CONSERVATIVE HYPERBOLIC MODEL FOR COMPRESSIBLE TWO-PHASE FLOW WITH DIFFERENT PHASE PRESSURES AND TEMPERATURES

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Abstract

A model for two-phase compressible flow is presented, the derivation of which is based on extended irreversible thermodynamics principles. An isentropic two-phase flow model proposed earlier and the hyperbolic model for heat transfer underlie the developed theory of this paper. The governing equations of the model form a hyperbolic system of differential equations in conservation-law form. A set of interfacial exchange processes such as pressure relaxation, interfacial friction, temperature relaxation and phase transition are taken into account by source terms in the balance equations. It is shown that the heat flux relaxation limit of governing equations can be written in the Baer-Nunziato form, in which the Furier thermal conductivity diffusion terms for each phase are included.

1 Introduction

Multi-phase flow modelling has undergone intensive developments in recent years because of its great importance for practical applications. In resent decades several theories of two-phase flow have been developed and exploited successfully for the needs of industry and environmental sciences.

In this paper we study a model of two-phase two-fluid flow in which the constituents of the mixture have different velocities, pressures and temperatures. Mathematical models for such a flow are still far from a final satisfactory state. The pioneering work [1] in this field has formulated two-phase flow governing equations as a set of two separate system of conservation laws for each phase coupled by interface exchange terms. This model has been applied to the study of many concrete problems, and some modifications and generalizations of this model has been made, in for example [2, 25, 8, 18] and references therein. Note that the governing equations of the Baer-Nunziato model are hyperbolic. This is a very important attribute of the model, which gives a theoretical basis for its mathematical study and for confidence in numerical solutions of the equations. But not all equations of Baer-Nunziato type model are in conservative form, and this results in difficulties in definition and studying discontinuous solutions [26, 13].

We propose a different approach to model two-phase flows, which is based on extended irreversible thermodynamics principles. This theory allows us to formulate classes of hyperbolic conservation equations using generating thermodynamic potentials and variables. Its core is a phenomenological approach in modelling of continuous media and using thermodynamic laws, which determine the structure of the governing equations. Extended thermodynamics has been developed in the last decades and has been successfully applied in

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different areas of continuum mechanics and mathematical physics [9, 6, 7, 11, 20, 12, 17, 21]. With respect to multi-phase compressible flow a general principle has been constructed, and the models for quasi-isentropic flows has been developed and applied to studying some specific problems [22, 23, 19, 24]. In this approach the mixture is supposed to be a continuum in which the two-phase character of the flow is taken into account. The complete set of governing equations is a hyperbolic system and consists of the mass, momentum and energy conservation laws for the mixture, which are completed by balance laws for additional mixture parameters, namely, for the volume fraction and relative velocity. Moreover all equations of the system have a conservative form. This fact is very attractive from the mathematical and numerical viewpoints, because it gives a straightforward way to develop the theory of discontinuous solutions and modern numerical methods. Note that the equations of the conservative model can be transformed to a system which is similar to the Baer-Nunziato form. The difference consists of the distinct definition of interfacial pressure and appearing lift forces in the phase momentum equations. These forces are not included into the traditional Baer-Nunziato model.

In this paper we present a generalization of the model [24] for the case of flow with two different phase temperatures. The design methodology of the model is based on the synthesis of the conservative isentropic model from [24] and hyperbolic heat transfer equations which have been formulated in [16]. The hyperbolic equations for heat conductivity are well-known and widely discussed in the literature, see for example [17]. This model is based on the Cattaneo equation for heat transfer, and the temperature and heat flux are used as variables. In the present paper we use phase entropies and entropy fluxes as physical variables which seems to be more convenient for our study and are typical variables in irreversible thermodynamics [4]. The resulting system of governing equations is also hyperbolic. Moreover, all equations are written in a conservative form. The use of a hyperbolic model for heat transfer in multi-phase flows seems to be reasonable because of the small scales and characteristic times of the processes.

The paper is organized as follows. In Section 2 we give a brief formulation of the conservative isentropic compressible two-phase flow model and its comparison with the Baer-Nunziato formulation. Section 3 describes the hyperbolic heat transfer equations, on the basis of which the sophisticated conservative model for two-phase flow with two temperatures is presented in Section 4. Also in Section 4 the diffusive limit of the hyperbolic heat flux relaxation is discussed. It is shown that such an approximation leads to equations, which are similar to the Baer-Nunziato equations with parabolic phase thermal conductivity terms. Conclusions are drawn in Section 5. Finally, in the Appendix a description of the generative system is given, which is formulated by the extended thermodynamics laws and generates the presented two-phase model.

2 Isentropic conservative model for two-phase flow

In this section we briefly describe the model for isentropic two-phase flow which has been proposed in [24]. Its derivation is based on the principles of extended thermodynamics and the governing equations are a hyperbolic system of differential equations in conservative form. We ignore here many possible dissipative processes and phase transition and take into account only interfacial friction and pressures relaxation of phases to the common pressure value.

2.1 The system of balance laws for two-phase isentropic flow

We consider processes in the Cartesian coordinate system, the tensor notation such as upper and lower indexes, and summation with respect to common index are used. Assume that the mixture is a continuum in which the two-phase character of flow is taken into account. Suppose that the state of each phase with a number $i = 1, 2$ is characterized by its volume fraction α_i, mass density ρ_i and velocity vector $u_i^k, k = 1, 2, 3$. For the volume fractions the saturation constraint $\alpha_1 + \alpha_2 = 1$ is assumed.

The set of physical parameters of state which is convenient in applying the extended thermodynamics for the derivation of the governing equations is as follows:

$$\alpha, \quad \rho, \quad c, \quad u^k, \quad w^k, \tag{1}$$

where $\alpha = \alpha_1$ is the volume fraction of the first phase, ρ is the mixture mass density, $c = c_1$ is the mass fraction of the first phase, u^k is the average velocity of the mixture, and w^k is the relative velocity of phases.

These parameters of state for the mixture are connected with the parameters of state for each phase by the relations

$$\alpha = \alpha_1, \quad \rho = \alpha_1\rho_1 + \alpha_2\rho_2, \quad c = c_1 = \frac{\alpha_1\rho_1}{\rho}, \quad u^k = c_1u_1^k + c_2u_2^k, \quad w^k = u_1^k - u_2^k. \tag{2}$$

Using (2) we can derive the relations expressing the individual parameters, such as mass densities and velocities, by the mixture parameters:

$$\rho_1 = \frac{c_1\rho}{\alpha_1} = \frac{c\rho}{\alpha}, \quad \rho_2 = \frac{c_2\rho}{\alpha_2} = \frac{(1-c)\rho}{(1-\alpha)}, \tag{3}$$

$$u_1^k = u^k + c_2w^k = u^k + (1-c)w^k, \quad u_2^k = u^k - c_1w^k = u^k - cw^k. \tag{4}$$

The system of governing equations for isentropic two-phase flow written in terms of the mixture parameters (2) can be derived with the help of extended thermodynamics principles (see Appendix) and looks as follows:

$$\begin{aligned}
&\frac{\partial\rho\alpha}{\partial t} + \frac{\partial\rho u^k\alpha}{\partial x^k} = -\phi,\\
&\frac{\partial\rho}{\partial t} + \frac{\partial\rho u^k}{\partial x^k} = 0,\\
&\frac{\partial\rho u^l}{\partial t} + \frac{\partial(\rho u^l u^k + p\delta^{kl} + \rho w^l E_{w^k})}{\partial x^k} = 0,\\
&\frac{\partial\rho c}{\partial t} + \frac{\partial(\rho u^k c + \rho E_{w^k})}{\partial x^k} = 0,\\
&\frac{\partial w_k}{\partial t} + \frac{\partial(u^l w_l + E_c)}{\partial x^k} = -(e_{klj}u^l\omega^j + \lambda_k).
\end{aligned} \tag{5}$$

