Limit laws for norms of i.i.d. samples with Weibull tails

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We study the limit distribution of l-norms $R_N(t) = \|\mathbf{X}_N\|_t$ (of order t) of samples $\mathbf{X}_N = (X_1, \dots, X_N)$ of i.i.d. positive random variables, as $t \to \infty$, $N \to \infty$. This problem, motivated by a recent work by Schlather (2001), is closely related to the behaviour of the sums $S_N(t) = \sum_{i=1}^N X_i^t$, which is of interest in its own right. It is assumed that the function $h(x) = -\log \mathsf{P}\{X_i > x\}$ is regularly varying at infinity with index $0 < \varrho < \infty$. The appropriate growth scale of N relative to t is of the form $\mathrm{e}^{\alpha t/\varrho}$. We show that there are two critical points, $\alpha_1 = 1$ and $\alpha_2 = 2$, below which the Law of Large Numbers and the Central Limit Theorem, respectively, break down. For $\alpha < 2$, we impose a slightly stronger condition of normalised regular variation of h. Here the limit laws for $S_N(t)$ appear to be stable, with characteristic exponent $\alpha \in (0, 2)$ and skewness parameter $\beta \equiv 1$. Limit laws for the norms $R_N(t)$ are also obtained. In particular, our results corroborate a conjecture by Schlather (2001) regarding the 'endpoints' $\alpha \to \infty$ and $\alpha \to 0+$ of the limits of $R_N(t)$.

Keywords: sums of independent random variables; weak limit theorems; central limit theorem; infinitely divisible laws; stable laws; l-norms

1. Introduction

Let us consider a family of random variables

$$R_N(t) := \left(\sum_{i=1}^N X_i^t \right)^{1/t}, \qquad t > 0, \tag{1.1}$$

where $X_1, X_2, \dots$ is a sequence of i.i.d. positive random variables. Note that $R_N(t)$ has the meaning of the l_t-norm (of order t) of the random vector $\mathbf{X}_N = (X_1, \dots, X_N)$:

$$R_N(t) = \|\mathbf{X}_N\|_t.$$

Our goal is to study the limiting distribution of $R_N(t)$ as both N and t tend to infinity.

Clearly, the asymptotic behaviour of $R_N(t)$ depends heavily on the relationship between the parameters N and t. If, for instance, one lets N tend to infinity with t fixed or growing slowly enough, then, under appropriate moment conditions, the usual Law of Large Numbers (LLN) and the Central Limit Theorem (CLT) should be valid. In contrast, if the growth rate of N is small enough as compared to t, then the asymptotic behaviour of $R_N(t)$ is dominated by the maximum of the sample $X_1, \ldots, X_N$. Therefore, one can expect that such a setting provides a tool to interpolate between the classical limit theorems concerning the bulk of the sample (i.e., the LLN and CLT) and limit theorems for extreme values.

One can also anticipate that the results will depend upon the structure of the upper distribution tail of the random variables X_i. In this paper, we will focus on a particular case where X_i are unbounded above and have the upper tail of the Weibull form:

$$\mathsf{P}\{X_i > x\} \approx \exp(-cx^{\varrho}) \qquad (x \to +\infty), \tag{1.2}$$

where $0 < \varrho < \infty$. More precisely, we will be assuming that the log-tail distribution function $h(x) := -\log \mathsf{P}\{X_i > x\}$ is regularly varying with index $\varrho \in (0, \infty)$ as $x \to +\infty$. For example, an exponential distribution is contained in this class with $\varrho = 1$.

Note that the above problem is closely related to the limiting behaviour of the partial sums

$$S_N(t) = \sum_{i=1}^{N} X_i^t, \tag{1.3}$$

which is of interest in its own right. As we will see, it is most convenient to obtain the limit of the sums $S_N(t)$ first, using the well-elaborated classical techniques, and then derive results for the norms $R_N(t) = S_n(t)^{1/t}$ using an elementary 'transfer' lemma (see Lemma 9.1 below).

Limits of the norms of the form (1.1) were first considered in the recent paper by Schlather (2001) with the aim to combine the CLT with limit theorems in extreme value theory. Qualitatively speaking, Schlather has demonstrated that under a suitable parametrisation of the functional relation between the norm order t and the sample size N, there exists a 'homotopy' for the limit distributions of $R_N(t)$ extending from the CLT to a limit law for extreme values. The situation where both N and t tend to infinity arises in Schlather (2001, Theorem 2.2, p. 864), where the random variables X_i are assumed to be bounded above and, in the sense of extreme value theory, belong to the domain of attraction of the Weibull distribution $\Psi_\alpha(x) = \exp(-(-x)^\alpha)$ ($\alpha > 0$, $x < 0$). In contrast, in Theorem

2.3 (Schlather 2001, p. 865), where X_i are unbounded and are in the domain of attraction of the Fréchet distribution $\Phi_\alpha(x) = \exp(-x^{-\alpha})$ ($\alpha > 0$, $x > 0$), the norm order t is supposed to be fixed.

Let us point out that for random variables X_i with Weibull tails of the form (1.2), the distribution of the maximum of the sample $X_1, \dots, X_N$ can be shown to converge (under a slightly more restrictive assumption of *normalised regular variation*, see below) to the Gumbel (double exponential) distribution $\Lambda(x) = \exp(-\mathrm{e}^{-x})$ ($x \in \mathbb{R}$). Note that in this case Schlather (2001) has obtained a partial result and only for exponential random variables (Theorem 2.4, p. 867). However, he has conjectured (p. 867) that in the general case of attraction to Λ, the weak limit of the properly centered and normalised $R_N(t)$ does exist; moreover, the endpoints of the parametric family of the limits should be represented by the normal distribution and the Gumbel distribution (heuristically, corresponding to the cases $N \gg t$ and $t \gg N$, respectively).

The results obtained in the present paper do corroborate this conjecture. Moreover, we have found explicitly the full spectrum of the limiting laws for the underlying sums $S_N(t)$ (and hence for $R_N(t)$). In particular, we have shown that the non-Gaussian limits of $S_N(t)$ are given by the family of stable laws $\mathcal{F}_\alpha$ with characteristic exponent $\alpha \in (0, 2)$ and skewness parameter $\beta \equiv 1$.

Another class of examples within this setting is provided by Ben Arous, Bogachev and Molchanov (2003), where the sum $S_N(t)$ has the terms of the form $X_i = \exp(\tilde{X}_i)$ and $\tilde{X}_i$ are either unbounded above and have Weibull tails of the form (1.2) with index $1 < \tilde{\varrho} < \infty$ (*case B*), or bounded above and have a similar exponential behaviour in the vicinity of the upper edge of the support, with $0 < \tilde{\varrho} < \infty$ (*case A*). Note that for the Weibull tail (1.2), we have

$$\mathsf{P}\{\tilde{X}_i > x\} = \mathsf{P}\{X_i > \mathrm{e}^x\} \approx \exp\left(-c\,\mathrm{e}^{\varrho x}\right),$$

which formally corresponds to case B in Ben Arous, Bogachev and Molchanov (2003) with a 'limiting' value $\tilde{\varrho} = \infty$.

As shown in Ben Arous, Bogachev and Molchanov (2003), under the assumption of normalised regular variation of $h(\cdot)$, in both cases A and B the random exponentials $\exp(\tilde{X}_i)$ belong to the domain of attraction of the double exponential distribution Λ, and the implications for the l_t-norms $R_N(t)$ are again in line with Schlather's conjecture. It is interesting that the family of the limiting distributions for $S_N(t)$ (and hence for $R_N(t)$) appears to be the same as in the present paper. Such universality is quite remarkable and should be studied in more detail.

2. Statement of the results

We will be working under the condition that the log-tail distribution function $h(x) := -\log \mathsf{P}\{X_i > x\}$ is regularly varying at infinity with index $\varrho \in (0,\infty)$ (we write $h \in R_\varrho$; see Assumption 2 below).

It follows that the moment generating function

$$m(t) := \mathsf{E}[X^t] \tag{2.1}$$

is well defined for all $t \geq 0$. Note that the expected value of the sum $S_N(t)$ is given by

$$\mathsf{E}[S_N(t)] = \sum_{i=1}^{N} \mathsf{E}[X_i^t] = N m(t),$$

suggesting that the moment function $m(t)$ should be relevant to the appropriate scale for the number of terms $N = N(t)$. However, Kasahara's exponential Tauberian theorem (see Lemma 3.2 below) shows that $m(t)$ grows like $\mathrm{e}^{(t/\varrho)\log t}$, while a suitable rate function should be chosen as $\mathrm{e}^{\alpha t/\varrho}$. We will see that the values $\alpha_1 = 1$ and $\alpha_2 = 2$ are critical with respect to this scale, in that the LLN and CLT break down below α_1 and α_2, respectively. Moreover, it will be shown that α plays the role of characteristic exponent in the limit laws.

