Target zone dynamics where the fundamental follows a SDE with periodic forcing

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Abstract

This paper investigates exchange rate and reserves dynamics in a target zone where the fundamental follows a stochastic differential equation with a periodic forcing function. Solutions to the model show that the target zone plays a stabilizing role but the dynamics of the exchange rate follows a family of S-shaped functions as time evolves. This dynamics causes the width of the target zone to be a varying function of time. It becomes what we can call a flexible target zone. The monetary authorities are then forced to hold, not only large reserves, but also varying reserves in order to properly defend the target zone.

1. Introduction

The introduction of the common currency the Euro in 2002 was considered the solution to the endemic problem of currency fluctuations and their negative effects on trade and financial markets in Europe. The Euro replaced the old monetary arrangement called the European Monetary System (EMS). The EMS introduced some form of stability into currency markets in Europe by establishing a target zone which formed upper and lower limits for a currency to fluctuate around a central parity. Despite the introduction of the Euro, target zones are still important to investigate. Accession countries to the EU are required to maintain their currencies in an explicit target zone if they want to join the Euro-zone. It is estimated that the joining of the Euro-zone by the big three Poland, Hungary and the Czech Republic will occur in 2009-11. Moreover, many currencies are under an implicit target zone such as the US dollar in mid-1980's [1]. This implicit target zone may also apply to the US dollar today as many argue that the US government is conducting a policy of weak dollar.

The stabilizing property of target zones was first established by a seminal paper by Paul Krugman [2]. Krugman showed that a target zone properly defended by monetary authorities with large amount of reserves reduced the fluctuations of the exchange rate relative to the pure floating regime. The specific dynamics followed by a currency in the target zone depends, in addition to the intervention by an able central bank, on the assumptions made on the type of dynamics followed by the fundamental in the foreign exchange market. The fundamental in the Krugman model is taken to be the velocity of money variable. It is assumed that the log of the fundamental follows an arithmetic Brownian motion. Krugman model and its extensions show that for a target zone with Brownian motion money demand, two important results arise for monetary management

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and reserve adequacy conditions for preventing speculative attacks on the exchange rate. First, money supply should be adjusted as the exchange rate reaches the target band. Second, for a credible monetary policy, the central bank reserves should be larger than a fraction of domestic credit. The exact fraction is determined by the drift and variance of the Brownian motion of money demand. These results show that monetary policy implications of target zones depend on the Brownian motion assumption for the fundamental.

This assumption on the dynamics of the fundamental has been modified in other models to include different types of stochastic processes. These are motivated by the empirical shortcomings of the original currency option model by Garman and Kohlhagen [3]. Many papers have constructed models for the valuation of free-floating currency options (in non-target zones) such as jump-diffusion processes, GARCH processes and regime-switching models [4]. There is also growing evidence on the presence of deterministic dynamics in the foreign exchange market. In a recent paper, it was shown that competition in the foreign exchange markets leads to the generation of chaotic butterfly attractors [5]. Moreover, in the presence of chartists, which use trends for trading in foreign exchange markets, currencies deviate from their fundamental value and follow a distinct cyclical motion as evidenced by empirical investigations in many foreign exchange markets [6].

In addition, to such evidence on nonlinearities and determinism in the foreign exchange markets, the dynamics of the velocity of money, an important ingredient in the dynamics of exchange rates, are shown to exhibit such determinism. Such dynamics vary from procyclical dynamics relative to the business cycle where money velocity

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dynamics agrees with the Keynesian speculative demand for money [7] to chaotic deterministic motion [8]

The implications of such deterministic dynamics on monetary policy and target zone dynamics have not been investigated. This paper studies the effect on exchange rate dynamics, monetary policy and reserves dynamics in target zones when the underlying fundamental is assumed to follow a stochastic differential equation with a periodic forcing function. Many interesting results are derived. First, solutions to the model show that the dynamics of the exchange rates follows a family of S-shaped functions as time evolves, in contrast to a single S-shaped function in the Krugman model. Second, and more interesting, the model shows that the width of the target zone becomes a varying function of time or a flexible target zone where band widening becomes part of credible policy. The design of the target zone should then allow for flexibility of the limits of allowed fluctuations of the currency around the central parity. Finally, the monetary authorities are then forced to hold, not only large reserves, but also varying reserves in order to properly defend the target zone.