The equations of the above system are the balance law for volume fraction, total mass conservation law, total momentum conservation law, mass fraction balance law and the balance law for the relative velocity respectively. Here e_{klj} is the unit pseudoscalar. Fluxes in the three last equations contain the derivatives of the equation of state (specific internal energy) E, which is supposed to be a known function of the parameters of state. Its derivation with the use of the equations of state for individual phase will be described below. The momentum flux contains the mixture pressure p, which is also defined by the derivative of the equation of state with respect to the mixture density:

$$p = \rho^2 E_\rho = \rho^2\frac{\partial E}{\partial\rho}. \tag{6}$$

The two source terms in the first and in the last equation of system (5) are the pressure relaxation term and interfacial friction term respectively. The variable ω_j is an auxiliary variable and its introduction is necessary to write the equation for the relative velocity in conservative form. Thereby two additional equations should be added to the system (5) in order to provide its compatibility [21]. The first one is a steady conservative equation

$$e^{jkl}\frac{\partial w_k}{\partial x^l} = \omega^j, \tag{7}$$

which defines ω_i as the vorticity of the relative velocity vector, and the second one is as follows

$$\frac{\partial \omega^k}{\partial t} + \frac{\partial(u^l\omega_k - u^k\omega^l + e^{klj}\lambda_j)}{\partial x^l} = 0. \tag{8}$$

We emphasize again that the vector variable ω^i is introduced only to write the equation for the relative velocity in conservative form. Actually one can use in the system (5) the equivalent nonconservative equation for the relative velocity

$$\frac{\partial w_k}{\partial t} + u^l\frac{\partial w_k}{\partial x_l} + \frac{\partial E_c}{\partial x_k} + w_l\frac{\partial u^l}{\partial x^k} = -\lambda_k, \tag{9}$$

instead of the conservative one. Nevertheless, the additional equations (7), (8) can be useful in studying discontinuous solutions.

It is reasonable to define the source terms in the following way:

$$\phi = \frac{\rho}{\tau^{(p)}}E_\alpha = \frac{\rho}{\tau^{(p)}}\frac{\partial E}{\partial \alpha}, \tag{10}$$

where $\tau^{(p)}$ is the pressure relaxation time which is assumed to be a function of parameters of state:

$$\lambda_k = \kappa E_{w^k} = \kappa\frac{\partial E}{\partial w^k}, \tag{11}$$

where κ is the interfacial friction coefficient, which also can be a function of parameters of state. Further we will see that such a definition will provide the positiveness of entropy production in more sophisticated models of two-phase flow.

The very important property of the system (5) is that its solution to the system satisfies the energy balance law. Its derivation can be reached by summing up all equations of the system (5) multiplying by the $E_\alpha, E + \rho E_\rho - u_i u_i/2 - \alpha E_\alpha - cE_c, u_l, E_c, \rho E_{w^k}$ respectively, and as a result we have

$$\frac{\partial \rho(E + u_l u^l/2)}{\partial t} + \frac{\partial(\rho u^k(E + u_l u^l/2) + \Pi^k)}{\partial x_k} = -\rho Q, \tag{12}$$

where Π_k is the energy flux

$$\Pi^k = u^k p + \rho u^k w^l E_{w^l} + \rho E_c E_{w_k} \tag{13}$$

and Q is the dissipative function producing the energy dissipation

$$Q = E_\alpha\phi + \rho E_{w_k}\lambda_k = \rho\left(\frac{1}{\tau}E_\alpha^2 + \kappa E_{w_k}E_{w^k}\right) \geq 0.$$

Note that energy dissipation appears because we consider isentropic processes. In [24] the model with mixture entropy has been considered in which the dissipation appears as entropy production source term in an entropy balance law. In such a model the flow is governed by the energy conservation law (with zero right hand side). In this paper we shall consider a more sophisticated model with two different phase entropies.

2.2 Equations in terms of individual phase parameters

The closing relations for the system (5) are the source terms ϕ, λ_k and the equation of state E. The source terms are defined by formulae (10), (11) in which the derivatives of the equation of state E_α, E_{w^k} are included. Only the pressure relaxation time $\tau^{(p)}$ and the interfacial friction coefficient κ should be defined empirically using physical assumptions and experimental data.

Now we describe how to define the equation of state for the mixture if the equations of state for each phase are known. First of all we assume that the equation of state for the mixture is a sum of thermodynamic energy e and kinematic energy of relative motion:

$$E(\alpha, \rho, c, w_1, w_2, w_3) = e(\alpha, \rho, c) + c(1-c)\frac{w_i w^i}{2}. \tag{14}$$

Let us suppose now that the equation of state (specific internal energy) for each phase is a known function of its mass density $e^i = e^i(\rho_i)$, $i = 1, 2$. We define the thermodynamic mixture energy as follows:

$$e(\alpha, \rho, c) = c_1 e^1(\rho_1) + c_2 e^2(\rho_2).$$

Using the relations connecting phase mass densities and parameters of the mixture we obtain

$$e(\alpha, \rho, c) = c e^1\left(\frac{c\rho}{\alpha}\right) + (1-c) e^2\left(\frac{(1-c)\rho}{(1-\alpha)}\right), \tag{15}$$

which gives us the thermodynamic equation of state for the mixture.

The definition of the equation of state for the mixture (14), (15) via equations of state for each phase gives the possibility to rewrite the system (5) in terms of individual parameters of each phase. To do this it is necessary to express the derivatives of the equation of state via derivatives with respect to individual phase parameters. First we obtain $E_\alpha = e_\alpha$, $E_\rho = e_\rho$, $E_c = e_c$, $E_{w^k} = c(1-c)w^k$. Further, using (2),(3) we obtain

$$\frac{\partial e}{\partial \alpha} = -\frac{\rho_1^2}{\rho}\frac{\partial e^1}{\partial \rho_1} + \frac{\rho_2^2}{\rho}\frac{\partial e^2}{\partial \rho_2} = \frac{p_2 - p_1}{\rho}, \tag{16}$$

$$\frac{\partial e}{\partial c} = e^1 + \frac{\rho c}{\alpha}\frac{\partial e^1}{\partial \rho_1} - e^2 - \frac{\rho(1-c)}{(1-\alpha)}\frac{\partial e^2}{\partial \rho_2} = e^1 + \frac{p^1}{\rho_1} - e^2 - \frac{p^2}{\rho_2}, \tag{17}$$

$$\frac{\partial e}{\partial \rho} = \frac{\alpha_1}{\rho^2}\rho_1^2\frac{\partial e^1}{\partial \rho_1} + \frac{\alpha_2}{\rho^2}\rho_2^2\frac{\partial e^2}{\partial \rho_2} = \frac{\alpha_1 p^1 + \alpha_2 p^2}{\rho^2}. \tag{18}$$

From (18) it follows that the mixture pressure is an average of constituents pressures: $p = \rho^2 E_\rho = \alpha_1 p^1 + \alpha_2 p^2$. Finally we obtain the system which is written in terms of parameters of state for individual phases (note that in some equations the mixture density ρ and velocity u^k are used for simplicity)

$$\begin{aligned}
&\frac{\partial \rho\alpha_1}{\partial t} + \frac{\partial \rho u^k \alpha_1}{\partial x^k} = -\phi, \\
&\frac{\partial(\alpha_1\rho_1 + \alpha_2\rho_2)}{\partial t} + \frac{\partial(\alpha_1\rho_1 u_1^k + \alpha_2\rho_2 u_2^k)}{\partial x^k} = 0, \\
&\frac{\partial(\alpha_1\rho_1 u_1^l + \alpha_2\rho_2 u_2^l)}{\partial t} + \frac{\partial(\alpha_1\rho_1 u_1^l u_1^k + \alpha_2\rho_2 u_2^l u_2^k + (\alpha_1 p^1 + \alpha_2 p^2)\delta_{lk})}{\partial x^k} = 0, \\
&\frac{\partial \alpha_1\rho_1}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1^k}{\partial x^k} = 0, \qquad (19) \\
&\frac{\partial(u_1^k - u_2^k)}{\partial t} + \frac{\partial(\frac{u_1^l u_1^l}{2} - \frac{u_2^l u_2^l}{2} + e^1 + \frac{p^1}{\rho_1} - e^2 - \frac{p^2}{\rho_2})}{\partial x^k} = -e_{klj} u^l \omega^j + \lambda^k,
\end{aligned}$$

where the source terms are

$$\phi = \frac{1}{\tau^{(p)}}(p_2 - p_1), \quad \lambda^k = \kappa c(1-c)w^k.$$

We see that the first four equations of the system (19) are well-known in the two-phase flow theory (the balance law for volume fraction, the total mass density conservation law, the total momentum conservation law, and the mass conservation law for the second phase). The fifth equation is the balance law for the relative velocity. It postulates that the relative velocity appears if there is a nonzero gradient of difference of phase enthalpies and/or kinetic energies.

