator, $\psi \mapsto [H, \psi]$, and the evolution with time is determined by the Heisenberg equation of motion,

$$\mathrm{i}\hbar \frac{d\psi}{dt} = [H, \psi] \ ,$$

where $\hbar$ is Planck's constant. By the canonical anticommutation relations, this dynamical equation is equivalent to a system of linear differential equations for the coefficients $v_i$ and $f_i$:

$$\begin{aligned} \mathrm{i}\hbar \dot{v}_i &= \sum_j \left( A_{ij} v_j + B_{ij} f_j \right) \ , \\ -\mathrm{i}\hbar \dot{f}_i &= \sum_j \left( \bar{B}_{ij} v_j + \bar{A}_{ij} f_j \right) \ . \end{aligned}$$

If these are assembled into a column vector $\mathbf{v}$, the evolution equation takes the form

$$\dot{\mathbf{v}} = X\mathbf{v} \ , \quad X = -\frac{\mathrm{i}}{\hbar} \begin{pmatrix} A & B \\ -\bar{B} & -\bar{A} \end{pmatrix} \ .$$

The matrix elements of $X$ obey the relations $B_{ij} = -B_{ji}$ (from $c_i c_j = -c_j c_i$) and $A_{ij} = \bar{A}_{ji}$ (from the physical requirement of self-adjointness of $H$).

To recast all this in concise mathematical terms, recall the grading of the Clifford algebra $C(W) = \mathbb{C} \oplus C^1(W) \oplus C^2(W) \oplus \ldots$. From the Clifford algebra perspective, the Hamiltonian $H$ is viewed as an operator in the degree-two component $C^2(W)$.

For our purposes it is therefore important to accumulate some information about $C^2(W)$. It is well known [**B**] that $C^2(W)$ is a Lie algebra with the commutator playing the role of the Lie bracket; in fact $C^2(W)$ is canonically isomorphic to the complex orthogonal Lie algebra $\mathfrak{so}(W,b)$ associated with the vector space $W = V \oplus V^*$ and its canonical symmetric complex bilinear form $b$. Here the Lie algebra $\mathfrak{so}(W,b)$ is defined to be the subspace of elements $E \in \mathrm{End}(W)$ satisfying the condition

$$b(E\psi,\tilde{\psi}) + b(\psi,E\tilde{\psi}) = 0 \,.$$

If $E$ is any endomorphism in $\mathrm{End}(V \oplus V^*)$, decompose it into blocks as

$$E = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} ,$$

where $\mathsf{A} \in \mathrm{End}(V)$, $\mathsf{B} \in \mathrm{Hom}(V^*,V)$, $\mathsf{C} \in \mathrm{Hom}(V,V^*)$ and $\mathsf{D} \in \mathrm{End}(V^*)$. Let the adjoint (or transpose) of $\mathsf{A} \in \mathrm{End}(V)$ be denoted by $\mathsf{A}^{\mathrm{t}} \in \mathrm{End}(V^*)$.

***Proposition 2.3***. — *An endomorphism* $E = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \in \mathrm{End}(V \oplus V^*)$ *lies in the complex orthogonal Lie algebra* $\mathfrak{so}(V \oplus V^*, b)$ *if and only if* $\mathsf{B},\mathsf{C}$ *are skew and* $\mathsf{D} = -\mathsf{A}^{\mathrm{t}}$.

*Proof.* — Consider first the case $\mathsf{B} = \mathsf{C} = 0$, and let $\psi = v + f$ and $\tilde{\psi} = \tilde{v} + \tilde{f}$. Then

$$\begin{aligned} b(E\psi,\tilde{\psi}) &= b(\mathsf{A}v - \mathsf{A}^{\mathrm{t}}f, \tilde{v}+\tilde{f}) = \tilde{f}(\mathsf{A}v) - \mathsf{A}^{\mathrm{t}}f(\tilde{v}) \\ &= \mathsf{A}^{\mathrm{t}}\tilde{f}(v) - f(\mathsf{A}\tilde{v}) = -b(v+f, \mathsf{A}\tilde{v} - \mathsf{A}^{\mathrm{t}}\tilde{f}) = -b(\psi, E\tilde{\psi}) \,. \end{aligned}$$

A similar calculation for the case $\mathsf{A} = 0$ gives

$$\begin{aligned} b(E\psi,\tilde{\psi}) &= b(\mathsf{B}f + \mathsf{C}f, \tilde{v}+\tilde{f}) = \tilde{f}(\mathsf{B}f) + \mathsf{C}v(\tilde{v}) \\ &= -f(\mathsf{B}\tilde{f}) - \mathsf{C}\tilde{v}(v) = -b(v+f, \mathsf{B}\tilde{f} + \mathsf{C}\tilde{v}) = -b(\psi, E\tilde{\psi}) \,. \end{aligned}$$

Since these two cases complement each other, the statement follows. □

By fixing orthonormal bases $c_1^\dagger, c_2^\dagger, \ldots$ of $V$ and $c_1, c_2, \ldots$ of $V^*$ as before, we assign matrices with matrix elements $A_{ij}$, $B_{ij}$, $C_{ij}$ to the linear operators $\mathsf{A}$, $\mathsf{B}$, $\mathsf{C}$. A straightforward computation using the canonical anticommutation relations then yields:

***Proposition 2.4***. — *The* $\mathbb{C}$*-linear mapping from* $\mathfrak{so}(W,b)$ *to* $C^2(W)$ *given by*

$$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & -\mathsf{A}^{\mathrm{t}} \end{pmatrix} \mapsto \tfrac{1}{2}\textstyle\sum_{ij} A_{ij}(c_i^\dagger c_j - c_j c_i^\dagger) + \tfrac{1}{2}\textstyle\sum_{ij} (B_{ij} c_i^\dagger c_j^\dagger + C_{ij} c_i c_j)$$

*is an isomorphism of Lie algebras.*

In addition to acting on itself by the commutator, the Lie algebra $C^2(W)$ acts (still by the commutator) on all of the components $C^k(W)$ of degree $k \geq 1$ of the Clifford algebra $C(W)$. In particular, $C^2(W)$ acts on the degree-one component $C^1(W)$. By the isomorphisms $C^2(W) \cong \mathfrak{so}(W,b)$ and $C^1(W) \cong W$, this action coincides with the fundamental representation of $\mathfrak{so}(W,b)$ on its defining vector space $W$. In other words, taking the commutator of the Hamiltonian $H \in C^2(W)$ with a field operator

$\psi \in C^1(W)$ yields the same answer as viewing $H$ as an element of $\mathfrak{so}(W,b)$, then applying $H = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & -\mathsf{A}^t \end{pmatrix}$ to the vector $\psi = v + f \in V \oplus V^*$ by

$$H \cdot (v+f) = (\mathsf{A}v + \mathsf{B}f) + (\mathsf{C}v - \mathsf{A}^t f) \;,$$

and finally reinterpreting the result as a field operator in $C^1(W)$.

The closure relation $[C^2(W), C^1(W)] \subset C^1(W)$ and the isomorphisms $C^1(W) \cong W$ and $C^2(W) \cong \mathfrak{so}(W,b)$ make it possible to reduce the dynamics of field operators to a dynamics on the Nambu space $W$. As we have seen, after reduction the generators $X \in \mathrm{End}(V \oplus V^*)$ of time evolutions of the physical system are of the special form

$$X = -\frac{\mathrm{i}}{\hbar} \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{B}^* & -\mathsf{A}^t \end{pmatrix} \;,$$

where $\mathsf{B} \in \mathrm{Hom}(V^*, V)$ is skew, and $\mathsf{A} = \mathsf{A}^* \in \mathrm{End}(V)$ is self-adjoint w.r.t. $\langle\, ,\rangle_V$.

***Proposition 2.5***. — *The one-parameter groups of time evolutions $t \mapsto \mathrm{e}^{tX}$ in the Nambu space model preserve both the canonical unitary structure $\langle\, ,\rangle$ and the canonical symmetric complex bilinear form $b$ of $W = V \oplus V^*$.*

*Proof*. — By Prop. 2.3 the generator $X$ is an element of the complex Lie algebra $\mathfrak{so}(W,b)$. Hence the exponential $U_t = \mathrm{e}^{tX}$ lies in the complex orthogonal Lie group $\mathrm{SO}(W,b)$, which is defined to be the set of solutions $g$ in $\mathrm{End}(W)$ of the conditions

$$b(g\psi, g\tilde{\psi}) = b(\psi, \tilde{\psi}) \;, \quad \text{and} \quad \mathrm{Det}(g) = 1 \;.$$

Since $\mathsf{A} = \mathsf{A}^*$, and $\mathsf{B}^* \in \mathrm{Hom}(V, V^*)$ is the adjoint of $\mathsf{B} \in \mathrm{Hom}(V^*, V)$, the generator $X$ is antihermitian with respect to the unitary structure of $W$. The exponentiated generator $U_t$ therefore lies in the unitary group $\mathrm{U}(W)$, which is to say that

$$\langle U_t \psi, U_t \tilde{\psi} \rangle = \langle \psi, \tilde{\psi} \rangle$$

for all real $t$. Thus $U_t$ preserves both $b$ and $\langle\, ,\rangle$. ☐

***Remark***. — In physical language, the invariance of $b$ under time evolutions means that the canonical anticommutation relations for fermionic field operators do not change with time. Invariance of $\langle\, ,\rangle$ means that probability in Nambu space is conserved. (If the quadratic Hamiltonian $H$ arises as the mean-field approximation to some many-fermion problem, the latter conservation law holds as long as quasiparticles do not interact and thereby are protected from decay into multi-particle states.)

We now distill the essence of the information conveyed in this section. The quantum theory of many-fermion systems is set up in a Hilbert space called the fermionic Fock space in physics (or the spinor representation in mathematics). The field operators of the physical system span a vector space $W = V \oplus V^*$, which generates a Clifford algebra $C(W)$ whose defining relations are the canonical anticommutation relations.

Since $[C^2(W), C^1(W)] \subset C^1(W)$, the discussion of the field operator dynamics for the important case of quadratic Hamiltonians $H \in C^2(W)$ can be reduced to a discussion on the Nambu space $W \cong C^1(W)$. Via this reduction, the vector space $W$ inherits two natural structures: the canonical symmetric complex bilinear form $b$ encoding the anticommutation relations, and a canonical unitary structure $\langle\, ,\rangle$ determined by the Hermitian scalar product of $V$. Both of these structures are invariant, i.e., are preserved by physical time evolutions. Under the reduction to $W$, the commutator action of $C^2(W)$ on $C^1(W)$ becomes the fundamental representation of $\mathfrak{so}(W, b)$ on $W$.

**2.2. Symmetry groups. —** Following Dyson, the classification of disordered fermion systems will be carried out in a setting that prescribes two pieces of data:

- One is given a Nambu space $W = V \oplus V^*$ equipped with its canonical unitary structure $\langle\, ,\rangle$ and canonical symmetric $\mathbb{C}$-bilinear form $b$.
- On $W$ there acts a group $G$ of unitary and antiunitary operators (the joint symmetry group of a multi-parameter family of fermionic quantum systems).

Given this setup, one is interested in the linear space of Hamiltonians $H$ with the property that they commute with the $G$-action on $W$, while preserving the invariant structures $b$ and $\langle\, ,\rangle$ of $W$ under time evolution by $\mathrm{e}^{-\mathrm{i}tH/\hbar}$. Such a space of Hamiltonians is of course reducible in general, i.e., the Hamiltonian matrices decompose into blocks. The goal of classification is to enumerate all the *symmetry classes*, i.e., all the types of irreducible block which occur in this way.

In the present subsection, we provide some information on what is meant by unitary and antiunitary symmetries in the present context. We begin by recalling the basic notion of a symmetry group in quantum Hamiltonian systems.

In classical mechanics the symmetry group $G_0$ of a Hamiltonian system is understood to be the group of symplectomorphisms that commute with the phase flow of the system. Examples are the rotation group for systems in a central field, and the group of Euclidean motions for systems with Euclidean invariance.

In passing from classical to quantum mechanics, one replaces the classical phase space by a complex Hilbert space $V$, and assigns to the symmetry group $G_0$ a (projective) representation by unitary $\mathbb{C}$-linear operators on $V$. While the consequences due to one-parameter continuous subgroups of $G_0$ are particularly clear from Noether's theorem, the components of $G_0$ *not* connected with the identity also play an important role. A prominent example is provided by the operator for space reflection. Its eigenspaces are the subspaces of states with positive and negative parity, and they reduce the matrix of any reflection-invariant Hamiltonian to two blocks.

Not all symmetries of a quantum mechanical system are of the canonical, unitary kind: the prime counterexample is the operation $g_T$ of inverting the time direction – called time reversal for short. In classical mechanics this operation reverses the sign of the symplectic structure of phase space; in quantum mechanics its algebraic properties reflect the fact that the time $t$ enters in the Dirac, Pauli, or Schrödinger equation as

$\mathrm{i}\hbar d/dt$: there, time reversal $g_T$ is represented by an *antiunitary* operator $T$, which is to say that $T$ is complex antilinear:

$$T(zv) = \bar{z}Tv \quad (z \in \mathbb{C},\ v \in V)\ ,$$

and preserves the Hermitian scalar product up to complex conjugation:

$$\langle v, \tilde{v} \rangle_V = \overline{\langle Tv, T\tilde{v} \rangle_V}\ .$$

Another example of such an operation is charge conjugation in relativistic theories. Further examples are provided by chiral symmetry transformations (see Sect. 2.3).

By the symmetry group $G$ of a quantum mechanical system with Hamiltonian $H$, one then means the group of all unitary and antiunitary transformations $g$ of $V$ that leave the Hamiltonian invariant: $gHg^{-1} = H$. It should be noted that finding the total symmetry group of a quantization of some Hamiltonian system is not always straightforward. The reason is that there may exist nonobvious quantum symmetries such as Hecke symmetries, which are of number-theoretic origin and have no classical limit. For our purposes, however, this complication will not be an issue. We take the group $G$ and its action on the Hilbert space to be *fundamental and given*, and then ask what is the linear space of Hamiltonians that commute with the $G$-action.

For technical reasons, we assume the group $G_0$ to be compact; this is an assumption that covers most (if not all) of the cases of interest in physics. The noncompact group of space translations can be incorporated, if necessary, by wrapping the system around a torus, whereby translations are turned into compact torus rotations.

What we have sketched – a symmetry group $G$ acting on a Hilbert space $V$ – is the framework underlying Dyson's classification. As was explained in Sect. 2.1, we wish to enlarge it so as to capture all examples that arise in disordered fermion physics.

For this, recall that in the Nambu space model for fermions, the Hilbert space is not $V$ but the space of field operators $W = V \oplus V^*$. The given $G$-representation on $V$ therefore needs to be extended to a representation on $W$. This is done by the condition that the pairing between $V$ and $V^*$ (and thus the pairing between fermion creation and annihilation operators) be preserved. In other words, if $U : V \to V$ and $A : V \to V$ are unitary resp. antiunitary operators, their induced representations on $V^*$ (which we still denote by the same symbols) are defined by requiring that

$$(Uf)(Uv) = f(v) = \overline{(Af)(Av)}$$

for all $v \in V$ and $f \in V^*$. In particular the $G_0$-representation on $V^*$ is the *dual* one,

$$U(f) = f \circ U^{-1}\ .$$

Equivalently, the $G$-representation on $W$ is defined so as to be compatible with particle-hole conjugation $C : W \to W$ in the sense that operations commute:

$$CU = UC\ , \quad \text{and} \quad CA = AC\ .$$

Indeed, if $f = Cv$ then $f(\tilde{v}) = \langle v, \tilde{v}\rangle$ and from the invariance of the pairing between $V$ and $V^*$ one infers the relations $\langle v, \tilde{v}\rangle = (Uf)(U\tilde{v}) = \langle U^{-1}C^{-1}UCv, \tilde{v}\rangle$ and $\langle v, \tilde{v}\rangle = \overline{(Af)(A\tilde{v})} = \langle A^{-1}C^{-1}ACv, \tilde{v}\rangle$.

While the framework so obtained is flexible enough to capture the situations that arise in the nonrelativistic quasiparticle physics of disordered metals, semiconductors and superconductors, it is still slightly too narrow to accommodate some much studied examples that have emerged from elementary particle physics. Let us explain this.

**2.3. The Euclidean Dirac operator. —** An important development in random-matrix physics over the last ten years was the formulation [**VZ**] and study of the so-called chiral ensembles, which model Dirac fermions in a random gauge field background, and lie beyond Dyson's 3-way classification. From the viewpoint of applications, these random-matrix models have the merit of capturing some universal features of the Dirac spectrum of quantum chromodynamics (QCD) in the low-energy limit. In the present subsection we will demonstrate that, but for one minor difference, they fit naturally into our fermionic Nambu space model with symmetries.

Let $M$ be a four-dimensional Euclidean space-time (more generally, $M$ could be a Riemannian 4-manifold with spin structure), and consider over $M$ a unitary spinor bundle $S$ twisted by a module $R$ for the action of some compact gauge group $K$. Denote by $V$ the Hilbert space of $L^2$-sections of the twisted bundle $S \otimes R$.

Now let $D_A$ be a self-adjoint Dirac operator for $V$ in a given gauge field background (or gauge connection) $A$. Although $D_A$ is not a Hamiltonian in the strict sense of the word, it has all the right mathematical attributes in the sense of Sect. 2.1; in particular it determines a Hermitian form, called the action functional, on differentiable sections $\psi \in V$. In physics notation this functional is written

$$\psi \mapsto \int_M \bar{\psi}(x) \cdot (D_A\psi)(x)\, d^4x\,, \quad D_A = \mathrm{i}\gamma^\mu(\partial_\mu - A_\mu)\,,$$

where $\gamma^\mu = \gamma(e^\mu)$ are the gamma matrices [i.e., the Clifford action $\gamma : T^*M \to \mathrm{End}(S)$ evaluated on the dual $e^\mu$ of an orthonormal coordinate frame $e_\mu$ of $TM$], the operators $\partial_\mu$ are the partial derivatives corresponding to the $e_\mu$, and $A_\mu(x) \in \mathrm{Lie}(K)$ are the components of the gauge field. If the physical situation calls for a mass, then one adds a complex number $\mathrm{i}m$ (times the unit operator on $V$) to the expression for $D_A$.

The Dirac operators of prime interest to low-energy QCD have zero (or small) mass. To express the massless nature of $D_A$ one introduces an object called the *chirality* operator $\Gamma$ in mathematics [**B**], or $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$ in physics. $\Gamma = \gamma_5$ is a section of $\mathrm{End}(S)$ which is self-adjoint and involutory ($\Gamma^2 = \mathrm{Id}$) and anticommutes with the Clifford action ($\Gamma\gamma^\mu + \gamma^\mu\Gamma = 0$). By the last property one has

$$\Gamma D_A + D_A \Gamma = 0$$

in the massless limit. This relation is called chiral symmetry in physics. Note, however, that chiral 'symmetry' *is not a symmetry* in the sense of the present paper. (Symmetries always *commute* with the Hamiltonian, never do they anticommute with it!) Nonetheless, we shall now recognize chiral symmetry as being equivalent to a true symmetry, by importing the Dirac operator into the Nambu space model as follows.

As before, take Nambu space to be the sum $W = V \oplus V^*$ equipped with its canonical unitary structure $\langle\, ,\, \rangle$ and symmetric complex bilinear form $b$. The antilinear bijection $C : V \to V^*$ and $C : V^* \to V$ is still defined by $\langle w_1, w_2 \rangle = b(Cw_1, w_2)$.

Now extend the Dirac operator $D_A \in \mathrm{i}\mathfrak{u}(V)$ to an operator $D_A$ that acts diagonally on $W = V \oplus V^*$, by requiring $D_A$ to satisfy the commutation law $C\,\mathrm{i}D_A = \mathrm{i}D_A C$, or equivalently $CD_A = -D_A C$. Thus,

$$D_A \in \mathrm{End}(V) \oplus \mathrm{End}(V^*) \hookrightarrow \mathrm{End}(W) \;,$$

and $D_A$ on $\mathrm{End}(V^*)$ is given by $-D_A^{\mathrm{t}}$. The diagonally extended operator $D_A$ lies in the intersection of $\mathfrak{so}(W, b)$ with $\mathrm{i}\mathfrak{u}(W)$ – as is required in order for the statement of Prop. 2.5 to carry over to the one-parameter group $t \mapsto \mathrm{e}^{\mathrm{i}tD_A}$. The property that $D_A$ does not mix $V$ and $V^*$ can be attributed to the existence of a $\mathrm{U}_1$ symmetry group that has $V$ and $V^*$ as inequivalent representation spaces.