The paper is organized as follows. Section 2 provides a brief summary of the original Krugman model. Section 3 introduces the periodically forced SDE for the fundamental and the exchange rate equation is then derived. The exchange rate and the zone width are then shown to follow time-dependent dynamics. Section four discuses the role of reserves in the defense of the target zone. Section five concludes.

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2. The Krugman model

In this section we will review the main results of the original Krugman model. The exchange rate is determined by the following logarithmic formula

$$x = m + f + \eta \frac{\varepsilon(dx)}{dt} \qquad (1)$$

where x, m and f are the logarithms of the exchange rate, money supply and money demand shock term respectively. The last term models the expectations about the future movements of the exchange rate where η is a positive parameter. The fundamental f is assumed to follow the following stochastic differential equation (SDE)

$$df = \mu dt + \sigma dW_t \qquad (2)$$

Equations (1) and (2) with proper boundary conditions form the dynamics of the exchange rate in the target zone. The solutions show that the target zone plays a stabilizing role. The increased stability of the currency (relative to the pure floating case) is reflected in the S-shaped curve followed by the exchange rate as a function of the fundamental within the target zone. Figure 1 shows a numerical solution of the Krugman model represented by equations (1) and (2).
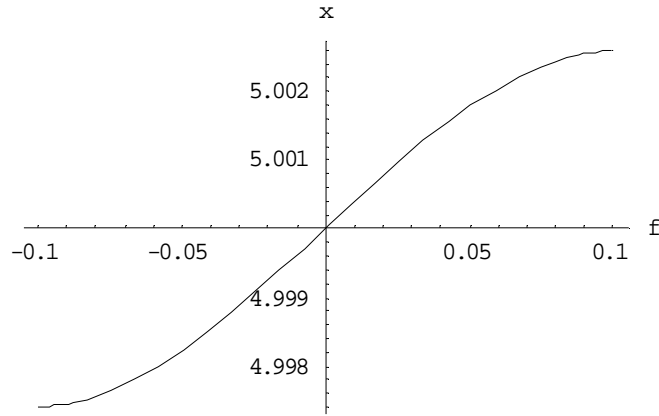
Figure 1. The S-shaped function representing the dynamics of the exchange rate, x, in the target zone. The parameters of the model are $\sigma = 0.5, m = 5$ and the boundary conditions $\frac{\partial x}{\partial f}(0.1) = \frac{\partial x}{\partial f}(-0.1) = 0.$

3. Dynamics under periodic forcing

Given the evidence on the presence of cycles and deterministic dynamics on the exchange rate and the velocity of money, the question becomes: How do we introduce such dynamics into the target zone model? The introduction of the cyclical dynamics is done here by assuming a periodic forcing function acting on the Krugman SDE. The resultant forced SDE is then used to describe the exchange rate in the target zone.

Differential equations describing the dynamics of systems can be transformed into their stochastic analogs through noise perturbation of parameters or variables [9]. The arithmetic Brownian motion process $df = \mu dt + \sigma dW$ can be derived from perturbing with white noise the simple differential equation describing linear trend dynamics

$$\frac{df}{dt} = \alpha \qquad (3)$$

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If α is perturbed by noise => α = μ + σ ε$_t$ where ε$_t$ is white noise. Hence the simple linear dynamics described by (3) is transformed into arithmetic Brownian motion given by

$$df = \mu dt + \sigma dW_t \tag{4}$$

where dW$_t$ = ε$_t$ dt is a Wiener process. Equation (4) is then the stochastic analog of (3). The stochastic analog of the simplest differential equation with periodic motion can be derived as follows. We assume that (3) is forced by a sinusoidal function sinωt then (3) becomes

$$\frac{dx}{dt} = \alpha + \beta \sin \omega t$$

In a similar fashion to the perturbation of (3), if α and β are noise perturbed so that

$$\alpha = \mu + \sigma \xi_t$$

$$\alpha = \delta + \gamma \xi_t$$

and the equation for the fundamental becomes

$$df = (\mu + \delta \sin \omega t)dt + (\sigma + \gamma \sin \omega t)dW \tag{5}$$