In the next subsection we see that system (19) can be written in the form of Baer-Nunziato type model with some differences. Further, the system (5), or equivalence (19), will be used as the basis for the model of two phases with two different temperatures.

2.3 Equations in a Baer-Nunziato type form

The system (19) can be rewritten in the Baer-Nunziato type form as it is for example in [25], in which two sets of balance equations for each phase are coupled by the interfacial momentum exchange terms. The transformation of equations is not difficult but cumbersome procedure. We provide the final system only, which is as follows:

$$\begin{aligned}
&\frac{\partial \rho\alpha_1}{\partial t} + \frac{\partial \rho\alpha_1 u^k}{\partial x^k} = \frac{p^1 - p^2}{\tau}, \\
&\frac{\partial \alpha_1\rho_1}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1^k}{\partial x^k} = 0, \\
&\frac{\partial \alpha_2\rho_2}{\partial t} + \frac{\partial \alpha_2\rho_2 u_2^k}{\partial x^k} = 0, \\
&\frac{\partial \alpha_1\rho_1 u_1^i}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1^i u_1^k}{\partial x_k} + \frac{\partial \alpha_1 p^1}{\partial x_i} = \hat{p}_I\frac{\partial \alpha_1}{\partial x_i} + f_i + \kappa\frac{(\alpha_1\alpha_2\rho_1\rho_2)^2}{\rho^3}(u_2^i - u_1^i), \\
&\frac{\partial \alpha_2\rho_2 u_2^i}{\partial t} + \frac{\partial \alpha_2\rho_2 u_2^i u_2^k}{\partial x_k} + \frac{\partial \alpha_2 p^2}{\partial x_i} = \hat{p}_I\frac{\partial \alpha_2}{\partial x_i} - f_i - \kappa\frac{(\alpha_1\alpha_2\rho_1\rho_2)^2}{\rho^3}(u_2^i - u_1^i).
\end{aligned} \tag{20}$$

Here, as before, $\rho = \alpha_1\rho + \alpha_2\rho_2$, $u^k = \frac{\alpha_1\rho_1}{\rho}u_1^k + \frac{\alpha_2\rho_2}{\rho}u_2^k$, and $\hat{p}_I$ is an *interfacial pressure*:

$$\hat{p}_I = \frac{\alpha_2\rho_2 p^1 + \alpha_1\rho_1 p^2}{\alpha_1\rho_1 + \alpha_2\rho_2}. \tag{21}$$

The definition of interfacial pressure in our model differs from the definition in the conventional Baer-Nunziato type model, in which $p_I = \alpha_1 p_1 + \alpha_2 p_2$. The more significant difference is the appearing terms f_i in the momentum equations which do not appear in the Baer-Nunziato model:

$$f_i = \rho c_1 c_2(u_1^k - u_2^k)\left(c_1\left(\frac{\partial u_2^i}{\partial x^k} - \frac{\partial u_2^k}{\partial x^i}\right) + c_2\left(\frac{\partial u_1^i}{\partial x^k} - \frac{\partial u_1^k}{\partial x^i}\right)\right). \tag{22}$$

These body forces arise for the flow with nonzero relative velocity and are caused by the phase vorticities. Such type of force is called lift force, see for example [5].

3 Hyperbolic heat transfer equations

The model for two-phase compressible flow with two temperatures proposed in this paper uses a hyperbolic heat transfer approach, which gives a finite speed of propagation of thermal perturbations. Such an approach in heat transfer modelling was founded by Cattaneo and has been studied by many researches, see for example [17]. The Cattaneo theory is usually formulated in terms of temperature T and heat flux J_k, and the governing equation for J_k is as follows

$$\tau \frac{\partial J_k}{\partial t} + \kappa \frac{\partial T}{\partial x^k} = -J_k. \tag{23}$$

Here ρ is the mass density of medium which is assumed to be incompressible, κ is the thermal conductivity coefficient, and τ is heat flux relaxation time.

For our purpose another set of variables is preferable, namely, entropy and entropy flux. These variables accepted in irreversible thermodynamics [4] have been used in [16]. We also presuppose the existence of a thermodynamic potential - specific internal energy E. Moreover we assume that the entropy flux is a derivative of the equation of state (specific internal energy) with respect of the conjugate variable which we call the thermal impulse. Such set of variables as the entropy and thermal impulse seem to be more convenient for models of complicated media. Finally we assume that heat transfer processes are governed by the following system

$$\begin{aligned} &\rho \frac{\partial S}{\partial t} + \frac{\partial E_{j^k}}{\partial x^k} = \sigma, \\ &\rho \frac{\partial j_k}{\partial t} + \frac{\partial T}{\partial x^k} = -\frac{E_{j^k}}{\kappa}. \end{aligned} \tag{24}$$

Here S is the specific entropy, j_k is the thermal impulse, $E(S, j_k)$ is the specific internal energy, $T = E_S$ is the temperature, σ is the entropy production.

Multiplying equations of the above system by E_S and E_{j_k} respectively and summing the results we obtain the energy balance law in the form

$$\rho \frac{\partial E}{\partial t} + \frac{\partial E_S E_{j^k}}{\partial x^k} = \sigma E_S - \frac{E_{j_k} E_{j^k}}{\kappa}. \tag{25}$$

From to the first law of thermodynamics, namely conservation of energy, we conclude that the right hand side in the latter equation must be zero and hence for the entropy production we have

$$\sigma = \frac{1}{T} \frac{E_{j_k} E_{j^k}}{\kappa} \geq 0.$$

Now we have to define the dependence of internal energy on S and j^k. The simplest one we can choose is

$$E = e(S) + \frac{A}{2} j_k j^k, \quad A = \frac{\rho\kappa}{\tau}.$$

The term $e(S)$ can be defined by integrating the equation $de = C(T)dT = TdS$, where $C(T)$ is the specific heat capacity.

It is important to note that the classical Fourier heat transfer law can be obtained assuming that the heat flux relaxation time τ is sufficiently small and that the time scale of the processes under consideration is much bigger than τ. Appealing to physical intuition we do not provide a proof of this statement. Some mathematical results concerning the relaxation limit of hyperbolic equations can be found in [15, 3]. So the relaxation limit can be obtained neglecting the term $\partial j_k / \partial t$ in the second equation of the system (24), and the entropy flux becomes

$$E_{j_k} = -\kappa \frac{\partial T}{\partial x^k}.$$

Hence the entropy equation takes the form

$$\rho \frac{\partial S}{\partial t} - \frac{\partial}{\partial x^k} \left(\kappa \frac{\partial T}{\partial x^k} \right) = \kappa \frac{1}{T} \sum_{k=1}^{3} \left(\frac{\partial T}{\partial x^k} \right)$$

and neglecting terms of second order with τ^2, the energy conservation equation (25) takes the form

$$\rho\frac{\partial E}{\partial t}-\frac{\partial}{\partial x^k}\left(\kappa T\frac{\partial T}{\partial x^k}\right)=0. \tag{26}$$

These equations are well-known in heat transfer Furier theory.

Note that if we denote $j^k=\frac{\tau}{\rho\kappa}J^k$ and take $A=\frac{\rho\kappa}{\tau}$ we obtain for J^k equation (23) directly.

4 Conservative model for two-phase flow with two temperatures

Now we can construct a more complete model for the flow of the mixture of two compressible phases with different velocities, pressures and temperatures. The model is based on the synthesis of two models presented in previous sections, isentropic model for two-phase flow and the hyperbolic equations for heat transfer. We shall consider mechanisms such as interfacial exchange, pressure relaxation, interfacial friction, interfacial temperature exchange and phase transition.

