

Esscher transforms and martingale measures in incomplete diffusion models

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Abstract

The minimal entropy and minimal martingale measures are shown to be related by an Esscher transform, involving the mean-variance trade-off, in an incomplete diffusion model containing a traded stock and a correlated non-traded stochastic factor. The coefficients of the diffusions are measurable with respect to the Brownian motion driving the non-traded factor, as is typical in stochastic volatility models. The result is motivated by an analysis of exponential indifference prices, and made rigorous by appealing to a representation equation for the q -optimal measure due to Hobson [14]. The result yields a new representation for the marginal price of a claim on the non-traded factor.

1 Introduction

In this note we relate the minimal martingale measure Q^M and the minimal entropy martingale measure Q^E in an incomplete Markovian model, using an Esscher transform [7]. The result is first motivated by a formal analysis of exponential indifference prices of a claim on the non-traded factor Y . The rigorous proof is obtained from a representation equation of Hobson [14] for the q -optimal measure $Q^{(q)}$, related to Q^M, Q^E by $Q^M = Q^{(0)}, Q^E = Q^{(1)}$ [15].

The model comprises a stock S whose logarithmic return is a diffusion with coefficients dependent on a correlated non-traded stochastic factor Y , as in Zariphopoulou [27]. Denote by W the Brownian motion driving Y , and let $\mathbb{G} := (\mathcal{G}_t)_{0 \leq t \leq T}$ denote the filtration generated by W . Let B denote the Brownian motion driving the traded asset

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S , and let $\rho \in [-1, 1]$ denote the correlation between B and W . The crucial feature of the model from our point of view is the following.

Property 1 *The correlation ρ is constant and the processes $\log S, Y$ are diffusions with coefficients adapted to \mathbb{G} .*

In particular, this implies that the so-called mean-variance trade-off process [18], K , is \mathbb{G} -adapted.

Let $\widehat{Q}^E, \widehat{Q}^M$ denote the projections of Q^E, Q^M onto \mathcal{G}_T . The main result (Theorem 1) is a characterization of the density of \widehat{Q}^E with respect to \widehat{Q}^M as an Esscher transform involving the mean-variance trade-off process K and a constant depending on the correlation ρ .

Zariphopoulou [27] has analyzed optimal investment under power utility in the model studied here, and Henderson [13] and Zariphopoulou and co-authors [16, 24, 25] have analyzed exponential valuation of claims in similar models. Monoyios [15] analyzes the dual to a primal portfolio problem across different preferences in the diffusion model studied here, yielding representations for the dual optimizer Q^* (which is the q -optimal measure $Q^{(q)}$ for $q \in \mathbb{R}$ depending on the utility function).

The dynamic programming solution of utility maximization problems in such models follows a technique called *distortion*. The Hamilton-Jacobi-Bellman (HJB) equation for the value function is solved by separating out dependence on initial wealth x and then applying a power transformation to the remaining function. For power or exponential utility the value function $u(x)$ has the form

$$u(x) = U(x)F^\delta,$$

where U is the utility function and F is a function of time and the initial value of Y . The value of the distortion power δ can be chosen to obtain a linear PDE for F and, by the Feynman-Kac Theorem, an expectation representation for F . With exponential utility, this expectation is taken with respect to \widehat{Q}^M .

Tehranchi [26] has extended the distortion solution to a non-Markovian scenario when the factors follow correlated Itô processes. This suggests that the results here may extend to that scenario, and this a topic for future research.

We use the distortion method under exponential utility, and with a random terminal endowment involving European claims on Y , to formally obtain a formula for Davis' [3] marginal indifference price \widehat{p} for the claim. This heuristic argument is given in Section 3. Theorem 1 is motivated by equating this representation with the classical representation for \widehat{p} as the expectation of the payoff under the optimal dual martingale measure Q^* . As is well-known, for exponential utility $Q^* = Q^E$ (see Delbaen *et al* [5], for example, and Becherer [1] for properties of exponential indifference prices in a general semimartingale setting).