For simplicity, we are going to assume that the drift terms $\mu = \delta = 0$. From Ito's lemma on stochastic processes given $x = x(f)$ then

$$dx = (\sigma + \gamma \sin \omega t)\frac{\partial x}{\partial f}dW + \frac{1}{2}(\sigma + \gamma \sin \omega t)^2 \frac{\partial^2 x}{\partial f^2}dt$$

Taking the expected value of dx

$$E[dx] = \frac{1}{2}(\sigma + \gamma \sin \omega t)^2 \frac{\partial^2 x}{\partial f^2}dt$$

Inserting $E[dx]$ into (1), we get the following differential equation for x

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$$x = m + f + \frac{1}{2}(\sigma + \gamma \sin \omega t)^2 \frac{\partial^2 x}{\partial f^2} \tag{6}$$

where for simplicity $\eta = 1$. The general solution of (6) is given by

$$x = m + f + Ae^{\lambda f} + Be^{-\lambda f} \tag{7}$$

where

$$\lambda(t) = \sqrt{\frac{2}{(\sigma + \gamma \sin \omega t)^2}} \tag{8}$$

Assuming symmetry when $m=0 \Rightarrow x=f=0$ then $B=-A$. Given that condition the solution can be written as

$$x = m + f + A(e^{\lambda f} - e^{-\lambda f})$$

The boundary conditions are given by the assumed behavior of the exchange rate x as it reaches the upper (or lower) band of the target zone. At the top of the band, the exchange rate x satisfies the following "smooth pasting" conditions

$$x(\bar{f}) = \bar{x}$$

$$\frac{dx}{df}(\bar{f}) = 0$$

Hence at the top of the band,

$$\bar{x} = \bar{f} + A(e^{\lambda \bar{f}} - e^{-\lambda \bar{f}})$$

$$1 + A\lambda(e^{\lambda \bar{f}} + e^{-\lambda \bar{f}}) = 0$$

giving

$$A = -\frac{1}{\lambda(e^{-\lambda \bar{f}} + e^{\lambda \bar{f}})} \tag{9}$$

Given that λ is a function of time then the solution to (8) represent a family of S-curves that have different values at different times $\{S_t : 0 \le t \le \infty\}$. Figure 2 shows a plot of

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such S-functions for different times. The plots are generated by a numerical solution of the differential equation (6) for various times t.
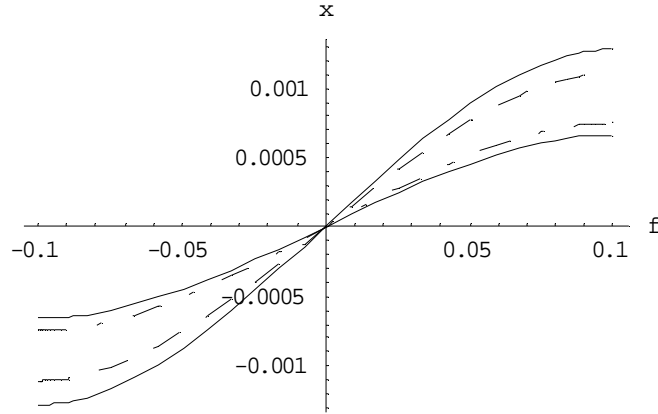

Figure 2. Different representations of the family of S-curves followed by x for different times t. The curves are drawn for t= 0, 1, 2 and 2.25 for the upper to lower S-curve respectively. The parameters are $\sigma = 0.5, \gamma = 0.5, \omega = 0.5$, and m=0. The boundary conditions are $\frac{\partial x}{\partial f}(0.1) = \frac{\partial x}{\partial f}(-0.1) = 0.$