To implement the chiral symmetry of the massless limit, extend the chirality operator $\Gamma$ to a diagonally acting endomorphism in $\mathrm{End}(V) \oplus \mathrm{End}(V^*)$ by $C\Gamma C^{-1} = \Gamma$. The extended operators still satisfy the chiral symmetry relation $\Gamma D_A + D_A \Gamma = 0$. Then define an antiunitary operator $T$ by $T := C\Gamma$. Note that this is *not* the operation of inverting the time but will still be called the 'time reversal' for short.

Because $D_A$ anticommutes with both $C$ and $\Gamma$, one has

$$T D_A T^{-1} = D_A \;.$$

Thus $T$ is a true symmetry of the (extended) Dirac operator in the massless limit.

Note that $CT = TC$ from $C\Gamma = \Gamma C$. As was announced above, the situation is the same as before but for one difference: while the time reversal in Sect. 2.2 was an operator $T : V \to V$ and $T : V^* \to V^*$, the present one is an operator $T : V \to V^*$ and $T : V^* \to V$. We refer to the latter type as *mixing*, and the former as *nonmixing*.

To summarize, physical systems modelled by the Euclidean (or positive signature) Dirac operator are naturally incorporated into the framework of Sects. 2.1 and 2.2. The Hilbert space $V$ here is the space of $L^2$-sections of a twisted spinor bundle over Euclidean space-time, and the role of the Hamiltonian is taken by the quadratic action functional of the Dirac fermion theory. When transcribed into the Nambu space $W = V \oplus V^*$, the chiral 'symmetry' of the massless theory can be expressed as a true antiunitary symmetry $T$, with the only new feature being that $T$ mixes $V$ and $V^*$.

The most general situation occurring in physics may exhibit, beside $T$, one or several other antiunitary symmetries. In the example at hand this happens if the representation space $R$ carries a complex bilinear form which is invariant under gauge transformations (see Sects. 6.2.2 and 6.2.3 for the details). The Dirac operator $D_A$ then has one extra antiunitary symmetry, say $T_1$, which is nonmixing. Forming the composition of $T_1$ with

$T$ we get a *mixing unitary symmetry* $P = TT_1 : V \leftrightarrow V^*$. This fact leads us to adopt the final framework described in the next subsection.

**2.4. The mathematical model. —** The following model is now well motivated.

We are given a Nambu space $(W, b, \langle\,,\,\rangle)$ carrying the action of a compact group $G$. The group $G_0$ is defined to be the subgroup of $G$ which acts by canonical unitary transformations, i.e., unitary transformations that preserve the decomposition $W = V \oplus V^*$. The full symmetry group $G$ is generated by $G_0$ and at most two distinguished antiunitary time-reversal operators. If there is just one, we denote it by $T$, and if there are two, by $T$ and $T_1$. In the latter case we adopt the convention that $T$ mixes, i.e., $T : V \to V^*$, while $T_1$ is nonmixing. The distinguished time-reversal symmetries always satisfy $T^2 = \pm\mathrm{Id}$ and $T_1^2 = \pm\mathrm{Id}$. In the case that there are two, it is assumed that they commute or anticommute, i.e., $T_1 T = \pm T T_1$. Consequently the unitary operator $P = TT_1$ (which mixes) also satisfies $P^2 = \pm\mathrm{Id}$. In the situation with $P$ present we let $G_1$ denote the $\mathbb{Z}_2$-extension of $G_0$ defined by $P$ and refer to it as the full group of unitary symmetries.

We emphasize that the original action of $G_0$ on $V$ has been extended to $W$ via its canonically induced action on $V^*$. In other words, if $f \in V^*$ then $g(f)(v) = f(g^{-1}(v))$. This is equivalent to requiring that a unitary operator $U \in G_0$ commutes with particle-hole conjugation $C : W \to W$. In fact we require that all operators of $G$ commute with $C$. Whereas the unitary operators preserve the Hermitian scalar product $\langle\,,\,\rangle$, for an antiunitary operator $A$ we have that $\langle Aw_1, Aw_2\rangle = \overline{\langle w_1, w_2\rangle}$ for all $w_1, w_2 \in W$.

If $U$ is an operator coming from $G_0$ and $T$ is a distinguished time-reversal symmetry, then $TUT^{-1}$ is unitary and nonmixing, i.e., it is in $G_0$. Thus, for the corresponding operator $g_T$ in $G$, we assume that $g_T$ normalizes $G_0$ and $g_T^2$ is in the center of $G_0$.

According to Prop. 2.5 the time evolutions of the physical system leave the structure of Nambu space invariant. The infinitesimal version of this statement is that the Hamiltonians $H$ lie in the intersection of the complex orthogonal Lie algebra $\mathfrak{so}(W, b)$ with $\mathrm{i}\mathfrak{u}(W)$, the Hermitian operators on $W$.

Let us summarize our situation in the language and notation introduced above.

***Definition 2.6***. — *The data in the Nambu space model for fermions with symmetries is* $(W, b, \langle\,,\,\rangle; G)$, *where the compact group* $G$ *is called the symmetry group of the system.* $G$ *is represented on* $W = V \oplus V^*$ *by unitary and antiunitary operators that preserve the structure of* $W$*; i.e., for every unitary* $U$ *and antiunitary* $A$ *one has*

$$\langle\psi, \tilde\psi\rangle = \langle U\psi, U\tilde\psi\rangle = \overline{\langle A\psi, A\tilde\psi\rangle}\,, \quad b(\psi, \tilde\psi) = b(U\psi, U\tilde\psi) = \overline{b(A\psi, A\tilde\psi)}$$

*for all* $\psi, \tilde\psi \in W$. *The space of 'good' Hamiltonians is the* $\mathbb{R}$*-vector space* $\mathcal{H}$ *of operators* $H$ *in* $\mathfrak{so}(W, b) \cap \mathrm{i}\mathfrak{u}(W)$ *that commute with the* $G$*-action:*

$$UHU^{-1} = H = AHA^{-1}\,.$$

*At the group level of time evolutions this means that*

$$U \mathrm{e}^{-\mathrm{i}tH/\hbar} = \mathrm{e}^{-\mathrm{i}tH/\hbar} U , \quad A \mathrm{e}^{-\mathrm{i}tH/\hbar} = \mathrm{e}^{+\mathrm{i}tH/\hbar} A ,$$

*for all unitary $U$, antiunitary $A$, $H \in \mathcal{H}$, and $t \in \mathbb{R}$.*

We remind the reader that the subgroup of unitary operators which preserves the decomposition $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^*$ is denoted by $G_0$, and the full group of unitaries by $G_1$.

Several further remarks are in order. First, for a unitary $U \in G_1$ (resp. antiunitary $A$), the compatibility of $b$ with the $G$-action is a consequence of Prop. 2.2 and the commutation law $CU = UC$ and $CA = AC$. Second, it is possible that the fermion system does not have any antiunitary symmetries and $G = G_0$. When some antiunitary symmetries are present, $G$ is generated by $G_0$ and one or at most two distinguished time-reversal symmetries as explained above. Third, motivated by the prime physics example of time reversal, we have assumed that the (one or two) distinguished time-reversal symmetries $T$ satisfy $T^2 = \pm \mathrm{Id}$. The reason for this can be explained as follows.

The operator $T$ has been chosen to represent some kind of *inversion* symmetry. Since this means that conjugation by $T^2$ represents the unit operator, $T^2$ must be a unitary multiple of the identity on any subspace of $\mathcal{W}$ which is irreducible under time evolutions of the fermion system. Thus for all practical purposes we may assume that $T$ is a projective involution, i.e., $T^2 = z \times \mathrm{Id}$ with $z$ a complex number of unit modulus.

***Proposition 2.7***. — *If a projective involution $T : \mathcal{W} \to \mathcal{W}$ of a unitary vector space $\mathcal{W}$ is antiunitary, then either $T^2 = +\mathrm{Id}_{\mathcal{W}}$ or $T^2 = -\mathrm{Id}_{\mathcal{W}}$.*

*Proof*. — A projective involution $T$ has square $T^2 = z \times \mathrm{Id}$ with $z \in \mathbb{C} \setminus \{0\}$. Since $T$ is antiunitary, $T^2$ is unitary, and hence $|z| = 1$. But an antiunitary operator is $\mathbb{C}$-antilinear, and therefore the associative law $T^2 \cdot T = T \cdot T^2$ forces $z$ to be real, leaving only the possibilities $T^2 = \pm \mathrm{Id}$. □

Since this work is meant to simultaneously handle symmetry at both the Lie algebra and Lie group level, a final word should be said about the notion that a bilinear form $F$ is respected by a transformation $B$. At the group level when $B$ is invertible and is regarded as being in $\mathrm{GL}(W)$, where $W$ is the underlying vector space of $F : W \times W \to \mathbb{C}$, this means that $B$ is an isometry in the sense that $F(Bw_1, Bw_2) = F(w_1, w_2)$ for all $w_1, w_2 \in W$. On the other hand, at the Lie algebra level where $B \in \mathrm{End}(W)$, this means that for all $w_1, w_2 \in W$ one has

$$\frac{d}{dt}\Big|_{t=0} F(\mathrm{e}^{tB} w_1, \mathrm{e}^{tB} w_2) = F(Bw_1, w_2) + F(w_1, Bw_2) = 0 .$$

## 3. Reduction to the case of $G_0 = \{\mathrm{Id}\}$

Recall that our main goal, e.g., on the Lie algebra level, is to describe the space of $G_0$-invariant endomorphisms which on a block in Nambu space are compatible with the unitary structure, time reversal and the symmetric $\mathbb{C}$-bilinear form.

Here we prove results which allow us to transfer this space to a certain space of $G_0$-equivariant homomorphisms. The unitary structure, time reversal and the bilinear form are essentially canonically transferred, and as before, compatibility with these structures is required. However, in the new setting $G_0$ acts trivially. This is of course an essential simplification.

**3.1. Spaces of equivariant homomorphisms. —** Throughout this section $\lambda \in \hat{G}_0$ denotes a fixed isomorphism class (i.e., an equivalence class of irreducible representations of $G_0$), and $\lambda^*$ denotes its dual. A *block* is determined by a choice of finite-dimensional $G_0$-invariant subspace $V = V_\lambda$ (in the given Hilbert space $\mathcal{V}$) such that all of its irreducible subrepresentations have isomorphism class $\lambda$.

The full group $G$ of (unitary and antiunitary) symmetries is generated by $G_0$ and at most two distinguished time-reversal symmetries. In this section it is assumed throughout that these time-reversal operators stabilize the truncated subspace $W = V \oplus V^*$ of Nambu space. The case where one or both time-reversal symmetries do not stabilize $W$, i.e., where a larger block is generated, is handled in later sections.

If $\langle\, ,\rangle_V$ is the initial unitary structure on $V$, one defines $C : V \to V^*$ by $C(v)(w) = \langle v, w\rangle_V$. Taking $C|_{V^*}$ to be the inverse of this map, one obtains the associated $\mathbb{C}$-antilinear isomorphism $C : W \to W$. All symmetries in $G$ are assumed to commute with $C$. We remind the reader that $G_0$ acts on $V^*$ by $g(f) = f \circ g^{-1}$.

Let $R$ be a fixed irreducible $G_0$-representation space which is in $\lambda$. Denote by $d$ its dimension. Of course $R^*$ is a representative of $\lambda^*$. We fix an antilinear bijection $\iota : R \to R^*$ which is defined by a $G_0$-invariant unitary structure $\langle\, ,\rangle_R$ on $R$.

In the sequel we will often make use of the following consequence of Schur's Lemma.

***Proposition 3.1***. — *If two irreducible $G_0$-representation spaces $R_1$ and $R_2$ are equivariantly isomorphic by $\psi : R_1 \to R_2$, then $\mathrm{Hom}_{G_0}(R_1, R_2) = \mathbb{C}\cdot\psi$, i.e., the linear space of $G_0$-equivariant homomorphisms from $R_1$ to $R_2$ has complex dimension one and every operator in it is some multiple of $\psi$.*

The following related statement was essential to Dyson's classification and will play a similarly important role in the present article.

***Lemma 3.2***. — *If an irreducible $G_0$-representation space $R$ is equivariantly isomorphic to its dual $R^*$ by an isomorphism $\psi : R \to R^*$, then $\psi$ is either symmetric or alternating, i.e., either $\psi(r)(t) = \psi(t)(r)$ or $\psi(r)(t) = -\psi(t)(r)$ for all $r, t \in R$.*

*Proof.* — It is convenient to think of $\psi$ as defining an invariant bilinear form $B(r,t) = \psi(r)(t)$ on $R$. We then decompose $B$ into its symmetric and alternating parts, $B = S + A$, where

$$S(r,t) = \tfrac{1}{2}\big(B(r,t) + B(t,r)\big) \quad \text{and} \quad A(r,t) = \tfrac{1}{2}\big(B(r,t) - B(t,r)\big) \, .$$

Both are $G_0$-invariant, and consequently their degeneracy subspaces are invariant. Since the representation space $R$ is irreducible, it follows that each is either nondegenerate or vanishes identically. But both being nondegenerate would violate the fact that up to a constant multiple there is only one equivariant isomorphism in $\mathrm{End}(R)$. Therefore $B$ is either symmetric or alternating as claimed. □

Now let $H := \mathrm{Hom}_{G_0}(R,V)$ be the space of $G_0$-equivariant linear mappings from $R$ to $V$. Its dual space is $H^* = \mathrm{Hom}_{G_0}(R^*,V^*)$. The key space for our first considerations is $(H \otimes R) \oplus (H^* \otimes R^*)$. Note that $G_0$ acts on it by

$$g(h \otimes r + f \otimes t) = h \otimes g(r) + f \otimes g(t) \, .$$

We can apply $h \in H$ to $r \in R$ to form $h(r) \in V$. Since $h$ is $G_0$-equivariant we have $g \cdot h(r) = h(g(r))$. The same goes for the corresponding objects on the dual side. Thus in our finite-dimensional setting the following is immediate.

***Proposition 3.3.*** — *If* $H = \mathrm{Hom}_{G_0}(R,V)$ *and* $H^* = \mathrm{Hom}_{G_0}(R^*,V^*)$ *the map*

$$\begin{aligned} \varepsilon : (H \otimes R) \oplus (H^* \otimes R^*) &\to V \oplus V^* = W \, , \\ h \otimes r + f \otimes t &\mapsto h(r) + f(t) \, , \end{aligned}$$

*is a* $G_0$*-equivariant isomorphism.*

Transferring the unitary structure from $W$ to $(H \otimes R) \oplus (H^* \otimes R^*)$ induces a unitary structure on $H \oplus H^*$. For this, note for example that for $h_1 \otimes r_1$ and $h_2 \otimes r_2$ in $H \otimes R$ we have

$$\langle h_1 \otimes r_1, h_2 \otimes r_2 \rangle_{H \otimes R} := \langle h_1(r_1), h_2(r_2) \rangle_V \, .$$

Observe that for $h_1$ and $h_2$ fixed, the right-hand side of this equality defines a $G_0$-invariant unitary structure on $R$ which is unique up to a multiplicative constant. Thus we define $\langle \, , \rangle_H$ by

$$\langle h_1 \otimes r_1, h_2 \otimes r_2 \rangle_{H \otimes R} = \langle h_1, h_2 \rangle_H \cdot \langle r_1, r_2 \rangle_R \, .$$

Given the fixed choice of $\langle \, , \rangle_R$ this definition is canonical.

We will in fact transfer all of our considerations for $V \oplus V^*$ to the space $H \oplus H^*$, the latter being equipped with the unitary structure defined as above. One of the key points for this is to understand how to express a $G_0$-invariant endomorphism

$$S \in \mathrm{End}_{G_0}(V \oplus V^*) \;\cong_\varepsilon\; \mathrm{End}_{G_0}(H \otimes R \oplus H^* \otimes R^*)$$

as an element of $\mathrm{End}(H \oplus H^*)$. Also, we must understand the role of time reversal.

In this regard the two cases $\lambda \neq \lambda^*$ and $\lambda = \lambda^*$ pose slightly different problems. Before going into these in the next sections, we note several facts which are independent of the case.

First, let $V_1$ and $V_2$ be vector spaces where $G_0$ acts trivially, and let $R_1$ and $R_2$ be arbitrary $G_0$-representation spaces.

***Proposition 3.4***. —

$$\mathrm{Hom}_{G_0}(V_1 \otimes R_1, V_2 \otimes R_2) = \mathrm{Hom}(V_1, V_2) \otimes \mathrm{Hom}_{G_0}(R_1, R_2) \ .$$

*Proof*. — Note that $\mathrm{Hom}(V_1 \otimes R_1, V_2 \otimes R_2) = \mathrm{Hom}(V_1, V_2) \otimes \mathrm{Hom}(R_1, R_2)$, and let $(\varphi_1, \ldots, \varphi_m)$ be a basis of $\mathrm{Hom}(V_1, V_2)$. Then for every element $S$ of $\mathrm{Hom}(V_1, V_2) \otimes \mathrm{Hom}(R_1, R_2)$ there are unique elements $\psi_1, \ldots, \psi_m$ so that $S = \sum \varphi_i \otimes \psi_i$. If $S$ is $G_0$-equivariant, then

$$S = g \circ S \circ g^{-1} = \sum \varphi_i \otimes (g \circ \psi_i \circ g^{-1}) \ ,$$

and the desired result follows from the uniqueness statement. □

Our second general remark concerns the way in which a distinguished time-reversal symmetry $T$ is transferred to an antilinear endomorphism of $H \otimes R \oplus H^* \otimes R^*$. Let us consider for example the case of mixing where it is sufficient to understand $T : H \otimes R \to H^* \otimes R^*$. For that purpose we view $\mathrm{End}(H \otimes R)$ as $\mathrm{End}(H) \otimes \mathrm{End}(R)$, let $(\varphi_1, \ldots, \varphi_m)$ be a basis of $\mathrm{End}(H)$ and write

$$\Gamma = CT = \sum \varphi_i \otimes \psi_i$$

for $\psi_1, \ldots, \psi_m \in \mathrm{End}(R)$. Now $T$ is equivariant in the sense that $T \circ g = a(g) \circ T$, where $a$ is the automorphism of $G_0$ determined by conjugation with $g_T$. Thus, since the $\mathbb{C}$-antilinear operator $C$ intertwines $G_0$-actions, the $\mathbb{C}$-linear mapping $\Gamma = CT$ is invariant with respect to the twisted conjugation $\Gamma \mapsto a(g)\Gamma g^{-1}$. Consequently, every $\psi_i$ is invariant with respect to this conjugation.

This means that the $\psi_i : R \to R$ are equivariant with respect to the original $G_0$-representation on the domain space and the new $G_0$-action, $v \mapsto a(g)(v)$, on the image space. But by Prop. 3.1, up to a constant multiple there is only one such element of $\mathrm{End}(R)$, i.e., we may assume that

$$\Gamma = \varphi \otimes \psi \ ,$$

where $\psi$ is unique up to a multiplicative constant.

Note further that $C$ is also of this factorized form. Indeed, we have

$$\langle h \otimes r, \cdot \rangle_{H \otimes R} = \langle h, \cdot \rangle_H \, \langle r, \cdot \rangle_R \ ,$$

and if $\gamma : H \to H^*$ is defined by $h \mapsto \langle h, \cdot \rangle_H$, then $C = \gamma \otimes \iota$. Furthermore, since $\Gamma$ and $C$ are pure tensors, so is $T = C\Gamma = T_H \otimes T_R$ , with the factors being antilinear mappings $T_H = \gamma \circ \varphi : H \to H^*$ and $T_R = \iota \circ \psi : R \to R^*$.

Of course we have only considered a piece of $T$, and that only in the case of mixing. However, exactly the same arguments apply to the other piece and also in the case of nonmixing. Thus we have the following observation.

***Proposition 3.5***. — *The induced map*

$$T : (H \otimes R) \oplus (H^* \otimes R^*) \to (H \otimes R) \oplus (H^* \otimes R^*) \,,$$

*is the sum $T = A_1 \otimes B_1 + A_2 \otimes B_2$ of pure tensors.*

In the case of mixing this means that $A_1 \otimes B_1$ is an antilinear mapping from $H \otimes R$ to $H^* \otimes R^*$ and vice versa for $A_2 \otimes B_2$.

If $T$ doesn't mix, then $A_1 \otimes B_1 : H \otimes R \to H \otimes R$ and $A_2 \otimes B_2 : H^* \otimes R^* \to H^* \otimes R^*$. In this case we impose the natural condition that the $A_i$ and $B_i$ be antiunitary. For later purposes we note that this condition determines the factors only up to multiplication by a complex number of unit modulus. Using the formula $C = \gamma \otimes \iota$ and the fact that $C$ commutes with $T$, one immediately computes $A_2 \otimes B_2$ from $A_1 \otimes B_1$ (or vice versa). The involutory property $T^2 = \pm \mathrm{Id}$ also adds strong restrictions. Of course there may be two distinguished time reversals, $T$ and $T_1$, and we require that they commute with $C$ and $T_1 T = \pm T T_1$. These properties are automatically transferred at this level, because the transfer process from $(H \otimes R) \oplus (H^* \otimes R^*)$ to $V \oplus V^*$ is an isomorphism.