The first two theorems state that $S_N(t)$ satisfies LLN and CLT in their conventional form providing that the number of terms N in the sum $S_N(t)$ grows fast enough (roughly speaking, $N \gg \mathrm{e}^{t/\varrho}$ and $N \gg \mathrm{e}^{2t/\varrho}$, respectively). More precisely, set

$$\alpha := \liminf_{t\to\infty} \frac{\varrho \log N}{t}\,. \tag{2.2}$$

Theorem 2.1. *Suppose that $h \in R_\varrho$ and $\alpha > 1$. Set*

$$S_N^*(t) := \frac{S_N(t)}{\mathsf{E}[S_N(t)]} = \frac{1}{N m(t)} \sum_{i=1}^{N} X_i^t.$$

Then

$$S_N^*(t) \xrightarrow{\ p\ } 1 \qquad (t \to \infty).$$

Theorem 2.2. *Suppose that* $h \in R_\varrho$ *and* $\alpha > 2$. *Then*

$$\frac{S_N(t) - \mathsf{E}[S_N(t)]}{\sqrt{\mathsf{Var}[S_N(t)]}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (t \to \infty),$$

where $\mathcal{N}(0,1)$ *is the standard normal distribution.*

Below the critical points, the rate of growth of $N = N(t)$ must be specified more accurately. Namely, we will require the following

Scaling Assumption. The number N of terms in the sum $S_N(t)$ satisfies the condition

$$\lim_{t\to\infty} N(t)\,\mathrm{e}^{-\alpha t/\varrho} = 1, \tag{2.3}$$

where $\alpha > 0$ is a parameter.

We also need to impose a slightly stronger condition on regularity of the log-tail distribution function h — that of *normalised regular variation,* $h \in NR_\varrho$. This property will be discussed in detail in Section 5. In particular, a function $h(x) \in NR_\varrho$ is ultimately *strictly increasing*, and hence its inverse $h^{-1}(t)$ exists for sufficiently large t. Let $\eta_1(t)$ be a (unique) solution of the equation

$$\varrho h(\eta_1(t)) = \alpha t. \tag{2.4}$$

Theorem 2.3. *Assume that* $h \in NR_\varrho$ *and the scaling condition (2.3) is fulfilled. Suppose that* $0 < \alpha < 2$ *and set*

$$B(t) := \eta_1(t)^t = \mathrm{e}^{t\log\eta_1(t)}, \tag{2.5}$$

$$A(t) := \begin{cases} Nm(t) & (1 < \alpha < 2), \\ Nm_1(t) & (\alpha = 1), \\ 0 & (0 < \alpha < 1), \end{cases} \tag{2.6}$$

where $\eta_1(t)$ *is given by (2.4),* $m(t)$ *is the moment function defined in (2.1) and* $m_1(t)$ *is a truncated moment function,*

$$m_1(t) := \mathsf{E}\big[X^t \mathbf{1}_{\{X\le\eta_1(t)\}}\big]. \tag{2.7}$$

Then, as $t \to \infty$,

$$\frac{S_N(t) - A(t)}{B(t)} \xrightarrow{d} \mathcal{F}_\alpha, \tag{2.8}$$

where $\mathcal{F}_\alpha$ is a stable law with characteristic exponent α and skewness parameter $\beta = 1$. The characteristic function ϕ_α of the law $\mathcal{F}_\alpha$ is given by

$$\phi_\alpha(u) = \begin{cases} \exp\left\{-\Gamma(1-\alpha)|u|^\alpha \exp\left(-\dfrac{i\pi\alpha}{2}\operatorname{sgn} u\right)\right\} & (\alpha \neq 1) \\ \exp\left\{iu(1-\gamma) - \dfrac{\pi}{2}|u|\left(1 + i\operatorname{sgn} u \cdot \dfrac{2}{\pi}\log|u|\right)\right\} & (\alpha = 1) \end{cases} \tag{2.9}$$

where $\gamma = 0.5772\ldots$ is the Euler constant.

Remark. For $1 < \alpha < 2$, we use the analytic continuation of the gamma function in (2.9), given by the formula $\Gamma(1-\alpha) = \Gamma(2-\alpha)/(1-\alpha)$.

At the critical points, $\alpha = 1$ and $\alpha = 2$, the Law of Large Numbers and the Central Limit Theorem, respectively, prove to be valid; however, the normalising constants require some truncation.

Theorem 2.4. *Under the hypotheses of Theorem 2.3, let $\alpha = 1$. Then*

$$\frac{S_N(t)}{N m_1(t)} \xrightarrow{p} 1, \tag{2.10}$$

where $m_1(t)$ is given by (2.7).

Theorem 2.5. *Under the hypotheses of Theorem 2.3, let $\alpha = 2$. Then*

$$\frac{S_N(t) - \mathsf{E}[S_N(t)]}{\sqrt{N m_2(t)}} \xrightarrow{d} \mathcal{N}(0,1),$$

where $m_2(t)$ is a truncated moment function of 'second order',

$$m_2(t) := \mathsf{E}\big[X^{2t}\mathbf{1}_{\{X \le \eta_1(t)\}}\big]. \tag{2.11}$$

Limit theorems for the norms $R_N(t)$ easily follow from the corresponding results for the sums $S_N(t)$.

Theorem 2.6. (a) *For $\alpha \ge 2$,*

$$\frac{t\sqrt{N}m(t)}{\sqrt{\tilde{m}_\alpha(t)}}\left(\frac{R_N(t)}{(Nm(t))^{1/t}} - 1\right) \xrightarrow{d} \mathcal{N}(0,1) \qquad (t \to \infty),$$

where $\tilde{m}_\alpha(t) = m(2t)$ *for* $\alpha > 2$ *and* $\tilde{m}_2(t) = m_2(t)$*, and the functions* $m(\cdot)$ *and* $m_2(\cdot)$ *are given by (2.1) and (2.11), respectively.*

(b) *For* $1 \le \alpha < 2$*,*

$$\frac{tN\tilde{m}_\alpha(t)}{\eta_1(t)^t}\left(\frac{R_N(t)}{(N\tilde{m}_\alpha(t))^{1/t}} - 1\right) \xrightarrow{d} \mathcal{F}_\alpha \qquad (t \to \infty),$$

where $\tilde{m}_\alpha(t) = m(t)$ *for* $1 < \alpha < 2$ *and* $\tilde{m}_1(t) = m_1(t)$*, with* $m_1(\cdot)$ *given by (2.7).*

(c) *For* $0 < \alpha < 1$*,*

$$t\left(\frac{R_N(t)}{\eta_1(t)} - 1\right) \xrightarrow{d} \log \mathcal{F}_\alpha \qquad (t \to \infty),$$

where $\log \mathcal{F}_\alpha$ *stands for the distribution of a random variable* $\log \zeta$ *with* ζ *having distribution* $\mathcal{F}_\alpha$*.*

3. Regular variation and Kasahara's Tauberian theorem

Using the log-tail distribution function

$$h(x) := -\log \mathsf{P}\{X > x\}, \qquad x \in (0, \infty), \tag{3.1}$$

the upper distribution tail is represented in the form

$$\mathsf{P}\{X > x\} = \mathrm{e}^{-h(x)}.$$

We now make our basic assumption on regularity of this tail.

Regularity Assumption. The function h is *regularly varying at infinity with index* $\varrho \in (0, \infty)$ (we write $h \in R_\varrho$). That is, for every constant $\kappa > 0$

$$\lim_{x\to\infty} \frac{h(\kappa x)}{h(x)} = \kappa^\varrho. \tag{3.2}$$

The following result, known as the *Uniform Convergence Theorem* (UCT) is a useful extension of the definition of regular variation (see Bingham *et al.* 1989, Theorem 1.5.2, p. 22).

Lemma 3.1 (UCT**).** *If* $h \in R_\varrho$ *with* $\varrho > 0$ *then (3.2) holds uniformly in* κ *on each interval* $(0, \kappa_1]$.

The link between the asymptotic behaviour of the functions h and m at infinity is characterised by an exponential Tauberian theorem by Kasahara (1978). Recall that a generalised inverse of h can be defined by $h^{\leftarrow}(y) := \inf\{x : h(x) > y\}$ (see Bingham *et al.* 1989, Sect. 1.5.7, p. 28). One can show that $h \in R_\varrho$ if and only if $h^{\leftarrow} \in R_{1/\varrho}$ (see Bingham *et al.* 1989, Theorem 1.5.12, p. 28).

Lemma 3.2 (Kasahara's Tauberian theorem**).** *Let $m(t)$ be given by (2.1) and $h(x)$ by (3.1). Then $h \in R_\varrho$ with $0 < \varrho < \infty$ if and only if*

$$\frac{\log m(t)}{t} - \log h^{\leftarrow}(t) \to -\frac{1+\log\varrho}{\varrho} \qquad (t \to \infty). \tag{3.3}$$

A useful implication of this result is

Lemma 3.3. *Suppose that $h \in R_\varrho$. Then for any constant $r > 0$*

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{m(rt)}{m(t)^r} = \frac{r}{\varrho} \log r. \tag{3.4}$$

Proof. Applying formula (3.3) and recalling that $h^{\leftarrow} \in R_{1/\varrho}$ we obtain

$$\log \frac{m(rt)}{m(t)^r} = rt \log \frac{h^{\leftarrow}(rt)}{h^{\leftarrow}(t)} + o(t) = rt \log r^{1/\varrho} + o(t),$$

and the lemma follows. $\square$

4. Proof of Theorems 2.1 and 2.2

In this section, the parameter α is defined by (2.2).