For $q \geq 1$ Hobson [14] has derived a martingale representation identity for the q -optimal measure in (not necessarily Markovian) models similar to ours, extending a result of Rheinländer for the case $q = 1$, and Monoyios [15] extends this result to $q < 1$. We use Hobson's result to rigorously prove Theorem 1. Under the hypothesis

of Property 1, the Hobson representation equation is a relation between \mathcal{G}_T -measurable random variables. Property 1 is restrictive, but there are still many interesting financial applications that fit into this framework, and understanding the relationship between martingale measures is a fruitful exercise, yielding an interpretation of the distortion solution as a by-product.

The rest of the paper is as follows. Section 2 describes the model and gives the main result. Using exponential utility, Section 3 states the distortion solution for the primal optimization problem of an agent with random endowment of claims on Y , and gives a heuristic derivation of a formula for the marginal price \hat{p} of the claim. Section 4 gives a rigorous proof of Theorem 1, based on the particular form of Hobson's [14] representation equation in this model, and rigorously obtains the formula of Section 3 for \hat{p} , valid for any \mathcal{G}_T -measurable claim, as an immediate consequence. Section 5 concludes.

2 A Markov model with unhedgeable risk

A traded asset $S := (S_t)_{0 \leq t \leq T}$ and a non-traded stochastic factor $Y := (Y_t)_{0 \leq t \leq T}$ follow

$$dS_t = \sigma(t, Y_t)S_t(\lambda(t, Y_t)dt + dB_t), \quad (1)$$

$$dY_t = a(t, Y_t)dt + b(t, Y_t)dW_t, \quad (2)$$

subject to initial conditions, under the physical measure P . The Brownian motions B and W have constant correlation $\rho \in [-1, 1]$. We write

$$W_t = \rho B_t + \bar{\rho} Z_t,$$

with $\bar{\rho} = \sqrt{1 - \rho^2}$, and $(B, Z) := (B_t, Z_t)_{0 \leq t \leq T}$ a two-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, with \mathbb{F} generated by (B, Z) . Denote by $\mathbb{G} := (\mathcal{G}_t)_{0 \leq t \leq T}$ the filtration generated by W .

The parameter functions λ, σ, a, b are such that unique strong solutions to the stochastic differential equations (1,2) exist. We make the following assumption throughout.

Assumption 1 *The coefficients λ, a, b are $C^{1,2}([0, T] \times \mathbb{R})$ functions satisfying, uniformly in t , $|f(t, y)| \leq C(1 + |y|)$ for $f = \lambda, a, b$ and a positive constant C . The volatility coefficient $\sigma(t, y)$ satisfies $\sigma(y) \geq \ell > 0$ for some positive constant ℓ and $(t, y) \in ([0, T] \times \mathbb{R})$. The diffusion coefficient b is uniformly elliptic: $\exists \epsilon > 0 : b^2(t, y) \geq \epsilon y^2, \forall y \in \mathbb{R}, t \in [0, T]$.*

The interest rate is zero, or equivalently S represents a discounted price. This entails no loss of generality if the interest rate is deterministic or depends only on Y and on time. The crucial feature of (1,2) for our purposes is that λ, a, b are progressively \mathbb{G} -measurable. In principle, therefore, one could allow σ to be \mathbb{F} -adapted.

The class \mathcal{M} of equivalent local martingale measures consists of measures $Q \sim P$ on \mathcal{F}_T with densities given by

$$\frac{dQ}{dP} = \mathcal{E}(-\lambda \cdot B - \psi \cdot Z)_T,$$

where \mathcal{E} is the Doléans exponential and $\psi := (\psi_t)_{0 \leq t \leq T}$ is an \mathbb{F} -adapted process satisfying $\int_0^T \psi_t^2 dt < \infty$ a.s. The so-called mean-variance trade-off process is the increasing process $K := (K_t)_{0 \leq t \leq T}$ given by $K_t := \int_0^t \lambda^2(u, Y_u) du < \infty$, P -a.s., with the finiteness property true by assumption. We assume $E(dQ/dP) = 1$ so that Q is a probability measure equivalent to P on \mathcal{F}_T . A sufficient condition for this to be true is the Novikov condition

$$E \exp \left(\frac{1}{2} \left(K_T + \int_0^T \psi_t^2 dt \right) \right) < \infty.$$