4.1 Generating system of conservative equations

First of all we define the set of parameters of state for each phase

$$\alpha_i, \quad \rho_i, \quad s_i, \quad u_i^k, \quad j_i^k,$$

where i is the phase number ($i=1,2$), α_i is the volume concentration of i-th phase, ρ_i is the mass density of i-th phase, s_i is the specific entropy of i-th phase, u_i^k is the velocity vector of i-th phase, j_i^k is the specific thermal impulse of i-th phase. It is supposed that the saturation constraint for volume fractions $\alpha_1+\alpha_2=1$ holds.

The extended thermodynamics principles allow to formulate the system of governing equations in terms of parameters of state for the mixture:

$$\alpha_1,\ \rho=\alpha_1\rho_1+\alpha_2\rho_2,\ c_1=\frac{\alpha_1\rho_1}{\rho},\ u^k=c_1u_1^k+c_2u_2^k,\ w^k=u_1^k-u_2^k,\ S_i=c_is_i,\ j_i^k. \tag{27}$$

Here ρ is the total mass density of the mixture, c_i is the mass concentration of i-th phase, u^k is the mixture velocity vector, w^k is the relative velocity vector, S_i is the partial entropy of i-th phase, j_i^k is the thermal impulse vector of i-th phase. It is obvious that the saturation constraint $c_1+c_2=1$ is valid also for phase mass fractions.

As in the Section 2 let us denote $\alpha=\alpha_1$, $c=c_1$. The basic generating system of governing equations in a conservative form is

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho u^k}{\partial x^k} = 0,\\
&\frac{\partial \rho\alpha}{\partial t} + \frac{\partial \rho\alpha u^k}{\partial x^k} = -\phi,\\
&\frac{\partial \rho c}{\partial t} + \frac{\partial (\rho c u^k + E_{w_k})}{\partial x^k} = -\psi,\\
&\frac{\partial \rho u^i}{\partial t} + \frac{\partial (\rho u^i u^k + p\delta^i_k + \rho w^i E_{w_k})}{\partial x^k} = 0,\\
&\frac{\partial w^k}{\partial t} + \frac{\partial (u^l w_l + E_c)}{\partial x^k} = -e_{klj} u^l \omega^j - \frac{1}{\rho}\lambda^k_0,\\
&\frac{\partial \rho j^i_1}{\partial t} + \frac{\partial (\rho j^i_1 u^k + E_{S_1}\delta^i_k)}{\partial x^k} = -\lambda^i_1,\\
&\frac{\partial \rho j^i_2}{\partial t} + \frac{\partial (\rho j^i_2 u^k + E_{S_2}\delta^i_k)}{\partial x^k} = -\lambda^i_2,\\
&\frac{\partial \rho S_1}{\partial t} + \frac{\partial (\rho S_1 u^k + E_{j^k_1})}{\partial x^k} = \Pi_1 - \pi_1,\\
&\frac{\partial \rho S_2}{\partial t} + \frac{\partial (\rho S_2 u^k + E_{j^k_2})}{\partial x^k} = \Pi_2 - \pi_2.
\end{aligned}
\tag{28}
$$

The equations of system (28) are the total mass conservation law, the balance law for the volume fraction, the balance law for the mass fraction, the total momentum conservation law, the balance law for the relative velocity, two balance laws for phase thermal impulses, and two balance laws for the phase entropies respectively. The pressure p is defined in the same way as in the Section 2: $p = \rho^2 E_\rho$.

The constitutive relations which should be defined for closure of the system (28) are the specific internal energy E and the source terms $\phi, \lambda^k_0, \lambda^i_1, \lambda^i_2, \lambda^i_3, \Pi_1, \Pi_2, \pi_1, \pi_2$.

The procedure of derivation of E by the known equations of state for each phase is similar to Subsection 2.2 and will be given below. The source terms are defined by the following way:

$$
\begin{aligned}
&\phi = \frac{1}{\tau^{(p)}} E_\alpha, \quad \psi = \frac{1}{\tau^{(c)}} E_c,\\
&\lambda^k_0 = \chi_{00} E_{w_k} + \chi_{01} E_{j^k_1} + \chi_{02} E_{j^k_2},\\
&\lambda^k_1 = \chi_{01} E_{w_k} + \chi_{11} E_{j^k_1},\\
&\lambda^k_2 = \chi_{02} E_{w_k} + \chi_{22} E_{j^k_2},\\
&\Pi_1 = \frac{1}{E_{S_1}}\chi_{11}\sum_{k=1}^{3}\left(E_{j^k_1} + \frac{\chi_{01}}{\chi_{11}} E_{w^k}\right)^2 + \frac{c_1}{E_{S_1}}\left(\chi_{00} - \frac{\chi_{01}^2}{\chi_{11}} - \frac{\chi_{02}^2}{\chi_{22}}\right)\sum_{k=1}^{3}(E_{w^k})^2\\
&\qquad + \frac{c_1}{E_{S_1}}\frac{E_\alpha^2}{\tau^{(p)}} + \frac{c_1}{E_{S_1}}\frac{E_c^2}{\tau^{(c)}},\\
&\Pi_2 = \frac{1}{E_{S_2}}\chi_{22}\sum_{k=1}^{3}\left(E_{j^k_2} + \frac{\chi_{02}}{\chi_{22}} E_{w^k}\right)^2 + \frac{c_2}{E_{S_2}}\left(\chi_{00} - \frac{\chi_{01}^2}{\chi_{11}} - \frac{\chi_{02}^2}{\chi_{22}}\right)\sum_{k=1}^{3} E_{w^k}^2\\
&\qquad + \frac{c_2}{E_{S_2}}\frac{E_\alpha^2}{\tau^{(p)}} + \frac{c_2}{E_{S_2}}\frac{E_c^2}{\tau^{(c)}},\\
&\pi_1 = \frac{1}{\tau^{(T)}}\frac{E_{S_1} - E_{S_2}}{E_{S_1}}, \quad \pi_2 = \frac{1}{\tau^{(T)}}\frac{E_{S_2} - E_{S_1}}{E_{S_2}}.
\end{aligned}
\tag{29}
$$

These source terms are responsible to the following physical processes: ϕ governs the relaxation of phase pressures to a common uniform state, ψ governs the rate of phase transition, λ_0 simulates the interfacial friction force, λ_1 and λ_2 simulate relaxation of the phase thermal impulses to the corresponding equilibrium states. Here a mutual dependence of thermal impulses and relative velocity from each other is taken into account. We also emphasize that coefficients χ_{ij} in formulae for λ^i_j are symmetric, which is a consequence of the Onsager principle [4]. Π_1 and Π_2 simulates the phase entropy production caused by all above mentioned relaxation processes, and finally π_1 and π_2 governs the phase energy exchange leading to the phase temperatures equalizing.

We emphasize that the source terms Π_1, Π_2 in the equations for partial entropies must be nonnegative and hence the coefficients

$$\left(\chi_{00} - \frac{\chi_{01}^2}{\chi_{11}} - \frac{\chi_{02}^2}{\chi_{22}}\right), \quad \chi_{11}, \quad \chi_{22}, \quad \tau^{(p)}. \quad \tau^{(c)}$$

must be nonnegative.

We also suppose that $\tau^{(T)}$ is nonnegative. This requirement is necessary to provide the positiveness of the production of total entropy $S = S_1 + S_2$, the equation for which can be obtained by a summation of the equations for partial entropies, namely

$$\frac{\partial \rho S}{\partial t} + \frac{\partial(\rho S u + E_{j_1^k} + E_{j_2^k})}{\partial x^k} = \Pi = \Pi_1 + \Pi_2 + \frac{1}{\tau^{(T)}} \frac{(E_{S_1} - E_{S_2})^2}{E_{S_1} E_{S_2}} \geq 0. \tag{30}$$

Note that all coefficients in the source terms can be functions of the parameters of state.

As in the model for isentropic two-phase flow (Section 2) an additional compatibility equation connected with the relative velocity vorticity should be added to system (28). These equations are identical to (7) and (8):

$$e^{jkl} \frac{\partial w_k}{\partial x^l} = \omega^j,$$

$$\frac{\partial \omega^k}{\partial t} + \frac{\partial(u^l \omega_k - u^k \omega^l + e^{klj} \lambda_0^j)}{\partial x_l} = 0,$$

and we emphasize again that the relative velocity vorticity ω^k is an auxiliary variable, but not a parameter of state.