Proof of Theorem 2.1. By Chebyshev's inequality, it suffices to check that for some $r > 1$

$$\lim_{t\to\infty} \mathsf{E}\,|S_N^*(t) - 1|^r = 0.$$

By an inequality of von Bahr and Esseen (1965), for any $r \in [1, 2]$

$$\mathsf{E}\,|S_N^* - 1|^r \le 2N^{1-r}\,\mathsf{E}\left(\frac{X^t}{m(t)} + 1\right)^r. \tag{4.1}$$

Furthermore, by the elementary inequality $(x+1)^r \le 2^{r-1}(x^r+1)$ ($x \ge 0$, $r \ge 1$), which easily follows from Jensen's inequality applied to x^r, the right-hand side of (4.1) is bounded from above by

$$2^r N^{1-r}\left(\mathsf{E}\left[\frac{X^{rt}}{m(t)^r}\right]+1\right) = O(1)\, N^{1-r}\,\frac{m(rt)}{m(t)^r}\,. \tag{4.2}$$

According to Lemma 3.3 and using the hypothesis of the theorem, we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{\log\left(N^{1-r}\, m(rt)/m(t)^r\right)}{t} &= \frac{(1-r)\alpha}{\varrho}+\frac{r}{\varrho}\log r\\ &\le \frac{(1-r)\alpha}{\varrho}+\frac{r}{\varrho}\,(r-1) = -\frac{(r-1)(\alpha-r)}{\varrho} < 0,\end{aligned} \tag{4.3}$$

providing r is such that $1 < r < \alpha$. It then follows that the right-hand side of (4.2) tends to zero as $t\to\infty$, and the theorem is proved. $\square$

Proof of Theorem 2.2. First of all, note that by Lemma 3.3,

$$\mathsf{Var}[X^t] = m(2t) - m(t)^2 \sim m(2t) \qquad (t\to\infty),$$

so one can replace the normalisation $\sqrt{N\,\mathsf{Var}[X^t]}$ with $\sqrt{Nm(2t)}$.

By the Lyapunov theorem (see Petrov 1995, Theorem 4.9, p. 126), we only need to check that for an appropriate $r > 1$

$$N^{1-r} m(2t)^{-r}\,\mathsf{E}|X^t - m(t)|^{2r} \to 0 \qquad (t\to\infty). \tag{4.4}$$

Arguing as in the proof of Theorem 2.1, one can show that the left-hand side of (4.4) is dominated by

$$2^{2r-1}N^{1-r}m(2t)^{-r}\left[m(2rt)+m(t)^{2r}\right] \sim 2^{2r-1}N^{1-r}\,\frac{m(2rt)}{m(2t)^r}. \tag{4.5}$$

Furthermore, analogously to (4.3) we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{\log\left(N^{1-r}m(2rt)/m(2t)^r\right)}{t} &= \frac{(1-r)\alpha}{\varrho}+\frac{2r}{\varrho}\log r\\ &\le \frac{(1-r)\alpha}{\varrho}+\frac{2r(r-1)}{\varrho} = -\frac{(r-1)(\alpha-2r)}{\varrho} < 0,\end{aligned}$$

if r is such that $1 < r < \alpha/2$. Hence, the right-hand side of (4.5) tends to zero as $t\to\infty$, and the theorem is proved. $\square$

5. Normalised regular variation

As mentioned in Section 2, to characterise the limiting behaviour of the sum $S_N(t)$ in the zone $\alpha \le 2$, we need slightly more regularity. From now on we impose the following

Normalised Regularity Assumption. The log-tail distribution function h is *normalised regularly varying* at infinity, $h \in NR_\varrho$ (with $0 < \varrho < \infty$). The latter means that for every $\varepsilon > 0$ the function $h(x)/x^{\varrho-\varepsilon}$ is ultimately increasing and the function $h(x)/x^{\varrho+\varepsilon}$ is ultimately decreasing (see Bingham *et al.* 1989, p. 24).

The next lemma provides an important characterisation of the class NR_ϱ (cf. Bingham *et al.* 1989, p. 15).

Lemma 5.1. *Let h be a positive (measurable) function. Then $h \in NR_\varrho$ if and only if h is absolutely continuous (and hence a.e. differentiable) and*

$$\frac{xh'(x)}{h(x)} \to \varrho \qquad (x \to \infty). \tag{5.1}$$

Integration of the relation (5.1) shows that the function $h \in NR_\varrho$ can be represented in the form

$$h(x) = h(0) + \int_0^x \frac{h(u)}{u} \big(\varrho + \varepsilon(u)\big)\, \mathrm{d}u, \tag{5.2}$$

where $\varepsilon(x) \to 0$ as $x \to \infty$.

The next lemma can be viewed as a refinement of the UCT (Lemma 3.1) for the case of normalised regular variation.

Lemma 5.2. *If $h \in NR_\varrho$ with $\varrho > 0$ then, uniformly in κ on each interval $[\kappa_0, \kappa_1] \subset (0, \infty)$,*

$$h(\kappa x) - h(x) = h(x)(\kappa^\varrho - 1)(1 + o(1)) \qquad (x \to \infty).$$

Proof. Suppose that $\kappa \ge 1$ (the case $\kappa \le 1$ is considered similarly). Using the representation (5.2), after the substitution $u = xy$ we have

$$\frac{h(\kappa x) - h(x)}{h(x)} = \int_1^\kappa \frac{h(xy)}{h(x)y} \big(\varrho + \varepsilon(xy)\big)\, \mathrm{d}y. \tag{5.3}$$

The UCT (Lemma 3.1) implies that the function under the integral sign in (5.3) tends to $\varrho y^{\varrho-1}$ uniformly on $[1, \kappa_1]$ as $x \to \infty$. Therefore, the integral (5.3) converges, uniformly in $\kappa \in [1, \kappa_1]$, to $\int_1^\kappa \varrho y^{\varrho-1}\, \mathrm{d}y = \kappa^\varrho - 1$. $\square$

For $\tau > 0$, denote

$$\eta_\tau \equiv \eta_\tau(t) := \tau^{1/t} h^{\leftarrow}(\alpha t/\varrho). \tag{5.4}$$

In particular, for $\tau = 1$ the expression (5.4) is reduced to

$$\eta_1 \equiv \eta_1(t) = h^{\leftarrow}(\alpha t/\varrho), \tag{5.5}$$

and we note

$$\eta_\tau(t) = \tau^{1/t}\eta_1(t). \tag{5.6}$$

Since $h \in NR_\varrho$, for all t large enough a usual inverse of h exists, and hence η_1 satisfies the equation (cf. (2.4))

$$\varrho\, h(\eta_1(t)) \equiv \alpha t. \tag{5.7}$$

The next lemma plays a crucial role in further calculations.

Lemma 5.3. *Uniformly in τ on each interval $[\tau_0, \tau_1] \subset (0, \infty)$,*

$$\lim_{t\to\infty} [h(\eta_\tau(t)) - h(\eta_1(t))] = \alpha \log \tau.$$

Proof. Note that

$$\kappa_\tau(t) := \frac{\eta_\tau(t)}{\eta_1(t)} = \tau^{1/t} \to 1 \qquad (t \to \infty),$$

uniformly in $\tau \in [\tau_0, \tau_1]$. Therefore, for all large enough t the function $\kappa_\tau(t)$ is uniformly bounded, $0 < \kappa_0 \le \kappa_\tau(t) \le \kappa_1 < \infty$. Applying Lemma 5.2, in the limit $t \to \infty$ we obtain, uniformly in τ,

$$\begin{aligned} h(\eta_\tau) - h(\eta_1) \sim h(\eta_1)(\kappa_\tau^\varrho - 1) &= \frac{\alpha t}{\varrho}\left(\tau^{\varrho/t} - 1\right) \\ &\sim \frac{\alpha t}{\varrho} \cdot \frac{\varrho \log \tau}{t} = \alpha \log \tau, \end{aligned}$$

where we also used the identity (5.7). □

6. Asymptotics of truncated moments

The goal of this section is to establish some estimates for truncated moments of X.