Finally, we prove an identity which is essential for transferring the complex bilinear form. For this we begin with

$$h \otimes r + f \otimes t \in (H \otimes R) \oplus (H^* \otimes R^*) \,,$$

apply $\varepsilon$ to obtain $h(r) + f(t)$, and then apply the linear function $f(t) \in V^*$ to the vector $h(r) \in V$. The result $f(t)(h(r))$ is to be compared to the product $f(h)\,t(r)$. Recall that the dimension of the vector space $R$ is denoted by $d$.

***Proposition 3.6***. —

$$f(t)(h(r)) = d^{-1} f(h)\,t(r) \,.$$

Before beginning the proof, which uses bases for the various spaces, we set the notation and prove a preliminary lemma. Let $m$ denote the multiplicity of the component $V$ and fix an identification

$$V \oplus V^* = R \oplus \ldots \oplus R \oplus R^* \oplus \ldots \oplus R^*$$

with $m$ summands of $R$ and $R^*$. Let $(e_1, \ldots, e_d)$ be a basis of $R$ and $(\vartheta_1, \ldots, \vartheta_d)$ be its dual basis. These define bases $(e_1^k, \ldots, e_d^k)$ and $(\vartheta_1^k, \ldots, \vartheta_d^k)$ of the corresponding $k$-th summands above. Let $I_R^k$ and $I_{R^*}^k$ be the respective identity mappings.

***Lemma 3.7***. —

$$I_{R^*}^\ell (I_R^k) = \delta_{k\ell}\, d \,.$$

*Proof.* — Expressing the operators in the bases, i.e.,

$$I_R^k = \sum_i \vartheta_i^k \otimes e_i^k \quad \text{and} \quad I_{R^*}^\ell = \sum_j e_j^\ell \otimes \vartheta_j^\ell \ ,$$

one has

$$I_{R^*}^\ell(I_R^k) = \sum_{i,j} \vartheta_i^k(e_j^\ell)\, \vartheta_j^\ell(e_i^k) = \sum_{i,j} \delta_{ij}^{k\ell} = \delta_{k\ell}\, d \ ,$$

which is the statement of the lemma. □

*Proof of Prop. 3.6.* — We expand $h \in H = \mathrm{Hom}_{G_0}(R,V)$ as $h = \sum h_k I_R^k$ , and $f \in H^* = \mathrm{Hom}_{G_0}(R^*,V^*)$ as $f = \sum f_\ell I_{R^*}^\ell$ . If $r = \sum r_i e_i$ and $t = \sum t_j \vartheta_j$ , then

$$h(r) = \sum_{i,k} h_k r_i\, e_i^k \quad \text{and} \quad f(t) = \sum_{j,\ell} f_\ell t_j\, \vartheta_j^\ell \ .$$

Thus

$$f(t)(h(r)) = \sum_{ijk\ell} \delta_{ij}^{k\ell} f_\ell h_k t_j r_i = \left(\sum_k f_k h_k\right) t(r) \ .$$

Prop. 3.6 now follows from the above lemma which implies that $f(h) = d \sum f_k h_k$.

**3.2. The case where $\lambda \neq \lambda^*$.** — Recall that our goal is to canonically transfer the data on $V \oplus V^*$ to $H \oplus H^*$, thus removing $G_0$ from the picture. In the case where $\lambda \neq \lambda^*$ this is a particularly simple task.

First, we apply Prop. 3.4 to transfer elements of $\mathrm{End}_{G_0}(V \oplus V^*)$. In the case at hand $\mathrm{Hom}_{G_0}(R,R^*)$ and $\mathrm{Hom}_{G_0}(R^*,R)$ are both zero, and both $\mathrm{End}_{G_0}(R)$ and $\mathrm{End}_{G_0}(R^*)$ are isomorphic to $\mathbb{C}$. Thus it follows from Prop. 3.4 that

$$\begin{aligned} \mathrm{End}_{G_0}(V \oplus V^*) &\cong \mathrm{End}_{G_0}(H \otimes R \oplus H^* \otimes R^*) \\ &\cong \mathrm{End}(H) \oplus \mathrm{End}(H^*) \hookrightarrow \mathrm{End}(H \oplus H^*) \ . \end{aligned}$$

We always normalize operators in $\mathrm{End}_{G_0}(H \otimes R)$ to the form $\varphi \otimes \mathrm{Id}_R$ and normalize operators in $\mathrm{End}_{G_0}(H^* \otimes R^*)$ in a similar way. Thus we identify $\mathrm{End}_{G_0}(V \oplus V^*)$ with $\mathrm{End}(H) \oplus \mathrm{End}(H^*)$ as a subspace of $\mathrm{End}(H \oplus H^*)$ and have the following result.

***Proposition 3.8.*** — *The condition that an operator in* $\mathrm{End}_{G_0}(V \oplus V^*)$ *respects the unitary structure on* $V \oplus V^*$ *is equivalent to the canonically transferred operator in* $\mathrm{End}(H \oplus H^*)$ *respecting the canonically transferred unitary structure on* $H \oplus H^*$.

Now let us turn to the condition of compatibility with a transferred time-reversal operator $T : H \otimes R \oplus H^* \otimes R^* \to H \otimes R \oplus H^* \otimes R^*$. There are a number of cases, depending on whether or not $T$ mixes and which of the conditions $T^2 = -\mathrm{Id}$ or $T^2 = \mathrm{Id}$ are satisfied. The arguments are essentially the same in every case. Let us first go through the details in one of them, the mixing case where $T^2 = -\mathrm{Id}$. To be consistent with the slightly more complicated discussion in the case where $\lambda = \lambda^*$, let us write this in matrix notation.

For $A \in \mathrm{End}(H)$ and $D \in \mathrm{End}(H^*)$, we regard

$$M = \begin{pmatrix} A \otimes \mathrm{Id}_R & 0 \\ 0 & D \otimes \mathrm{Id}_{R^*} \end{pmatrix}$$

as the associated transformation in $\mathrm{End}_{G_0}(H \otimes R \oplus H^* \otimes R^*)$. To construct the transferred time-reversal operator recall the statement of Prop. 3.5. In the setting under consideration $T$ squares to minus the identity; it is therefore expressed as

$$T = \begin{pmatrix} 0 & -\alpha^{-1} \otimes \beta^{-1} \\ \alpha \otimes \beta & 0 \end{pmatrix} ,$$

where $\alpha : H \to H^*$ and $\beta : R \to R^*$ are complex antilinear. Note that since $\alpha \otimes \beta = z\alpha \otimes z^{-1}\beta$, the mappings $\alpha$ and $\beta$ are determined only up to a common multiplicative constant $z \in \mathbb{C} \setminus \{0\}$. Conjugation of $M$ in $\mathrm{End}_{G_0}(H \otimes R \oplus H^* \otimes R^*)$ by $T$ yields

$$TMT^{-1} = \begin{pmatrix} \alpha^{-1} D \alpha \otimes \mathrm{Id}_R & 0 \\ 0 & \alpha A \alpha^{-1} \otimes \mathrm{Id}_{R^*} \end{pmatrix} .$$

Clearly, compatibility of $M$ with $T$ here means that $D = \alpha A \alpha^{-1}$.

Formulating this in a less detailed way gives the appropriate statement: conjugation of $M$ in $\mathrm{End}_{G_0}(H \otimes R \oplus H^* \otimes R^*)$ by $T$ yields the same compatibility condition as conjugating

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 0 & \mp\alpha^{-1} \\ \alpha & 0 \end{pmatrix} .$$

Here the sign in front of $\alpha^{-1}$ is arbitrary. For definiteness we choose it in such a way that the transferred time-reversal operator has the same involutory property $T^2 = -\mathrm{Id}$ or $T^2 = \mathrm{Id}$ as the original operator; in the case under consideration this means that we choose the minus sign.

***Proposition 3.9***. — *There is a transferred time-reversal operator $T : H \oplus H^* \to H \oplus H^*$ which satisfies either $T^2 = -\mathrm{Id}$ or $T^2 = \mathrm{Id}$. It mixes if and only if the original operator mixes, and a canonically transferred mapping in $\mathrm{End}(H \oplus H^*)$ commutes with it if and only if the original mapping in $\mathrm{End}_{G_0}(V \oplus V^*)$ commutes with the original time-reversal operator.*

*Proof.* — It only remains to handle the case of nonmixing, e.g., when $T^2 = -\mathrm{Id}$. As we have seen, $T : H \otimes R \to H \otimes R$ is a pure tensor:

$$T|_{H\otimes R} = \alpha \otimes \beta ,$$

which gives $T^2|_{H\otimes R} = \alpha^2 \otimes \beta^2 = -\mathrm{Id}_H \otimes \mathrm{Id}_R$ in the case at hand. Since the induced map $\beta : R \to R$ is antiunitary by convention, we have $\beta^2 = z \times \mathrm{Id}_R$ with $|z| = 1$. Associativity ($\beta^2 \cdot \beta = \beta \cdot \beta^2$) then implies $z = \pm 1$. Unlike the case of mixing, $\beta$ now plays a role through its parity. If $\beta^2 = +\mathrm{Id}_R$, the transferred time-reversal operator $\alpha$ on $H$ still satisfies $\alpha^2 = -\mathrm{Id}_H$. On the other hand, if $\beta^2 = -\mathrm{Id}_R$ we have $\alpha^2 = +\mathrm{Id}_H$ instead. Thus the involutory property $T^2 = \pm\mathrm{Id}$ is passed on to the transferred time-reversal operator, but depending on the involutory character of $\beta$ the parity may change. □

We remind the reader that two distinguished time-reversal symmetries may be present. The above shows that both can be transferred with appropriate involutory properties.

Further, it must be shown that they can be transferred (along with $C$) so that $TC = CT$, $T_1C = CT_1$, and $T_1T = \pm TT_1$ still hold. Even if there is just one such operator, it must be shown that the transferred operator can be chosen to satisfy $TC = CT$. Since the discussion for this is the same as in the case where $\lambda = \lambda^*$, we postpone it to Sect. 3.4.

Finally, we turn to the problem of transferring the complex bilinear form on $V \oplus V^*$ to $H \oplus H^*$. If $b$ denotes the pullback by $\varepsilon$ of the canonical symmetric bilinear form on $V \oplus V^*$, then by Prop. 3.6

$$b(h_1 \otimes r_1 + f_1 \otimes t_1, h_2 \otimes r_2 + f_2 \otimes t_2) = d^{-1}(f_2(h_1)t_2(r_1) + f_1(h_2)t_1(r_2)) \; .$$

Now in this case, i.e., where $\lambda \neq \lambda^*$, the $G_0$-invariant endomorphisms are acting on $H \otimes R \oplus H^* \otimes R^*$ by $\begin{pmatrix} A \otimes \mathrm{Id}_R & 0 \\ 0 & D \otimes \mathrm{Id}_{R^*} \end{pmatrix}$, where

$$A \oplus D \in \mathrm{End}(H) \oplus \mathrm{End}(H^*) \hookrightarrow \mathrm{End}(H \oplus H^*) \; .$$

Inserting the operator $A \oplus D$ into the above expression for $b$ we have the following fact involving the canonical symmetric bilinear form $s$ on $H \oplus H^*$,

$$s(h_1 + f_1, h_2 + f_2) = f_1(h_2) + f_2(h_1) \; .$$

***Proposition 3.10***. — *A map in* $\mathrm{End}_{G_0}(V \oplus V^*)$ *respects the canonical symmetric bilinear form if and only if the transferred map in* $\mathrm{End}(H) \oplus \mathrm{End}(H^*) \hookrightarrow \mathrm{End}(H \oplus H^*)$ *respects the canonical symmetric bilinear form* $s$ *on* $H \oplus H^*$.

In summary, we have shown that if $\lambda \neq \lambda^*$, then all relevant structures on $V \oplus V^*$ transfer to data of essentially the same type on $H \oplus H^*$ (the only exception being that the parity of the transferred time-reversal operator may be reversed). In this case $\mathrm{End}_{G_0}(V \oplus V^*)$ is canonically isomorphic to $\mathrm{End}(H) \oplus \mathrm{End}(H^*) \hookrightarrow \mathrm{End}(H \oplus H^*)$. An operator in $\mathrm{End}_{G_0}(V \oplus V^*)$ respects the original structures if and only if the corresponding operator in $\mathrm{End}(H \oplus H^*)$ respects the transferred structures on $H \oplus H^*$. The latter are the transferred unitary structure, induced time reversal and the symmetric bilinear form $s$.

**3.3. The case where $\lambda = \lambda^*$. —** Throughout this section it is assumed that $\lambda = \lambda^*$, and $\psi : R \to R^*$ is a $G_0$-equivariant isomorphism. Thus we have the identification

$$\begin{aligned} H \otimes R \oplus H^* \otimes R &\cong H \otimes R \oplus H^* \otimes R^*, \\ h \otimes r + f \otimes t &\mapsto h \otimes r + f \otimes \psi(t) \; . \end{aligned}$$

Applying Prop. 3.4 to each component of an operator in $\mathrm{End}_{G_0}(H \otimes R \oplus H^* \otimes R)$ it follows that

$$\mathrm{End}_{G_0}(H \otimes R \oplus H^* \otimes R^*) \cong \mathrm{End}(H \oplus H^*) \; .$$

We therefore identify $\mathrm{End}(H \oplus H^*)$ with $\mathrm{End}_{G_0}(H \otimes R \oplus H^* \otimes R^*) = \mathrm{End}_{G_0}(V \oplus V^*)$ by the mapping

$$M = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{A} \otimes \mathrm{Id}_R & \mathsf{B} \otimes \psi^{-1} \\ \mathsf{C} \otimes \psi & \mathsf{D} \otimes \mathrm{Id}_{R^*} \end{pmatrix} .$$

Recall the induced unitary structure which is defined, e.g., on $H \otimes R$ by

$$\langle h_1 \otimes r_1, h_2 \otimes r_2 \rangle_{H \otimes R} := \langle h_1(r_1), h_2(r_2) \rangle_V = \langle h_1, h_2 \rangle_H \langle r_1, r_2 \rangle_R .$$

It is easy to verify that this defines a unitary structure on $H \oplus H^*$ with the desired property: a map in $\mathrm{End}_{G_0}(V \oplus V^*)$ preserves the given unitary structure on $V \oplus V^*$ if and only if the transferred map $M$ preserves the induced unitary structure on $H \oplus H^*$.

Now let us consider time reversal. For example, take the case of nonmixing where $T_1 : H \otimes R \to H \otimes R$. Using Prop. 3.5 we have

$$T_1 = \begin{pmatrix} \alpha \otimes \beta & 0 \\ 0 & \tilde{\alpha} \otimes \tilde{\beta} \end{pmatrix} ,$$

and conjugating the transformation $\begin{pmatrix} \mathsf{A} \otimes \mathrm{Id}_R & \mathsf{B} \otimes \psi^{-1} \\ \mathsf{C} \otimes \psi & \mathsf{D} \otimes \mathrm{Id}_{R^*} \end{pmatrix}$ at the level of operators on $H \otimes R \oplus H^* \otimes R^*$ yields

$$\begin{pmatrix} \alpha \mathsf{A} \alpha^{-1} \otimes \mathrm{Id}_R & \alpha \mathsf{B} \tilde{\alpha}^{-1} \otimes \beta \psi^{-1} \tilde{\beta}^{-1} \\ \tilde{\alpha} \mathsf{C} \alpha^{-1} \otimes \tilde{\beta} \psi \beta^{-1} & \tilde{\alpha} \mathsf{D} \tilde{\alpha}^{-1} \otimes \mathrm{Id}_{R^*} \end{pmatrix} .$$

Now, as has been mentioned in Sect. 3.1, the equivariant antiunitary maps $\beta$ and $\tilde{\beta}$ are only unique up to multiplicative constants of unit modulus. They will be chosen in the next subsection so that the distinguished time-reversal operator(s) and the unitary structure $C$ commute. These choices having been made, we choose $\psi$ so that $\tilde{\beta}\psi\beta^{-1} = \psi$. In this way, in the case where $T_1$ is nonmixing as above, conjugation of the matrix $M$ by $T_1$ is given by

$$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \mapsto \begin{pmatrix} \alpha \mathsf{A} \alpha^{-1} & \alpha \mathsf{B} \tilde{\alpha}^{-1} \\ \tilde{\alpha} \mathsf{C} \alpha^{-1} & \tilde{\alpha} \mathsf{D} \tilde{\alpha}^{-1} \end{pmatrix} . \tag{1}$$

Thus the transferred time-reversal operator is simply given by $T_1 = \alpha \oplus \tilde{\alpha}$ on $H \oplus H^*$.

If $T^2 = \varepsilon_T \mathrm{Id}$ (with $\varepsilon_T = \pm 1$), the compatibility condition for the mixing operator

$$T = \begin{pmatrix} 0 & \alpha^{-1} \otimes \beta^{-1} \\ \varepsilon_T \, \alpha \otimes \beta & 0 \end{pmatrix}$$

is $\beta \psi^{-1} \beta = \varepsilon_\beta \psi$ (with $\varepsilon_\beta = \pm 1$). If this holds, conjugation of $M$ by $T$ is given by

$$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \mapsto \begin{pmatrix} \alpha^{-1} \mathsf{D} \alpha & \varepsilon_\alpha \, \alpha^{-1} \mathsf{C} \alpha^{-1} \\ \varepsilon_\alpha \, \alpha \mathsf{B} \alpha & \alpha \mathsf{A} \alpha^{-1} \end{pmatrix} . \tag{2}$$

with $\varepsilon_\alpha = \varepsilon_\beta \varepsilon_T$. In this case the appropriate transferred operator is given by

$$T = \begin{pmatrix} 0 & \alpha^{-1} \\ \varepsilon_\alpha \alpha & 0 \end{pmatrix} .$$

Given the (essentially unique) choices of the tensor-product representations of $T$, $T_1$ and $C$ which are defined by $T_1 T = \pm T T_1$ and by the conditions that $T$ and $T_1$ commute with $C$, we show in Sect. 3.4 that there is a unique choice of $\psi$ so that both of these compatibility conditions hold.

If we are in the nonmixing case $\beta : R \to R$, and it so happens that $\beta$ is $G_0$-invariant, then the two alternatives for the involutory property of $T$ can be distinguished by the type of the unitary representation $R$ as follows. Defining $\iota : R \to R^*$ by $r \mapsto \langle r, \cdot \rangle_R$ as before, consider the unitary mapping $\psi : R \to R^*$ given as the composition $\psi = \iota \circ \beta$. Since $\beta$ is $G_0$-invariant, $\psi$ is $G_0$-equivariant, and the statement of Lemma 3.2 applies. Using the antiunitarity of $\beta$ one has

$$\psi(r)(t) = \langle \beta r, t \rangle_R = \overline{\langle \beta^2 r, \beta t \rangle_R} = \psi(t)(\beta^2 r) ,$$

and therefore the following statement is immediate.

***Lemma 3.11***. — *The parity of an antiunitary and $G_0$-invariant mapping $\beta : R \to R$ is determined by the parity of the irreducible $G_0$-representation space $R$; i.e., $\beta$ satisfies $\beta^2 = \mathrm{Id}_R$ resp. $\beta^2 = -\mathrm{Id}_R$ if $R$ carries an invariant $\mathbb{C}$-bilinear form which is symmetric resp. alternating.*

If $\beta^2 = \mathrm{Id}_R$, the transferred time reversal satisfies $T^2 = -\mathrm{Id}$ or $T^2 = \mathrm{Id}$ if the original time reversal has these properties. On the other hand, if $\beta^2 = -\mathrm{Id}_R$, then the properties are reversed; e.g., if $T^2 = -\mathrm{Id}$ on the original space, then transferred time reversal satisfies $T^2 = \mathrm{Id}$. We again remind the reader that we must check that the transferred time-reversal operator(s) and $C$ can be compatibly chosen. It turns out that there is in fact just enough freedom in the choice of the constants to achieve this (see Sect. 3.4).

***Example***. — An example of particular importance in physics is the transfer of the (true) time reversal $T$ in the case where all spin rotations are symmetries. On fundamental grounds, $T$ is a (nonmixing) operator which commutes with the spin-rotation group $\mathrm{SU}_2$ and satisfies $T^2 = (-1)^n \mathrm{Id}$ on quantum mechanical states with spin $S = n/2$.

Let $V = H \otimes \mathbb{C}^{n+1}$ be the tensor product of a vector space $H$ with the spin $n/2$ representation space of $\mathrm{SU}_2$. For simplicity assume that there are no further symmetries.