The solution to system (28) satisfies an additional total energy conservation law, which can be derived by the summing up of all equations of this system multiplied by the corresponding factors: $E + \rho E_\rho - u_l u^l/2 - \alpha E_\alpha - c E_c - j_1^k E_{j_1^k} - j_2^k E_{j_2^k} - S_1 E_{S_1} - S_2 E_{S_2}$, E_α, E_c, u^k, ρE_{w^k}, $E_{j_1^k}$, $E_{j_2^k}$, E_{S_1}, E_{S_2}. The equation itself has the following form

$$\frac{\partial \rho \left(E + \frac{u_l u^l}{2}\right)}{\partial t} + \frac{\partial \left(\rho u^k \left(E + \frac{u_l u^l}{2} + \frac{p}{\rho} + w^l E_{w^l}\right) + E_c E_{w^k} + E_{j_i^k} E_{S_i}\right)}{\partial x^k} = 0. \tag{31}$$

Below we describe how to choose closing relationships, namely, equation of state and coefficients in the source terms.

4.2 Equation of state

Suppose that we know the internal energy $e^i (i = 1, 2)$ for each phase as a function of its mass density ρ_i, specific entropy s_i and specific thermal impulse j_i^k: $e^i = e^i(\rho_i, s_i, j_i^k)$. The dependence on the thermal impulse j_i^k can be taken as in Section 3: $e^i = e_0^i(\rho_i, s_i) + A^i \frac{j_l^k j_l^k}{2}$, where A^i is a constant.

As in Section 2 we take the specific internal energy of the mixture as a sum of thermodynamic internal energy and kinematic energy

$$E(\rho,\alpha,c,S_1,S_2,j_1^k,j_2^k,w^k) = e(\rho,\alpha,c,S_1,S_2,j_1^k,j_2^k) + c(1-c)\frac{w_k w^k}{2}.$$

The thermodynamic energy e we define using known e_1, e_2 by the formula

$$e(\rho,\alpha,c,S_1,S_2,j_1^k,j_2^k) = c_1 e_1(\rho_1,s_1,j_1^k) + c_2 e_2(\rho_2,s_2,j_2^k).$$

Now we derive the dependence of the parameters of state for each individual phase on the parameters of state for the mixture using formulae (27). This dependence is as follows

$$u_1^k = u^k + (1-c)w^k, \quad u_2^k = u^k - cw^k, \quad \rho_1 = \frac{c\rho}{\alpha}, \quad \rho_2 = \frac{(1-c)\rho}{(1-\alpha)},$$

$$s_1 = \frac{S_1}{c_1}, \quad s_2 = \frac{S_2}{c_2}, \quad j_1^k = j_1^k, \quad j_2^k = j_2^k.$$

Using these formulae we can derive the derivatives of the equation of state for the mixture $E_\alpha, E_\rho, E_c, E_{w^k}, E_{S_1}, E_{j_1^k}, E_{S_2}, E_{j_2^k}$ by the derivatives of equations of state for individual phases $e^1_{\rho_1}, e^1_{s_1}, e^1_{j_1^k}$ and $e^2_{\rho_2}, e^2_{s_2}, e^2_{q_2^k}$.

To do this we can use the following set of thermodynamic identities

$$de = d(c_1 e^1 + c_2 e^2) = (e^1 - e^2)dc + c_1 de^1 + c_2 de^2,$$

$$de^1 = e^1_{\rho_1} d\rho_1 + e^1_{s_1} ds_1 + e^1_{q_1^k} dq_1^k,$$

$$de^2 = e^2_{\rho_2} d\rho_2 + e^2_{s_2} ds_2 + e^2_{q_2^k} dq_2^k,$$

$$d\rho_1 = \frac{c_1}{\alpha_1} d\rho - \frac{\rho_1}{\alpha_1} d\alpha + \frac{\rho}{\alpha_1} dc, \quad d\rho_2 = \frac{c_2}{\alpha_2} d\rho + \frac{\rho_2}{\alpha_2} d\alpha - \frac{\rho}{\alpha_2} dc,$$

$$ds_1 = \frac{1}{c_1} dS_1 - \frac{s_1}{c_1} dc, \quad ds_2 = \frac{1}{c_2} dS_2 + \frac{s_2}{c_2} dc.$$

These identities lead us to the final thermodynamic identity

$$de(\rho,\alpha,c,S_1,S_2,j_1^k,j_2^k) = e_\rho d\rho + e_\alpha d\alpha + e_c dc + e_{S_1} dS_1 + e_{S_2} dS_2 + e_{j_1^k} dj_1^k + e_{j_2^k} dj_2^k =$$

$$\frac{1}{\rho^2}(\alpha_1 \rho_1^2 e^1_{\rho_1} + \alpha_2 \rho_2^2 e^2_{\rho_2}) d\rho + \frac{1}{\rho}(\rho_1^2 e^1_{\rho_1} - \rho_2^2 e^2_{\rho_2}) d\alpha +$$
$$((e^1 + \rho_1 e^1_{\rho_1} - s_1 e^1_{s_1}) - (e^2 + \rho_2 e^2_{\rho_2} - s_2 e_{s_2}))dc +$$
$$e^1_{s_1} dS_1 + c_1 e^1_{j_1^k} dj_1^k + e^2_{s_2} dS_2 + c_2 e^2_{j_2^k} dj_2^k,$$

from which we obtain

$$E_\rho = \frac{1}{\rho^2}(\alpha_1 \rho_1^2 e^1_{\rho_1} + \alpha_2 \rho_2^2 e^2_{\rho_2}) = \frac{1}{\rho^2}(\alpha_1 p^1 + \alpha_2 p^2), \tag{32}$$

$$E_\alpha = e_\alpha = \frac{1}{\rho}(\rho_1^2 e^1_{\rho_1} - \rho_2^2 e^2_{\rho_2}) = \frac{p^1 - p^2}{\rho}, \tag{33}$$

$$E_c = e_c + (1-2c)\frac{w^2}{2} = ((e^1 + p^1/\rho_1 - s_1 e^1_{s_1}) - (e^2 + p^2/\rho_2 - s_2 e^2_{s_2})) + (1-2c)\frac{w^2}{2}, \tag{34}$$

$$E_{S_1} = e^1_{s_1} = T^1, \quad E_{S_2} = e^2_{s_2} = T^2, \quad E_{j_1^k} = c_1 e^1_{j_1^k}, \quad E_{j_2^k} = c_2 e^2_{j_2^k}. \tag{35}$$

Here T^1, T^2 are temperatures of phases. It is interesting to note that the derivative e_c of thermodynamic part of equation of state with respect to c is simply the difference of chemical potentials of phases μ^i:

$$e_c = \mu^1 - \mu^2, \quad \mu^i = e^i + \frac{p^i}{\rho_i} - s_i T^i.$$

Now we can write the system of governing equations in terms of individual parameters of state for each phase.

4.3 Governing equations in terms of individual parameters of state

The system of conservative balance equations (28) can be rewritten using individual parameters of each phase $\alpha_1, \alpha_2, \rho_1, \rho_2, u_1^k, u_2^k, s_1, s_2, j_1^k, j_2^k$ (recall that the volume fraction of the second phase can be determined via the volume fraction of the first phase: $\alpha_2 = 1 - \alpha_1$). To do this it is necessary to use formulae (27),(29) and (32)-(35). The resulting system is as follows (note that we left the equation for α as in (28)):