Lemma 6.1. *For any $\tau > 0$ and $p > 0$*

$$\lim_{t\to\infty} \eta_1(t)^{-\alpha t} \mathrm{e}^{\alpha t/\varrho}\, \mathsf{E}\big[X^{pt}\big(\mathbf{1}_{\{X\le\eta_\tau(t)\}} - \mathbf{1}_{\{X\le\eta_1(t)\}}\big)\big] = \begin{cases} \dfrac{\alpha}{p-\alpha}\,(\tau^{p-\alpha}-1), & p\neq\alpha, \\ \alpha\log\tau, & p=\alpha. \end{cases} \tag{6.1}$$

Proof. The case $\tau = 1$ is obvious. Suppose that $\tau > 1$. Integration by parts yields

$$\begin{aligned} \mathsf{E}\big[X^{pt}\mathbf{1}_{\{\eta_1<X\le\eta_\tau\}}\big] &= \int_{\eta_1}^{\eta_\tau} x^{pt}\,\mathrm{d}\big(1-\mathrm{e}^{-h(x)}\big) = -\int_{\eta_1}^{\eta_\tau} x^{pt}\,\mathrm{d}\mathrm{e}^{-h(x)} \\ &= \eta_1^{pt}\mathrm{e}^{-h(\eta_1)} - \eta_1^{pt}\tau^p\mathrm{e}^{-h(\eta_\tau)} + \int_{\eta_1}^{\eta_\tau} \mathrm{e}^{-h(x)}\,\mathrm{d}x^{pt}. \end{aligned} \tag{6.2}$$

Using the substitution $x = \eta_y \equiv \eta_1 y^{1/t}$ and identity (5.7) we obtain

$$\eta_1^{-pt}\mathrm{e}^{\alpha t/\varrho}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{\eta_1<X\le\eta_\tau\}}\big] = 1 - \tau^p\mathrm{e}^{h(\eta_1)-h(\eta_\tau)} + \int_1^\tau \mathrm{e}^{h(\eta_1)-h(\eta_y)}\,\mathrm{d}y^p. \tag{6.3}$$

By Lemma 5.3, $h(\eta_1) - h(\eta_y) \to -\alpha\log y$ as $t\to\infty$, uniformly in $y \in [1,\tau]$. Hence, the right-hand side of (6.3) tends to

$$1 - \tau^p\mathrm{e}^{-\alpha\log\tau} + \int_1^\tau \mathrm{e}^{-\alpha\log y}\,\mathrm{d}y^p = 1 - \tau^{p-\alpha} + p\int_1^\tau y^{p-\alpha-1}\,\mathrm{d}y,$$

and (6.1) follows. The case $0 < \tau < 1$ is considered analogously. □

Lemma 6.2. (i) *For any $\tau > 0$ and each $p > \alpha$,*

$$\lim_{t\to\infty} \mathrm{e}^{\alpha t/\varrho}\eta_1(t)^{-pt}\; \mathsf{E}\big[X^{pt}\mathbf{1}_{\{X\le\eta_\tau(t)\}}\big] = \frac{\alpha}{p-\alpha}\,\tau^{p-\alpha}. \tag{6.4}$$

(ii) *For any $\tau > 0$ and each $p < \alpha$,*

$$\lim_{t\to\infty} \mathrm{e}^{\alpha t/\varrho}\eta_1(t)^{-pt}\; \mathsf{E}\big[X^{pt}\mathbf{1}_{\{X>\eta_\tau(t)\}}\big] = \frac{\alpha}{\alpha-p}\,\tau^{p-\alpha}. \tag{6.5}$$

Proof. In view of Lemma 6.1 it suffices to consider the case $\tau = 1$.

(i) Set $\theta := \mathrm{e}^{-(\alpha+\delta)/(p\varrho)}$ with $\delta > 0$, then

$$\eta_1^{-pt}\mathrm{e}^{\alpha t/\varrho}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{X\le\theta\eta_1\}}\big] \le \mathrm{e}^{\alpha t/\varrho}\theta^{pt} = \mathrm{e}^{-\delta t/\varrho} = o(1). \tag{6.6}$$

Further, similarly as in the proof of Lemma 6.1 (see (6.3)), integrating by parts and using the substitution $x = \eta_y$ and identity (5.7) we obtain

$$\eta_1^{-pt}\mathrm{e}^{\alpha t/\varrho}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{\theta\eta_1<X\le\eta_1\}}\big] = O(1)\,\mathrm{e}^{-\delta t/\varrho} - 1 + \int_0^1 f_t(y)\,\mathrm{d}y^p, \tag{6.7}$$

where $f_t(y) := \mathrm{e}^{h(\eta_1)-h(\eta_y)}\,\mathbf{1}_{\{\theta^t<y\}}$. Noting that $\theta^t \to 0$ and using Lemma 5.3, it is easy to see that $\lim_{t\to\infty} f_t(y) = y^{-\alpha}$ for each $y > 0$. We also note that $\eta_y/\eta_1 = y^{1/t} \in [\theta, 1]$ for $y \in [\theta^t, 1]$, and so Lemma 5.2 implies that for any $\varepsilon > 0$, all t large enough and all $y \in [\theta^t, 1]$

$$\begin{aligned} h(\eta_1) - h(\eta_y) &\le h(\eta_1)(1 - y^{\varrho/t})(1+\varepsilon) \\ &\le \frac{\alpha t}{\varrho}\cdot\frac{(-\varrho)\log y}{t}\,(1+\varepsilon) = -\alpha(1+\varepsilon)\log y. \end{aligned}$$

It follows that $f_t(y)$ is bounded by the function $y^{-\alpha(1+\varepsilon)}$, which is integrable on $[0, 1]$ with respect to $\mathrm{d}y^p$ if $p > \alpha$ and ε is sufficiently small. Hence, by Lebesgue's dominated convergence theorem the limit of (6.7) equals

$$-1 + \int_0^1 y^{-\alpha}\,\mathrm{d}y^p = -1 + \frac{p}{p-\alpha} = \frac{\alpha}{p-\alpha}\,, \tag{6.8}$$

in accord with (6.4).

(ii) We start by showing that for any $\theta > 1$

$$\lim_{t\to\infty} \mathrm{e}^{\alpha t/\varrho}\eta_1^{-pt}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{X>\theta\eta_1\}}\big] = 0. \tag{6.9}$$

Indeed, using that $p - \alpha < 0$ we have

$$\mathsf{E}\big[X^p\mathbf{1}_{\{X>\theta\eta_1\}}\big] \le (\theta\eta_1)^{(p-\alpha)t}\,\mathsf{E}[X^{\alpha t}] = \theta^{(p-\alpha)t}\eta_1^{(p-\alpha)t}m(\alpha t).$$

By the Kasahara theorem (see (3.3)),

$$\frac{1}{\alpha t}\log\big(\mathrm{e}^{\alpha t/\varrho}\eta_1^{-\alpha t}m(\alpha t)\big) = \log\frac{h^{\leftarrow}(\alpha t)}{\eta_1(t)} - \frac{\log\varrho}{\varrho} + o(1). \tag{6.10}$$

Recall that the inverse function h^{-1} exists (see Section 2), so equation (5.5) reduces to $\eta_1(t) = h^{-1}(\alpha t/\varrho)$. Using that $h^{-1} \in R_{1/\varrho}$, it is easy to see that the right-hand side of (6.10) tends to 0 as $t \to \infty$. Hence,

$$\mathrm{e}^{\alpha t/\varrho}\eta_1^{-pt}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{X>\theta\eta_1\}}\big] = \mathrm{e}^{-Ct(1+o(1))} = o(1),$$

where $C := (\alpha - p)\log\theta > 0$, and (6.9) follows.

Similarly to (6.7) integration by parts gives

$$\eta_1^{-pt}\mathrm{e}^{\alpha t/\varrho}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{\eta_1<X\le\theta\eta_1\}}\big] = -\theta^{pt}\mathrm{e}^{h(\eta_1)-h(\theta\eta_1)} + 1 + \int_1^\infty \bar{f}_t(y)\,\mathrm{d}y^p, \quad (6.11)$$

where $\bar{f}_t(y) := \mathrm{e}^{h(\eta_1)-h(\eta_y)}\,\mathbf{1}_{\{1<y\le\theta^t\}}$. Using Lemma 5.2 and the identity (5.7) we note that

$$\lim_{t\to\infty}\frac{h(\eta_1)-h(\theta\eta_1)}{t} = -\frac{\alpha}{\rho}\,(\theta^\varrho - 1) \le -\alpha\log\theta,$$

so the first term in (6.11) is estimated by $\mathrm{e}^{-(\alpha-p)t\log\theta\,(1+o(1))} = o(1)$. Next, noting that $\theta^t \to \infty$ and using Lemma 5.3, we obtain $\lim_{t\to\infty}\bar{f}_t(y) = y^{-\alpha}$ for each $y>1$. Moreover, similarly to the proof of part (i) one can show that the integral in (6.11) converges to

$$\int_1^\infty y^{-\alpha}\,\mathrm{d}y^p = \frac{p}{\alpha-p}\,. \quad (6.12)$$

The limit (6.5) now follows, and the proof is complete. □

In the case $p=\alpha$ not covered by Lemma 6.2, we obtain one crude estimate that will nevertheless be sufficient for our purposes below.

