Characterization of optimal dual measures via distortion

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Abstract

We derive representations for the optimal dual martingale measure Q^* associated with the dual to a primal utility maximization problem in a class of incomplete diffusion models containing a traded stock and a non-traded correlated stochastic factor. Using a distortion power solution [29] for the primal problem, an explicit solution is obtained for the dual value function, yielding representations for the dual measure and a novel solution for a dual stochastic control problem. The optimal measure is recast as the q-optimal measure $Q^{(q)}$, with $q \leq 1$ for HARA and CARA utilities, which we treat in a unified manner using the parameter q. We extend the Hobson [13] representation equation for $Q^{(q)}$ to $q < 1$. A similar program is applied to a problem with random endowment under exponential preferences, yielding representations for indifference prices and the dual minimizer, and an extension of the Hobson representation equation to the problem with random endowment.

1 Summary

This paper derives representations for optimal dual martingale measures in a class of incomplete Markovian models, of the type analyzed in Zariphopoulou [29]. The optimal measure Q^* is associated with the the dual to a primal utility maximization problem, for which explicit solutions are available via a PDE technique called *distortion*.

The market comprises a stock S and a non-traded risk factor Y, the source of incompleteness in the model. The processes S, Y are diffusions on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, driven by Brownian motions B, W with fixed correlation ρ. Let $\mathbb{G} := (\mathcal{G}_t)_{0 \leq t \leq T}$ denote the filtration generated by the Brownian motion W

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driving Y. Crucially, the coefficients of the stochastic differential equations for $\log S, Y$ are progressively $\mathbb{G}$-measurable.

Let X denote the wealth process associated with a self-financing portfolio strategy π over $[0,T]$, with $X_0 = x$, and let $U(\cdot)$ be a power or exponential utility function. In the setting of this paper, the maximal utility

$$u(x) := \sup_{\pi \in \mathcal{A}} E\left[U(X_T)\right], \tag{1}$$

where $\mathcal{A}$ denotes some class of admissible strategies, has a representation as a *distortion power solution* [29, 28] of the form

$$u(x) = U(x)F^{\delta} = U(x)\left(E^{\widehat{P}^M}\xi^{1/\delta}\right)^{\delta}. \tag{2}$$

In (2), δ is called the distortion power and F is a function which solves a linear parabolic PDE for suitable choice of δ. By the Feynman-Kac formula F has an expectation representation, in terms of some $\mathcal{G}_T$-measurable random variable ξ and a measure $\widehat{P}^M$ under which Y has modified drift.

Numerous papers [29, 18, 11, 27, 16] have established and applied (2) in similar settings to ours, and Tehranchi [28] has extended the result to non-Markovian Itô dynamics. Some of our results may extend to the non-Markovian case and this is a line of possible future research, which would require probabilistic techniques as opposed to the PDE methods that we utilise.

Our program is to investigate the ramifications of (2) for the solution to the dual problem. Let $\mathcal{M}$ denote the set of local martingale measures Q for S. The dual problem to (1) has value function v defined by

$$v(\eta) := \inf_{Q \in \mathcal{M}} EV\left(\eta \frac{dQ}{dP}\right), \tag{3}$$

where $V(\eta) := \sup_{x \in \mathrm{dom}(U)}\left(U(x) - x\eta\right)$ is the convex conjugate of $U(\cdot)$. As a method for solving (1) the dual approach has proved very powerful in abstract settings with minimal assumptions on the price processes, culminating in the work of Kramkov and Schachermayer [15, 23]. Our emphasis is on using the explicit solution (2) and the well-established conjugacy of the value functions u, v, to relate the optimizer Q^* in (3) to $\widehat{P}^M, F, \delta, \xi$ in (2).