Our Nambu space is already in the form $V \oplus V^* = (H \otimes R) \oplus (H^* \otimes R^*)$. Thus the reduced space is $H \oplus H^*$. Let the time-reversal operator on $V = H \otimes \mathbb{C}^{n+1}$ be written $T = \alpha \otimes \beta$. The $\mathrm{SU}_2$-representation space with spin $n/2$, $\mathbb{C}^{n+1}$, is known to have parity $+1$ (symmetric invariant form) for $n$ even, and $-1$ (alternating invariant form) for $n$ odd. By Lemma 3.11 this implies $\beta^2 = (-1)^n \mathrm{Id}$. The situation on the dual space $V^*$ is the same. Thus in this case the transferred time-reversal operator $\alpha : H \oplus H^* \to H \oplus H^*$ always satisfies $\alpha^2 = +\mathrm{Id}_{H \oplus H^*}$, independent of the spin. □

Now let us turn to the problem of transferring the complex bilinear form. For this Lemma 3.2 is an essential fact. Earlier we identified $H\otimes R\oplus H^*\otimes R^*$ with $V\oplus V^*$ by the map $\varepsilon : h\otimes r + f\otimes t \mapsto h(r)+f(t)$. Using this along with Prop. 3.6 we now transfer the canonical symmetric bilinear form on $V\oplus V^*$ to $H\oplus H^*$. For this let $s$ (resp. $a$) denote the canonical symmetric (resp. alternating) form on $H\oplus H^*$.

***Proposition 3.12***. — *Depending on $\psi$ being symmetric or alternating, a transferred map in* $\mathrm{End}(H\oplus H^*)$ *respects the canonical symmetric form $s$ or alternating form $a$ if and only if the original endomorphism in* $\mathrm{End}_{G_0}(V\oplus V^*)$ *respects the canonical symmetric complex bilinear form on $V\oplus V^*$.*

*Proof*. — We give the proof for the case where $\psi$ is alternating. The proof in the symmetric case is completely analogous.

Let $M=\begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{pmatrix}\in \mathrm{End}(H\oplus H^*)$ act as a $G_0$-invariant operator

$$\begin{pmatrix}\mathsf{A}\otimes\mathrm{Id}_R & \mathsf{B}\otimes\psi^{-1}\\ \mathsf{C}\otimes\psi & \mathsf{D}\otimes\mathrm{Id}_{R^*}\end{pmatrix}$$

on $H\otimes R\oplus H^*\otimes R^*$ and let $b$ be the symmetric complex bilinear form on this space which is induced from the canonical symmetric form on $V\oplus V^*$. We assume that $M\in \mathrm{GL}(H\oplus H^*)$ and give the proof in terms of the isometry property $b(Mv,Mw)=b(v,w)$. Let us do this in a series of cases. First, for $h_1\otimes r_1$ and $h_2\otimes r_2$ in $H\otimes R$,

$$\begin{aligned}
&b(M(h_1\otimes r_1),M(h_2\otimes r_2))\\
=\;& b(\mathsf{A}h_1\otimes r_1+\mathsf{C}h_1\otimes\psi(r_1),\mathsf{A}h_2\otimes r_2+\mathsf{C}h_2\otimes\psi(r_2))\\
=\;& \mathsf{C}h_2(\mathsf{A}h_1)\,\psi(r_2)(r_1)/d+\mathsf{C}h_1(\mathsf{A}h_2)\,\psi(r_1)(r_2)/d\\
=\;& a(\mathsf{A}h_1+\mathsf{C}h_1,\mathsf{A}h_2+\mathsf{C}h_2)\,\psi(r_2)(r_1)/d\ .
\end{aligned}$$

When $M$ is the identity this becomes

$$b(h_1\otimes r_1,h_2\otimes r_2)=a(h_1,h_2)\,\psi(r_2)(r_1)/d\ .$$

Therefore $b(h_1\otimes r_1,h_2\otimes r_2)=b(M(h_1\otimes r_1),M(h_2\otimes r_2))$ if and only if $a(h_1,h_2)=a(M(h_1),M(h_2))$. For $f_1\otimes t_1$, $f_2\otimes t_2\in H^*\otimes R^*$ the discussion is analogous.

For $h\otimes r\in H\otimes R$ and $f\otimes t\in H^*\otimes R^*$ we have a similar calculation:

$$\begin{aligned}
&b(M(h\otimes r),M(f\otimes t))\\
=\;& b(\mathsf{A}h\otimes r+\mathsf{C}h\otimes\psi(r),\mathsf{B}f\otimes\psi^{-1}(t)+\mathsf{D}f\otimes t)\\
=\;& \mathsf{D}f(\mathsf{A}h)\,t(r)/d+\mathsf{C}h(\mathsf{B}f)\,\psi(r)(\psi^{-1}(t))/d\\
=\;& a(M(h),M(f))\,t(r)/d\ .
\end{aligned}$$

Of course the analogous identity holds for $b(M(f\otimes t),M(h\otimes r))$. □

***Remark***. — To avoid making sign errors and misidentifications in later computations, we find it helpful to transfer the particle-hole conjugation operator $C$ along with the complex bilinear form. This is done by insisting that the statement of Lemma 2.2 remains true after the transfer. Thus the relation $b(Cw_1, w_2) = \langle w_1, w_2 \rangle$ continues to hold in all cases. By an almost identical variant of the computation that led to Lemma 3.11, the transferred operator $C$ has parity $C^2 = +\mathrm{Id}$ or $C^2 = -\mathrm{Id}$ depending on whether the transferred bilinear form is symmetric or alternating. □

**3.4. Precise choice of time-reversal transfer. —** Recalling the situation of this section, we have assumed that the distinguished time-reversal operator(s) stabilize the initial block $V \oplus V^*$, and we have transferred all structures to the space $(H \otimes R) \oplus (H^* \otimes R^*)$ which is isomorphic to $V \oplus V^*$.

The time-reversal operator(s) $T$ and the operator $C$ are given by $(2 \times 2)$-matrices of pure tensors on this space. The space of endomorphisms $B$ that commute with the $G_0$-action is identified with $\mathrm{End}(H \oplus H^*)$ or $\mathrm{End}(H) \oplus \mathrm{End}(H^*)$ depending on whether or not $\lambda = \lambda^*$. The good Hamiltonians $B$ anticommute with $C$, and commute with the time-reversal operator(s) $T$. If the matrices of pure tensors representing the antiunitary operators $C$ and $T$ have entries $\gamma \otimes \delta$, this means that $B$ anticommutes (resp. commutes) with the matrices defined by the operators $\gamma$. Although the pure tensor decomposition is not unique, this statement is independent of that decomposition.

It has been shown above that the transferred operators $T$ and $C$ on $H \oplus H^*$, i.e., those defined by the operators $\gamma$, can be chosen with the desired involutory properties. It will now be shown that there is just enough freedom to insure that $TC = CT$, $T_1 C = CT_1$, and $T_1 T = \pm T T_1$ still hold after transferral. After these conditions have been met, we show as promised that $\psi : R \to R^*$ can be chosen in a unique way so that the compatibility conditions of Sect. 3.3 hold, i.e., so that it makes sense to define the transferred operators by the first factors of the tensor-product representations.

We carry this out in the case where $\lambda = \lambda^*$ and two distinguished time-reversal operators are present. All other cases are either subcases of this or are much simpler.

The operator $C$ always mixes. We will always choose it to be of the form $C = \gamma \otimes \iota : H \otimes R \to H^* \otimes R^*$ and $C = \gamma^{-1} \otimes \iota^{-1} : H^* \otimes R^* \to H \otimes R$. Of course this is in the case where $b$ is symmetric. If $b$ is alternating, then we have $C^2 = -\mathrm{Id}$, and we make the necessary sign change.

Here we restrict to the case where $T^2 = T_1^2 = \mathrm{Id}$. The various other involutory properties make no difference in the argument. Just as in the case of $C$ we choose $T = \alpha \otimes \beta : H \otimes R \to H^* \otimes R^*$ and $T = \alpha^{-1} \otimes \beta^{-1} : H^* \otimes R^* \to H \otimes R$. Similarly, we choose $T_1 = \alpha_1 \otimes \beta_1 : H \otimes R \to H \otimes R$ and $T_1 = \alpha_2 \otimes \beta_2 : H^* \otimes R^* \to H^* \otimes R^*$.

On $(H \otimes R) \oplus (H^* \otimes R^*)$, the operators $T$ and $T_1$ commute with $C$, and we have $T_1 T = \pm T T_1$. We now choose the tensor representations so that the same relations hold for the induced operators on the first factors.

If $\alpha$, $\alpha_1$, $\alpha_2$, and $\gamma$ are any choices for the first factors of the tensor-product representations of $T$, $T_1$ and $C$, then there exist constants $c_1$, $c_2$ and $c_3$ so that $\alpha_2 \alpha = c_1 \alpha \alpha_1$ (from $T T_1 = \pm T_1 T$), $\gamma \alpha_1 = c_2 \alpha_2 \gamma$ (from $CT_1 = T_1 C$), and $\gamma \alpha^{-1} \gamma = c_3 \alpha$ ($CT = TC$).

Let $\tilde\alpha = \xi\alpha$, $\tilde\gamma = \eta\gamma$, and $\tilde\alpha_i = z_i\alpha$ (for $i = 1,2$), where $\xi$, $\eta$ and $z_i$ are complex numbers yet to be determined. Just as the $c_i$, these constants are of modulus one.

The scaled operators satisfy $\tilde\alpha_2\tilde\alpha = \chi_1 c_1\tilde\alpha\tilde\alpha_1$, $\tilde\gamma\tilde\alpha_1 = \chi_2 c_2\tilde\alpha_2\tilde\gamma$, and $\tilde\gamma\tilde\alpha^{-1}\tilde\gamma = \chi_3 c_3\tilde\alpha$, where $\chi_1 = \xi^{-2}z_1z_2$, $\chi_2 = \eta^2(z_1z_2)^{-1}$, and $\chi_3 = \xi^{-2}\eta^2$. Observe that the characters $\chi_i$ satisfy the relation $\chi_1\chi_2 = \chi_3$, and that, e.g., $\chi_2$ and $\chi_3$ are independent.

The constants $c_i$ satisfy an analogous relation. For this first use $\gamma\alpha_1\gamma^{-1} = c_2\alpha_2$ and $\gamma\alpha^{-1}\gamma = c_3\alpha$ to obtain $\gamma\alpha_1\alpha^{-1}\gamma = (c_2/c_3)\alpha_2\alpha$. Then compose both sides of this equation with the inverse of $\alpha_1$ on the right and use the relation $\alpha_2\alpha\alpha_1^{-1} = c_1\alpha$ to obtain $\gamma\alpha_1\alpha^{-1}\gamma\alpha_1^{-1} = (c_1c_2/c_3)\alpha$. Now $\gamma\alpha_1^{-1} = (c_2\alpha_2)^{-1}\gamma$. Thus

$$\gamma\alpha_1\alpha^{-1}\gamma\alpha_1^{-1} = \gamma\alpha_1\alpha^{-1}\alpha_2^{-1}c_2^{-1}\gamma = c_2^{-1}\gamma c_1\alpha^{-1}\gamma = (c_1c_2)^{-1}c_3\alpha\,,$$

and hence $c_1c_2/c_3 = c_3/c_1c_2$, i.e., $c_1^2c_2^2 = c_3^2$.

Since $\chi_2$ and $\chi_3$ are independent, we can choose the scaling numbers so that $c_2 = c_3 = 1$, thereby arranging that $CT = TC$ and $CT_1 = T_1C$ still hold after transferral. To preserve these relations we must now keep $\chi_2$ and $\chi_3$ fixed at unity, which from $\chi_1\chi_2 = \chi_3$ implies that $\chi_1 = 1$. Since $c_3^2 = c_1^2c_2^2$, we then conclude that $c_1$ takes one of the two values $\pm 1$, and further scaling does not change this constant.

In summary we have the following result.

***Proposition 3.13***. — *The transferred operators $T_1$, $T$ and $C$ can be chosen so that $T_1C = CT_1$, $TC = CT$, and $T_1T = \pm TT_1$. Assuming that the time-reversal operators have been transferred to commute with $C$ in this way, the relation $T_1T = \pm TT_1$ is automatic and further scaling does not change the sign. Furthermore, the $\mathbb{C}$-linear isomorphism $\psi : R \to R^*$ can be chosen to meet the compatibility conditions which determine the conjugation rules (1) and (2).*

*Proof*. — It remains to prove that $\psi$ can be chosen as stated. For the nonmixing operator $T_1$ the compatibility condition is $\beta_2\psi\beta_1^{-1} = \psi$. Given some choice of $\psi$ (which we will modify) there is a constant $c \in \mathbb{C}$ so that $\beta_2\psi\beta_1^{-1} = c\psi$. This constant $c$ is unimodular since $\beta_1$ and $\beta_2$ are antiunitary. To satisfy the compatibility condition, replace $\psi$ by $\xi\psi$, where $\xi\bar\xi^{-1}c = 1$. Note that this choice of $\xi$ only determines its argument.

Turning to the compatibility condition $\beta\psi^{-1}\beta = \varepsilon_\beta\psi$ for the mixing operator $T$, we start from $c\psi = \beta\psi^{-1}\beta$ for some other $c \in \mathbb{C}$, and use the $\mathbb{C}$-antilinearity of $\beta$ to deduce $\psi^{-1} = \bar c\beta^{-1}\psi\beta^{-1}$. Multiplying expressions gives $c = \bar c \in \mathbb{R}$. Then, rescaling $\psi$ to $\xi\psi$, the compatibility condition is achieved by setting $\varepsilon_\beta := c/|c|$ and solving $|\xi|^2 = |c|$. Since this rescaling (with $\xi \in \mathbb{R}$) does not affect the compatibility condition for the nonmixing operator, we have determined the desired isomorphism $\psi$. □

Finally, since $C$ is a pure tensor, it follows from our representation of the transferred bilinear form $b$ that $cb(Ch_1, h_2) = \langle h_1, h_2\rangle$ for some constant $c$. Thus we replace $b$ by $cb$ and obtain the following final transferred setup on $H \oplus H^*$:

- The canonical bilinear form $b$ which is either symmetric or alternating.

- A unitary structure $\langle\, ,\, \rangle$ which is compatible with $b$ in the sense that $b(Ch_1, h_2) = \langle h_1, h_2 \rangle$. The operator $C : H \leftrightarrow H^*$ satisfies either $C^2 = \mathrm{Id}$ or $C^2 = -\mathrm{Id}$, depending on $b$ being symmetric or alternating.
- Either zero, one, or two time-reversal operators. They are antiunitary and commute with $C$. In the case of two, $T$ is mixing and $T_1$ is nonmixing. In the case of one, both mixing and nonmixing are allowed. The same involutory properties hold as before transfer, but signs might change, i.e., if $T^2 = \mathrm{Id}$ holds before transfer, then it is possible that $T^2 = -\mathrm{Id}$ afterwards. Furthermore, $T_1 T = \pm T T_1$, and consequently the unitary product $P := T T_1$ satisfies $P^2 = \pm\mathrm{Id}$.

In the following sections all of the symmetric spaces which occur in our basic model will be described, using the transferred setup. This means that we describe the subspace of Hermitian operators in $\mathrm{End}(H) \oplus \mathrm{End}(H^*)$ or $\mathrm{End}(H \oplus H^*)$ which are compatible with $b$ and the $T$-symmetries. We first handle the case of one or no time-reversal operator (Sect. 4), and then carry out the classification when both $T$ and $T_1$ are present (Sect. 5). The final classification result, Theorem 1.1, then follows.

## 4. Classification: at most one distinguished time reversal

This section is devoted to giving a precise statement of Theorem 1.1 and its proof in the case where at most one distinguished time-reversal symmetry is present. Combining this with the results of Sect. 3, we obtain a precise description of the blocks that occur in the model motivated and described in Sects. 1 and 2.

**4.1. Statement of the main result. —** Throughout this section $V$ denotes a finite-dimensional unitary vector space. The associated space $W = V \oplus V^*$ is equipped with the canonically induced unitary structure $\langle\, ,\, \rangle$ and $\mathbb{C}$-antilinear map $C : V \to V^*$, $v \mapsto \langle v, \cdot \rangle$. The results of the previous section allow us to completely eliminate $G_0$ from the discussion so that it is only necessary to consider the following data:

- *The relevant space $E$ of endomorphisms.* This is either the full space $\mathrm{End}(W)$ or $\mathrm{End}(V) \oplus \mathrm{End}(V^*)$ embedded as usual in $\mathrm{End}(W)$.
- *The canonical complex bilinear form $b : W \times W \to \mathbb{C}$.* This is either the symmetric form $s$ which is given by

$$s(v_1 + f_1, v_2 + f_2) = f_1(v_2) + f_2(v_1)\ ,$$

  or the alternating form $a$ which is given by

$$a(v_1 + f_1, v_2 + f_2) = f_1(v_2) - f_2(v_1)\ .$$

  Equivalently, $C : V \to V^*$ is extended to a $\mathbb{C}$-antilinear mapping $C : W \to W$ by $C^2 = +\mathrm{Id}$ resp. $C^2 = -\mathrm{Id}$, and $b(Cw_1, w_2) = \langle w_1, w_2 \rangle$ holds in all cases.
- *The antiunitary mapping $T : W \to W$*, which satisfies either $T^2 = -\mathrm{Id}$ or $T^2 = \mathrm{Id}$. We say that $T$ is nonmixing if $T|_V : V \to V$ and $T|_{V^*} : V^* \to V^*$. If $T|_V : V \to V^*$,

then we refer to $T$ as mixing. In all cases $T$ commutes with $C$. We also include the case where $T$ is not present.

Fixing one of these properties each, we refer to $(V,E,b,T)$ as *block data*; e.g., $E = \mathrm{End}(W)$, $b = s$, $T^2 = -\mathrm{Id}$ and $T$ being nonmixing would be such a choice.

Our main result describes the symmetric spaces associated to given block data. Let us state this at the Lie algebra level, where for convenience of formulation we only consider the case of trace-free operators. In order to state this result, it is necessary to introduce some notation.

Given block data $(V,E,b,T)$, let $\mathfrak{g}$ be the subspace of $E$ of antihermitian operators $A$ which are compatible with $b$ in the sense that

$$b(Aw_1,w_2) + b(w_1,Aw_2) = 0$$

for all $w_1,w_2 \in W$. It will be shown that $\mathfrak{g}$ is a Lie subalgebra of $E$ which is invariant under conjugation $A \mapsto TAT^{-1}$ with $T$. This defines a Lie algebra automorphism

$$\theta : \mathfrak{g} \to \mathfrak{g} , \quad A \mapsto TAT^{-1} ,$$

which is usually called a *Cartan involution*. If $\mathfrak{k} := \mathrm{Fix}(\theta) = \{A \in \mathfrak{g} : \theta(A) = A\}$ and $\mathfrak{p}$ is the space $\{A \in \mathfrak{g} : \theta(A) = -A\}$ of antifixed points, then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is called the associated *Cartan decomposition*.

The space $\mathcal{H} = \mathcal{H}(V,E,b,T)$ of Hermitian operators which are compatible with the block data is $i\mathfrak{p}$, which is identified with the infinitesimal version $\mathfrak{p} = \mathfrak{g}/\mathfrak{k}$.

In order to give a smooth statement of our classification result, we recall that the Lie algebras $\mathfrak{su}_n$, $\mathfrak{usp}_{2n}$, and $\mathfrak{so}_{2n}$ are commonly referred to as being of type *A*, *C*, and *D*, respectively. By an irreducible *ACD*-symmetric space of compact type one means an (irreducible) compact symmetric space of any of these Lie algebras. With a slight exaggeration we use the same terminology in Theorem 4.1 below. The exaggeration is that the case $\mathfrak{so}_{2n}/(\mathfrak{so}_p \oplus \mathfrak{so}_q)$ with $p$ and $q$ odd must be excluded in order for that theorem to be true. For the overall statement of Theorem 1.1 there is no danger of misinterpretation, as the case where $p$ and $q$ are odd does occur in the situation where two distinguished time-reversal symmetries are present (see Sect. 5).

***Theorem 4.1***. — *Given block data $(V,E,b,T)$, the space $\mathcal{H} = \mathcal{H}(V,E,b,T) \cong \mathfrak{g}/\mathfrak{k}$ is the infinitesimal version of an irreducible ACD-symmetric space of compact type. Conversely, the infinitesimal version of any irreducible ACD-symmetric space of compact type can be constructed in this way.*

There are several remarks which should be made concerning this statement. First, as we have already noted, in order to give a smooth formulation, we have reduced to trace-free operators. As will be seen in the proof, there are several cases where without this assumption $\mathfrak{g}$ would have a one-dimensional center.