Under $Q \in \mathcal{M}$, $dS_t = \sigma(t, Y_t) S_t dB_t^Q$ and Y satisfies

$$dY_t = [a(t, Y_t) - b(t, Y_t)(\rho\lambda(t, Y_t) + \bar{\rho}\psi_t)] dt + b(t, Y_t) dW_t^Q,$$

where $W_t^Q = \rho B_t^Q + \bar{\rho} Z_t^Q$ and $(B^Q, Z^Q) := (B_t^Q, Z_t^Q)_{0 \leq t \leq T}$ is a two-dimensional Q -Brownian motion defined by $B_t^Q := B_t + \int_0^t \lambda(u, Y_u) du$ and

$$Z_t^Q := Z_t + \int_0^t \psi_u du. \quad (3)$$

Naturally, the traded asset S is a local Q -martingale, while the Q -drift of Y is arbitrary and parametrized by the integrand ψ in (3). The space \mathcal{M} is in one-to-one correspondence with the set of integrands ψ provided $E(dQ/dP) = 1$, and we write $Q = Q^\psi$ whenever we need to emphasize dependence on ψ .

The minimal martingale measure is $Q^M = Q^0$ corresponding to $\psi_t = 0, 0 \leq t \leq T$ in (3). Originally defined by Föllmer and Schweizer [8] in a quadratic hedging context, Q^M has subsequently appeared naturally in many other situations; see Schweizer [23] for more details. For models with continuous price trajectories, the most general characterization of Q^M is due to Schweizer [22], who shows that Q^M minimizes the reverse relative entropy $H(P, Q)$ over all $Q \in \mathcal{M}$, which means that Q^M is the q -optimal measure for $q = 0$ [15].

The relative entropy $H(Q, P)$ of $Q \in \mathcal{M}$ with respect to P is

$$H(Q, P) := \begin{cases} E \left(\frac{dQ}{dP} \log \frac{dQ}{dP} \right), & \text{if } Q \ll P \text{ on } \mathcal{F}_T, \\ +\infty, & \text{otherwise.} \end{cases}$$

The minimal entropy martingale measure Q^E is defined by

$$Q^E := \arg \min_{Q \in \mathcal{M}} H(Q, P).$$

There are close links between relative entropy and hedging under exponential utility, as espoused in [5, 9, 19, 20] for example, and Q^E is the q -optimal measure for $q = 1$ [11, 14, 15].

Let $\widehat{Q}^E, \widehat{Q}^M$ denote the projections of Q^E, Q^M onto the sigma-algebra \mathcal{G}_T :

$$\begin{aligned} \frac{d\widehat{Q}^E}{dP} &:= E \left[\frac{dQ^E}{dP} \middle| \mathcal{G}_T \right], \\ \frac{d\widehat{Q}^M}{dP} &:= E \left[\frac{dQ^M}{dP} \middle| \mathcal{G}_T \right], \end{aligned}$$

satisfying, for $i = E, M$, $\widehat{Q}^i(A) = Q^i(A), \forall A \in \mathcal{G}_T$, implying $E^{\widehat{Q}^i} G = E^{Q^i} G$ for any \mathcal{G}_T -measurable random variable G for which the expectations exist.

Our main result is an Esscher transform relation [7] between \widehat{Q}^E and \widehat{Q}^M , involving the mean-variance trade-off process K and a constant depending on the correlation ρ . The proof is deferred to Section 4.