To describe the relationship between Q^* and (2) we re-parametrize the dual problem (3) in two ways. First, we recast the dual minimizer Q^* as the *q-optimal measure* $Q^{(q)}$ [13]. For $q \in \mathbb{R}$, with a value dependent on the choice of utility function, $Q^{(q)}$ minimizes a convex statistical distance between Q and P. The parameter q provides a unified framework for describing different optimal measures. For $q \geq 1$, Hobson [13] has shown that determining $Q^{(q)}$ can be reduced to finding a solution to a martingale representation equation, extending results of Rheinländer [21] for the case $q = 1$. It is conjectured that this representation equation is valid for all $q \in \mathbb{R}$. Our analysis leads first to simple relations between q and δ ($\delta = 1/(1 - q\rho^2)$), and between the optimal distance $H_q(Q^{(q)}, P)$ and F (Proposition 2), and we show that Hobson's representation equation extends to $q < 1$, re-casting it into the simple form $\int_0^T d\left(\log F(t, Y_t)\right) = c/\delta$, where c is a constant.

Second, we recast (3) as a *stochastic control problem* over control processes ψ in the Girsanov exponential for dQ/dP. For $Q \in \mathcal{M}$, dQ/dP takes the form

$$\frac{dQ}{dP} = \mathcal{E}\left(-(\lambda \cdot B) - (\psi \cdot Z)\right)_T,$$

where $\mathcal{E}$ is the Doléans exponential, B, Z are independent Brownian motions (with B driving S), λ is the Sharpe ratio for S, and ψ is an arbitrary adapted process, with λ, ψ satisfying suitable regularity conditions such that $E(dQ/dP) = 1$. The set Ψ of such processes is in one-to-one correspondence with the set $\mathcal{M}$ of martingale measures, and we write $Q \equiv Q^{\psi}$ to emphasise this when necessary. The dual problem becomes equivalent to the stochastic control problem of optimizing a functional of the form

$$E^{P^{q,\psi}} \Theta^{(q)} \left(K_T + \int_0^T \psi_t^2 dt \right) =: E^{P^{q,\psi}} C^{q,\psi} \tag{4}$$

over control processes $\psi \in \Psi$, where $\Theta^{(q)}$ is a function whose form depends on q. The process K in (4) is the so-called mean-variance trade-off, defined by $K_t = \int_0^T \lambda_s^2 ds, 0 \le t \le T$, and the measure $P^{q,\psi}$ depends on both q and ψ. In particular, for power utility $\Theta^{(q)}$ is an exponential function and (4) is an example of a risk-sensitive control problem [7].

The distortion solution (2) implies an interesting representation for the solution to the control problem (4). It turns out that the measure appearing in (2) is $\widehat{P}^M = \widehat{P}^{q,0}$, the projection of $P^{q,0}$ onto the sigma field $\mathcal{G}_T$, and the expectation in (2) is given by

$$F = E^{\widehat{P}^M} \Phi^{(q)}(C^{q,0}),$$

where $\Phi^{(q)}$ is a function dependent on q. Moreover, the optimal control ψ^* in (4) is $\mathbb{G}$-adapted and satisfies a relation of the form (Theorem 1)

$$\Phi^{(q)} \left(E^{\widehat{P}^{q,\psi^*}} C^{q,\psi^*} \right) = E^{\widehat{P}^{q,0}} \Phi^{(q)}(C^{q,0}). \tag{5}$$

The recasting of (3) as a control problem also produces an expression for the optimal control process ψ^* in (4) and (5) in terms of the process $F(t, Y_t)$ (Proposition 3). We also recast the classical relation between $Q^* \equiv Q^{(q)}$ and the optimal terminal wealth X_T^*, $U'(X_T^*) = \eta(dQ^*/dP)$, as a relation between the optimal control processes π^*, ψ^* in (1) and (4).

Finally, we specialize to exponential utility and apply the same program to a basis risk model, where the terminal wealth is modified by a random endowment of a claim on Y. We derive representations for indifference prices and an extension of the Hobson-Rheinländer [13, 21] representation equation for the dual minimizer of the problem with random endowment depending on Y_T (Theorem 3).

Related results have appeared in other guises in the literature. In stochastic volatility models under exponential preferences, corresponding to $q = 1$, Benth and Karlsen [2] give a PDE representation of the minimal entropy martingale measure. Proposition 3 is a similar PDE characterization for general $q \leq 1$, by virtue of the PDE satisfied by F.