$$\begin{aligned}
&\frac{\partial(\alpha_1\rho_1 + \alpha_2\rho_2)}{\partial t} + \frac{\partial(\alpha_1\rho_1 u_1^k + \alpha_2\rho_2 u_2^k)}{\partial x^k} = 0,\\
&\frac{\partial \rho\alpha}{\partial t} + \frac{\partial \rho\alpha u^k}{\partial x^k} = -\phi,\\
&\frac{\partial \alpha_1\rho_1}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1^k}{\partial x^k} = -\psi,\\
&\frac{\partial(\alpha_1\rho_1 u_1^i + \alpha_2\rho_2 u_2^i)}{\partial t} + \frac{\partial(\alpha_1\rho_1 u_1^i u_1^k + \alpha_2\rho_2 u_2^i u_2^k + \alpha_1 p^1 + \rho\alpha_2 p^2)}{\partial x^k} = 0,\\
&\frac{\partial(u_1^k - u_2^k)}{\partial t} + \frac{\partial(u_1^i u_1^i/2 - u_2^i u_2^i/2 + \mu^1 - \mu^2)}{\partial x^k} = -e_{klj}u_k\omega_j - \frac{1}{\rho}\lambda_0^k, \qquad (36)\\
&\frac{\partial \rho j_1^i}{\partial t} + \frac{\partial(\rho j_1^i u^k + e^1_{s_1}\delta_{ik})}{\partial x^k} = -\lambda_1^i,\\
&\frac{\partial \rho j_2^i}{\partial t} + \frac{\partial(\rho j_2^i u^k + e^2_{s_2}\delta_{ik})}{\partial x^k} = -\lambda_2^i,\\
&\frac{\partial \alpha_1\rho_1 s_1}{\partial t} + \frac{\partial(\alpha_1\rho_1 s_1 u^k + c_1 e^1_{j_1^k})}{\partial x^k} = \Pi_1 - \pi_1,\\
&\frac{\partial \alpha_2\rho_2 s_2}{\partial t} + \frac{\partial(\alpha_2\rho_2 s_2 u^k + c_2 e^2_{j_2^k})}{\partial x} = \Pi_2 - \pi_2.
\end{aligned}$$

As in Section 2, this system must be supplemented by the compatibility relations (7), (8) for the relative velocity vorticity ω_j.

The source terms in the above equations can also be written in terms of individual parameters of phases:

$$\phi = \frac{1}{\tau^{(p)}}\frac{p_1 - p_2}{\rho}, \quad \psi = \frac{\mu^1 - \mu^2}{\tau^{(c)}},$$

$$\lambda_0^k = \chi_{00}c(1-c)(u_1^k - u_2^k) + \chi_{01}c_1 e^1_{j_1^k} + \chi_{02}c_2 e^2_{j_2^k},$$

$$\lambda_1^i = \chi_{01}c(1-c)(u_1^i - u_2^i) + \chi_{11}c_1 e^1_{j_1^i},$$

$$\lambda_2^i = \chi_{02}c(1-c)(u_1^i - u_2^i) + \chi_{22}c_2 e^2_{j_2^i},$$

$$\Pi_1 = \frac{1}{T^1}\chi_{11}\sum_{k=1}^{3}\left(c_1 e^1_{j_1^k} + \frac{\chi_{01}}{\chi_{11}}c_1 c_2(u_1^k - u_2^k)\right)^2 + \qquad (37)$$

$$\frac{c_1}{T^1}\left(\chi_{00}-\frac{\chi_{01}^2}{\chi_{11}}-\frac{\chi_{02}^2}{\chi_{22}}\right)\sum_{k=1}^{3}(c_1c_2(u_1^k-u_2^k))^2+\frac{c_1}{T^1}\frac{(p^1-p^2)^2}{\rho^2\tau^{(p)}}+\frac{c_1}{T^1}\frac{(\mu^1-\mu^2)^2}{\rho^2\tau^{(c)}},$$

$$\Pi_2=\frac{1}{T^2}\chi_{22}\sum_{k=1}^{3}\left(c_2e^2_{j_2^k}+\frac{\chi_{02}}{\chi_{22}}c_1c_2(u_1^k-u_2^k)\right)^2+$$

$$\frac{c_2}{T^2}\left(\chi_{00}-\frac{\chi_{01}^2}{\chi_{11}}-\frac{\chi_{02}^2}{\chi_{22}}\right)\sum_{k=1}^{3}(c_1c_2(u_1^k-u_2^k))^2+\frac{c_2}{T^2}\frac{(p^1-p^2)^2}{\rho^2\tau^{(p)}}+\frac{c_2}{T^2}\frac{(\mu^1-\mu^2)^2}{\rho^2\tau^{(c)}},$$

$$\pi_1=\frac{1}{\tau^{(T)}}\frac{T^1-T^2}{T^1},\quad \pi_2=\frac{1}{\tau^{(T)}}\frac{T^2-T^1}{T^2}.$$

Finally we can rewrite the total mixture entropy balance equation (30) end energy conservation law (31) in terms of individual parameters:

$$\frac{\partial(\alpha_1\rho_1s_1+\alpha_2\rho_2s_2)}{\partial t}+\frac{\partial((\alpha_1\rho_1s_1+\alpha_2\rho_2s_2)u^k+c_1e^1_{j_1^k}+c_2e^2_{j_2^k})}{\partial x^k}=$$

$$\Pi=\Pi_1+\Pi_2+\frac{1}{\tau^{(T)}}\frac{(T^1-T^2)^2}{T^1T^2}\geq 0. \tag{38}$$

$$\frac{\partial}{\partial t}\left(\alpha_1\rho_1(e^1+\frac{u_1^iu_1^i}{2})+\alpha_2\rho_2(e^2+\frac{u_2^iu_2^i}{2})\right)+$$

$$\frac{\partial}{\partial x^k}\left(\alpha_1\rho_1u_1^k\left(e^1+\frac{p^1}{\rho_1}+\frac{u_1^iu_1^i}{2}\right)+\alpha_2\rho_2u_2^k\left(e^2+\frac{p^2}{\rho_2}+\frac{u_2^iu_2^i}{2}\right)+c_1T^1e^1_{j_k}+c_2T^2e^2_{j_k}\right)=0, \tag{39}$$

It is interesting to note that the difference of chemical potentials of phases is presented in the system (28) twice, namely in the relative velocity flux and in the source term which is responsible for the phase transition. The thermodynamic theory uses equality of phase chemical potentials as phase equilibrium condition [14]. In our model such an equality is also responsible for the kinematic equilibrium of phases. In other words the chemical potentials difference causes the relative motion of the phases.

There is a set of coefficients in the source terms which should be defined, and, in particular, the coefficients $\chi_{00},\chi_{01},\chi_{02},\chi_{11},\chi_{22}$ in the source terms $\lambda_0^i,\lambda_1^i,\lambda_2^i$. We take

$$\chi_{00}=\kappa+\frac{\chi_{01}^2}{\chi_{11}}+\frac{\chi_{02}^2}{\chi_{22}},\chi_{01}=-\frac{\rho s_1}{c_1\kappa_1},\chi_{02}=\frac{\rho s_2}{c_2\kappa_2},\chi_{11}=\frac{1}{c_1\kappa_1},\chi_{22}=\frac{1}{c_2\kappa_2}, \tag{40}$$

where κ is an interfacial friction coefficient, κ_1,κ_2 are the coefficients of thermal conductivity for phases.

The motivation for this choice we explain in the next section, in which the diffusive limit of the heat flux is discussed. It turns out that the equations, which are derived assuming $\tau_i^{(T)}$ sufficiently small, look similar to the Baer-Nunziato equations in which the phases parabolic thermal conductivity is taken into account.

4.4 Diffusive heat transfer limit

In Section 3 it was observed that the hyperbolic heat transfer equations give a parabolic approximation of heat transfer processes if the heat flux relaxation time is small. In case of two-phase flow we have two interacting heat transfer processes for each phase with two different heat flux relaxation times. Here we study the relaxation limit in both phases assuming that two heat flux relaxation times are sufficiently small. We also suppose that the characteristic scale of the process is much larger than the relaxation times.

Let us consider the equations from the system (36) for phase thermal impulses

$$\frac{\partial \rho j^i_m}{\partial t} + \frac{\partial(\rho j^i_m u^k + e^m_{s_m}\delta_{mk})}{\partial x^k} = -\lambda^i_m, \quad m = 1, 2$$

where λ^i_m is defined by (37) with coefficients (40).