Two interesting results emerge. First, the family of curves has the standard shape S as in the original time independent formulation. Hence, a target zone will allow the stabilization of the exchange rate. Second, as time evolves, the width of the band itself oscillates. The central bank has then, in addition to intervention to prevent x from exceeding $\bar{x}$, to widen (or reduce) periodically the bandwidth of the target zone. This will allow the target zone to be credible. The bandwidth of the target zone is given by

$$\bar{x} = \bar{f} - \frac{1}{\lambda(t)} \frac{(e^{\lambda \bar{f}} - e^{-\lambda \bar{f}})}{(e^{\lambda \bar{f}} + e^{-\lambda \bar{f}})}$$
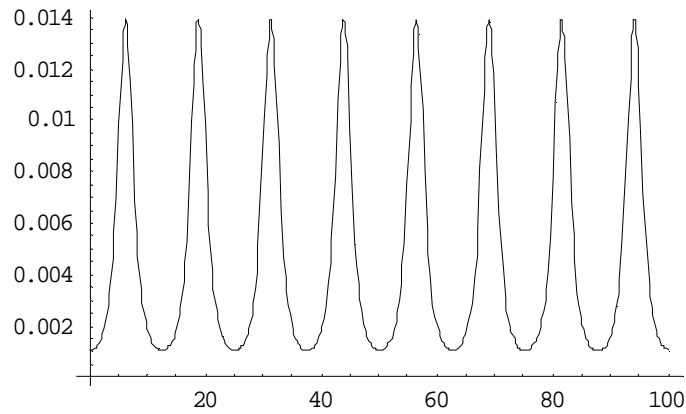
Figure 3. Bandwidth of the Target Zone as a function of time. The parameters are $\sigma = 0.5, \gamma = 0.3, \omega = 0.5, \bar{f} = 0.1, m = 0$.

4.   Speculative attacks and reserve dynamics

Given the solution to the model in section 3, it is of interest to investigate the impact of the new dynamics represented by the time-dependent S-curve and target width on the conduct of monetary policy and the necessary reserve requirement for maintaining a stable target zone. The effect on reserves is particularly important as foreign exchange reserves play an important role as shock absorbers in target zones or in any exchange rate regime. A proper assessment of reserve needs for the defense of the target zone becomes crucial. If the central bank holds too little reserves given exchange rate dynamics then the target zone will collapse. Different models have concentrated on the analysis of reserves in target zones. A Markovian model of the target zone shows that that the increase in reserves causes an increase in the probability of the target zone being maintained and decreases hence the probability of its collapse. Many factors contribute to the amount of reserves needed for the defense of the target zone. The threshold ratio defined as the ratio of reserves to domestic credit plus reserves is a decreasing function of the drift and volatility of the exchange rate [10]. This relationship

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between the dynamics of the exchange rate and the ability of authorities to maintain the target zone is also investigated in [11] which shows that the ERM crisis of 1992 which forced Britain and Italy to abandon it can be explained by assuming the existence of an additional feedback between the exchange rate and the dynamics of the fundamentals. Also uncertainty about the amount of reserves available to monetary authorities may have an effect on target zone sustainability. In [12], it is shown that the actual amount of reserves needed for the defense of the zones in the presence of uncertainty is less than that with no uncertainty about the state of the reserves. Moreover, if policy makers have sufficiently weak credibility then the target zone may add to the volatility of the exchange rate producing what is called an inverted S-curve [13].

The design of monetary policy in a target zone model depends on the dynamics of the pre attack exchange rate within the target band. This is given by

$$x = m + f + Ae^{\lambda(t)f} \tag{10}$$

where A<0. Suppose that monetary authorities are willing to buy and sell foreign exchange in order to defend the target zone. The monetary authorities have at their disposal a certain amount of foreign exchange reserves. In the following we will assume that the reserves are large enough to defend the currency against speculative attack. The initial money supply in the economy is given by the sum of reserves and domestic credit

$$m = \ln(R + D)$$

where R=reserves and D=domestic credit. We know that if reserves are large then the central bank is able to ensure the smooth pasting result discussed in the previous section. The critical level of reserves needed for proper defense of the target zone is determined by deriving the amount of money loss generated by a speculative attack on the currency. Again to ensure smooth pasting we know that

$$\frac{\partial x}{\partial f} = 1 + \lambda(t)Ae^{\lambda(t)\bar{f}} = 0 \tag{11}$$