Monoyios [17] uses the representation equation of Hobson [13] to derive Esscher transform relations between the minimal entropy measure $Q^E = Q^{(1)}$ and the minimal martingale measure $Q^{(0)} = Q^M$, providing a probabilistic interpretation of the distortion method. Stoikov and Zariphopoulou [27] use the distortion solution for exponential utility in conjunction with well known duality results under exponential preferences (see [5, 8, 22]) to link Q^E to Q^M, and their Corollary 3.1 emerges here as a special case ($q = 1$) of Proposition 2. Stoikov and Zariphopoulou [27] focus on characterizing the risk monitoring strategies associated with exponential indifference pricing. Our focus is on the optimal measure $Q^{(q)}$, on unifying results across different preferences using the parameter q, and on deriving novel representations for the solution to the dual problem (4).

The remainder of the paper is as follows. The next section describes the model and the primal and dual problems, and the distortion solution for the primal problem is reviewed. Section 3 gives the main results, using the distortion solution in conjunction with the dual problem to derive representations for the dual optimizer $Q^{(q)}$. Section 4 applies the same program to a problem with random endowment under exponential preferences, and Section 5 concludes.

2 The model

A traded asset $S := (S_t)_{0 \leq t \leq T}$ and a non-traded stochastic factor $Y := (Y_t)_{0 \leq t \leq T}$ follow

$$\begin{aligned} dS_t &= \sigma(t, Y_t) S_t \left(\lambda(t, Y_t) dt + dB_t \right), && (6) \\ dY_t &= a(t, Y_t) dt + b(t, Y_t) dW_t, && (7) \end{aligned}$$

subject to initial conditions, with (6,7) written under the physical measure P. The Brownian motions B, W have constant correlation $\rho \in [-1, 1]$. We write $W_t = \rho B_t + \bar{\rho} Z_t$, with $\bar{\rho} = \sqrt{1 - \rho^2}$, and $(B, Z) := (B_t, Z_t)_{0 \leq t \leq T}$ a two-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}, P)$. Using the Brownian filtration $\mathbb{F}$ generated by (B, Z) means that both S and Y are assumed observable. Denote by $\mathbb{G} := (\mathcal{G}_t)_{0 \leq t \leq T}$ be the filtration generated by W.

The parameter functions λ, σ, a, b are such that unique strong solutions to the stochastic differential equations (6,7) exist. We make the following assumption throughout.

Assumption 1 *The coefficients* λ, a, b *are* $C^{1,2}([0, T] \times \mathbb{R})$ *functions satisfying, uniformly in* t, $|f(t, y)| \leq C(1 + |y|)$ *for* $f = \lambda, a, b$ *and a positive constant* C. *The volatility coefficient* $\sigma(t, y)$ *satisfies* $\sigma(y) \geq \ell > 0$ *for some positive constant* ℓ *and* $(t, y) \in ([0, T] \times \mathbb{R}$. *The diffusion coefficient* b *is uniformly elliptic:* $\exists \epsilon > 0 : b^2(t, y) \geq \epsilon y^2, \forall y \in \mathbb{R}, t \in [0, T]$.

The interest rate is zero, or equivalently S represents a discounted price. This entails no loss of generality if the interest rate is deterministic or depends only on Y and on time.

2.1 Local martingale measures

The class $\mathcal{M}$ of local martingale measures consists of measures $Q \sim P$ with density processes

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} \equiv M_t := \mathcal{E}\left(-\lambda \cdot B - \psi \cdot Z\right)_t, \quad 0 \le t \le T, \tag{8}$$

where $\mathcal{E}$ is the Doléans exponential, and ψ is an $\mathbb{F}$-adapted process. We assume $\int_0^T \lambda_t^2 dt < \infty$ and $\int_0^T \psi_t^2 dt < \infty$ a.s., and that λ, ψ are such that $EM_T = 1$, so that any $Q \in \mathcal{M}$ is a probability measure equivalent to P on $\mathcal{F}_T$. The mean-variance trade-off process is the increasing process $K := (K_t)_{0 \le t \le T}$ given by