Lemma 6.3. *As* $t\to\infty$,

$$b_\alpha(t) := \eta_1(t)^{-\alpha t}\mathrm{e}^{\alpha t/\varrho}\,\mathsf{E}\big[X^{\alpha t}\mathbf{1}_{\{X\le\eta_1(t)\}}\big] \to +\infty. \quad (6.13)$$

Proof. For any $\delta\in(0,1)$, using Lemma 6.1 we have

$$b_\alpha(t) \ge \eta_1(t)^{-\alpha t}\mathrm{e}^{\alpha t/\varrho}\,\mathsf{E}\big[X^{\alpha t}\mathbf{1}_{\{\eta_\delta<X\le\eta_1\}}\big] \to -\alpha\log\delta, \quad (6.14)$$

hence

$$\liminf_{t\to\infty} b_\alpha(t) \ge -\alpha\log\delta.$$

Setting $\delta\downarrow 0$, we obtain $\lim_{t\to\infty} b_\alpha(t) = +\infty$, as claimed. □

For convenience of future references, let us record a few more estimates for truncated moments. Denote

$$m_\alpha(t) := \mathsf{E}\big[X^{\alpha t}\mathbf{1}_{\{X\le\eta_1(t)\}}\big] \tag{6.15}$$

and set

$$\tilde{Y} \equiv \tilde{Y}(t) := \frac{X^t}{(Nm_\alpha(t))^{1/\alpha}}. \tag{6.16}$$

From (6.16) it is seen that the inequality $\tilde{Y}(t) > \tau$ is equivalent to $X > \eta_{\alpha,\tau}(t)$, where

$$\tilde{\eta}_{\alpha,\tau} \equiv \tilde{\eta}_{\alpha,\tau}(t) := \tau^{1/t}(Nm_\alpha(t))^{1/\alpha t}. \tag{6.17}$$

Using the scaling condition (2.3) and Lemma 6.3, one can check that

$$\log\tilde{\eta}_{\alpha,t} = \log\eta_1 + \frac{\log b_\alpha}{\alpha t}\big(1+o(1)\big).$$

Hence, for all sufficiently large t

$$\tilde{\eta}_{\alpha,\tau}(t) > \eta_1(t). \tag{6.18}$$

Lemma 6.4. *For any p such that $0\le p<\alpha$ and each $\tau>0$*

$$\lim_{t\to\infty} N\,\mathsf{E}\big[\tilde{Y}(t)^p\mathbf{1}_{\{\tilde{Y}(t)>\tau\}}\big] = 0. \tag{6.19}$$

Proof. Substituting (6.16) and using inequality (6.18) we have

$$\begin{aligned} N\,\mathsf{E}\big[\tilde{Y}^p\mathbf{1}_{\{\tilde{Y}>\tau\}}\big] &\le \frac{N}{(Nm_\alpha)^{p/\alpha}}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{X>\eta_1\}}\big]\\ &\sim \frac{\alpha}{\alpha-p}\cdot b_\alpha^{-p/\alpha}\to 0 \qquad (t\to\infty), \end{aligned} \tag{6.20}$$

where we also used (2.3), (6.20) and Lemmas 6.2(ii) and 6.3. ☐

Denote

$$y_\alpha \equiv y_\alpha(t) := \frac{\eta_1(t)^t}{(Nm_\alpha(t))^{1/\alpha}}\,, \tag{6.21}$$

so that $\tilde{Y} > y_\alpha$ if and only if $X>\eta_1$. From (2.3) and Lemma 6.3 it follows that

$$y_\alpha(t) \sim b_\alpha(t)^{-1/\alpha} \to 0 \qquad (t\to\infty).$$

Lemma 6.5. *Suppose that* $p > 0$. *Then for any* $\tau > 0$

$$N\,\mathsf{E}\big[\tilde{Y}(t)^p \mathbf{1}_{\{y_\alpha(t)<\tilde{Y}(t)\le\tau\}}\big] \to 0 \qquad (t\to\infty). \tag{6.22}$$

Proof. Picking a number q such that $0 < q < \min\{\alpha, p\}$, the left-hand side of (6.22) is estimated from above by

$$N\tau^{p-q}\,\mathsf{E}\big[\tilde{Y}^q \mathbf{1}_{\{y_\alpha<\tilde{Y}\}}\big] = \frac{N\tau^{p-q}}{(Nm_\alpha)^{q/\alpha}}\,\mathsf{E}\big[X^{qt}\mathbf{1}_{\{X>\eta_1\}}\big] \to 0,$$

as was shown above (see (6.20)). □

7. Proof of Theorem 2.3

7.1. Convergence to an infinitely divisible law

As the first step towards the proof of Theorem 2.3, we establish convergence to an infinitely divisible law. Denote

$$Y_i \equiv Y_i(t) := \frac{X_i^t}{B(t)}, \qquad i = 1, 2, \dots \tag{7.1}$$

According to classical theorems on weak convergence of sums of independent random variables (see Petrov 1975, Ch. IV, § 2, Theorem 8, p. 81–82; cf. also Theorem 7, p. 80–81), in order for the sum

$$S_N^*(t) := \sum_{i=1}^N Y_i(t) - A^*(t)$$

to converge in distribution to an infinitely divisible law with characteristic function

$$\phi(u) = \exp\left\{iau - \frac{\sigma^2u^2}{2} + \int_{|x|>0}\left(\mathrm{e}^{iux} - 1 - \frac{iux}{1+x^2}\right)\mathrm{d}L(x)\right\}, \tag{7.2}$$

it is sufficient that the following three conditions are fulfilled:

1) In all points of its continuity, the function $L(\cdot)$ satisfies

$$L(x) = \begin{cases} \displaystyle\lim_{t\to\infty} N\,\mathsf{P}\{Y\le x\} & \text{for } x<0,\\ \displaystyle -\lim_{t\to\infty} N\,\mathsf{P}\{Y> x\} & \text{for } x>0. \end{cases} \tag{7.3}$$

2) The constant σ^2 is given by

$$\sigma^2 = \lim_{\tau\to 0+} \lim_{t\to\infty} N \operatorname{Var}\bigl[Y \mathbf{1}_{\{Y\le\tau\}}\bigr] \tag{7.4}$$

3) For each $\tau > 0$, the constant a satisfies the identity

$$\lim_{t\to\infty} \bigl\{N \operatorname{E}\bigl[Y \mathbf{1}_{\{Y\le\tau\}}\bigr] - A^*(t)\bigr\} = a + \int_0^\tau \frac{x^3}{1+x^2}\,\mathrm{d}L(x) - \int_\tau^\infty \frac{x}{1+x^2}\,\mathrm{d}L(x). \tag{7.5}$$

Theorem 7.1. *Suppose that* $0 < \alpha < 2$*. Then, as* $t \to \infty$*,*

$$\frac{S_N(t) - A(t)}{B(t)} \xrightarrow{d} \mathcal{F}_\alpha,$$

where $A(t)$ *and* $B(t)$ *are defined in (2.6) and (2.5), respectively, and* $\mathcal{F}_\alpha$ *is an infinitely divisible law with characteristic function*

$$\phi_\alpha(u) = \exp\left\{iau + \alpha \int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \frac{dx}{x^{\alpha+1}}\right\}, \tag{7.6}$$

where the constant a *is given by*

$$a = \begin{cases} \dfrac{\alpha\pi}{2\cos\frac{\alpha\pi}{2}} & (\alpha \ne 1), \\ 0 & (\alpha = 1). \end{cases} \tag{7.7}$$

7.2. Proof of Theorem 7.1

The proof is broken down into several steps according to formulas (7.3), (7.4) and (7.5).

Proposition 7.2. *The function* L *defined in (7.3) is given by*

$$L(x) = \begin{cases} 0, & x < 0, \\ -x^{-\alpha}, & x > 0. \end{cases} \tag{7.8}$$

Proof. Since $Y \ge 0$, it is clear that $L(x) \equiv 0$ for $x < 0$. For $x > 0$, using (7.1), (2.3) and Lemma 5.3 we obtain

$$\begin{aligned} N \operatorname{P}\{Y > x\} &= N \operatorname{P}\bigl\{X > x^{1/t}\eta_1\bigr\} \sim \mathrm{e}^{\alpha t/\varrho} \operatorname{P}\{X > \eta_x\} \\ &= \mathrm{e}^{h(\eta_1)-h(\eta_x)} \to \mathrm{e}^{-\alpha\log x} = x^{-\alpha}, \end{aligned}$$

and (7.8) is proved. $\square$

Proposition 7.3. *For σ^2 defined in (7.4), for all $\alpha \in (0,2)$ we have $\sigma^2 \equiv 0$.*

Proof. Since $0 \le \mathsf{Var}\big[Y\,\mathbf{1}_{\{Y\le\tau\}}\big] \le \mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y\le\tau\}}\big]$, it suffices to prove that

$$\lim_{\tau\to 0+}\lim_{t\to\infty} N\,\mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y\le\tau\}}\big] = 0.$$

Let us fix $\tau > 0$. Recalling (7.1) and (2.3) and using Lemma 6.2(i) with $p = 2$, we obtain, as $t \to \infty$,

$$N\,\mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y\le\tau\}}\big] \sim \mathrm{e}^{\alpha t/\varrho}\eta_1^{-2t}\,\mathsf{E}\big[X^{2t}\mathbf{1}_{\{X\le\eta_\tau\}}\big] \sim \frac{\alpha}{2-\alpha}\,\tau^{2-\alpha}. \tag{7.9}$$

As $\tau \to 0+$, the right-hand side of (7.9) tends to zero, since $2-\alpha > 0$. □

Proposition 7.4. *Let $B(t)$ and $A(t)$ be specified by (2.5) and (2.6), respectively, and set $A^*(t) := A(t)/B(t)$. Then the limit*

$$D_\alpha(\tau) := \lim_{t\to\infty}\big\{N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y\le\tau\}}\big] - A^*(t)\big\} \tag{7.10}$$

exists for all $\alpha \in (0,2)$ and is given by

$$D_\alpha(\tau) = \begin{cases} \dfrac{\alpha}{1-\alpha}\,\tau^{1-\alpha} & (\alpha \ne 1),\\ \log\tau & (\alpha = 1). \end{cases} \tag{7.11}$$

Proof. 1) Let $0 < \alpha < 1$. Using the scaling condition (2.3) and applying Lemma 6.2(i) with $p = 1$, we obtain

$$N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y\le\tau\}}\big] = \frac{N}{\eta_1^t}\,\mathsf{E}\big[X^t\,\mathbf{1}_{\{X\le\eta_\tau\}}\big] \sim \frac{\alpha}{1-\alpha}\,\tau^{1-\alpha} \qquad (t\to\infty), \tag{7.12}$$

in accord with (7.11).