Following the assumption concerning the smallness of κ_1, κ_2 and the characteristic time scale of processes, we can neglect the time derivative and convective term in the equation for q^i_m. The resulting equation is

$$\lambda^k_m = -\frac{\partial e^m_{s_m}}{\partial x^k} = -\frac{\partial T^m}{\partial x^k}, \quad m = 1, 2.$$

Now with the use of the formulae for coefficients (37), (40) we obtain

$$c_1 e^1_{j^k_1} = \rho s_1 c_1 c_2 (u^k_1 - u^k_2) - c_1\kappa_1 \frac{\partial T^1}{\partial x^k}, \quad c_2 e^2_{j^k_2} = -\rho s_2 c_1 c_2 (u^k_1 - u^k_2) - c_2\kappa_2 \frac{\partial T^1}{\partial x^k}. \tag{41}$$

Finally, the phases entropies balance equations are transformed to

$$\begin{aligned} \frac{\partial \alpha_1\rho_1 s_1}{\partial t} + \frac{\partial \alpha_1\rho_1 s_1 u^k_1}{\partial x^k} - \frac{\partial}{\partial x^k} c_1\kappa_1 \frac{\partial T^1}{\partial x^k} &= \Pi_1 - \pi_1, \\ \frac{\partial \alpha_2\rho_2 s_2}{\partial t} + \frac{\partial \alpha_2\rho_2 s_2 u^k_2}{\partial x^k} - \frac{\partial}{\partial x^k} c_2\kappa_2 \frac{\partial T^2}{\partial x} &= \Pi_2 - \pi_2. \end{aligned} \tag{42}$$

Expressions (41) should be also substituted to entropy production terms Π_k, π_k, which are defined by (37) and became (π_k does not change)

$$\Pi_1 = \frac{c_1\kappa_1}{T^1}\sum_{k=1}^{3}\left(\frac{\partial T^1}{\partial x^k}\right)^2 + \frac{c_1\kappa}{T^1}\sum_{k=1}^{3}(c_1c_2(u^k_1 - u^k_2))^2 + \frac{c_1}{T^1}\frac{(p^1 - p^2)^2}{\rho^2\tau^{(p)}} + \frac{c_1}{T^1}\frac{(\mu^1 - \mu^2)^2}{\rho^2\tau^{(c)}},$$

$$\Pi_2 = \frac{c_2\kappa_2}{T^2}\sum_{k=1}^{3}\left(\frac{\partial T^2}{\partial x^k}\right)^2 + \frac{c_2\kappa}{T^2}\sum_{k=1}^{3}(c_1c_2(u^k_1 - u^k_2))^2 + \frac{c_2}{T^2}\frac{(p^1 - p^2)^2}{\rho^2\tau^{(p)}} + \frac{c_2}{T^2}\frac{(\mu^1 - \mu^2)^2}{\rho^2\tau^{(c)}},$$

$$\pi_1 = \frac{1}{\tau^{(T)}}\frac{T^1 - T^2}{T^1}, \quad \pi_2 = \frac{1}{\tau^{(T)}}\frac{T^2 - T^1}{T^2}. \tag{43}$$

So the diffusive limit leads to the traditional parabolic equations for phase heat transfer.

It is interesting to look at the transformed relative velocity equation, because the original balance law contains a source term in which a thermal impulses are included. Substituting (41) into the source term in the equation for the relative velocity we obtain

$$\frac{\partial(u^k_1 - u^k_2)}{\partial t} + \frac{\partial(u^i_1u^i_1/2 - u^i_2u^i_2/2 + \mu^1 - \mu^2)}{\partial x^k} =$$

$$- s_1\frac{\partial T^1}{\partial x^k} + s_2\frac{\partial T^2}{\partial x^k} - \kappa c_1c_2(u^k_1 - u^k_2) - e_{klj}u_k\omega_j.$$

We see that although the heat flux relaxation times are small, the influence of heat transfer on the relative motion is finite.

The latter equation can be transformed to

$$\left(\frac{\partial u^k_1}{\partial t} + u^i_1\frac{\partial u^k_1}{\partial x^i} + \frac{1}{\rho_1}\frac{\partial p_1}{\partial x^k}\right) - \left(\frac{\partial u^k_2}{\partial t} + u^i_2\frac{\partial u^k_2}{\partial x^i} + \frac{1}{\rho_2}\frac{\partial p_2}{\partial x^k}\right) = -\kappa c_1c_2(u^k_1 - u^k_2),$$

which can be used to derive individual phase momentum equations. It turns out that they are identical to the phases momentum equations derived in the Section 2.

So we conclude that the diffusive limit of the system for two-phase flow (36) consists of the balance laws for phase volume fraction α, mass and momentum balance equations for two phases, and balance equations for partial entropies (42). The full system is similar to the equations of the Baer-Nunziato model, in which the phase thermal conductivity is taken into account:

$$
\begin{aligned}
&\frac{\partial \rho\alpha_1}{\partial t} + \frac{\partial \rho\alpha_1 u^k}{\partial x^k} = \frac{p^1 - p^2}{\tau}, \\
&\frac{\partial \alpha_1\rho_1}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1^k}{\partial x^k} = -\frac{\mu_1 - \mu_2}{\tau^{(c)}}, \\
&\frac{\partial \alpha_2\rho_2}{\partial t} + \frac{\partial \alpha_2\rho_2 u_2^k}{\partial x^k} = \frac{\mu_1 - \mu_2}{\tau^{(c)}}, \\
&\frac{\partial \alpha_1\rho_1 u_1^i}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1^i u_1^k}{\partial x_k} + \frac{\partial \alpha_1 p^1}{\partial x_i} = \hat{p}_I \frac{\partial \alpha_1}{\partial x_i} + f_i + \kappa \frac{(\alpha_1\alpha_2\rho_1\rho_2)^2}{\rho^3}(u_2^i - u_1^i), \\
&\frac{\partial \alpha_2\rho_2 u_2^i}{\partial t} + \frac{\partial \alpha_2\rho_2 u_2^i u_2^k}{\partial x_k} + \frac{\partial \alpha_2 p^2}{\partial x_i} = \hat{p}_I \frac{\partial \alpha_2}{\partial x_i} - f_i - \kappa \frac{(\alpha_1\alpha_2\rho_1\rho_2)^2}{\rho^3}(u_2^i - u_1^i), \\
&\frac{\partial \alpha_1\rho_1 s_1}{\partial t} + \frac{\partial \alpha_1\rho_1 s_1 u_1^k}{\partial x^k} - \frac{\partial}{\partial x} c_1\kappa_1 \frac{\partial T^1}{\partial x^k} = \Pi_1 - \pi_1, \\
&\frac{\partial \alpha_2\rho_2 s_2}{\partial t} + \frac{\partial \alpha_2\rho_2 s_2 u_2^k}{\partial x^k} - \frac{\partial}{\partial x} c_2\kappa_2 \frac{\partial T^2}{\partial x} = \Pi_2 - \pi_2,
\end{aligned}
\tag{44}
$$

where Π_i, π_i are defined by (43), p_I is defined by (21), and f_i is defined by (22).

Note that the phase entropy balance equations are presented in the latter system, but it is not difficult to derive phase energies balance equations which are used in the Baer-Nunziato model and looks as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\alpha_1\rho_1\left(e^1 + \frac{u_1^i u_1^i}{2}\right)\right) + \frac{\partial}{\partial x^k}\left(\alpha_1\rho_1 u_1^k\left(e^1 + \frac{u_1^i u_1^i}{2}\right) + \alpha_1 p^1 u_1^k - c_1\kappa_1 T^1 \frac{\partial T^1}{\partial x^k}\right) \\
&\qquad = -p^1 \frac{\partial \alpha_1}{\partial t} + (\hat{p}_I - p_1) u_1^k \frac{\partial \alpha_1}{\partial x^k} + u_1^k f_k - \frac{T^1 - T^2}{\tau^{(T)}}, \\
&\frac{\partial}{\partial t}\left(\alpha_1\rho_1\left(e^2 + \frac{u_2^i u_2^i}{2}\right)\right) + \frac{\partial}{\partial x^k}\left(\alpha_2\rho_2 u_2^k\left(e^2 + \frac{u_2^i u_2^i}{2}\right) + \alpha_2 p^2 u_2^k - c_2\kappa_2 T^2 \frac{\partial T^2}{\partial x^k}\right) \\
&\qquad = -p^2 \frac{\partial \alpha_2}{\partial t} + (\hat{p}_I - p^2) u_2^k \frac{\partial \alpha_2}{\partial x^k} + u_2^k f_k - \frac{T^2 - T^1}{\tau^{(T)}}.
\end{aligned}
$$

So the diffusive limit of the proposed conservative equations for two-phase flow can be written in the form of the Baer-Nunziato type equations, which are not in a conservative form.