$$K_t := \int_0^t \lambda^2(u, Y_u) du < \infty, \quad 0 \le t \le T.$$

The P-dynamics of M are

$$dM_t = -\lambda(t, Y_t) M_t dB_t - \psi_t M_t dZ_t. \tag{9}$$

Under $Q \in \mathcal{M}$, $dS_t = \sigma(t, Y_t) S_t dB_t^Q$, and Y follows

$$dY_t = \left[a(t, Y_t) - b(t, Y_t)(\rho\lambda(t, Y_t) + \bar{\rho}\psi_t)\right] dt + b(t, Y_t) dW_t^Q,$$

where B^Q, W^Q are Q-Brownian motions with correlation ρ. We write $W^Q = \rho B^Q + \bar{\rho} Z^Q$, where (B^Q, Z^Q) is a two-dimensional Q-Brownian motion given by $dB_t^Q = dB_t + \lambda(t, Y_t)dt$, $dZ_t^Q = dZ_t + \psi_t dt$. The traded asset S is a local Q-martingale, and the Q-drift of Y is arbitrary and parametrized by the integrand ψ in (8). Provided $EM_T = 1$, the space $\mathcal{M}$ is in one-to-one correspondence with the set Ψ of integrands ψ, and we write $Q \equiv Q^\psi$ whenever we need to emphasize dependence on ψ.

The minimal martingale measure of Föllmer and Schweizer [6] is $Q^M := Q^0$, corresponding to $\psi_t = 0, 0 \le t \le T$.

2.1.1 The q-optimal measure

For $q \in \mathbb{R}$ and $Q \in \mathcal{M}$ define a distance $H_q(Q, P)$ between Q and P by

$$H_q(Q, P) := \begin{cases} E\left[(dQ/dP)^q\right], & q \in \mathbb{R} \setminus \{0, 1\}, \\ E\left[(-1)^{1+q}(dQ/dP)^q \log(dQ/dP)\right], & q \in \{0, 1\}, \end{cases} \tag{10}$$

provided the expectations are finite, otherwise $H_q(Q, P) = \infty$. Note that $H_q(Q, P)$ is not necessarily symmetric in Q, P.

Definition 1 *The q-optimal measure $Q^{(q)}$ is the measure which minimizes (maximizes, for $0 < q < 1$)*[1] $H_q(Q, P)$.

We shall identify $Q^{(q)}$ with the dual optimizer Q^* shortly. The relative entropy between Q and P is $\mathcal{H}(Q, P) := H_1(Q, P)$ and the minimal entropy measure is $Q^E := Q^{(1)}$. The reverse relative entropy between Q and P is $\mathcal{H}(P, Q) = H_0(Q, P)$. For continuous asset price processes the minimal reverse relative entropy measure is equal to the minimal martingale measure, $Q^{(0)} = Q^M$, as shown by Schweizer [24].

Denote the projections of Q^M, Q^E onto the sigma-field $\mathcal{G}_T$ by $\widehat{Q}^M, \widehat{Q}^E$:

$$\begin{aligned} \frac{d\widehat{Q}^M}{dP} &:= E\left[\left.\frac{dQ^M}{dP}\right| \mathcal{G}_T\right], \\ \frac{d\widehat{Q}^E}{dP} &:= E\left[\left.\frac{dQ^E}{dP}\right| \mathcal{G}_T\right], \end{aligned}$$

satisfying, for $i = M, E$, $\widehat{Q}^i(A) = Q^i(A), \forall A \in \mathcal{G}_T$, implying $E^{\widehat{Q}^i} G = E^{Q^i} G$, for any $\mathcal{G}_T$-measurable random variable G for which the expectations exist.

2.2 Distortion measures

For $q \in \mathbb{R}$ and $\psi \in \Psi$ define measures $P^{q,\psi} \ll \mathbb{P}$ by

$$\frac{dP^{q,\psi}}{dP} := \mathcal{E}\left(-q\lambda \cdot B - q\psi \cdot Z\right)_T. \tag{11}$$

For $q = 1$, $P^{1,\psi} = Q^\psi \in \mathcal{M}$, and for $q = 0$, $P^{0,\psi} = P$, for any $\psi \in \Psi$. We call the set of measures $\widehat{P}^{q,\psi}$ *distortion measures*, and denote this class by $\mathcal{D} \equiv \mathcal{D}^{q,\psi}$. Provided $E(dP^{q,\psi}/dP) = 1$, then $P^{q,\psi}$ is a probability measure equivalent to P on $\mathcal{F}_T$. For fixed q, $\mathcal{D}^{q,\psi}$ is in one-to-one correspondence with $\mathcal{M}$, and for $q = 1$ we have $\mathcal{D}^{1,\psi} = \mathcal{M}$.