2) Let $1 < \alpha < 2$. Similarly to (7.12), application of Lemma 6.2(ii) with $p = 1$ yields

$$\begin{aligned} N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y\le\tau\}}\big] - A^*(t) &= \frac{N}{\eta_1^t}\Big(\mathsf{E}\big[X^t\,\mathbf{1}_{\{X\le\eta_\tau\}}\big] - \mathsf{E}\big[X^t\big]\Big)\\ &= -\frac{N}{\eta_1^t}\,\mathsf{E}\big[X^t\,\mathbf{1}_{\{X>\eta_\tau\}}\big]\\ &\sim \frac{\alpha}{1-\alpha}\,\tau^{1-\alpha} \qquad (t\to\infty). \end{aligned}$$

3) Let $\alpha = 1$. Similarly as above, we obtain using condition (2.3) and Lemma 6.1 (with $\alpha = 1$):

$$\begin{aligned} N\,\mathsf{E}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big] - A^*(t) &= N\eta_1^{-t}\Big(\mathsf{E}\big[X^t\mathbf{1}_{\{X\le\eta_\tau\}}\big] - \mathsf{E}\big[X^t\mathbf{1}_{\{X\le\eta_1\}}\big]\Big) \\ &\to \log\tau \qquad (t\to\infty), \end{aligned}$$

and the proof of (7.11) is complete. □

Proposition 7.5. *The parameter a defined in (7.7) satisfies the identity* (7.5) *with $L(\cdot)$ specified by (7.8), that is,*

$$D_\alpha(\tau) = a + \alpha\int_0^\tau \frac{x^{2-\alpha}}{1+x^2}\,\mathrm{d}x - \alpha\int_\tau^\infty \frac{x^{-\alpha}}{1+x^2}\,\mathrm{d}x \qquad (\tau>0), \tag{7.13}$$

where $D_\alpha(\tau)$ is given by (7.11).

Proof. 1) Let $0<\alpha<1$. Observe that

$$\int_0^\tau \frac{x^{2-\alpha}}{1+x^2}\,\mathrm{d}x = \frac{1}{1-\alpha}\,\tau^{1-\alpha} - \int_0^\tau \frac{x^{-\alpha}}{1+x^2}\,\mathrm{d}x.$$

Taking into account (7.11) and (7.7), we see that equation (7.13) amounts to

$$\int_0^\infty \frac{x^{-\alpha}}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2\cos\frac{\alpha\pi}{2}}\,, \tag{7.14}$$

which is true by a formula in Gradshteyn and Ryzhik (1994, #3.241(2)).

2) For $1<\alpha<2$, we note that

$$\int_\tau^\infty \frac{x^{-\alpha}}{1+x^2}\,\mathrm{d}x = \frac{\tau^{1-\alpha}}{\alpha-1} - \int_\tau^\infty \frac{x^{2-\alpha}}{1+x^2}\,\mathrm{d}x,$$

and hence, in view of (7.11) and (7.7), equation (7.13) is reduced to

$$\frac{\pi}{2\cos\frac{\alpha\pi}{2}} + \int_0^\infty \frac{x^{2-\alpha}}{1+x^2}\,\mathrm{d}x = 0, \tag{7.15}$$

which again follows from Gradshteyn and Ryzhik (1994, #3.241(2)).

3) Finally, for $\alpha=1$ equation (7.13) takes the form

$$\log\tau = \int_0^\tau \frac{x}{1+x^2}\,\mathrm{d}x - \int_\tau^\infty \frac{1}{(1+x^2)x}\,\mathrm{d}x. \tag{7.16}$$

The integrals on the right of (7.16) are easily evaluated to yield

$$\frac{1}{2}\log(1+x^2)\Big|_0^{\tau} - \frac{1}{2}\log\frac{x^2}{1+x^2}\Big|_{\tau}^{\infty} = \log\tau,$$

and this completes the proof of Proposition 7.5. □

Proof of Theorem 7.1. Gathering the results of Propositions 7.2, 7.3, 7.4 and 7.5, which identify the ingredients of the limit characteristic function ϕ_α, we conclude that Theorem 7.1 is true. □

7.3. Stability of the limit law

In this section, we show that the infinitely divisible law $\mathcal{F}_\alpha$ with the characteristic function (7.6) is in fact stable.

Theorem 7.6. *The characteristic function ϕ_α determined by Theorem 7.1 corresponds to a stable probability law with exponent $\alpha \in (0, 2)$ and skewness parameter $\beta = 1$, and can be represented in a canonical form (2.9).*

Remark. Formula (7.8) and Proposition 7.3 imply that ϕ_α corresponds to a stable law (see Ibragimov and Linnik 1971, Theorem 2.2.1, p. 39–40). We give a direct proof of this fact by reducing ϕ_α to the canonical form (2.9), which allows us to identify all the parameters explicitly.

Proof of Theorem 7.6. According to general theory (see, e.g., Zolotarev 1957, p. 441), the characteristic function of a stable law with characteristic exponent $\alpha \in (0, 2)$ admits a canonical representation

$$\phi_\alpha(u) = \begin{cases} \exp\left\{i\mu u - b|u|^\alpha\Big(1 - i\beta\,\mathrm{sgn}\,u\cdot\tan\dfrac{\pi\alpha}{2}\Big)\right\} & (\alpha \neq 1), \\ \exp\left\{i\mu u - b|u|\Big(1 + i\beta\,\mathrm{sgn}\,u\cdot\dfrac{2}{\pi}\,\log|u|\Big)\right\} & (\alpha = 1), \end{cases} \tag{7.17}$$

where μ is a real constant, $b > 0$ and $-1 \le \beta \le 1$.

1) Suppose that $0 < \alpha < 1$. It is easy to verify that, due to (7.7) and (7.14), the characteristic function (7.6) can be rewritten in the form

$$\phi_\alpha(u) = \exp\left\{\alpha\int_0^\infty \frac{\mathrm{e}^{iux} - 1}{x^{\alpha+1}}\,dx\right\}. \tag{7.18}$$

The integral in (7.18) can be evaluated (see Ibragimov and Linnik 1971, p. 43–44):

$$\int_0^\infty \frac{\mathrm{e}^{iux}-1}{x^{\alpha+1}}\,dx = -\frac{\Gamma(1-\alpha)}{\alpha}\,|u|^\alpha\,\mathrm{e}^{-(i\pi\alpha/2)\,\mathrm{sgn}\,u},$$

and (7.17) follows with $\mu = 0,\ b = \Gamma(1-\alpha)\,\cos(\pi\alpha/2) > 0,\ \beta = 1$.

2) Let now $1 < \alpha < 2$. Using (7.15), we can rewrite (7.6) in the form

$$\phi_\alpha(u) = \exp\left\{\alpha\int_0^\infty (\mathrm{e}^{iux}-1-iux)\,\frac{dx}{x^{\alpha+1}}\right\}. \tag{7.19}$$

The integral in (7.19) is given by (see Ibragimov and Linnik 1971, p. 44–45)

$$\int_0^\infty (\mathrm{e}^{iux}-1-iux)\,\frac{dx}{x^{\alpha+1}} = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}\,|u|^\alpha\,\mathrm{e}^{(i\pi\alpha/2)\,\mathrm{sgn}\,u},$$

which yields $\mu = 0,\ b = -\Gamma(2-\alpha)/(\alpha-1)\cdot\cos(\pi\alpha/2) > 0,\ \beta = 1$.