5 Conclusions

A system of governing equations for two-phase compressible flow with two different pressures and temperatures of phases is proposed. The derivation of the system is based on extended irreversible thermodynamics principles and on the synthesis of earlier proposed conservative model for isentropic two-phase flow and a hyperbolic heat transfer model. The system is hyperbolic and is written in conservation-law form.

The phase interaction is modelled by the source terms added to conservation equations and include the phase pressure relaxation, phase transition, interfacial friction, and phases temperatures equalizing.

The diffusive limit in the heat transfer equations in the proposed model can be transformed to the well-known Baer-Nunziato type of governing equations in which traditional Furier thermal conductivity is taken into account.

Because of its conservation form the presented model seems to be promising in in applying known mathematical means and the development of modern numerical methods for studying different problems.

6 Appendix: Thermodynamically compatible system of balance laws generating two-phase flow model

An *Extended Irreversible Thermodynamics* theory has been applied to the formulation of general classes of conservation equations for the processes in complicated media in the presence of electromagnetic fields and external forces, see, for example [11, 12, 17]. This theory gives an elegant way to write the system of conservation laws in terms of generating potentials and variables. Such a formulation is based on the requirement for the system to be thermodynamically compatible, in other words the complete system admits an additional conservation law corresponding to the first law of thermodynamics. Each system written in terms of canonical variables and generating potentials can be reduced to a symmetric hyperbolic system, and together with its conservative form it allows us to apply known mathematical means to study solutions of various initial-boundary problems. In [11, 20] the class of thermodynamically compatible systems generated by only one potential has been formulated. This class includes many system of governing equations of continuum mechanics. If dissipation is not included into the governing equations, then the equations can be derived by variational principle in Lagrangean coordinates [10]. After passing to Eulerean coordinates dissipative processes can be included in the governing equations. Each concrete system of continuum mechanics can be derived from the thermodynamically compatible system by assigning a physical meaning to the canonical variables and special a choice of generating potential.

In this paper the governing equations of two-phase flow are designed on the basis of representatives of the class of thermodynamically compatible system presented below. Note that we do not consider here possible dissipative processes. We give this description only to give an explanation how the equations of the paper have been derived. Details concerning the inclusion of dissipation can be found in [21].

$$
\begin{aligned}
&\frac{\partial L_{q_i}}{\partial t} + \frac{\partial (u_k L)_{q_i}}{\partial x_k} = 0, \quad i = 1,2,\\
&\frac{\partial L_{u_i}}{\partial t} + \frac{\partial [(u_k L)_{u_i} + z_k L_{z_i} - \delta_{ik} z_\alpha L_{z_\alpha}]}{\partial x_k} = 0,\\
&\frac{\partial L_{z_k}}{\partial t} + \frac{\partial (u_\alpha L_{z_\alpha} + n)}{\partial x_k} = 0,\\
&\frac{\partial L_n}{\partial t} + \frac{\partial (u_k L_n + z_k)}{\partial x_k} = 0,\\
&\frac{\partial L_{\theta^n}}{\partial t} + \frac{\partial ((u_k L)_{\theta^n} + v_k^n)}{\partial x_k} = 0, \quad n = 1,2,\\
&\frac{\partial L_{v_i^n}}{\partial t} + \frac{\partial ((u_k L)_{v_i^n} + \theta \delta_{ik})}{\partial x_k} = 0, \quad n = 1,2,\\
&\frac{\partial L_{z_k}}{\partial x_\alpha} - \frac{\partial L_{z_\alpha}}{\partial x_k} = 0.
\end{aligned}
\tag{45}
$$

Here $L(q_i, u_i, z_k, n, \theta^n, v_i^n)$ is the generating potential assumed to be a convex function. The last steady equation must be added to the system in order to provide the compatibility of the system (45). Actually this steady equation is a consequence of the equation for L_{z_k} and can be obtained by the differentiating the equation for L_{z_k} with respect to x_α and substraction the equation for L_{z_α} differentiated with respect to x_k. After that we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial L_{z_k}}{\partial x_\alpha} - \frac{\partial L_{z_\alpha}}{\partial x_k}\right) = 0,$$

and if in the initial data $\partial L_{z_k}/\partial x_\alpha - \partial L_{z_\alpha}/\partial x_k = 0$, then this equality holds for every time $t > 0$.

As was noted for the thermodynamically compatible system an additional (energy) conservation law exists, in our case it looks as follows

$$\frac{\partial}{\partial t}(q_i L_{q_i} + u_i L_{u_i} + z_k L_{z_k} + nL_n + \theta^n L_{\theta^n} + v_i^n L_{v_i^n} - L) +$$

$$\frac{\partial}{\partial x_k}(u_k(q_i L_{q_i} + u_i L_{u_i} + nL_n + \theta^n L_{\theta^n} + v_i^n L_{v_i^n}) + u_l z_k L_{z_l} + z_k n + \theta^n v_k^n) = 0.$$

To derive this conservation law it is necessary to sum all equations of the system (45) multiplied respectively by

$$q_i,\ u_i,\ z_k,\ n,\ \theta^n,\ v_i^n,\ 2u_i z_k.$$

Finally, the symmetric hyperbolic system can be derived from the (45). To perform this we rewrite the evolution equations in the equivalent form, which is simply obtained by adding $z_k\left(\frac{\partial L_{z_k}}{\partial x_i} - \frac{\partial L_{z_i}}{\partial x_k}\right) = 0$ to the equation for u_i and $u_i\left(\frac{\partial L_{z_k}}{\partial x_i} - \frac{\partial L_{z_i}}{\partial x_k}\right) = 0$ to the equation for z_k. After that we obtain the following system:

$$\begin{aligned}
&\frac{\partial L_{q_i}}{\partial t} + \frac{\partial (u_k L)_{q_i}}{\partial x_k} = 0,\\
&\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} + L_{z_i}\frac{\partial z_k}{\partial x_k} - L_{z_\alpha}\frac{\partial z_\alpha}{\partial x_i} = 0,\\
&\frac{\partial L_n}{\partial t} + \frac{\partial (u_k L)_n}{\partial x_k} + \frac{\partial z_k}{\partial x_k} = 0,\\
&\frac{\partial L_{z_l}}{\partial t} + \frac{\partial (u_k L)_{z_l}}{\partial x_k} + L_{z_\alpha}\frac{\partial u_\alpha}{\partial x_l} - L_{z_l}\frac{\partial u_k}{\partial x_k} + \frac{\partial n}{\partial x_l} = 0.\\
&\frac{\partial L_{\theta^n}}{\partial t} + \frac{\partial (u_k L)_\theta}{\partial x_k} + \frac{\partial v_k^n}{\partial x_k} = 0,\\
&\frac{\partial L_{v_k^n}}{\partial t} + \frac{\partial (u_k L)_{v_k^n}}{\partial x_k} + \delta_{ik}\frac{\partial \theta^n}{\partial x_k} = 0,
\end{aligned}$$

It is clear that the quasilinear form of the latter system is symmetric, and if the generating potential L is a convex function, then the system is symmetric hyperbolic.

This system is a prototype of the generating system (28) of Subsection 4.1. Actually in order to give a physical meaning to above formal thermodynamically compatible system we have to identify all variables as a physical variables and to define the generating potential L. One can prove that the system (28) can be obtained taking the derivatives of generating potential as

$$L_{q_1} = \rho,\ L_{q_2} = \rho\alpha,\ L_{u_i} = \rho u_i,\ L_n = \rho c,\ L_{z_k} = w^k,\ L_{\theta^n} = \rho S_n,\ L_{v_k^n} = \rho j_k^n,$$

the variables as

$$q_1 = E + \rho E_\rho - \frac{u_i u_i}{2} - \alpha E_\alpha - cE_c - j_k^n E_{j_k^n} - \theta^n E_{\theta^n},\ q_2 = E_\alpha,\ u^i,$$

$$n = E_c,\ z_l = \rho E_{w^l},\ \theta^n = E_{S_n},\ v_k^n = E_{j_k^n},$$

and finally the generating potential as

$$L = \rho^2 E_\rho - \rho w^k E_{w^k}.$$

An explanation how to incorporate some kinds of dissipative processes into the system (45) can be found in [21].

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