Denote the projection of $P^{q,\psi}$ onto $\mathcal{G}_T$ by $\widehat{P}^{q,\psi}$:

$$\frac{d\widehat{P}^{q,\psi}}{dP} := E\left[\left.\frac{dP^{q,\psi}}{dP}\right| \mathcal{G}_T\right] = \mathcal{E}(-q(\rho\lambda + \bar{\rho}\psi) \cdot W)_T.$$

Under $\widehat{P}^{q,\psi}$ (and indeed under $P^{q,\psi}$) the stochastic factor Y has dynamics

$$dY_t = [a(t, Y_t) - qb(t, Y_t)(\rho\lambda(t, Y_t) + \bar{\rho}\psi_t)]\, dt + b(Y_t) dW_t^{\widehat{P}^{q,\psi}}, \tag{12}$$

for a $\widehat{P}^{q,\psi}$-Brownian motion $W^{\widehat{P}^{q,\psi}}$.

The measure involved in the expectation representation for F in the distortion power solution (2) is the "minimal" distortion measure, $\widehat{P}^M := \widehat{P}^{q,0}$, corresponding to $\psi = 0$. Note that for $q = 1$, $\widehat{P}^M = \widehat{Q}^M$, the projection of the minimal martingale measure Q^M onto $\mathcal{G}_T$.

[1] The definition of $H_q(Q, P)$ could be modified so that we are always *minimizing* a convex functional of dQ/dP, but we avoid this in order to work with a simple L^q-norm formula for all $q \notin \{0, 1\}$.

2.3 The primal and dual problems

An agent trades a dynamic self-financing portfolio involving the traded asset. The portfolio wealth process $X := (X_t)_{0\le t\le T}$ satisfies

$$dX_t = \pi_t \sigma(t, Y_t) \left(\lambda(t, Y_t)dt + dB_t\right), \tag{13}$$

where $\pi := (\pi_t)_{0\le t\le T}$ is the wealth invested in the stock, representing the agent's trading strategy.

The agent has risk preferences expressed via a concave utility function $U(x)$. We consider three cases:

$$U(x) = \begin{cases} x^\gamma/\gamma, & \gamma < 1, \gamma \neq 0, \quad x \in \mathbb{R}^+ \quad \text{(power utility)}, \\ -\exp(-\alpha x), & \alpha > 0, \quad x \in \mathbb{R} \quad \text{(exponential utility)}, \\ \log x, & x \in \mathbb{R}^+ \quad \text{(logarithmic utility)}. \end{cases} \tag{14}$$

Given an initial time $t \in [0, T]$, the objective is to maximize expected utility of terminal wealth at time T:

$$J(t, x, y; \pi) = E[U(X_T)|X_t = x, Y_t = y],$$

The agent's primal value function is

$$u(t, x, y) := \sup_{\pi \in \mathcal{A}} J(t, x, y; \pi), \tag{15}$$

where $\mathcal{A}$ denotes a set of admissible trading strategies. A trading strategy is an adapted process $\pi := (\pi_t)_{0\le t\le T}$ satisfying $\int_0^T \sigma^2(t, Y_t)\pi_t^2 dt < \infty$ almost surely. Let $\mathcal{A}_0$ denote the set of trading strategies. For power and logarithmic utility, an admissible trading strategy is one whose wealth process satisfies $X \geq 0$, a.s. For exponential utility, where negative wealth is allowed, the definition of admissibility is more subtle, as discussed by Schachermayer [23]. We make the following definitions, along the lines of [4, 23, 19].