Theorem 1 *In the model described by (1,2) the projections $\widehat{Q}^E, \widehat{Q}^M$ of the minimal entropy measure Q^E and the minimal martingale measure Q^M onto \mathcal{G}_T are related by the Esscher transform*

$$\frac{d\widehat{Q}^E}{d\widehat{Q}^M} = \frac{\exp(\theta K_T)}{E^{\widehat{Q}^M} \exp(\theta K_T)},$$

where

$$\theta = -\frac{1}{2} (1 - \rho^2),$$

and K_T is the mean-variance trade-off at T .

Esscher transforms have a long history in actuarial pricing, and have been used by some authors to define a possible pricing measure in incomplete markets. See, for example [2, 10, 6].

3 Exponential indifference pricing

We give a heuristic derivation of a formula for the marginal price of a claim on Y , motivating the result in Theorem 1.

Suppose a European option on asset Y pays $h(Y_T)$. An agent trades a self-financing portfolio involving the traded asset S . The portfolio wealth process X satisfies

$$dX_t = \sigma(t, Y_t) \pi_t (\lambda(t, Y_t) dt + dB_t),$$

where π_t is the wealth invested in the traded asset S at time $t \in [0, T]$. The agent has preferences described by the exponential utility function

$$U(x) = -\exp(-\gamma x), \quad x \in \mathbb{R}, \quad \gamma > 0. \quad (4)$$

The objective is to maximize expected utility of terminal wealth at time T , with an additional random endowment of n units of the claim payoff:

$$J(t, x, y; \pi) = \mathbb{E}[U(X_T + nh(Y_T)) | X_t = x, Y_t = y],$$

where we assume $nh(Y_T)$ is bounded below. The agent's primal value function is

$$u(t, x, y) := \sup_{\pi \in \mathcal{A}} J(t, x, y; \pi), \quad (5)$$

with $u(T, x, y) = U(x + nh(y))$. We write $u^{(n)}(t, x, y) \equiv u(t, x, y)$ when we need to emphasize dependence on n . We denote the set of admissible trading strategies by \mathcal{A} . A

trading strategy is an adapted process $\pi := (\pi_t)_{0 \leq t \leq T}$ satisfying $\int_0^T \sigma^2(t, Y_t) \pi_t^2 dt < \infty$ almost surely. When wealth can become negative the definition of admissibility is subtle, as discussed by Schachermayer [21]. We make the following definitions, along the lines of [4, 21, 17].

$$\begin{aligned} \mathcal{A}_b &= \{ \pi \in \mathcal{A}_0 : X_t \geq a \in \mathbb{R} \text{ a.s. } \forall t \in [0, T] \}, \\ \mathcal{U}_b &= \{ \Gamma \in L^0(\Omega, \mathcal{F}_T, P) : \Gamma \leq X_T + nh(Y_T) \text{ for } \pi \in \mathcal{A}_b \text{ and } E|U(\Gamma)| < \infty \} \\ \mathcal{U} &= \{ U(\Gamma) : \Gamma \in \mathcal{U}_b \}^c \\ \mathcal{A} &= \{ \pi \in \mathcal{A}_0 : U(X_T) \in \mathcal{U} \}, \end{aligned}$$

where $\{\dots\}$ denotes the closure in $L^1(\Omega, \mathcal{F}_T, P)$. The point is that we first bound the portfolio wealth from below, to eliminate doubling strategies [12], but the resulting class \mathcal{A}_b is not big enough to guarantee finding the optimal strategy by searching only within it, so this class is suitably enlarged. See Schachermayer [21] or Owen [17] for more details. This subtlety will not overly concern us here, as our goal is to formally use the solution to the optimization problem to heuristically motivate subsequent results.