Moreover at $\bar{f}$, the exchange rate is

$$\bar{x} = m + \bar{f} + Ae^{\lambda(t)\bar{f}} \tag{12}$$

Combing (11) and (12), we get the following value of the exchange rate

$$\bar{x} = m + f - \frac{1}{\lambda(t)}$$

However, if the speculative attack was successful in collapsing the targets zone then the exchange rate will be free floating and given by

$$\bar{x} = \tilde{m} + f - \frac{1}{\lambda(t)}$$

where $\tilde{m}$ =money supply in the post-attack state. Hence, at maximum speculative attack, the money supply change will be

$$\tilde{m} - m = -\frac{1}{\lambda(t)} \tag{13}$$

Now, given the relationship between the money supply and reserves then after simple calculations, we get the relationship between the pre and post-attack money supply

$$\tilde{m} - m = -\ln(1 + \frac{R}{D}) \tag{14}$$

Equating (13) and (14) we get the limiting conditions on the amount of reserves needed to defend the target zone

$$\frac{R}{D} \geq e^{1/\lambda(t)} - 1$$

Now by substituting the value for $\lambda(t)$ we get

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$$\frac{R}{D} \geq e^{\sqrt{\dfrac{1}{\dfrac{2}{\eta(\sigma+\gamma\sin\omega t)^2}}}} - 1$$

or the lower limit of reserves is given by

$$R \approx (e^{\sqrt{\dfrac{1}{\dfrac{2}{\eta(\sigma+\gamma\sin\omega t)^2}}}} - 1).D \qquad (15)$$
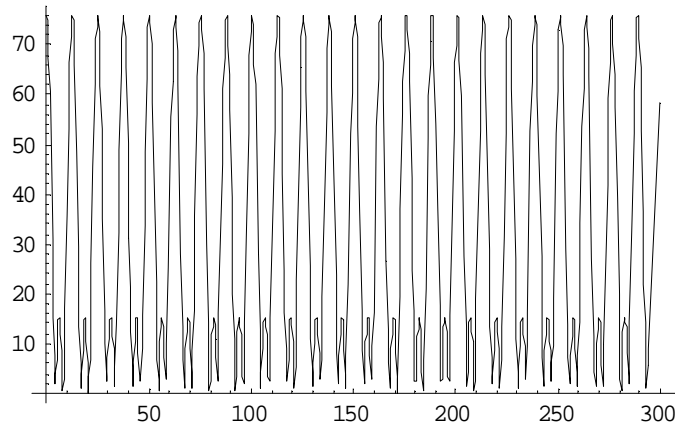

Figure 4. Reserves needed for a credible defense of the target zone as a function of time. Parameters are $\sigma = 0.3, \gamma = 0.5, w = 0.5, D = 10.$

This result shows that the monetary authorities must hold different levels of reserves at the different points in time in order to ensure an adequate level of reserves that prevents a speculative attack on the currency. The reserve fluctuations amplitude and frequency are determined by frequency of the periodic forcing w and the volatilities σ and γ.

5. Concluding remarks

This paper has shown that target zone dynamics under periodic forcing of the SDE of the fundamental have important implications for target zone design and the conduct of

monetary policy. The target zone remains a stabilizing instrument even under noisy cyclical exchange rates as evidenced by the S-shaped dynamics of the currency in the target zone. However, the target zone must be flexible with the band width variable with time. Moreover, the reserves needed to defend the currency must vary. This will allow the monetary authorities to hold optimal amounts of reserves in every time period. The results also pave the way for further research in this area that could incorporate more complex dynamics such as chaotic motion in exchange rates [14] and how to control chaos within a target zone. Other interesting extension is to investigate if target zones can defend the currency in the presence of log-periodic crash dynamics as shown by Sornette to exist for currencies in crisis [15].

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