$$\begin{aligned} \mathcal{A}_b &= \{\pi \in \mathcal{A}_0 : X_t \geq a \in \mathbb{R} \text{ a.s. } \forall t \in [0, T]\}, \\ \mathcal{U}_b &= \left\{\Gamma \in L^0(\Omega, \mathcal{F}_T, P) : \Gamma \leq X_T \text{ for } \pi \in \mathcal{A}_b \text{ and } E|U(\Gamma)| < \infty\right\} \\ \mathcal{U} &= \{U(\Gamma) : \Gamma \in \mathcal{U}_b\}^c \\ \mathcal{A} &= \{\pi \in \mathcal{A}_0 : U(X_T) \in \mathcal{U}\}, \end{aligned} \tag{16}$$

where $\{\ldots\}$ denotes the closure in $L^1(\Omega, \mathcal{F}_T, P)$. The point is that we first bound the portfolio wealth from below, to eliminate doubling strategies [10], but the resulting class $\mathcal{A}_b$ is not big enough to guarantee finding the optimal strategy by searching only within it, so this class is suitably enlarged.

The convex conjugate $V(\cdot)$ of the utility function $U(\cdot)$ is

$$V(\eta) := \sup_{x \in \mathrm{dom}(U)} \left(U(x) - x\eta\right), \quad \eta > 0,$$

the Legendre transform of $-U(-x)$. For the utility functions in (14), $V : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$V(\eta) = \begin{cases} -\left(\eta^q/q\right), & q = -\left(\frac{\gamma}{1-\gamma}\right), \quad U(x) = x^\gamma/\gamma, \\ (\eta/\alpha)\left(\log\left(\eta/\alpha\right) - 1\right), & U(x) = -\exp(-\alpha x), \\ -(1 + \log \eta), & U(x) = \log x. \end{cases} \tag{17}$$

The dual value function is defined by

$$v(t,\eta,y) := \inf_{\mathbb{Q}\in\mathcal{M}} E\left[V\left(\eta\frac{M_T}{M_t}\right) \middle| Y_t = y\right], \tag{18}$$

with $M_t = (dQ/dP)|_{\mathcal{F}_t}$ given in (8). In Section 3 we shall write (18) as a stochastic control problem over control processes $\psi \in \Psi$.

For $t = 0$ the measure that achieves the infimum in (18) is the dual optimizer Q^*. Kramkov and Schachermayer [15] relate the dual optimizer to the optimal wealth process in a general semimartingale setting, extending the seminal work of Karatzas *et al* [14]. The culmination of this theory is that the dual value function is the convex conjugate of the primal value function:

$$v(t,\eta,y) := \sup_{x\in\mathrm{dom}(U)} (u(t,x,y) - x\eta). \tag{19}$$

Write $u(x) \equiv u(0,x,y)$ and $v(\eta) \equiv v(0,\eta,y)$, so that $u(x)$ and $v(\eta)$ are conjugate, implying $u'(x) = \eta$ (equivalently, $x = -v'(\eta)$). The optimal terminal wealth X_T^* and the dual optimizer Q^* are related by $U'(X_T^*) = \eta(dQ/dP)$ [15].

From the explicit formulae (17) for V, combined with (18), (19) and (10) at $t = 0$, we relate $v(\eta)$ to $H_q(Q,P)$:

$$v(\eta) = \begin{cases} V(\eta)H_q(Q^{(q)},P), & U(x) = x^\gamma/\gamma, \qquad q = -\left(\frac{\gamma}{1-\gamma}\right), \\ V(\eta) + (\eta/\alpha)\,H_1(Q^{(1)},P), & U(x) = -\exp(-\alpha x), \\ V(\eta) + H_0(Q^{(0)},P), & U(x) = \log x. \end{cases} \tag{20}$$

The dual optimizer is therefore the q-optimal measure, with the value of $q \in (-\infty, 1]$ varying with the utility function according to

$$q = \begin{cases} -\left(\frac{\gamma}{1-\gamma}\right), \quad q < 1, q \neq 0, & U(x) = x^\gamma/\gamma, \\ 1, & U(x) = -\exp(-\alpha x), \\ 0, & U(x) = \log x. \end{cases} \tag{21}$$

Remark 1 *The logarithmic utility case is degenerate. In this case the primal value function is of the form $u(x) = U(x) + F^\delta$ with $\delta = 1$, and the measure in the expectation representation for F is the physical measure P. We include this case for completeness as it covers the case $q = 0$.*

2.4 Distortion power solution

We state the well-known distortion power solution for the primal optimization problem (15). We sketch the proof, since we shall have recourse to use some of the PDEs later on. Rigorous analysis of the optimality of the solution is given in [29, 27]. Our main goal is to use the solution to make inferences about the solution to the dual problem.