The *indifference price* $p(t, x, y) \equiv p^{(n)}(t, x, y)$ per claim, for a random endowment of n claims, is defined by

$$u^{(n)}(t, x - np^{(n)}(t, x, y), y) = u^{(0)}(t, x, y). \quad (6)$$

The *marginal price* $\hat{p}(t, x, y)$ of the claim corresponds to a price which essentially solves (6) as $n \rightarrow 0$. The original definition of Davis [3] used a “zero marginal rate of substitution” argument and subsequent papers, for example [1], have shown that it arises as the $n \rightarrow 0$ limit (alternatively, the $\gamma \rightarrow 0$ limit) of $p^{(n)}$:

$$\hat{p}(t, x, y) = \lim_{n \rightarrow 0} p^{(n)}(t, x, y). \quad (7)$$

With exponential utility, provided the payoff satisfies suitable integrability conditions, \hat{p} is also given by the Q^E -expectation of the payoff:

$$\hat{p} = E^{Q^E} h(Y_T). \quad (8)$$

Conditions on $h(Y_T)$ for validity of (8) are discussed by Becherer [1], and amount to the payoff having an exponential moment. For exponential utility, it is well known that $p^{(n)}$ and \hat{p} do not depend on x .

3.1 Distortion power solution

Using the well-known distortion method [27, 25, 13, 26] we obtain a closed form expectation representation for the value function $u^{(n)}(t, x, y)$. We merely state the result, as the distortion method is well established. Our goal in this section is to use the solution to heuristically derive a representation for the marginal price of the claim on Y . This will be equated with the well-known representation for \hat{p} as the Q^E -expectation of the payoff $h(Y_T)$. A rigorous analysis verifying the regularity and optimality of the distortion solution follows the same lines as [24, 25, 27].

Proposition 1 (Distortion power solution) *With exponential utility (4), the value function $u \equiv u^{(n)}$ in (5) is given by*

$$u(t, x, y) = U(x) (F(t, y))^{1/(1-\rho^2)}, \quad (9)$$

where the function $F \equiv F^{(n)} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^+$ has the stochastic representation

$$F^{(n)}(t, y) = E^{\widehat{Q}^M} \left[\exp \left(-(1-\rho^2) \left(\frac{1}{2} \int_t^T \lambda^2(u, Y_u) du + \gamma n h(Y_T) \right) \right) \middle| Y_t = y \right]. \quad (10)$$

Remark 1 *The expectation in (10) is taken with respect to the projection of the minimal measure Q^M onto \mathcal{G}_T . We can replace \widehat{Q}^M by Q^M in (10), since these measures give the same moments for \mathcal{G}_T -measurable random variables.*

3.2 Indifference price formulae

Using Proposition 1 and the definition (6) of the indifference price per claim gives the following formula for $p^{(n)}$:

$$p^{(n)}(t, y) = -\frac{1}{\gamma(1-\rho^2)n} \log \left(\frac{F^{(n)}(t, y)}{F^{(0)}(t, y)} \right), \quad (11)$$

with $F^{(n)}(t, y)$ given by (10).

We use a formal perturbative analysis of the formula for $p^{(n)} \equiv p^{(n)}(0, y)$ to obtain a formula for the marginal price $\widehat{p} \equiv \widehat{p}(0, y)$. Using (10) at $t = 0$ along with a formal Taylor series expansion applied to the exponential in (10) and the logarithm in (11) gives the indifference price $p^{(n)}$ as

$$p^{(n)} = \frac{E^{\widehat{Q}^M} [\exp(\theta K_T) h(Y_T)]}{E^{\widehat{Q}^M} \exp(\theta K_T)} + O(n),$$

where $O(n)$ denotes terms involving n and higher powers of n . Using (7) the marginal price \widehat{p} is obtained on taking the limit $n \rightarrow 0$, as

$$\widehat{p} = \frac{E^{\widehat{Q}^M} [\exp(\theta K_T) h(Y_T)]}{E^{\widehat{Q}^M} \exp(\theta K_T)}. \quad (12)$$

The derivation of (12) is purely formal, and will be made rigorous by other methods in the next section. Its value to us is that when equated with the classical representation (8) for \widehat{p} , it motivates a relation between Q^M and Q^E (or, more precisely, between their projections onto the sigma field generated by Y_T). Equating (12) with (8) gives the statement of Theorem 1 for the projections of Q^E, Q^M onto the sigma field generated by Y_T .