3) If $\alpha = 1$, by the substitution $y = |u|x$ in (7.7) we get

$$\phi_1(u) = \exp\left\{-|u|\int_0^\infty \frac{1-\cos y}{y^2}\,dy - iu\int_0^\infty \left(\sin y - \frac{u^2 y}{u^2+y^2}\right)\frac{dy}{y^2}\right\}. \tag{7.20}$$

It is well known (see Gradshteyn and Ryzhik 1994, #3.782(2)) that

$$\int_0^\infty \frac{1-\cos y}{y^2}\,dy = \frac{\pi}{2}. \tag{7.21}$$

To evaluate the second integral in (7.20), let us represent it in the form

$$\int_0^\infty \left(\frac{\sin y}{y} - \frac{1}{1+y}\right)\frac{dy}{y} + \int_0^\infty \left(\frac{1}{1+y} - \frac{u^2}{u^2+y^2}\right)\frac{dy}{y}. \tag{7.22}$$

It is known that (see Gradshteyn and Ryzhik 1994, #3.781(1))

$$\int_0^\infty \left(\frac{\sin y}{y} - \frac{1}{1+y}\right)\frac{dy}{y} = 1-\gamma, \tag{7.23}$$

where γ is the Euler constant. Furthermore, note that

$$\int_0^\infty \left(\frac{1}{1+y} - \frac{u^2}{u^2+y^2}\right)\frac{dy}{y} = \frac{1}{2}\log\frac{u^2+y^2}{(1+y)^2}\bigg|_0^\infty = -\log|u|. \tag{7.24}$$

Returning to (7.22), from (7.23) and (7.24) we get

$$\int_0^\infty \left(\sin y - \frac{u^2 y}{u^2+y^2}\right)\frac{dy}{y^2} = 1-\gamma-\log|u|. \tag{7.25}$$

Therefore, substituting expressions (7.21) and (7.25) into (7.20), we obtain a required canonical form (7.17) with $\mu = 1-\gamma,\ b = \pi/2,\ \beta = 1$. $\square$

8. Proof of Theorems 2.4 and 2.5

Proof of Theorem 2.4. The result follows from Theorem 2.3 (for $\alpha = 1$). Indeed, according to (2.3), (2.5), (2.6) and Lemma 6.3 we have

$$A^*(t) := \frac{A(t)}{B(t)} = N\eta_1(t)^{-t} m_1(t) \sim b_1(t) \to \infty \qquad (t \to \infty).$$

Therefore, dividing (2.8) by $A^*(t) \to \infty$ we obtain $S_N(t)/A(t) = 1 + o_p(1)$ as $t \to \infty$, which is in agreement with (2.10). □

Proof of Theorem 2.5. Denote

$$Y_i \equiv Y_i(t) := \frac{X_i^t}{\sqrt{Nm_2(t)}} \tag{8.1}$$

(cf. (6.16)). According to a classical CLT for independent summands (see Petrov 1975, Ch. IV, § 4, Theorem 18, p. 95), it suffices to check that for any $\tau > 0$ the following three conditions are satisfied as $t \to \infty$:

$$N\,\mathsf{P}\{Y(t) > \tau\} \to 0, \tag{8.2}$$

$$N\Big(\mathsf{E}\big[Y(t)^2 \mathbf{1}_{\{Y(t)\le\tau\}}\big] - \big(\mathsf{E}\big[Y(t)\mathbf{1}_{\{Y(t)\le\tau\}}\big]\big)^2\Big) \to 1, \tag{8.3}$$

$$N\,\mathsf{E}\big[Y(t)\mathbf{1}_{\{Y(t)>\tau\}}\big] \to 0. \tag{8.4}$$

Firstly, note that conditions (8.2) and (8.4) are guaranteed by Lemma 6.4 (with $p = 0$ and $p = 1$, respectively). To check (8.3), let us first show that

$$\frac{\big(\mathsf{E}\big[Y\mathbf{1}_{\{Y\le\tau\}}\big]\big)^2}{\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\le\tau\}}\big]} = \frac{\big(\mathsf{E}\big[X^t\mathbf{1}_{\{X\le\eta_{2,\tau}\}}\big]\big)^2}{\mathsf{E}\big[X^{2t}\mathbf{1}_{\{X\le\eta_{2,\tau}\}}\big]} \to 0 \qquad (t\to\infty), \tag{8.5}$$

where $\eta_{2,\tau}$ is defined in (6.17). Indeed, taking into account inequality (6.18) and Lemma 3.3 (with $r = 2$), the ratio in (8.5) is estimated from above by

$$\frac{(\mathsf{E}[X^t])^2}{\mathsf{E}\big[X^{2t}\mathbf{1}_{\{X\le 2t\}}\big]} = \frac{m(t)^2}{m_2(t)} = \mathrm{e}^{-t\,(2/\varrho)\log\varrho\,(1+o(1))} = o(1). \tag{8.6}$$

Hence, condition (8.3) amounts to

$$N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\le\tau\}}\big] \to 1 \qquad (t\to\infty). \tag{8.7}$$

Noting that, according to (8.1), (6.21) and (6.15),

$$N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\le y_2\}}\big]=\frac{1}{m_2}\,\mathsf{E}\big[X^{2t}\mathbf{1}_{\{X\le \eta_1\}}\big]\equiv 1,$$

we can rewrite (8.7) in the form $N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{y_2<Y\le\tau\}}\big]\to 0$. But the latter is true by Lemma 6.5, and (8.3) follows. □

9. Limit theorems for l_t-norms

In this section, we derive the limit distribution of the random variables

$$R_N(t):=S_N(t)^{1/t}=\left(\sum_{i=1}^{N}X_i^t\right)^{1/t}.$$

First, let us prove a general 'transfer' lemma.

Lemma 9.1. *Let* $\{S(t),\ t\ge 0\}$ *be a family of positive random variables, such that for some (non-negative) functions* $A(t)$ *and* $B(t)$*,*

$$S^*(t):=\frac{S(t)-A(t)}{B(t)}\xrightarrow{d}\mathcal{F}\qquad(t\to\infty). \tag{9.1}$$

Set $R(t):=S(t)^{1/t}$ *and* $A^*(t):=A(t)/B(t)$*.*

(a) *If* $A^*(t)\to\infty$ *as* $t\to\infty$*, then*

$$tA^*(t)\left(\frac{R(t)}{A(t)^{1/t}}-1\right)\xrightarrow{d}\mathcal{F}\qquad(t\to\infty).$$

(b) *If* $A(t)\equiv 0$ *then*

$$t\left(\frac{R(t)}{B(t)^{1/t}}-1\right)\xrightarrow{d}\log\mathcal{F}\qquad(t\to\infty).$$

Proof. (a) Note that $S(t)$ can be represented as

$$S(t)=A(t)\left(1+\frac{S^*(t)}{A^*(t)}\right), \tag{9.2}$$

whence

$$R(t) = S(t)^{1/t} = A(t)^{1/t} \exp\left(\frac{1}{t}\log\left(1 + \frac{S^*(t)}{A^*(t)}\right)\right). \tag{9.3}$$

The condition $A^*(t) \to \infty$ implies that $S^*(t)/A^*(t) = o_p(1)$, hence

$$\begin{aligned}\exp\left(\frac{1}{t}\log\left(1 + \frac{S^*(t)}{A(t)}\right)\right) &= \exp\left(\frac{S^*(t)}{tA^*(t)}\,(1 + o_p(1))\right)\\ &= 1 + \frac{S^*(t)}{tA^*(t)}\,(1 + o_p(1)).\end{aligned}$$

Substituting this into (9.3) yields

$$tA^*(t)\left(\frac{R(t)}{A(t)^{1/t}} - 1\right) = S^*(t)(1 + o_p(1)) \xrightarrow{d} \mathcal{F} \qquad (t \to \infty).$$

(b) We have $S(t) = S^*(t)\,B(t)$, whence

$$\frac{R(t)}{B(t)^{1/t}} = \exp\left(\frac{\log S^*(t)}{t}\right) = 1 + \frac{\log S^*(t)}{t}\,(1 + o_p(1)).$$

Therefore,

$$t\left(\frac{R(t)}{B(t)^{1/t}} - 1\right) = \log S^*(t)\,(1 + o_p(1)),$$

which converges weakly to $\log \mathcal{F}$ as $t \to \infty$. □

Applying this lemma to the sums $S_N(t)$, we obtain

Proof of Theorem 2.6. In view of Lemma 9.1, the assertions of the theorem will follow from the limit theorems for the sum $S_N(t)$ obtained in Sections 3, 7 and 8, according as the function $A^*(t) = A(t)/B(t)$ tends to infinity or vanishes as $t \to \infty$.

(a) For $\alpha \geq 2$, the CLT is valid (see Theorems 2.2 and 2.5), so we have weak convergence of the form (9.1) with $A(t) = Nm(t)$ and $B(t)$ given by (2.5). Clearly, $m_2(t) \leq \mathsf{E}[X^{2t}] = m(2t)$, and hence for all $\alpha \geq 2$

$$\begin{aligned}\liminf_{t\to\infty} \frac{\log A^*(t)}{t} &\geq \liminf_{t\to\infty} \left(\frac{\log N}{2t} + \frac{1}{2t}\log\frac{m(t)^2}{m(2t)}\right)\\ &= \frac{\alpha}{2\varrho} - \frac{\log 2}{\varrho} \geq \frac{1 - \log 2}{\varrho} > 0,\end{aligned} \tag{9.4}$$

where we used (2.2) and Lemma 3.3 with $r = 2$. Therefore, $A^*(t) \to \infty$, and application of Lemma 9.1(a) proves part (a).

(b) If $1 \le \alpha < 2$ then, according to Theorem 2.3, $B(t) = \eta_1(t)^t$ and $A(t)$ is defined in (2.6). Noting that $m(t) \ge m_1(t)$ and using (2.3) and Lemma 6.5, we obtain

$$A^*(t) \ge \frac{N m_1(t)}{\eta_1(t)^t} \sim b_1(t) \to \infty.$$

Hence, Lemma 9.1(a) applies and part (b) is proved.