Proposition 1 (Distortion Power Solution) *Let $q \le 1$ be given by (21). The value function (15) is given by*

$$u(t,x,y) = \begin{cases} U(x)\,(F(t,y))^{\delta}\,, & q \le 1, q \neq 0, \\ U(x) + (F(t,y))^{\delta}\,, & q = 0, \end{cases} \tag{22}$$

where

$$\delta := \frac{1}{1-q\rho^2}, \tag{23}$$

and $F(t,y)$ has the stochastic representation

$$F(t,y) = \begin{cases} E^{\widehat{P}^M}\left[\exp\left(-(q/2\delta)(K_T - K_t)\right) \middle| Y_t = y\right], & q \le 1, q \neq 0, \\ E^{\widehat{P}^M}\left[(K_T - K_t)/2 \middle| Y_t = y\right], & q = 0, \end{cases} \tag{24}$$

with the measure $\widehat{P}^M$ defined by

$$\frac{d\widehat{P}^M}{dP} = \mathcal{E}[-q\rho(\lambda \cdot W)]_T,$$

so that the $\widehat{P}^M$-dynamics for Y are

$$dY_t = (a(t,Y_t) - qb(t,Y_t)\rho\lambda(t,Y_t))\,dt + b(t,Y_t)dW_t^{\widehat{P}^M},$$

corresponding to $\psi_t = 0$ in (12).

The function $F : [0,T] \times \mathbb{R} \to \mathbb{R}^+$ satisfies

$$F_t + (a - qb\rho\lambda)F_y + \frac{1}{2}b^2 F_{yy} = \begin{cases} (q/2\delta)\lambda^2 F, & q \le 1, q \neq 0, \\ -\lambda^2/2, & q = 0, \end{cases} \tag{25}$$

with boundary condition

$$F(T,y) = \begin{cases} 1, & q \le 1, q \neq 0, \\ 0, & q = 0. \end{cases}$$

Proof We give the proof for $q \neq 0$. The proof for $q = 0$ follows the same reasoning.

The Bellman dynamic programming PDE for $u(t,x,y)$ is

$$\max_{\pi}\left(u_t + \sigma\lambda\pi u_x + a u_y + \frac{1}{2}\sigma^2\pi^2 u_{xx} + \frac{1}{2}b^2 u_{yy} + \rho\sigma b\pi u_{xy}\right) = 0.$$

Performing the maximization gives the optimal control $\pi^*(t,x,y)$ in feedback form as the Markov control

$$\pi^*(t,x,y) = -\frac{(\lambda u_x + \rho b u_{xy})}{\sigma u_{xx}}. \tag{26}$$

The optimal trading strategy $\pi^* := (\pi_t^*)_{0\le t\le T}$ is given by $\pi_t^* = \pi^*(t, X_t^*, Y_t)$, where X_t^* is the wealth process (13) with $\pi_t = \pi_t^*$. Inserting (26) into the Bellman equation gives

$$u_t + a u_y + \frac{1}{2}b^2 u_{yy} - \frac{1}{2u_{xx}}\left(\lambda u_x + \rho b u_{xy}\right)^2 = 0, \quad u(T,x,y) = U(x). \tag{27}$$

Let $q \leq 1, q \neq 0$. Seek a separable solution to (27) of the form (22), in turn for power and exponential preferences. Then it is easy to verify that choosing δ as in the Proposition results in F satisfying the linear PDE (25) with the given boundary conditions.

The proof is completed by establishing that the proposed solution is a viscosity solution of the HJB equation, and then using its growth and regularity properties to verify its optimality (see Zariphopoulou [29] and Stoikov and Zariphopoulou [27]). The value function is in fact a classical solution of the HJB equation, provided the conditions of Assumption 1 (and in particular the condition that b is uniformly elliptic) are satisfied [20].