Secondly, recall that one of the important cases of a compact symmetric space is that of a compact Lie group $K$ with the geodesic inversion symmetry at the identity being defined by $k \mapsto k^{-1}$. Usually one equips $K$ with the action of $G = K \times K$ defined by left- and right-multiplication, and views the symmetric space as $G/K$, where the isotropy group $K$ is diagonally embedded in $G$. The infinitesimal version is then $(\mathfrak{k} \oplus \mathfrak{k})/\mathfrak{k}$, and the automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ is defined by $(X_1, X_2) \mapsto (X_2, X_1)$. In this setting one speaks of symmetric spaces of type II.

In our case the classical compact Lie algebras do indeed arise from appropriate block data, but in the situation where $T$ does not leave the original space $W$ invariant. In that setting, $T$ maps $W = W_1 = V_1 \oplus V_1^*$ to $W_2 = V_2 \oplus V_2^*$, which has different $G_0$-representations from those in $W_1$. Thus the relevant block is $W_1 \oplus W_2$. Using the results of the previous section, in this case we also remove $G_0$ from the picture.

Nevertheless, we are left with a situation where the block is $W_1 \oplus W_2$ and $T : W_1 \to W_2$. Thus we wish to allow situations of this type, i.e., where $V \oplus V^*$ is not $T$-invariant, to be allowed block data. These cases are treated separately in Sect. 4.4.

The case where the symmetric space is just the compact group associated to $\mathfrak{g}$ also arises when $T$ is not present, i.e., when there is no condition which creates isotropy.

Finally, as has already been indicated in Sect. 1, the appropriate homogeneous space version of Theorem 4.1 is given by replacing the infinitesimal symmetric space $\mathfrak{g}/\mathfrak{k}$ by the Cartan-embedded symmetric space $M \cong G/K$. Here $G$ is the simply connected group associated to $\mathfrak{g}$, a mapping $\theta : G \to G$ is defined as the Lie group automorphism whose derivative at the identity is the Cartan involution of the Lie algebra, and $M$ is the orbit of $e \in G$ of the twisted $G$-action given by $x \mapsto gx\theta(g)^{-1}$.

**4.2. The associated symmetric space. —** In this and the next subsection we work in the context of simple block data $(V, E, b, T)$ where $W = V \oplus V^*$ is $T$-invariant.

In the present subsection we prove the first half of Theorem 4.1, namely that $H \cong \mathfrak{g}/\mathfrak{k}$ is an infinitesimal version of a classical symmetric space of compact type. This essentially amounts to showing that all the involutions which are involved commute.

Let $\sigma : E \to E$ be the $\mathbb{C}$-antilinear Lie-algebra involution that fixes the Lie algebra of the unitary group in $E$. If the adjoint operation $A \mapsto A^*$ is defined by

$$\langle A w_1, w_2 \rangle = \langle w_1, A^* w_2 \rangle \ ,$$

then $\sigma(A) = -A^*$. The transformations $S \in E$ which are isometries of the canonical bilinear form satisfy

$$b(Sw_1, Sw_2) = b(w_1, w_2)$$

for all $w_1, w_2 \in W$. Thus the appropriate Lie algebra involution is the $\mathbb{C}$-linear automorphism

$$\tau : E \to E \ , \quad A \mapsto -A^{\mathrm{t}} \ ,$$

where $A \mapsto A^{\mathrm{t}}$ is the adjoint operation defined by

$$b(Aw_1, w_2) = b(w_1, A^{\mathrm{t}} w_2) \ .$$

Finally, let $\theta : E \to E$ be the $\mathbb{C}$-antilinear map defined by $A \mapsto TAT^{-1}$.

***Proposition 4.2***. — *The operations $A \mapsto A^*$ and $A \mapsto A^{\rm t}$ are related by $A^* = CA^{\rm t}C^{-1}$.*

*Proof.* — From $b(Cw_1, w_2) = \langle w_1, w_2\rangle$ and the definition of $A \mapsto A^*$ we have

$$b(Aw_1, w_2) = \langle C^{-1}Aw_1, w_2\rangle = \langle C^{-1}w_1, (C^{-1}AC)^* w_2\rangle = b(w_1, C^{-1}A^*Cw_2) ,$$

i.e., $A^{\rm t} = C^{-1}A^*C$, independent of the case $b = s$ or $b = a$. □

***Proposition 4.3***. — *The involutions $\sigma$, $\tau$ and $\theta$ commute.*

*Proof.* — Using $\langle C^{-1}A^*Cw_1, w_2\rangle = \langle w_1, C^{-1}ACw_2\rangle$ along with $A^{\rm t} = C^{-1}A^*C$ we have

$$(A^{\rm t})^* = C^{-1}AC = (A^*)^{\rm t} ,$$

and consequently $\sigma\tau = \tau\sigma$.

Since $T$ is antiunitary, one immediately shows from the definition of $A^*$ that

$$\langle w_1, TA^*T^{-1}w_2\rangle = \langle TAT^{-1}w_1, w_2\rangle .$$

In other words,

$$\theta(\sigma(A)) = -TA^*T^{-1} = -(TAT^{-1})^* = \sigma(\theta(A)) .$$

Finally, since $\theta(A) = TAT^{-1}$ and $T$ commutes with $C$, it follows that $\theta\tau = \tau\theta$. □

Let $\mathfrak{s} := \mathrm{Fix}(\tau)$. Since $\theta$ and $\sigma$ commute with $\tau$, it follows that they restrict to $\mathbb{C}$-antilinear involutions of the complex Lie algebra $\mathfrak{s}$. We denote these restrictions by the same letters. For future reference let us summarize the relevant formulas.

***Proposition 4.4***. — *For $A \in \mathfrak{s}$ it follows that*

$$\sigma(A) = CAC^{-1} \quad \textit{and} \quad \theta(A) = TAT^{-1}.$$

*The parity of $C$ is $C^2 = +\mathrm{Id}$ for $b = s$ symmetric, and $C^2 = -\mathrm{Id}$ for $b = a$ alternating.*

The space $\mathfrak{g}$ of antihermitian operators in $E$ that respect $b$ is therefore the Lie algebra of $\sigma$-fixed points in $\mathfrak{s}$. Since $\sigma$ defines the unitary Lie algebra in $E$, it follows that $\mathfrak{g}$ is a compact real form of $\mathfrak{s}$. Let us explicitly describe $\mathfrak{s}$ and $\mathfrak{g}$.

If $E = \mathrm{End}(W)$ and $b = s$ is symmetric, then $\mathfrak{s}$ is the complex orthogonal Lie algebra $\mathfrak{so}(W, s) \cong \mathfrak{so}_{2n}(\mathbb{C})$. If $E = \mathrm{End}(W)$ and $b = a$ is alternating, then $\mathfrak{s}$ is the complex symplectic Lie algebra $\mathfrak{sp}(W, a) \cong \mathfrak{sp}_{2n}(\mathbb{C})$. If $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$, then in both cases for $b$ it follows that its isometry group $S$ is $\mathrm{SL}_{\mathbb{C}}(V)$ acting diagonally by its defining representation on $V$ and its dual representation on $V^*$. In this case we have $\mathfrak{s} = \mathfrak{sl}(V) \cong \mathfrak{sl}_n(\mathbb{C})$. Note that this is a situation where we have used the trace-free condition to eliminate the one-dimensional center.

For the discussion of $\mathfrak{g}$ it is important to note that since $\sigma(A) = CAC^{-1}$, it follows that $\mathfrak{g}$ just consists of the elements of $\mathfrak{s}$ which commute with $C$.

In the symmetric case $b = s$, where $C$ defines a real structure on $W$, it is appropriate to consider the set of real points $W_{\mathbb{R}} = \{v + Cv : v \in V\}$. Thinking in terms of isometries, we regard $G = \exp(\mathfrak{g})$ as being the group of $\mathbb{R}$-linear isometries of the restriction of $b = s$ to $W_\mathbb{R}$ which are extended complex linearly to $W$. Note that in this case $b|_{W_\mathbb{R}} = 2\,\mathrm{Re}\,\langle\, ,\,\rangle$, and that every $\mathbb{R}$-linear transformation of $W_\mathbb{R}$ which preserves $\mathrm{Re}\,\langle\, ,\,\rangle$ extends $\mathbb{C}$-linearly to a unitary transformation of $W$. Thus, if $E = \mathrm{End}(W)$ and $b = s$, then $\mathfrak{g}$ is naturally identified with $\mathfrak{so}(W_\mathbb{R}, s|_{W_\mathbb{R}}) \cong \mathfrak{so}_{2n}(\mathbb{R})$.

In the alternating case $b = a$, if $E = \mathrm{End}(W)$, then as in the previous case, since $\sigma$ defines $\mathfrak{u}(W) \subset E$, it follows that its set $\mathfrak{g}$ of fixed points in $\mathfrak{s}$ is a compact real form of $\mathfrak{s}$. Since $\mathfrak{s}$ is the complex symplectic Lie algebra $\mathfrak{sp}(W,a) \cong \mathfrak{sp}_{2n}(\mathbb{C})$, it follows that $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{usp}_{2n}$ of the unitary symplectic group.

It is perhaps worth mentioning that $C$ for $b = a$ defines a quaternionic structure on the complex vector space $W$. Thus the condition $A = CAC^{-1}$ defines the subalgebra $\mathfrak{gl}_n(\mathbb{H})$ in $\mathrm{End}(W)$. The further condition $A = -A^*$ shows that $\mathfrak{g}$ can be identified with the algebra of quaternionic isometries, another way of seeing that $\mathfrak{g} \cong \mathfrak{usp}_{2n}$.

Finally, in the case where $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$ we have already noted that $\mathfrak{s} = \mathfrak{sl}(V)$ which is acting diagonally. It is then immediate that in both the symmetric and alternating cases $\mathfrak{g} = \mathfrak{su}(V) \cong \mathfrak{su}_n$. Of course $\mathfrak{g}$ acts diagonally as well.

Let us summarize these results.

***Proposition 4.5***. — *In the case where* $E = \mathrm{End}(W)$ *the following hold:*

- *If* $b = s$ *is symmetric, then* $\mathfrak{g} \cong \mathfrak{so}_{2n}(\mathbb{R})$.
- *If* $b = a$ *is alternating, then* $\mathfrak{g} \cong \mathfrak{usp}_{2n}$.

*If* $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$, *then* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{su}_n$ *and acts diagonally.*

Since $\theta$ commutes with $\sigma$, it stabilizes $\mathfrak{g}$. Hence, $\theta|_\mathfrak{g}$ is a Cartan involution which defines a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of $\mathfrak{g}$ into its $(\pm 1)$-eigenspaces. The fixed subspace $\mathfrak{k} = \{A \in \mathfrak{g} : \theta(A) = A\}$ is a subalgebra and $\mathfrak{g}/\mathfrak{k}$ is the infinitesimal version of a symmetric space of compact type.

Recall that, given block data $(V,E,b,T)$, the associated space

$$\mathcal{H} = \mathcal{H}(V,E,b,T) \cong i\mathfrak{p}$$

of structure-preserving Hamiltonians has been identified with $\mathfrak{p} = \mathfrak{g}/\mathfrak{k}$. Thus we have proved the first part of Theorem 4.1. The second part is proved in the next section by going through the possibilities in Prop. 4.5 along with the various possibilities for $T$.

It should be noted that if $T = \pm C$, then $\mathfrak{g} = \mathfrak{k}$, i.e., the symmetric space is just a point. Such a degenerate situation, where the set of Hamiltonians is empty, never occurs in a well-posed physics setting.

**4.3. Concrete description: symmetric spaces of type I. —** Here we describe the possibilities for each set of block data $(V,E,b,T)$ under the assumption that $W = V \oplus V^*$ is $T$-invariant. The results are stated in terms of the *ACD*-symmetric spaces, with $n := \dim_\mathbb{C} V$. The methods of proof of showing which symmetric spaces arise also show

how to explicitly construct them. In the present subsection, all of these are compact irreducible classical symmetric spaces of type I in the notation of [**H**].

*4.3.1. The case* $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$. — Under the assumption $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$ it follows that $\mathfrak{g}$ is just the unitary Lie algebra $\mathfrak{su}(V) \cong \mathfrak{su}_n$ which is acting diagonally on $W = V \oplus V^*$. This is independent of $b$ being symmetric or alternating. Thus we need only consider the various possibilities for $T$. If $T$ is not present, the symmetric space is $\mathfrak{g} = \mathfrak{su}_n$.

1. $T^2 = -\mathrm{Id}$, nonmixing: $\mathfrak{su}_n/\mathfrak{usp}_n$.
   Since $T$ is nonmixing and satifies $T^2 = -\mathrm{Id}$, it follows that $\langle Tv_1, v_2\rangle = a(v_1, v_2)$ is a $\mathbb{C}$-linear symplectic structure on $V$ which is compatible with $\langle v_1, v_2\rangle$. Thus the dimension $n$ of $V$ must be even here. The facts that $\mathfrak{g}$ is acting diagonally as $\mathfrak{su}(V)$ and that the elements of $\mathfrak{k}$ are precisely those which commute with $T$, imply that $\mathfrak{k} = \mathfrak{usp}_n$ as announced.
2. $T^2 = \mathrm{Id}$, nonmixing: $\mathfrak{su}_n/\mathfrak{so}_n$.
   Since $T$ and $\mathfrak{g}$ are acting diagonally, as in the previous case it is enough to only discuss the matter on $V$. In this case $T$ defines a real structure on $V$ with $V_{\mathbb{R}} = \{v + Tv : v \in V\}$, and the unitary isometries which commute with $T$ are just those transformations which stabilize $V_{\mathbb{R}}$ and preserve the restriction of $\langle\, , \rangle$. Since $\langle x, y\rangle_{V_{\mathbb{R}}} = \mathrm{Re}\,\langle x, y\rangle_V$ for $x, y \in V_{\mathbb{R}}$, it follows that $\mathfrak{k} = \mathfrak{so}(V_{\mathbb{R}}) \cong \mathfrak{so}_n(\mathbb{R})$.
3. $T^2 = \pm\mathrm{Id}$, mixing: $\mathfrak{su}_n/\mathfrak{s}(\mathfrak{u}_p \oplus \mathfrak{u}_q)$.
   Here it is convenient to introduce the unitary operator $P = CT$, which satisfies $P^2 = \mathrm{Id}$ or $P^2 = -\mathrm{Id}$, depending on the parity of $T$. Denote the eigenvalues of $P$ by $u$ and $-u$. Since $P$ does not mix, the condition that a diagonally acting unitary operator commutes with $T$ (or equivalently, with $P$) is just that it preserves the $P$-eigenspace decomposition $V = V_u \oplus V_{-u}$. Since the two eigenspaces $V_u$ and $V_{-u}$ are $\langle\, , \rangle$-orthogonal, we have $\mathfrak{k} = \mathfrak{s}\left(\mathfrak{u}(V_u) \oplus \mathfrak{u}(V_{-u})\right)$, and the desired result follows with $p = \dim V_u$ and $q = \dim V_{-u}$.
   In the case $P^2 = -\mathrm{Id}$, if there existed a subspace $V_{\mathbb{R}}$ of real points that was stabilized by $P$, then $P$ would be a complex structure of $V_{\mathbb{R}}$ and the dimensions of $V_u$ and $V_{-u}$ would have to be equal. In general, however, no such space $V_{\mathbb{R}}$ exists and the dimensions $p$ and $q$ are arbitrary.

*4.3.2. The case* $E = \mathrm{End}(W)$, $b = s$. — In this case we have the advantage that we may restrict the entire discussion to the set of real points

$$W_{\mathbb{R}} = \mathrm{Fix}(C) = \{v + Cv : v \in V\}\ .$$

Thus $\mathfrak{k}$ is translated to being the Lie algebra of the group of isometries of $2\,\mathrm{Re}\,\langle\, , \rangle$ on $V$. Here the Lie algebra $\mathfrak{g}$ is $\mathfrak{so}(W_{\mathbb{R}})$. Thus in the case where $T$ is not present, the symmetric space is $\mathfrak{so}_{2n}(\mathbb{R})$.

1. $T^2 = -\mathrm{Id}$, nonmixing or mixing: $\mathfrak{so}_{2n}(\mathbb{R})/\mathfrak{u}_n$.
   Independent of whether or not it mixes, $T|_{W_\mathbb{R}} : W_\mathbb{R} \to W_\mathbb{R}$ is a complex structure on $W_\mathbb{R}$. A transformation in $\mathrm{SO}(W_\mathbb{R})$ commutes with $T$ if and only if it is holomorphic. Since $\mathrm{Re}\langle\, ,\rangle$ is $T$-invariant, this condition defines the unitary subalgebra $\mathfrak{k} \cong \mathfrak{u}_n$ in $\mathfrak{g} \cong \mathfrak{so}_{2n}(\mathbb{R})$.
2. $T^2 = \mathrm{Id}$, nonmixing: $\mathfrak{so}_{2n}(\mathbb{R})/(\mathfrak{so}_n(\mathbb{R}) \oplus \mathfrak{so}_n(\mathbb{R}))$.
   Since $T|_{W_\mathbb{R}} : W_\mathbb{R} \to W_\mathbb{R}$, we have the decomposition $W_\mathbb{R} = W_\mathbb{R}^+ \oplus W_\mathbb{R}^-$ into the $(\pm 1)$-eigenspaces of $T$. We still identify $\mathfrak{g}$ with the Lie algebra of the group of isometries of $W_\mathbb{R}$ equipped with the restricted form $\mathrm{Re}\langle\, ,\rangle$. The subalgebra $\mathfrak{k}$, which is fixed by $\theta : X \mapsto TXT^{-1}$, is the stabilizer $\mathfrak{so}(W_\mathbb{R}^+) \oplus \mathfrak{so}(W_\mathbb{R}^-)$ of the above decomposition. Now let us compute the dimensions of the eigenspaces. In the case at hand $T$ defines a real structure on both $V$ and $V^*$. Since $C$ commutes with $T$, it follows that $\mathrm{Fix}(T) = V_\mathbb{R} \oplus V_\mathbb{R}^*$ is $C$-invariant. Thus $W_\mathbb{R}^+ = \{v + Cv : v \in V_\mathbb{R}\}$. A similar argument shows that $W_\mathbb{R}^- = \{v + Cv : v \in \mathrm{i}V_\mathbb{R}\}$.
3. $T^2 = \mathrm{Id}$, mixing: $\mathfrak{so}_{2n}(\mathbb{R})/(\mathfrak{so}_{2p}(\mathbb{R}) \oplus \mathfrak{so}_{2q}(\mathbb{R}))$.
   The exact same argument as above shows that $\mathfrak{k} = \mathfrak{so}(W_\mathbb{R}^+) \oplus \mathfrak{so}(W_\mathbb{R}^-)$. It only remains to show that the eigenspaces are even-dimensional. For this we consider the unitary operator $P = CT$ which leaves both $V$ and $V^*$ invariant. Its $(+1)$-eigenspace $W_{+1}$ is just the complexification of $W_\mathbb{R}^+$. The intersections of $W_{+1}$ with $V$ and $V^*$ are interchanged by $C$, and therefore $\dim_\mathbb{C} W_{+1} =: 2p$ is even. Of course the same argument holds for $W_{-1}$.

*4.3.3. The case $E = \mathrm{End}(W)$, $b = a$.* — Since in this case $\mathfrak{g}$ is the Lie algebra of antihermitian endomorphisms which respect the alternating form $a$ on $W$, it follows that $\mathfrak{g} \cong \mathfrak{usp}_{2n}$. Thus if $T$ is not present the associated symmetric space is $\mathfrak{usp}_{2n}$.

If $T$ is present, we let $P := CT$. The unitary operator $P$ always commutes with $T$, and from $a(w_1, w_2) = \langle C^{-1}w_1, w_2\rangle$ one infers that $a(Pw_1, Pw_2) = a(w_1, w_2)$ in all cases, independent of $T$ being mixing or not.

The classification spelled out below follows from the fact that commutation with $T$ is equivalent to preservation of the $P$-eigenspace decomposition of $W$.

1. $T^2 = -\mathrm{Id}$, nonmixing: $\mathfrak{usp}_{2n}/(\mathfrak{usp}_n \oplus \mathfrak{usp}_n)$.
   In this case $P^2 = \mathrm{Id}$, and $T^2 = -\mathrm{Id}$ forces $n$ to be even. Let $W$ be decomposed into $P$-eigenspaces as $W = W_{+1} \oplus W_{-1}$. If $w_1 \in W_{+1}$ and $w_2 \in W_{-1}$, then

$$a(w_1, w_2) = a(Pw_1, Pw_2) = -a(w_1, w_2) = 0 \, ,$$

   and we see that $W_{+1}$ and $W_{-1}$ are $a$-orthogonal. The mixing operator $P$ is traceless. Therefore the dimensions of $W_{+1}$ and $W_{-1}$ are equal, and both of them are symplectic subspaces of $W$. The fact that the decomposition $W = W_{+1} \oplus W_{-1}$ is also $\langle\, ,\rangle$-orthogonal therefore implies that $\mathfrak{k} = \mathfrak{usp}(W_{+1}) \oplus \mathfrak{usp}(W_{-1})$.