4 Proofs

We prove Theorem 1 from the representation equation of Hobson [14] for the q -optimal measure $Q^{(q)}$. For $q \in \mathbb{R} \setminus \{0, 1\}$, $Q^{(q)}$ minimizes (maximizes, for $0 < q < 1$) the L^q norm $E[(dQ/dP)^q]$ over $Q \in \mathcal{M}$. For $q = 1$, $Q^{(1)} = Q^E$, minimizing the relative entropy $H(Q, P)$ between Q and P , and for $q = 0$, $Q^{(0)} = Q^M$, minimizing the reverse relative entropy $H(P, Q)$ (see [15] for instance). We need a couple of lemmas specifying the solution to Hobson's representation equation in our model. Write

$$\frac{dQ^{(q)}}{dP} = \mathcal{E}(-\lambda \cdot B - \psi^* \cdot Z)_T, \quad (13)$$

for some optimal dual process ψ^* , in general \mathbb{F} -adapted.

In the Markovian model of this paper, it turns out that ψ^* is given by a \mathbb{G} -adapted process related to the distortion function $F^{(0)}(t, Y_t)$ in (10) for $n = 0$. The following lemma is proved in [15] for general q (we give the version for $q = 1$), from an analysis of the HJB equation for the dual problem, and may also be derived from the results in [14].

Lemma 1 *Under the conditions of Assumption 1, the process ψ^* in (13) is given by*

$$\psi_t^* = -\frac{b(t, Y_t)}{\bar{\rho}} \frac{\partial}{\partial y} \log F^{(0)}(t, Y_t), \quad t \in [0, T], \quad (14)$$

where $F^{(0)}(t, y)$ is the function (10) in the distortion solution (9) for $n = 0$.

For $q = 1$, and in the Markovian model (1,2), Hobson's representation equation is as follows.

Lemma 2 *Define processes $(N, L) = (N_t, L_t)_{0 \leq t \leq T}$ by*

$$\begin{aligned} N_t &:= \int_0^t \left(\frac{\rho}{\bar{\rho}} \right) \psi_u^* (dB_u + \lambda(u, Y_u) du), \\ L_t &:= \int_0^t \psi_u^* dZ_u, \end{aligned}$$

with ψ^* given by (14). Then N_T, L_T satisfy

$$\frac{1}{2} K_T = N_T + L_T + \frac{1}{2} [L]_T - \frac{1}{\bar{\rho}^2} \log F^{(0)}, \quad (15)$$

where $F^{(0)} \equiv F^{(0)}(0, y)$ is defined in (10).

Proof This is proved in Monoyios [15]. It follows from substituting (14) in (15) and using the PDE for $F^{(0)}(t, y)$, obtained from (10) and the Feynman-Kac Theorem, to show that (15) is satisfied. □

Proof of Theorem 1 Define a Brownian motion W^\perp independent of the \mathbb{G} -adapted Brownian motion W , via

$$\begin{aligned} W &:= \rho B + \bar{\rho} Z, \\ W^\perp &:= \bar{\rho} B - \rho Z. \end{aligned}$$

Re-write (13) for $q = 1$ in terms of W, W^\perp , to obtain

$$\frac{dQ^E}{dP} = \mathcal{E} \left[-((\rho\lambda + \bar{\rho}\psi^*) \cdot W) - ((\bar{\rho}\lambda - \rho\psi^*) \cdot W^\perp) \right]_T.$$

The processes in the stochastic exponential are \mathbb{G} -adapted, except for the Brownian motion W^\perp , which is independent of \mathbb{G} . We condition on \mathcal{G}_T to obtain

$$\frac{d\widehat{Q}^E}{dP} = \exp \left\{ -((\rho\lambda + \bar{\rho}\psi^*) \cdot W)_T - \frac{1}{2} \int_0^T (\rho\lambda_t + \bar{\rho}\psi_t^*)^2 dt \right\}. \quad (16)$$