(c) In the case $0 < \alpha < 1$, the assertion of the theorem readily follows from Lemma 9.1(b), since by Theorem 2.3 we have $A(t) \equiv 0$, so that $A^*(t) \equiv 0$. $\square$

10. Discussion and an example

In order to clarify the link with the setting in Schlather (2001), let us show that under our conditions, the random variables X_i belong to the domain of attraction of the Gumbel (double exponential) distribution Λ.

Proposition 10.1. *Assume that* $h \in NR_\varrho$. *Denote* $X_{1,n} := \max\{X_1, \dots, X_n\}$, *and set*

$$a_n := h^{\leftarrow}(\log n), \qquad b_n := \frac{a_n}{\varrho \log n}. \tag{10.1}$$

Then

$$\lim_{n\to\infty} \mathsf{P}\left\{\frac{X_{1,n} - a_n}{b_n} \le x\right\} = \exp(-\mathrm{e}^{-x}), \qquad x \in \mathbb{R}. \tag{10.2}$$

Proof. It is not difficult to verify available sufficient conditions for convergence of the maximum's distribution to Λ (see, e.g., Galambos 1978, Theorem 2.1.3, p. 52). However, it is even simpler to prove (10.2) directly. Indeed, setting $L_n(x) := a_n + x b_n$ we have

$$\mathsf{P}\left\{\frac{X_{1,n} - a_n}{b_n} \le x\right\} = \Big(\mathsf{P}\{X \le a_n + x b_n\}\Big)^n = \Big(1 - \mathrm{e}^{-h(L_n(x))}\Big)^n. \tag{10.3}$$

Note that, according to (10.1),

$$L_n(0) = a_n = h^{\leftarrow}(\log n) \to +\infty \qquad (n \to \infty), \tag{10.4}$$

since $h^{\leftarrow}(x) \in R_{1/\varrho}$, and

$$\frac{b_n}{a_n} = \frac{1}{\varrho \log n} \to 0 \qquad (n \to \infty).$$

Therefore,

$$\kappa_n(x) := \frac{L_n(x)}{L_n(0)} = \frac{a_n + xb_n}{a_n} = 1 + \frac{xb_n}{a_n} \to 1 \qquad (n \to \infty). \tag{10.5}$$

Recalling (10.3), it is then easy to see that (10.2) is reduced to

$$h(L_n(x)) - \log n \to x \qquad (n \to \infty). \tag{10.6}$$

Furthermore, since $h \in NR_\varrho$, a usual inverse h^{-1} exists and so (10.4) implies that $h(L_n(0)) = \log n$. Hence, (10.6) takes the form

$$h(L_n(x)) - h(L_n(0)) \to x \qquad (n \to \infty). \tag{10.7}$$

To show (10.7), we use (10.5) and apply Lemma 5.2 to obtain

$$\begin{aligned} h(L_n(x)) - h(L_n(0)) &\sim h(L_n(0))\big(\kappa_n(x)^\varrho - 1\big) \\ &= \log n \left(\left(1 + \frac{xb_n}{a_n}\right)^\varrho - 1\right) \\ &\sim \log n \cdot \frac{\varrho x b_n}{a_n} = x, \end{aligned}$$

according to the choice of b_n (see (10.1)). Thus, (10.7) is proved. □

As mentioned in the Introduction, in the case of attraction to the double exponential distribution Schlather (2001) considered a concrete example of the random variables X_i with the unit exponential distribution (therefore, fitting in the class of distributions (1.2) with $\varrho = 1$). Namely, in our notation he has shown (Theorem 2.4, p. 867) that under the scaling $N = \mathrm{e}^{\alpha t}$ the limit distribution of $R_N(t)$ is Gaussian if $\alpha > 2$ and non-Gaussian if $2\log 2 < \alpha < 2$.

Note that our results (see Theorem 2.6) show that $\alpha = 2$ is indeed a critical point, in that a Gaussian law breaks down for $\alpha < 2$. (However, the value $\alpha = 2\log 2$ does not seem to play any special role.) Furthermore, it is not difficult to check that our results corroborate a general conjecture by Schlather (2001, p. 867) asserting (in our terms) that in the case of attraction to Λ there exist functions $a(t)$, $b(t)$ such that, under an appropriate scaling $t = cp(N)$, $a(t)/b(t) = p(N)$, the distribution of $\big(R_N(t) - a(t)\big)/b(t)$ weakly converges to a distribution which, in turn, tends to Λ as $c \to +\infty$ and, properly re-centered and re-normalised, to $\mathcal{N}(0,1)$ as $c \to 0+$. Comparing this conjecture with our Theorem 2.6, one can see that the role of c is played by $1/\alpha$, so that $c \to +\infty$ is equivalent to $\alpha \to 0+$.

As a result, normality in the limit $c \to 0+$ (that is, $\alpha \to +\infty$) is obvious from Theorem 2.6(a). To obtain the limit as $c \to +\infty$ (that is, $\alpha \to 0+$), note that in Schlather's terms Theorem 2.6(c) takes the form

$$\frac{R_N(t) - B(t)^{1/t}}{B(t)^{1/t}/(\alpha t)} \xrightarrow{d} \alpha \log \zeta_\alpha \qquad (t \to \infty),$$

where ζ_α has the distribution $\mathcal{F}_\alpha$. An application of a general result by Zolotarev (1957, Theorem 5, p. 447–448; see also Ben Arous, Bogachev and Molchanov 2003, Proposition 8.29, p. 47) provides a required limit theorem for the stable distribution $\mathcal{F}_\alpha$ as its parameter α tends to zero.

Lemma 10.2. *Let a random variable ζ_α have the stable distribution $\mathcal{F}_\alpha$ determined by (2.9). Then, as $\alpha \to 0+$, the distribution of $\alpha \log \zeta_\alpha$ weakly converges to the double exponential distribution,*

$$\lim_{\alpha \to 0+} \mathsf{P}\{\alpha \log \zeta_\alpha \le x\} = \exp(-\mathrm{e}^{-x}), \qquad x \in \mathbb{R}. \tag{10.8}$$

Proof. By Theorem 5 in Zolotarev (1957) we have, as $\alpha \to 0+$,

$$\mathsf{P}\{\alpha \log \zeta_\alpha \le x\} = \mathsf{P}\{\zeta_\alpha \le \mathrm{e}^{1/\alpha}\} \sim \frac{1-\beta}{2} + \frac{1+\beta}{2} \exp\bigl(-b\,\mathrm{e}^{-x}\bigr), \tag{10.9}$$

where, according to (2.9), $\beta \equiv 1$ and $b = \Gamma(1-\alpha)\cos(\pi\alpha/2) \to 1$. Hence, the right-hand side of (10.9) tends to $\exp(\mathrm{e}^{-x})$ as $\alpha \to 0$. $\square$

Example 10.3. Let us specify Theorem 2.6 in the case where X has a unit exponential distribution, that is,

$$\mathsf{P}\{X > x\} = \mathrm{e}^{-x}, \qquad x \ge 0.$$

Therefore, $\varrho = 1$ and $h(x) = x$. The moment function $m(t)$ defined in (2.1) is given by

$$m(t) = \int_0^\infty x^t\,\mathrm{e}^{-x}\,\mathrm{d}x = \Gamma(t+1),$$

and the known Stirling asymptotic formula yields

$$m(t) \sim \sqrt{2\pi}\,t^{t+1/2}\,\mathrm{e}^{-t} \qquad (t \to \infty). \tag{10.10}$$

Furthermore, from (5.5) it is seen that $\eta_1(t) = \alpha t$ and

$$m_\alpha(t) = \int_0^{\alpha t} x^{\alpha t}\,\mathrm{e}^{-x}\,\mathrm{d}x \sim \frac{1}{2}\,\Gamma(\alpha t + 1) \sim \sqrt{\frac{\pi}{2}}\,(\alpha t)^{\alpha t + 1/2}\,\mathrm{e}^{-\alpha t}. \tag{10.11}$$

Hence, Theorem 2.6 implies the following:

(a) *For* $\alpha \geq 2$,

$$\frac{\pi^{1/4}\, \mathrm{e}^{\alpha t/2}\, t^{5/4}}{2^t} \left(\frac{R_N(t)}{(N\Gamma(t+1))^{1/t}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \sigma_\alpha^2),$$

where $\sigma_\alpha^2 = 1$ *for* $\alpha > 2$ *and* $\sigma_2^2 = 1/2$.

(b) *For* $1 < \alpha < 2$,

$$\frac{\sqrt{2\pi}\, \mathrm{e}^{(\alpha-1)t}\, t^{3/2}}{C_\alpha\, \alpha^t} \left(\frac{R_N(t)}{(N\tilde{m}_\alpha(t))^{1/t}} - 1 \right) \xrightarrow{d} \mathcal{F}_\alpha,$$

where

$$\tilde{m}_\alpha(t) = \begin{cases} m(t), & 1 < \alpha < 2, \\ m_1(t), & \alpha = 1, \end{cases} \qquad C_\alpha = \begin{cases} 1, & 1 < \alpha < 2, \\ 2, & \alpha = 1, \end{cases}$$

with $m_1(\cdot)$ *given by (10.11).*

(c) *For* $0 < \alpha < 1$,

$$R_N(t) - \alpha t \xrightarrow{d} \alpha \log \mathcal{F}_\alpha.$$

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