2. $T^2 = -\mathrm{Id}$, mixing: $\mathfrak{usp}_{2n}/(\mathfrak{usp}_{2p} \oplus \mathfrak{usp}_{2q})$.
   Here, using the same argument as in the previous case, one shows that the $P$-eigenspace decomposition $W = W_{+1} \oplus W_{-1}$ still is a direct sum of $a$-orthogonal, complex symplectic subspaces. Since these are also $\langle\, , \rangle$-orthogonal, it follows that $\mathfrak{k} = \mathfrak{usp}(W_{+1}) \oplus \mathfrak{usp}(W_{-1})$. Note that in the present case the nonmixing operator $P$ stabilizes the decomposition $W = V \oplus V^*$. Thus, since $P$ commutes with $C$, it follows that $W_{+1} = V_{+1} \oplus V_{+1}^*$ and $W_{-1} = V_{-1} \oplus V_{-1}^*$.

3. $T^2 = \mathrm{Id}$, mixing or nonmixing: $\mathfrak{usp}_{2n}/\mathfrak{u}_n$.
   In this case $P^2 = -\mathrm{Id}$. Here $a(Pw_1, Pw_2) = a(w_1, w_2)$ implies that the $P$-eigenspace decomposition $W = W_{+\mathrm{i}} \oplus W_{-\mathrm{i}}$ is Lagrangian. (This means in particular $\dim W_{+\mathrm{i}} = \dim W_{-\mathrm{i}}$.) Thus its stabilizer in $\mathfrak{sp}(W)$ is the diagonally acting $\mathfrak{gl}(W_{+\mathrm{i}})$. Since the decomposition is $\langle\, , \rangle$-orthogonal, it follows that $\mathfrak{k} = \mathfrak{u}(W_{+\mathrm{i}}) \cong \mathfrak{u}_n$.

**4.4. Concrete description: symmetric spaces of type II. —** Recall the original situation where the symmetry group $G_0$ is still in the picture. As described in Sect. 1 we select from the given Hilbert space a basic finite-dimensional $G_0$-invariant subspace $V$ which is composed of irreducible subrepresentations all of which are equivalent to a fixed irreducible representation $R$.

Although the initial block of interest is $W = V \oplus V^*$, it is possible that it is not $T$-invariant and that it must be expanded. Let us formalize this situation by denoting the initial block by $W_1 = V_1 \oplus V_1^*$. We then let $P = CT$ and regard this as a unitary isomorphism

$$P : W_1 \to W_2,$$

where $W_2 = V_2 \oplus V_2^*$ is another initial block.

For $i \in \{1, 2\}$, let $R_i$ be the $G_0$-representation on $V_i$ which induces the representation on $W_i$. The map $P$ is equivariant, but only with respect to the automorphism $a$ of $G_0$ which is defined by $g_T$-conjugation: $P \circ g = a(g) \circ P$.

Now two situations arise. If $R_1 \cong R_2$ or $R_1^* \cong R_2$, then we may build a new block $W = V \oplus V^*$ which *is* $T$-invariant so that the results of the previous section can be applied: if $R_1 \cong R_2$, then we let $V := V_1 \oplus V_2$ and if $R_1 \cong R_2^*$, then $V := V_1 \oplus V_2^*$.

We assume now that neither $R_1 \cong R_2$ nor $R_1 \cong R_2^*$, and consider the expanded block $W = W_1 \oplus W_2$. Recall that $W_1$ and $W_2$ are in the Nambu space $\mathcal{W}$ which decomposes as a direct sum of nonisomorphic representation spaces that are orthogonal with respect to both the unitary structure and the canonical symmetric form. Thus the decomposition $W = W_1 \oplus W_2$ is orthogonal with respect to both of these structures.

Under the assumption at hand it is immediate that

$$\mathrm{End}_{G_0}(W) = \mathrm{End}_{G_0}(W_1) \oplus \mathrm{End}_{G_0}(W_2)\ .$$

Thus we are in a position to apply the results of Sect. 3.

To do so in the case where $R_1 \cong R_1^*$, we let $\psi_1 : R_1 \to R_1^*$ denote an equivariant isomorphism, and organize the notation so that $P : V_1 \to V_2$. Of course $R_1$ and $R_2$ are abstract representations, but we now choose realizations of them in $V_1$ and $V_2$ so that $\psi_2 := P\psi_1 P^{-1} : R_2 \to R_2^*$ makes sense. Since

$$\begin{aligned} & P\psi_1 P^{-1}(g(v_2)) = P(\psi_1(a^{-1}(g)P^{-1}(v_2))) \\ = \quad & P(a^{-1}(g)\psi_1(P^{-1}(v_2))) = g(P\psi_1 P^{-1}(v_2)) \ , \end{aligned}$$

it follows that $\psi_2 : R_2 \to R_2^*$ is a $G_0$-equivariant isomorphism.

Assume for simplicity that $\psi_1$ is even, i.e., that $\psi_1(v_1)(\tilde{v}_1) = \psi_1(\tilde{v}_1)(v_1)$. Then

$$\begin{aligned} & \psi_2(v_2)(\tilde{v}_2) = P\psi_1 P^{-1}(v_2)(\tilde{v}_2) = \psi_1(P^{-1}(v_2))(P^{-1}(\tilde{v}_2)) \\ = \quad & \psi_1(P^{-1}(\tilde{v}_2))(P^{-1}(v_2)) = P\psi_1 P^{-1}(\tilde{v}_2)(v_2) = \psi_2(\tilde{v}_2)(v_2) \ . \end{aligned}$$

The computation in the case where $\psi_1$ is odd is the same except for a sign change. Thus $\psi_1$ and $\psi_2$ have the same parity.

Now let $E_i$ (for $i = 1, 2$) be the relevant space of endomorphisms that was produced by our analysis of $W_i$ in Sect. 3. Recall that this is either the space $\mathrm{End}(H_i) \oplus \mathrm{End}(H_i^*)$ or $\mathrm{End}(H_i \oplus H_i^*)$. Let $\mathfrak{g}_i$ be the Lie algebra of the group of unitary transformations which preserve $b_i$. The key points now are that the unitary structure on $E := E_1 \oplus E_2$ is the direct sum structure, the complex bilinear form on $E$ is $b = b_1 \oplus b_2$, and the parity of $b_1$ is the same as that of $b_2$. Thus $\mathfrak{g}_1 \cong \mathfrak{g}_2$.

For the statement of our main result in this case, let us recall that the infinitesimal versions of symmetric spaces of type II are of the form $\mathfrak{g} \oplus \mathfrak{g}/\mathfrak{g}$, where the isotropy algebra is embedded diagonally.

***Proposition 4.6***. — *If $R_1$ is neither isomorphic to $R_2$ nor to $R_2^*$, then the infinitesimal symmetric space associated to the $T$-invariant block data is a type-II ACD-symmetric space of compact type. Specifically, the classical Lie algebras $\mathfrak{su}_n$, $\mathfrak{so}_{2n}(\mathbb{R})$, and $\mathfrak{usp}_{2n}$ arise in this way.*

*Proof*. — Identify $\mathfrak{g}_1$ and $\mathfrak{g}_2$ by the isomorphism $P$. Call the resulting Lie algebra $\mathfrak{g}$. The transformations that commute with $T$ are those in the diagonal in $\mathfrak{g} \oplus \mathfrak{g}$. Thus the associated infinitesimal version of the symmetric space is of type II. The fact that the only Lie algebras which occur are those in the statement has been proved in 4.2. □

This completes the proof of Theorem 4.1. In closing we underline that under the assumptions of Prop. 4.6 the odd-dimensional orthogonal Lie algebra does not appear as a type-II space; only the even-dimensional one does.

## 5. Classification: two distinguished time-reversal symmetries

Here we describe in detail the situation where both of the distinguished time-reversal operators $T$ and $T_1$ are present. As would be expected, there are quite a few cases. The work will be carried out in a way which is analogous to our treatment of the case where only one time-reversal operator was present. In the first part (Sect. 5.1) we operate under the assumption that the initial truncated space $V \oplus V^*$ is invariant under both of the distinguished operators. In the second part (Sect. 5.2) we handle the general case where bigger blocks must be considered.

**5.1. The case where $V \oplus V^*$ is $G$-invariant. —** Throughout, $T$ is mixing, $T_1$ is nonmixing and $P := TT_1$. Our strategy in Sects. 5.1.3 and 5.1.4 will be to first compute the operators which are $b$-isometries, are unitary and commute with $P$. This determines the Lie algebra $\mathfrak{g}$ and its action on $V \oplus V^*$. Then $\mathfrak{k}$ is determined as the subalgebra of operators which commute with $T$ or $T_1$, whichever is most convenient for the proof. The space of Hamiltonians is identified with $\mathfrak{g}/\mathfrak{k}$ as before.

In the case of $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$, where $\mathfrak{g}$ acts diagonally, the answer for $\mathfrak{g}/\mathfrak{k}$ does not depend on the involutory properties of $C$, $T$, and $T_1$ individually, but only on those of the nonmixing operators $CP = CTT_1$ and $T_1$. The pertinent Sects. 5.1.1 and 5.1.2 are organized accordingly.

*5.1.1. The case* $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$, $(CP)^2 = \mathrm{Id}$. — Recall that in the case of $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$ it follows that the $b$-isometry group is $\mathrm{SL}_{\mathbb{C}}(V)$ acting diagonally. Thus the Lie algebra $\mathfrak{g}$ consists of those elements of the unitary algebra $\mathfrak{su}(V)$ which commute with the mixing unitary symmetry $P$. Equivalently, $\mathfrak{g}$ is the subalgebra of $\mathfrak{su}(V)$ defined by commutation with the antiunitary operator $CP$.

In the present case $CP$ defines a real structure on $V$, and we have the $\mathfrak{g}$-invariant decomposition $V = V_{\mathbb{R}} \oplus \mathrm{i}V_{\mathbb{R}}$. Since the unitary structure $\langle\, ,\, \rangle$ is compatible with this real structure, it follows that $\mathfrak{g} = \mathfrak{so}(V_{\mathbb{R}})$. Our argumentation is based around $T_1$. If it anticommutes with $P$, then we replace $P$ by i$P$ so that it commutes. Of course this has the effect of changing to the case $(CP)^2 = -\mathrm{Id}$ which is, however, handled below. Hence, in both cases we may assume that $P$ and $T_1$ commute.

1. $T_1^2 = \mathrm{Id}$: $\mathfrak{so}_n/(\mathfrak{so}_p \oplus \mathfrak{so}_q)$.
   The space of $CP$-real points $V_{\mathbb{R}}$ is $T_1$-invariant and splits into a sum $V_{\mathbb{R}}^+ \oplus V_{\mathbb{R}}^-$ of $T_1$- eigenspaces. The Lie algebra $\mathfrak{k}$ is the stabilizer of this decomposition, which is $\langle\, ,\, \rangle$-orthogonal. Thus $\mathfrak{k} = \mathfrak{so}(V_{\mathbb{R}}^+) \oplus \mathfrak{so}(V_{\mathbb{R}}^-)$.
   Observe that in this case $n$ can be any even or odd number and that $p$ and $q$ are arbitrary with the condition that $n = p + q$.
2. $T_1^2 = -\mathrm{Id}$: $\mathfrak{so}_{2n}/\mathfrak{u}_n$.
   In this case $T_1$ is a complex structure on $V_{\mathbb{R}}$ which is compatible with the unitary structure. Thus $\mathfrak{k} = \mathfrak{u}(V_{\mathbb{R}}, T_1)$ and the desired result follows with $2n = \dim_{\mathbb{C}} V$.

*5.1.2. The case* $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$, $(CP)^2 = -\mathrm{Id}$. — The first remarks made at the beginning of Sect. 5.1.1 still apply: $\mathfrak{g}$ is the subalgebra of the diagonally acting $\mathfrak{su}(V)$ which commutes with the antiunitary operator $CP$. But now $CP$ defines a $\mathbb{C}$-bilinear symplectic structure on $W = V \oplus V^*$ by $a(w_1, w_2) := \langle CPw_1, w_2 \rangle$. Actually $CP$ is already defined on $V$ and transported to $V^*$ by $C$. Thus $\mathfrak{g} = \mathfrak{usp}(V)$.

1. $T_1^2 = -\mathrm{Id}$: $\mathfrak{usp}_{2n}/(\mathfrak{usp}_{2p} \oplus \mathfrak{usp}_{2q})$.
   In this case $\Gamma := CT : V \to V$ is a unitary operator which satisfies $\Gamma^2 = \mathrm{Id}$, and

which defines the eigenspace decomposition $V = V^+ \oplus V^-$. This decomposition is both $a$- and $\langle\,,\rangle$-orthogonal, and consequently $\mathfrak{k} = \mathfrak{usp}(V^+) \oplus \mathfrak{usp}(V^-)$. Note that there is no condition on $p$ and $q$ other than $p+q=n$.

2. $T_1^2 = \mathrm{Id}$: $\mathfrak{usp}_{2n}/\mathfrak{u}_n$.
Let $V_\mathbb{R}$ be the $T_1$-real points of $V$. Then $\mathfrak{k}$ is the stabilizer of $V_\mathbb{R}$ in $\mathfrak{g} = \mathfrak{usp}(V)$. Here the symplectic structure $a$ on $V$ restricts to a real symplectic structure $a_\mathbb{R}$ on $V_\mathbb{R}$. Since the unitary structure $\langle\,,\rangle$ is compatible with this structure, $\mathfrak{k}$ is the maximal compact subalgebra $\mathfrak{u}_n$ of the associated real symplectic algebra.

*5.1.3. The case $E = \mathrm{End}(V \oplus V^*)$, $b = s$.* — Recall that in this case $C^2 = \mathrm{Id}$, and the $b$-isometry group of $W = V \oplus V^*$ is $\mathrm{SO}(W)$. Before going into the various cases, let us remark on the relevance of whether or not time-reversal operators commute with $P$.

If $P^2 = u^2\mathrm{Id}$, where either $u = \pm 1$ or $u = \pm \mathrm{i}$, we consider the $P$-eigenspace decomposition $W = W_u \oplus W_{-u}$. Note $\dim W_u = \dim W_{-u}$ from $\mathrm{Tr}\,P = 0$. The Lie algebra $\mathfrak{g} \subset \mathfrak{so}(W)$ of operators which preserve $b = s$ and commute with $P$ is $\mathfrak{so}_\mathbb{R}(W_u) \oplus \mathfrak{so}_\mathbb{R}(W_{-u})$.

An antiunitary operator which commutes with $P$ preserves the decomposition $W = W_u \oplus W_{-u}$ if $u = \pm 1$, and exchanges the summands if $u = \pm \mathrm{i}$. Similarly, if it anticommutes with $P$, then it exchanges the summands in $W = W_{+1} \oplus W_{-1}$ and preserves the decomposition $W = W_{+\mathrm{i}} \oplus W_{-\mathrm{i}}$. For this reason, as will be clear from the first case below, the sign of $TT_1 = \pm T_1 T$ has no bearing on our classification.

1. $T^2 = T_1^2 = \mathrm{Id}$: $(\mathfrak{so}_n/(\mathfrak{so}_p \oplus \mathfrak{so}_q)) \oplus (\mathfrak{so}_n/(\mathfrak{so}_p \oplus \mathfrak{so}_q))$.
Suppose first that $P^2 = \mathrm{Id}$, giving the $P$-eigenspace decomposition $W = W_{+1} \oplus W_{-1}$. Each of the time-reversal operators commutes with $P$. To determine $\mathfrak{k}$ we consider the unitary operator $\Gamma = CT_1$ which is a mixing $b$-isometry satisfying $\Gamma P = P\Gamma$ and $\Gamma^2 = \mathrm{Id}$. Thus $W_{+1}$ further decomposes into a direct sum $W_{+1} = W_{+1}^{+1} \oplus W_{+1}^{-1}$ of $\Gamma$-eigenspaces, which are orthogonal with respect to both $b$ and $\langle\,,\rangle$. The same discussion holds for $W_{-1}$. The stabilizer of this refined decomposition is $\mathfrak{k} = \big(\mathfrak{so}_\mathbb{R}(W_{+1}^{+1}) \oplus \mathfrak{so}_\mathbb{R}(W_{+1}^{-1})\big) \oplus \big(\mathfrak{so}_\mathbb{R}(W_{-1}^{+1}) \oplus \mathfrak{so}_\mathbb{R}(W_{-1}^{-1})\big)$. From $\mathrm{Tr}\,P = \mathrm{Tr}\,\Gamma = 0$ one infers $\dim W_{+1}^{+1} = \dim W_{-1}^{-1} = p$ and $\dim W_{+1}^{-1} = \dim W_{-1}^{+1} = q$.
Now consider the case where $P^2 = -\mathrm{Id}$ but the time-reversal operators *anti*commute with each other and hence with $P$. In this situation the $P$-eigenspace decomposition $W = W_{+\mathrm{i}} \oplus W_{-\mathrm{i}}$ is still $T$-invariant. Therefore we are in exactly the same situation as above, and of course obtain the same result.
This happens in all cases below. Thus, for the remainder of this section we assume that the time-reversal operators commute with $P$.

2. $T^2 = T_1^2 = -\mathrm{Id}$: $(\mathfrak{so}_{2n}/(\mathfrak{so}_n \oplus \mathfrak{so}_n)) \oplus (\mathfrak{so}_{2n}/(\mathfrak{so}_n \oplus \mathfrak{so}_n))$.
The situation is exactly the same as that above, except that $\Gamma = CT_1$ now satisfies $\Gamma^2 = -\mathrm{Id}$. Since $\Gamma$ preserves the sets of $C$-real points of $W_{+1}$ and $W_{-1}$, $\Gamma$ defines a complex structure of these real vector spaces. Therefore we have the additional condition $\dim W_{+1}^{+\mathrm{i}} = \dim W_{+1}^{-\mathrm{i}}$ on the dimensions of the $\Gamma$-eigenspaces.

3. $T^2 = -T_1^2$: $(\mathfrak{so}_n \oplus \mathfrak{so}_n)/\mathfrak{so}_n$.
   The argument to be given is true independent of whether $T^2 = \mathrm{Id}$ or $T^2 = -\mathrm{Id}$. As usual we consider the $P$-eigenspace decomposition $W = W_{+\mathrm{i}} \oplus W_{-\mathrm{i}}$. Since $P$ is an isometry of both $b$ and $\langle\,,\rangle$, the decomposition is $b$- and $\langle\,,\rangle$-orthogonal. Thus $\mathfrak{g} = \mathfrak{so}_{\mathbb{R}}(W_{+\mathrm{i}}) \oplus \mathfrak{so}_{\mathbb{R}}(W_{-\mathrm{i}})$. Now $T$ is antilinear and commutes with $P$. Thus it permutes the $P$-eigenspaces, i.e., $T : W_{+\mathrm{i}} \to W_{-\mathrm{i}}$. Since $\mathfrak{k}$ consists of those operators in $\mathfrak{g}$ that commute with $T$, and $T$ is compatible with both the unitary structure and the bilinear form $b$, it follows that $(A,B) \in \mathfrak{g}$ is in $\mathfrak{k}$ if and only if $B = TAT^{-1}$. In other words, after applying the obvious automorphism, $\mathfrak{k}$ is the diagonal in $\mathfrak{g} \cong \mathfrak{so}_n \oplus \mathfrak{so}_n$.

*5.1.4. The case* $E = \mathrm{End}(V \oplus V^*)$, $b = a$. — Recall that in this case $C^2 = -\mathrm{Id}$, and the $b$-isometry group of $W = V \oplus V^*$ is $\mathrm{Sp}(W)$. For the same reasons as indicated above we may assume that the time-reversal operators commute with $P$.

1. $T^2 = T_1^2 = \mathrm{Id}$: $(\mathfrak{usp}_{2n}/\mathfrak{u}_n) \oplus (\mathfrak{usp}_{2n}/\mathfrak{u}_n)$.
   Observe that the $P$-eigenspace decomposition $W = W_{+1} \oplus W_{-1}$ is $a$- and $\langle\,,\rangle$-orthogonal and that therefore $\mathfrak{g} = \mathfrak{usp}(W_{+1}) \oplus \mathfrak{usp}(W_{-1})$. Let the dimension be denoted by $\dim_{\mathbb{C}}(W_{+1}) = \dim_{\mathbb{C}}(W_{-1}) = 2n$.
   Now $T$ defines real structures on $W_{+1}$ and $W_{-1}$, and these are compatible with $a$. Hence in both cases the restriction $a_{\mathbb{R}}$ to the set $W^{\mathbb{R}}_{\pm 1}$ of fixed points of $T$ is a real symplectic structure. The algebra $\mathfrak{k}$ consists of the pairs $(A,B)$ of operators in $\mathfrak{g}$ which stabilize $W^{\mathbb{R}}_{+1} \oplus W^{\mathbb{R}}_{-1}$. This means that $A$, e.g., is in the maximal compact subalgebra of the real symplectic Lie algebra determined by $a_{\mathbb{R}}$ on $W^{\mathbb{R}}_{+1}$, i.e., in a unitary Lie algebra isomorphic to $\mathfrak{u}_n$. A similar statement holds for $B$.