Similarly, from

$$\frac{dQ^M}{dP} = \mathcal{E}[-(\lambda \cdot B)]_T,$$

we obtain

$$\frac{d\widehat{Q}^M}{dP} = \exp \left(-\rho(\lambda \cdot W)_T - \frac{1}{2} \rho^2 K_T \right),$$

which is (16) with ψ^* set to zero. Combining these results we obtain

$$\frac{d\widehat{Q}^E}{d\widehat{Q}^M} = \exp \left\{ -((\rho\lambda + \bar{\rho}\psi^*) \cdot W)_T - \frac{1}{2} \bar{\rho} \int_0^T (\bar{\rho}\psi_t^* + 2\rho\lambda(t, Y_t)) dt \right\}. \quad (17)$$

Now write the representation equation (15) in terms of (W, W^\perp) to obtain

$$(\psi^* \cdot W)_T = \frac{1}{2} \left(\int_0^T \psi_t^* (\bar{\rho}\psi_t^* - 2\rho\lambda(t, Y_t)) dt - \bar{\rho} K_T \right) - \frac{1}{\bar{\rho}} \log F^{(0)}.$$

Note that this contains only \mathcal{G}_T -measurable terms. Use this equation to eliminate the integral $(\psi^* \cdot W)_T$ from (17), and the result follows on recalling that, from (10), $F^{(0)} = E^{\widehat{Q}^M} \exp(\theta K_T)$.

□

Property 1 means that the relevant connection between the measures Q^E, Q^M is expressed in terms of their projections onto the sigma-algebra \mathcal{G}_T . The result may be recast into the equivalent form

$$E^{Q^M} \left[\frac{dQ^E}{dQ^M} \middle| \mathcal{G}_T \right] = \frac{\exp(\theta K_T)}{E^{Q^M} \exp(\theta K_T)}.$$

An immediate corollary of Theorem 1 is the following representation for the marginal price of a European claim on Y .

Corollary 1 *The marginal price of a \mathcal{G}_T -measurable claim G has the representation*

$$\widehat{p} = \frac{E^{\widehat{Q}^M} [\exp(\theta K_T) G]}{E^{\widehat{Q}^M} [\exp(\theta K_T)]}.$$

Proof The result is immediate on writing $\hat{p} = E^{Q^E} G = E^{\hat{Q}^E} G$ and using Theorem 1 to express the price as a \hat{Q}^M -expectation. □

Naturally, the expectations in Corollary 1 may also be written with respect to Q^M .

5 Conclusion

The crucial feature of the model studied here is the fact that the mean-variance trade-off K and the non-traded factor Y are progressively measurable with respect to \mathbb{G} , the sigma-field generated by the Brownian motion W driving Y . This has enabled us to derive an Esscher transform relation between the projections of Q^E, Q^M onto \mathcal{G}_T . The result is obtained from the Hobson-Rheinländer representation equation (15) for the q -optimal measure, along with the fact that in the Markovian scenario studied here, the process ψ^* solving (15) is expressed in terms of the distortion function $F(t, y)$, so that ψ^* is progressively \mathbb{G} -measurable. The PDE satisfied by F plays a role in showing that (15) is indeed solved by ψ^* given in (14). It may be possible to relax the Markovian assumption, and recent work by Tehranchi [26] showing that the distortion solution is valid in a non-Markovian setup, lends weight to this conjecture. This would require probabilistic methods.

A further obvious question that arises is whether similar results extend to the q -optimal measure for $q \neq 1$. Using similar methods to those for $q = 1$, we have been able to obtain the following result for general q . Let $Q^{(q)}$ denote the q -optimal measure, with $Q^M = Q^{(0)}$. Then we have:

$$\left\{ E^{P^M} \left[\left(\frac{dQ^{(q)}}{dQ^{(0)}} \right)^q \middle| \mathcal{G}_T \right] \right\}^\Delta = \frac{\exp(\theta K_T)}{E^{P^M} \exp(\theta K_T)},$$

where $\theta = -\frac{1}{2}q(1 - q\rho^2)$ and $\Delta = (1 - q\rho^2)/q\bar{\rho}^2$. This result reduces to Theorem 1 for $q = 1$, but is clearly not as sharp as the main theorem.

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