2. $T^2 = T_1^2 = -\mathrm{Id}$: $(\mathfrak{usp}_{2n}/(\mathfrak{usp}_{2p} \oplus \mathfrak{usp}_{2q}) \oplus (\mathfrak{usp}_{2n}/(\mathfrak{usp}_{2p} \oplus \mathfrak{usp}_{2q}))$.
   The argument made above still shows that $\mathfrak{g} = \mathfrak{usp}(W_{+1}) \oplus \mathfrak{usp}(W_{-1})$.
   Now, to determine $\mathfrak{k}$ we consider the operator $\Gamma := CT_1$ which stabilizes this decomposition and satisfies $\Gamma^2 = \mathrm{Id}$. Thus the further condition to be satisfied in order for an operator to be in $\mathfrak{k}$ is that the $\Gamma$-eigenspace decomposition of each summand must be stabilized, i.e., $\mathfrak{k} = \oplus_{\varepsilon,\delta=\pm 1}\mathfrak{usp}(W^{\delta}_{\varepsilon})$. The dimensions must match pairwise because $\mathrm{Tr}\,P = \mathrm{Tr}\,\Gamma = 0$.

3. $T^2 = -T_1^2$: $\mathfrak{su}_n/\mathfrak{so}_n$.
   The answer for $\mathfrak{g}/\mathfrak{k}$ is the same for the two cases $T^2 = \mathrm{Id}$ or $T^2 = -\mathrm{Id}$.
   In either case it follows from $a(w_1,w_2) = a(Pw_1,Pw_2)$ that the summands of the $P$-decomposition $W = W_{+\mathrm{i}} \oplus W_{-\mathrm{i}}$ are $a$-Lagrangian. Thus an $a$-isometry stabilizes the decomposition if and only if it is a $\mathbb{C}$-linear transformation acting diagonally, and consequently $\mathfrak{g} = \mathfrak{su}(W_{+\mathrm{i}})$ (which is acting diagonally as well).
   Without loss of generality we may assume that $T^2 = \mathrm{Id}$ (or else we replace $T$ by $T_1$). Then $T$ is a real structure which permutes the $P$-eigenspaces. Thus the diagonal action $(w^+,w^-) \mapsto (Bw^+,Bw^-)$ commutes with $T$ if and only if $TBT^{-1} = B$. Since $T$ is compatible with the initial unitary structure, if follows that $B$ is in the

associated real orthogonal group. For example, if unitary coordinates are chosen so that $T$ is given by $(z,w) \mapsto (\bar{w},\bar{z})$, then $TBT^{-1} = B$ simply means that $B = \bar{B}$.

**5.2. Building bigger blocks. —** Before $G_0$-reduction we must determine the basic block associated to the $G_0$-representation space $V$. This has been adequately discussed in all cases with the exception of the one where there are two time-reversal operators. Here we handle that case by reducing it to the situation where there is only one.

Write the initial block as $V_1 \oplus V_1^*$ and build a diagram consisting of the four spaces $V_i \oplus V_i^*$, $i = 1,\dots,4$, with the maps $T$, $T_1$, and $P$ emanating from each of them. To be concrete, $T : V_1 \oplus V_1^* \to V_2 \oplus V_2^*$ defines $V_2$, $T_1 : V_1 \oplus V_1^* \to V_3 \oplus V_3^*$ defines $V_3$, and $T_1 : V_2 \oplus V_2^* \to V_4 \oplus V_4^*$ defines $V_4$. The relation $P = TT_1$ defines the remaining maps. At this point there is no need to discuss mixing.

We also underline that, by the nature of the basic model, any two spaces $V_i \oplus V_i^*$ and $V_j \oplus V_j^*$ are either disjoint in the big Nambu space or are equal.

Let us now complete the proof of our classification result, Theorem 1.1, by running through the various cases which occur in the present setting where the initial block must be extended. We only sketch this, because given how the extended block case was handled in the setting of one distinguished time-reversal symmetry (Sect. 4.4) and the detailed classification results above, the proof requires no new ideas or methods.

*1) $V_1 \oplus V_1^*$ is $T$-invariant and is not $T_1$-invariant.* — Here it is only necessary to consider $P : W_1 = V_1 \oplus V_1^* \to V_3 \oplus V_3^* = W_3$. If $\mathfrak{g}$ is the Lie algebra of unitary operators which commute with the $G_0$-action and respect the $b$-structure on $V_1 \oplus V_1^*$, then the further condition of compatibility with $P$ means that the algebra in the present case is $\mathfrak{g}$ acting diagonally via $P$ on $W_1 \oplus W_3$. Thus we have reduced to the case of only one time-reversal operator on $W_1$, which has been classified above.

Note that this argument has nothing to do with whether or not $T$ is mixing. Hence, in this and all of the following cases there is no need to differentiate between $T$ and $T_1$.

*2) $V_1 \oplus V_1^*$ is neither $T$- nor $T_1$-invariant.* — Consider the diagram introduced above where all the spaces $W_i = V_i \oplus V_i^*$ occur. If any of the $W_i$ is invariant by either $T$ or $T_1$, then we change our perspective, replace $W_1$ by that space and apply the above argument. Thus we may assume that no $W_i$ is stabilized by either $T$ or $T_1$. It is still possible, however, that $W_1 = W_4$, and in that case it follows that $W_2 = W_3$.

*2.1) $W_1 = W_4$.* — Here both $W_1$ and $W_4$ are $P$-invariant. We leave it to the reader to check that $P$ can be transferred to the level of $\mathrm{End}(H) \oplus \mathrm{End}(H^*)$ or $\mathrm{End}(H \oplus H^*)$ just as we transferred the time-reversal operators. Thus, e.g., it is enough to know the Lie algebra of operators $\mathfrak{g}$ on $W_1$ which are compatible with the unitary structure, are $b$-isometries and are compatible with $P$. This has been computed in Sect. 5.1. Of course we did this in the case where $V \oplus V^*$ is $T$- and $T_1$-invariant, but the compatibility with $P$ had nothing to do with time reversal.

In the present case both $T$ and $T_1$ exchange $W_1$ and $W_2$. Thus our symmetric space is $(\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{g}$.

*2.2) The spaces $W_i$ are pairwise disjoint.* — Here we will go through a number of subcases, depending on whether or not there exist (equivariant) isomorphisms between various spaces. Such an isomorphism is of course assumed to be unitary and to commute with $C$; in particular it is a $b$-isometry.

*2.2.1)* $W_1 \cong W_4$. — If $\varphi$ is the isomorphism which does this, then $T\varphi T^{-1} =: \psi$ is an isomorphism of $W_2$ and $W_3$. Using these isomorphisms, we build $W := W_1 \oplus W_4$ and $\tilde{W} := W_2 \oplus W_3$ which are of our initial type; they are stabilized by $P$ and exchanged by $T$. Thus, as in *2.1)*, if $\mathfrak{g}$ is the Lie algebra of operators on $W$ which are compatible with the unitary structure, are $b$-isometries and are compatible with $P$, then our symmetric space is $(\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{g}$.

*2.2.2)* $W_1 \cong W_2$. — For the reasons given above, $W_3 \cong W_4$ and we build $W$ and $\tilde{W}$ as in that case. In the present situation $P$ exchanges $W$ and $\tilde{W}$. We must then consider two subcases during our procedure for identifying $\mathfrak{g}$.

The simplest case is where $W$ and $\tilde{W}$ are not isomorphic. In that setting the Lie algebra $\mathfrak{g}$ of unitary operators on $W$ which commute with the $G_0$-action and are compatible with $b$ acts diagonally on $W \oplus \tilde{W}$. This is exactly our algebra of interest.

Thus in this case we can forget $\tilde{W}$, and regard $\mathfrak{g}$ as acting on $W$. Here $T$ stabilizes $W$ and thus the associated symmetric space is $\mathfrak{g}/\mathfrak{k}$, where $\mathfrak{k}$ consists of the operators in $\mathfrak{g}$ which commute with $T$. This situation has been classified above; in particular, only classical irreducible symmetric spaces of compact type occur.

Our final case occurs under the assumption $W_1 \cong W_2$ in the situation where $W$ and $\tilde{W}$ are isomorphic. Here we view an operator which commutes with the $G_0$-action as a matrix

$$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} .$$

Compatibility with $P$ can then be interpreted as $\mathsf{B}$ and $\mathsf{D}$ being determined from $\mathsf{A}$ and $\mathsf{C}$ by $P$-conjugation. In this notation $\mathsf{A} : W \to W$ and $\mathsf{C} : W \to \tilde{W}$. But we may also regard $\mathsf{C}$ as an operator on $W$ which is transferred to a map from $W$ to $\tilde{W}$ by the isomorphism at hand. Therefore the Lie algebra of interest can be identified with the set of pairs $(\mathsf{A},\mathsf{C})$ of operators on $W$ which are compatible with the unitary and $b$-structures and commute with the $G_0$-action on $W$. Hence the associated symmetric space is the direct sum $\mathfrak{g}/\mathfrak{k} \oplus \mathfrak{g}/\mathfrak{k}$ where $\mathfrak{k}$ is determined by compatibility with $T : W \to W$, i.e., a direct sum of two copies of an arbitrary example that occurs with only one $T$-symmetry.

## 6. Physical realizations

We now illustrate Theorem 1.1 by the two large sets of examples that were already referred to in Sect. 2: (i) fermionic quasiparticle excitations in disordered normal- and superconducting systems, and (ii) Dirac fermions in a stochastic gauge field background. In each case we fix a specific Nambu space $W$, and show how a variety of symmetric spaces (each corresponding to a symmetry class) is realized by *varying the group of unitary and antiunitary symmetries*, $G$.

The invariable nature of $W$ is a principle imposed by physics: electrons, e.g., have electric charge $e = -1$ and spin $S = 1/2$ and these properties cannot ever be changed. What can be changed, however, by varying the experimental conditions, are the symmetries of the Hamiltonian governing the specific situation at hand. For example, turning on an external magnetic field breaks time-reversal symmetry, adding spin-orbit scatterers to the system breaks spin-rotation symmetry, lowering the temperature enhances the pairing forces that may lead to a spontaneous breakdown of the global $\mathrm{U}_1$ charge symmetry, and so on.

**6.1. Quasiparticles in metals and superconductors. —** The setting here is the one already described in Sect. 2.1: given the complex Hilbert space $V$ of single-electron states, we form the Nambu space $W = V \oplus V^*$ of electron field operators. On $W$ we then have the canonical symmetric bilinear form $b$, the particle-hole conjugation operator $C : W \to W$, and the canonical unitary structure $\langle \, , \rangle$.

The complex Hilbert spaces $V$ and $V^*$ are to be viewed as representation spaces of a $\mathrm{U}_1$ group, which is the global $\mathrm{U}_1$ gauge degree of freedom of electrodynamics. Indeed, creating or annihilating one electron amounts to adding one unit of negative or positive electric charge to the fermion system. In representation-theoretic terms, this means that $V$ carries the fundamental representation of the $\mathrm{U}_1$ gauge group while $V^*$ carries the antifundamental one. Thus $z \in \mathrm{U}_1$ here acts on $V$ by multiplication with $z$, and on $V^*$ by multiplication with $\bar{z}$.

Extra structure arises from the fact that electrons carry spin 1/2, which implies that $V$ is a tensor product of spinor space, $\mathbb{C}^2$, with the Hilbert space $X$ for the orbital motion in real space. The spin-rotation group $\mathrm{Spin}_3 = \mathrm{SU}_2$ acts trivially on $X$ and by the spinor representation on the factor $\mathbb{C}^2$. (In a framework more comprehensive than is of relevance to the disordered systems setting developed here, the spinor representation would enter as a projective representation of the rotation group $\mathrm{SO}_3$, and $\mathrm{SO}_3$ would act on the factor $X$ by rotations in the three-dimensional Euclidean space.) On physical grounds, spin rotations must preserve the canonical anticommutation relations as well as the unitary structure of $V$. Therefore, by Prop. 2.2 spin rotations commute with the particle-hole conjugation operator $C$.

Another symmetry operation of importance for present purposes is time reversal. As always in quantum mechanics, time reversal is implemented as an antiunitary operator $T$ on the single-electron Hilbert space $V$. Its algebraic properties are influenced by the spin 1/2 nature of the electron: fundamental physics considerations dictate $T^2 = -\mathrm{Id}$. A closely related condition is that time reversal commutes with spin rotations. $T$ extends to an operation on $W$ by $CT = TC$.

In physics one uses the word *quasiparticle* for the excitations that are created by acting with a fermionic field operator on a many-fermion ground state.

*6.1.1. Class D.* — In the general context of quasiparticle excitations in metals and superconductors, this is the fundamental class where *no* symmetries are present.

A concrete realization takes place in superconductors where the order parameter transforms under spin rotations as a spin triplet, $S = 1$ (i.e., the adjoint representation of $SU_2$), and transforms under $SO_2$-rotations of two-dimensional space as a $p$-wave (the fundamental representation of $SO_2$). A recent candidate for a quasi-2d (or layered) spin-triplet $p$-wave superconductor is the compound $Sr_2RuO_4$ [**M, E**]. (A noncharged analog is the $A$-phase of superfluid $^3$He [**VW**].) Time-reversal symmetry in such a system may be broken spontaneously, or else can be broken by an external magnetic field creating vortices in the superconductor. Further realizations proposed in the recent literature include double-layer fractional quantum Hall systems at half filling [**R**] (more precisely, a mean-field description for the composite fermions of such systems), and a network model for the random-bond Ising model [**SF**].

The time-evolution operators $U = \mathrm{e}^{-\mathrm{i}tH/\hbar}$ in this class are constrained only by the requirement that they preserve both the unitary structure and the symmetric bilinear form of $W$. If $W_{\mathbb{R}}$ is the set of real points $\{v + Cv : v \in V\}$, we know from Prop. 4.5 that the space of time evolutions is a real orthogonal group $\mathrm{SO}(W_{\mathbb{R}})$. In Cartan's notation this is called a symmetric space of the $D$ family. The Hamiltonians $H$ are such that $\mathrm{i}H \in \mathfrak{so}(W_{\mathbb{R}})$; this means that the Hamiltonian matrices are imaginary skew in a suitably chosen basis (called Majorana fermions in physics).

Note that since $W_{\mathbb{R}}$ is a real form of $(X \otimes \mathbb{C}^2) \oplus (X \otimes \mathbb{C}^2)^*$, the dimension of $W_{\mathbb{R}}$ must be a multiple of four (for spinless particles it would only be a multiple of two).

*6.1.2. Class D*III. — Let now time reversal be a symmetry of the quasiparticle system. This means that magnetic fields and scattering by magnetic impurities are absent. On the other hand, spin-rotation invariance is again required to be broken.

Known realizations of this situation exist in gapless superconductors, say with spin-singlet pairing, but with a sufficient concentration of spin-orbit impurities to cause strong spin-orbit scattering [**SF**]. In order for quasiparticle excitations to exist at low energy, the spatial symmetry of the order parameter should be $d$-wave (more precisely, a time-reversal invariant combination of the angular momentum $l = +2$ and $l = -2$ representations of $SO_2$). A noncharged realization occurs in the $B$-phase of $^3$He [**VW**], where the order parameter is spin-triplet without breaking time-reversal symmetry. Another candidate are heavy-fermion superconductors [**S**], where spin-orbit scattering often happens to be strong owing to the presence of elements with large atomic weights such as uranium and cerium.

Time-reversal invariance constrains the set of good Hamiltonians $H$ by $H = THT^{-1}$. Since $T^2 = -\mathrm{Id}$ for spin 1/2 particles, we are dealing with the case treated in 4.3.2.1. The space of time evolutions therefore is $\mathrm{SO}(W_{\mathbb{R}})/\mathrm{U}(V)$, which is a symmetric space of the $D$III family. The standard form of the Hamiltonians in this class is

$$H = \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix} , \tag{3}$$

where $Z \in \mathrm{Hom}(V^*, V)$ is skew. (Note again that the dimension of $W_{\mathbb{R}}$ is a multiple of four, and would be a multiple of two for particles with spin zero).

*6.1.3. Class C.* — Next let the spin of the quasiparticles be conserved, and let time-reversal symmetry be broken instead. Thus magnetic fields (or some equivalent $T$-breaking agent) are now present, while the effect of spin-orbit scattering is absent. The symmetry group of the physical system then is $G = G_0 = \mathrm{Spin}_3 = \mathrm{SU}_2$.

This situation is realized in spin-singlet superconductors in the vortex phase [**S4**]. Prominent examples are the cuprate (or high-$T_c$) superconductors [**T**], which are layered and exhibit $d$-wave symmetry in their copper-oxide planes. It has been speculated that some of these superconductors break time-reversal symmetry spontaneously, by the generation of an order-parameter component $\mathrm{i}d_{xy}$ or $\mathrm{i}s$ [**S3**]. Other realizations of this class include network models of the spin quantum Hall effect [**G**].

Following the general strategy of Sect. 3, we eliminate $G_0 = \mathrm{SU}_2$ from the picture by transferring from $V \oplus V^*$ to the reduced space $X \oplus X^*$. In the process the bilinear form $b$ undergoes a change of parity. To see this let $R = \mathbb{C}^2$ (a.k.a. spinor space) be the fundamental representation space of $\mathrm{SU}_2$. $R$ is isomorphic to $R^*$ by $\psi : r \mapsto \langle \mathrm{i}\sigma_2 \overline{r}, \cdot \rangle_R$ where $\sigma_2$ is the second Pauli matrix. This isomorphism $\psi : R \to R^*$ is alternating. Therefore, by Prop. 3.12 the symmetric bilinear form of $V \oplus V^*$ gets transferred to the alternating form $a$ of $X \oplus X^*$.

From Prop. 4.5 we then infer that the space of time evolutions is $\mathrm{USp}(X \oplus X^*)$ — a symmetric space of the $C$ family. The standard form of the Hamiltonians here is

$$H = \begin{pmatrix} A & B \\ B^* & -A^{\mathrm{t}} \end{pmatrix} ,$$

with self-adjoint $A \in \mathrm{End}(X)$ and complex symmetric $B \in \mathrm{Hom}(X^*, X)$.

*6.1.4. Class CI.* — The next class is obtained by taking spin rotations as well as the time reversal $T$ to be symmetries of the quasiparticle system. Thus the symmetry group is $G = G_0 \cup TG_0$ with $G_0 = \mathrm{Spin}_3 = \mathrm{SU}_2$.

Like in the previous symmetry class, physical realizations are provided by the low-energy quasiparticles of unconventional spin-singlet superconductors [**T**]. The difference is that the superconductor must now be in the Meissner phase where magnetic field are expelled by screening currents. In the case of superconductors with several low-energy points in the first Brillouin zone, scattering off hard impurities is needed to break additional conservation laws that would otherwise emerge (see Sect. 6.1.5).

To identify the relevant symmetric space, we again transfer from $V \oplus V^*$ to the reduced space $X \oplus X^*$. As before, the bilinear form $b$ changes parity from symmetric to alternating under this reduction. In addition now, time reversal has to be transferred. As was explained in the example following Lemma 3.11, the time-reversal operator changes its involutory character from $T^2 = -\mathrm{Id}_{V \oplus V^*}$ to $T^2 = +\mathrm{Id}_{X \oplus X^*}$.

In the language of Sect. 4 the block data are $V = X$, $E = \mathrm{End}(V \oplus V^*)$, $b = a$, $T$ nonmixing, and $T^2 = \mathrm{Id}$. This case was treated in 4.3.3.2. From there, we know that the space of time evolutions is $\mathrm{USp}(X \oplus X^*)/\mathrm{U}(X)$ – a symmetric space in the $C$I family. The standard form of the Hamiltonians in this class is the same as that given in (3) but now with $Z \in \mathrm{Hom}(X^*, X)$ complex symmetric.

*6.1.5. Class A*III. — This class is commonly associated with random-matrix models for the low-energy Dirac spectrum of quantum chromodynamics with massless quarks (see Sect. 6.2.1). Here we review an alternative realization, which has recently been identified [**A3**] in $d$-wave superconductors with soft impurity scattering.

To construct this realization one starts from class $C$I, i.e. from quasiparticles in a superconductor with time-reversal invariance and conserved spin, and enlarges the symmetry group by imposing another $\mathrm{U}_1$ symmetry, generated by a Hermitian operator $Q$ with $Q^2 = \mathrm{Id}$. The physical reason for the extra conservation law is approximate momentum conservation in a disordered quasiparticle system with a dispersion law that has Dirac-type low-energy points at four distinct places in the Brillouin zone.

Thus beyond the spin-rotation group $\mathrm{SU}_2$ there now exists a one-parameter group of unitary symmetries $\mathrm{e}^{\mathrm{i}\theta Q}$. The operators $\mathrm{e}^{\mathrm{i}\theta Q}$ are defined on $V$, and are diagonally extended to $W = V \oplus V^*$. They are characterized by the property that they commute with particle-hole conjugation $C$, time reversal $T$, and the spin rotations $g \in \mathrm{SU}_2$.

The reduction to standard block data is done in two steps. In the first step, we eliminate the spin-rotation group $\mathrm{SU}_2$. From the previous section, the transferred data are known to be $E = \mathrm{End}(X \oplus X^*)$, $b = a$, $T$ nonmixing, and $T^2 = \mathrm{Id}$.

The second step is to reduce by the $\mathrm{U}_1$ group generated by $Q$. For this consider the $\mathbb{C}$-linear operator $J := \mathrm{i}Q$ with $J^2 = -\mathrm{Id}$, and let the $J$-eigenspace decomposition of $X$ be written $X = X_{+\mathrm{i}} \oplus X_{-\mathrm{i}}$. There is a corresponding decomposition $X^* = {X^*}_{+\mathrm{i}} \oplus {X^*}_{-\mathrm{i}}$. Since $J$ commutes with $T$, a complex structure is defined by it on the set of $T$-real points of $X$. Therefore $\dim X_{+\mathrm{i}} = \dim X_{-\mathrm{i}}$. Another consequence of $JT = TJ$ is that the $\mathbb{C}$-antilinear operator $T$ exchanges $X_{+\mathrm{i}}$ with $X_{-\mathrm{i}}$. Thus $T$ is mixing with respect to the decompositions $X = X_{+\mathrm{i}} \oplus X_{-\mathrm{i}}$ and $X^* = {X^*}_{+\mathrm{i}} \oplus {X^*}_{-\mathrm{i}}$. The $\mathbb{C}$-antilinear operator $C$ maps $X_{\pm\mathrm{i}}$ to ${X^*}_{\mp\mathrm{i}}$.

The fully reduced block data now are $V := X_{+\mathrm{i}} \oplus {X^*}_{+\mathrm{i}}$, $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$, $b = a$, $T$ mixing, and $T^2 = \mathrm{Id}$. The finite-dimensional version of this case was treated in 4.3.1.3. Our answer for the space of time-evolution operators was $\mathrm{SU}_{p+q}/\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$, which is a symmetric space in the $A$III family.

Unlike the general case handled in 4.3.1.3, it here follows from the fundamental physics definition of particle-hole conjugation $C$ and time reversal $T$ that the operator $CT$ stabilizes a real subspace $V_{\mathbb{R}}$. We also have $(CT)^2 = -\mathrm{Id}$. Therefore, the operator $CT$ defines a complex structure of $V_{\mathbb{R}}$, and hence the integers $p$ and $q$, which are the dimensions of the $CT$-eigenspaces in $V$, must be equal.

*6.1.6. Class A.* — At this point a new symmetry requirement is brought into play: conservation of the electric charge. Thus the global $\mathrm{U}_1$ gauge transformations of electrodynamics are now decreed to be symmetries of the quasiparticle system. This means that the system no longer is a superconductor, where $U_1$ gauge symmetry is spontaneously broken, but is a metal or normal-conducting system. If all further symmetries are broken (time reversal by a magnetic field or magnetic impurities, spin rotations by spin-orbit scattering, etc.), the symmetry group is $G = G_0 = U_1$.

All states (actually, field operators) in $V$ have the same electric charge. Thus the irreducible $U_1$ representations which they carry all have the same isomorphism class, say $\lambda$. States in $V^*$ carry the opposite charge and belong to the dual class $\lambda^*$. Since $\lambda \neq \lambda^*$, we are in the situation of Sect. 4.3.1, where $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$. With $T$ being absent, the space of time evolutions is $\mathrm{U}(V)$ acting diagonally on $V \oplus V^*$.

In random-matrix theory, and in the finite-dimensional case where $\mathrm{U}(V) \cong \mathrm{U}_N$, one refers to these matrix spaces as the circular Wigner-Dyson class of unitary symmetry. The Hamiltonians in this class are represented by complex Hermitian matrices.

If we make the restriction to traceless Hamiltonians, the space of time evolutions becomes $\mathrm{SU}_N$, which is a type-II irreducible symmetric space of the *A* family.

*6.1.7. Class A*II. — Beyond charge conservation or $U_1$ gauge symmetry, time reversal $T$ is now required to be a symmetry of the quasiparticle system. Physical realizations of this case occur in metallic systems with spin-orbit scattering. The pioneering experimental work (of the weak localization phenomenon in this class) was done on disordered magnesium films with gold impurities.

The block data now is $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$, $b = s$, $T$ nonmixing, $T^2 = -\mathrm{Id}$. This case was considered in 4.3.1.1. The main point there was that time reversal $T$ defines a $\mathbb{C}$-linear symplectic structure $a$ on $V$ by $a(v_1, v_2) = \langle T v_1, v_2 \rangle$. Conjugation by $T$ therefore fixes a unitary symplectic group $\mathrm{USp}(V)$ inside of $\mathrm{U}(V)$, and the space of good time evolutions is $\mathsf{G}/\mathsf{K} = \mathrm{U}(V)/\mathrm{USp}(V)$. In the finite-dimensional setting where $\mathsf{G}/\mathsf{K} \cong \mathrm{U}_{2N}/\mathrm{USp}_{2N}$, this is called the circular Wigner-Dyson class of symplectic symmetry in random-matrix theory. The Hamiltonians in this class are represented by Hermitian matrices whose matrix entries are real quaternions. The irreducible part $\mathrm{SU}_{2N}/\mathrm{USp}_{2N}$, obtained by restricting to traceless Hamiltonians, is a type-I symmetric space in the *A*II family.

*6.1.8. Class A*I. — The next class is the Wigner-Dyson class of orthogonal symmetry. In the present quasiparticle setting it is obtained by imposing spin-rotation symmetry, $U_1$ gauge (or charge) symmetry and time-reversal symmetry all at once.

Important physical realizations are by disordered metals in zero magnetic field. Families of quantum chaotic billiards also belong to this class.

The group of unitary symmetries here is $G_0 = U_1 \times SU_2$. We eliminate the spin-rotation group $SU_2$ from the picture by transferring from $V = X \otimes \mathbb{C}^2$ to the reduced space $X$. Again, the involutory character of $T$ is reversed in the process: the transferred time reversal satisfies $T^2 = +\mathrm{Id}$. The parity of the bilinear form also changes, from symmetric to alternating; however, this turns out to be irrelevant here, as there is still the $U_1$ charge symmetry and we are in the situation $\lambda \neq \lambda^*$.

The block data now is $E = \mathrm{End}(X) \oplus \mathrm{End}(X^*)$, $b = a$, $T$ nonmixing, $T^2 = \mathrm{Id}$. According to 4.3.1.2 these yield (the Cartan embedding of) $\mathrm{U}(X)/\mathrm{O}(X)$ as the space of good time evolutions. The irreducible part $\mathrm{SU}(X)/\mathrm{SO}(X)$, or $\mathrm{SU}_N/\mathrm{SO}_N$ in the finite-dimensional setting, is a symmetric space in the $A$I family. The Hamiltonian matrices in this class can be arranged to be real symmetric.

**6.2. The Euclidean Dirac operator for chiral fermions. —** We now explore the physical examples afforded by Dirac fermions in a random gauge field background. These examples include the Dirac operator of quantum chromodynamics, i.e., the theory of strong $\mathrm{SU}_3$ gauge interactions between elementary particles called quarks.

The mathematical setting for this has already been described in Sect. 2.3. Recall that one is given a twisted spinor bundle $S \otimes R$ over Euclidean space-time, and that $V$ is taken to be the Hilbert space of $L^2$-sections of that bundle. One is interested in the Dirac operator $D_A$ in a gauge field background $A$ and in the limit of zero mass:

$$D_A = \mathrm{i}\gamma^\mu(\partial_\mu - A_\mu) \ .$$

We extend the self-adjoint operator $D_A$ diagonally from $V$ to the fermionic Nambu space $W = V \oplus V^*$ by the condition $D_A = -CD_AC^{-1}$. The chiral 'symmetry' $\Gamma D_A + D_A\Gamma = 0$, where $\Gamma = \gamma_5$ is the chirality operator, then becomes a true symmetry $D_A = TD_AT^{-1}$ with an antiunitary operator $T = C\Gamma = \Gamma C$, which mixes $V$ and $V^*$.

*6.2.1. Class A*III. — Let now the complex vector space $R = \mathbb{C}^N$ be the fundamental representation space for the gauge group $\mathrm{SU}_N$ with $N \geq 3$. ($N$ is called the number of colors in this context.) Quantum chromodynamics is the special case $N = 3$.

The fact that the extended Dirac operator $D_A$ acts diagonally on $W = V \oplus V^*$ is attributed to a symmetry group $G_0 = \mathrm{U}_1$ which has $V$ and $V^*$ as inequivalent representation spaces. For a generic gauge-field configuration there exist no further symmetries; thus the total symmetry group is $G = G_0 \cup TG_0$.

The block data here is $V = V$, $E = \mathrm{End}(V) \oplus \mathrm{End}(V^*)$, $b = s$, $T$ mixing, $T^2 = \mathrm{Id}$, which is the case considered in 4.3.1.3. If $n = \dim V$, we have

$$\mathfrak{p} \cong \mathfrak{su}_n/\mathfrak{s}(\mathfrak{u}_p \oplus \mathfrak{u}_q) \ .$$

The difference of integers $p - q$ is to be identified with the difference between the number of right and left zero modes of $D_A^2$. ('Right' and 'left' in this context pertain to the $(+1)$- and $(-1)$-eigenspaces of the chirality $\Gamma = \gamma_5$.) The latter number is a topological invariant called the index of the Dirac operator.

*6.2.2. Class BD*I. — We retain the framework from before, but now consider the gauge group $\mathrm{SU}_2$, where the number of colors $N = 2$. In this case the massless Dirac operator $D_A$ has an additional antiunitary symmetry [**V1**], which emerges as follows.

Recall that the unitary $\mathrm{SU}_2$-representation space $R = \mathbb{C}^2$ is isomorphic to the dual representation space $R^*$ by a $\mathbb{C}$-linear mapping $\psi : R \to R^*$. Combining the inverse of this with $\iota : R \to R^*$ defined by $\iota(r) = \langle r, \cdot \rangle_R$, we obtain a $\mathbb{C}$-antilinear mapping

$\beta := \psi^{-1} \circ \iota : R \to R$. The map $\beta$ thus defined commutes with the $SU_2$-action on $R$. By Lemma 3.11 it satisfies $\beta^2 = -\mathrm{Id}_R$ since $\psi$ is alternating.

Now, on the (untwisted) spinor bundle $S$ over Euclidean space-time $M$ there exists a $\mathbb{C}$-antilinear operator $\alpha$, called *charge conjugation* in physics, which anticommutes with the Clifford action $\gamma : T^*M \to \mathrm{End}(S)$; thus $\alpha i\gamma = i\gamma\alpha$. Since $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$, this implies that $\alpha$ commutes with $\gamma_5 = \Gamma$ and stabilizes the $\Gamma$-eigenspace decomposition $S = S_+ \oplus S_-$ into half-spinor components $S_\pm$. The charge conjugation operator has square $\alpha^2 = -\mathrm{Id}_S$.

For the case of three or more colors, the existence of $\alpha$ is of no consequence from a symmetry perspective, as the fundamental and antifundamental representations of $SU_N$ are inequivalent for $N \geq 3$. For $N = 2$, however, we also have $\beta$, and $\alpha$ combines with it to give an antiunitary symmetry $T_1 = \alpha \otimes \beta$. Indeed,

$$T_1 D_A T_1^{-1} = (\alpha \otimes \beta) D_A (\alpha \otimes \beta) = \alpha(i\gamma^\mu)\alpha^{-1} \otimes \beta(\partial_\mu - A_\mu)\beta^{-1} .$$

Since gauge transformations $g(x) \in SU_2$ commute with $\beta$, so do the components $A_\mu(x) \in \mathfrak{su}_2$ of the gauge field. Thus $\beta A_\mu \beta^{-1} = A_\mu$, and since $\alpha(i\gamma)\alpha^{-1} = i\gamma$, we have

$$T_1 D_A T_1^{-1} = D_A .$$

Note that the antiunitary symmetry $T_1 : V \to V$ is nonmixing, and $T_1^2 = \mathrm{Id}$. As usual, the extension to an operator $T_1 : W \to W$ is made by requiring $CT_1 = T_1 C$.

Thus we now have two antiunitary symmetries, $T$ and $T_1$. Because $T$ is mixing and $T_1$ nonmixing, the unitary operator $P = TT_1 = T_1 T$ mixes $V$ with $V^*$. Since $T^2 = T_1^2 = \mathrm{Id}$, and $(CP)^2 = \mathrm{Id}$, this is the case treated in 5.1.1.1, where we found

$$\mathfrak{p} \cong \mathfrak{so}(V_\mathbb{R}) / (\mathfrak{so}(V_\mathbb{R}^+) \oplus \mathfrak{so}(V_\mathbb{R}^-)) .$$

After truncation to finite dimension this is $\mathfrak{so}_{p+q}/(\mathfrak{so}_p \oplus \mathfrak{so}_q)$. The difference $p - q$ still has a topological interpretation as the index of the Dirac operator.

Although our considerations explicitly referred to the case of the gauge group being $SU_2$, the only specific feature we used was the existence of an alternating isomorphism $\psi : R \to R^*$. The same result therefore holds for any gauge group representation $R$ where such an isomorphism exists. In particular it holds for the fundamental representation of the whole series of symplectic groups $USp_{2N}$ (which includes $SU_2 \cong USp_2$).

*6.2.3. Class* CII. — Now take $R$ to be the adjoint representation of any compact Lie (gauge) group $K$ with semisimple Lie algebra. This case is called ‘adjoint fermions’ in physics. A detailed symmetry analysis of it was presented in [**HV**].

The Cartan-Killing form on $\mathrm{Lie}(K)$,

$$B(X,Y) = \mathrm{Tr}\,\mathrm{ad}(X)\mathrm{ad}(Y) ,$$

is nondegenerate, invariant, complex bilinear, and symmetric. $B$ therefore defines an isomorphism $\psi : R \to R^*$ by $\psi(X) = B(X, \cdot)$. Since $B$ is symmetric, so is $\psi$.

The change in parity of $\psi$ reverses the parity of the antiunitary operator $\beta = \psi^{-1} \circ \iota$, which now satisfies $\beta^2 = +\mathrm{Id}_R$. By $\alpha^2 = -\mathrm{Id}$ this translates to $T_1^2 = (\alpha \otimes \beta)^2 = -\mathrm{Id}$.

Thus we now have two antiunitary symmetries $T$ and $T_1$ with $T^2 = \mathrm{Id} = -T_1^2$, and $(CP)^2 = (CTT_1)^2 = -\mathrm{Id}$. This case was handled in 5.1.2.1 where we found

$$\mathfrak{p} \cong \mathfrak{usp}(V)/(\mathfrak{usp}(V^+) \oplus \mathfrak{usp}(V^-)) .$$

In a finite-dimensional setting this would be $\mathfrak{usp}_{2p+2q}/(\mathfrak{usp}_{2p} \oplus \mathfrak{usp}_{2q})$.

In summary, the physical situation is ruled by a mathematical trichotomy: the isomorphism $\psi : R \to R^*$ is either symmetric, or alternating, or does not exist. The corresponding symmetry class of the massless Dirac operator is $C$II, $BD$I, or $A$III, respectively. As was first observed by Verbaarschot [**V**], this is the same trichotomy that ruled Dyson's threefold way.

**Acknowledgment**. This work was carried out under the auspices of the Deutsche Forschungsgemeinschaft, SFB/TR12. Major portions of the article were prepared while M.R.Z. was visiting the Institute for Advanced Study (Princeton, USA) and the Newton Institute for Mathematical Sciences (Cambridge, UK). The support of these institutions is gratefully acknowledged.

## References

[A] Altland, A. and Zirnbauer, M.R.: Nonstandard symmetry classes in mesoscopic normal-/superconducting hybrid systems, Phys. Rev. B **55**, 1142-1161 (1997)

[A3] Altland, A., Simons, B.D. and Zirnbauer, M.R.: Theories of low-energy quasiparticle states in disordered $d$-wave superconductors, Phys. Rep. **359**, 283-354 (2002)

[B] Berline, N., Getzler, E. and Vergne, M.: *Heat kernels and Dirac operators* (Springer-Verlag, Berlin, 1992)

[C] Caselle, M. and Magnea, U.: Random-matrix theory and symmetric spaces, Phys. Rep. **394**, 41-156 (2004)

[D] Dyson, F.J.: The threefold way: algebraic structure of symmetry groups and ensembles in quantum mechanics, J. Math. Phys. 3, 1199-1215 (1962)

[E] Eremin, I., Manske, D., Ovchinnikov, S.G. and Annett, J.F.: Unconventional superconductivity and magnetism in $Sr_2RuO_4$ and related materials, Ann. Physik **13**, 149-174 (2004)

[G] Gruzberg, I.A., Ludwig, A.W.W. and Read, N.: Exact exponents for the spin quantum Hall transition, Phys. Rev. Lett. **82**, 4524-4527 (1999)

[H] Helgason, S.: *Differential geometry, Lie groups and symmetric spaces* (Academic Press, New York, 1978)

[HV] Halasz, M.A. and Verbaarschot, J.J.M.: Effective Lagrangians and chiral random-matrix theory, Phys. Rev. D **51**, 2563-2573 (1995)

[K] Katz, N.M. and Sarnak, P.: *Random matrices, Frobenius eigenvalues, and monodromy* (American Mathematical Society, Providence, R.I., 1999)

[M] Mackenzie, A.P. and Maeno, Y.: The superconductivity of $Sr_2RuO_4$ and the physics of spin-triplet pairing, Rev. Mod. Phys. **75**, 657-712 (2003)

[R] Read, N. and Green, D.: Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect, Phys. Rev. B **61**, 10267-10297 (2000)

[S] Stewart, G.S.: Heavy-fermions systems, Rev. Mod. Phys. **56**, 755-787 (1984)

[S3] Senthil, T., Marston, J.B. and Fisher, M.P.A.: Spin quantum Hall effect in unconventional superconductors, Phys. Rev. B **60**, 4245-4254 (1999)

[S4] Senthil, T., Fisher, M.P.A., Balents, L. and Nayak, C.: Quasiparticle transport and localization in high-$T_c$ superconductors, Phys. Rev. Lett. **81**, 4704-4707 (1998)

[SF] Senthil, T. and Fisher, M.P.A.: Quasiparticle localization in superconductors with spin-orbit scattering, Phys. Rev. B **61**, 9690-9698 (2000)

[T] Tsuei, C.C. and Kirtley, J.R.: Pairing symmetry in the cuprate superconductors, Rev. Mod. Phys. **72**, 969-1016 (2000)

[V] Verbaarschot, J.J.M.: The spectrum of the QCD Dirac operator and chiral random-matrix theory: the threefold way, Phys. Rev. Lett. **72**, 2531-2533 (1994)

[V1] Verbaarschot, J.J.M.: The spectrum of the Dirac operator near zero virtuality for $N_c = 2$, Nucl. Phys. B **426**, 559-574 (1994)

[VW] Vollhardt, D. and Wölfle, P.: *The superfluid phases of Helium 3* (Taylor & Francis, London, 1990)

[VZ] Verbaarschot, J.J.M. and Zahed, I.: Spectral density of the QCD Dirac operator near zero virtuality, Phys. Rev. Lett. **70**, 3852-3855 (1993)

[Z] Zirnbauer, M.R.: Riemannian symmetric superspaces and their origin in random-matrix theory, J. Math. Phys. **37**, 4986-5018 (1996)

---

*June 10, 2004*

P. HEINZNER, A. HUCKLEBERRY, Fakultät für Mathematik, Ruhr-Universität Bochum, Germany

M.R. ZIRNBAUER, Institut für Theoretische Physik, Universität zu Köln, Germany

*E-mail :* heinzner@cplx.ruhr-uni-bochum.de, ahuck@cplx.ruhr-uni-bochum.de, zirn@thp.uni-koeln.de