□

Remark 2 *If $q\rho^2 = 1$ (that is, if $q = \rho^2 = 1$, given $q \leq 1$), the value function is given by*

$$u(t, x, y) = U(x) \exp\left(-G(t, y)\right),$$

where G has the stochastic representation

$$G(t, y) = E^{\widehat{P}^M}\left[\frac{1}{2}q(K_T - K_t)\middle| Y_t = y\right], \quad q = \rho^2 = 1, \tag{28}$$

and satisfies

$$G_t + (a - qb\rho\lambda)G_y + \frac{1}{2}b^2 G_{yy} + \frac{1}{2}q\lambda^2 = 0, \quad q = \rho^2 = 1, \quad G(T, y) = 0.$$

The solution in this case can be obtained by letting $q\rho^2 \to 1$, and hence $\delta \to \infty$, in (22), and using a simple asymptotic analysis based on Taylor series. An alternative approach is to analyze the PDE for F before *setting δ to a specific value. This is left as an exercise for the reader.*

Remark 3 *In [13] the case $q \geq 1$ is analyzed. It is noted that if $\rho^2 > 1/q$ then the q-optimal measure may not exist beyond a certain time horizon. This corresponds to the situation that arises if $\delta < 0$ in the stochastic representation (24), which then might not be finite. This does not arise in our model, where $q \leq 1$, and the q-optimal measure always exists.*

The distortion solution has been obtained by Zariphopoulou [29] for power utility and by various authors [11, 16, 18, 26, 27] for exponential utility. Proposition 1 gives a unified stochastic representation for F, modulo the appropriate value for q. The logarithmic case is degenerate since in that case $q = 0$ implies $\delta = 1$, and the measure $\widehat{P}^M$ is the physical measure P. For exponential utility $q = 1$ and $\widehat{P}^M = \widehat{Q}^M$, the projection of the minimal martingale measure Q^M onto the sigma-field $\mathcal{G}_T$. Of course, for any $\mathcal{G}_T$-measurable random variable G, $\widehat{Q}^M$ and Q^M give the same moments.

3 Representations for the dual optimizer

This section gives our main results, which are ramifications of Proposition 1. We begin with a simple lemma.

Lemma 1 *The dual value function* $v(t,\eta,y)$ *is given in terms of the distortion function* $(F(t,y))^{\delta}$ *by*

$$v(t,\eta,y)=\begin{cases}(F(t,y))^{\delta}\,V\left(\eta/(F(t,y))^{\delta}\right), & q\leq 1, q\neq 0,\\ (F(t,y))^{\delta}+V(\eta), & q=0.\end{cases} \tag{29}$$

Proof We prove the result for $q\neq 0$. The result for $q=0$ follows similar reasoning. The value of x achieving the supremum in the Legendre transform (19) is x^* satisfying

$$u_x(x^*)=\eta,$$

where we have suppressed dependence on (t,y). Using this in (22) we get

$$U'(x^*)F^{\delta}=\eta,$$

or

$$x^*=I\left(\frac{\eta}{F^{\delta}}\right),$$

where $I(\cdot)=(U'(\cdot))^{-1}$ is the inverse of the gradient of the utility function U. Inserting the last expression for x^* into (19) and using $u(x^*)=U(x^*)F^{\delta}$ gives

$$\begin{aligned} v(\eta) &= u(x^*)-x^*\eta \\ &= F^{\delta}\left[U(I(\eta/F^{\delta}))-(\eta/F^{\delta})I(\eta/F^{\delta})\right], \end{aligned}$$

and the result follows from the identity

$$V(\eta)=U(I(\eta))-\eta I(\eta).$$

□

Using the specific form (17) of $V(\cdot)$ for each utility function gives Lemma 1 in the following explicit form.

Corollary 1 *The dual value function* $v(t,\eta,y)$ *is related to* $F(t,y)$ *and* δ *by*

$$v(t,\eta,y)=\begin{cases}V(\eta)\,(F(t,y))^{\delta(1-q)}, & q<1, q\neq 0,\\ V(\eta)-(\eta/\alpha)\log\left((F(t,y))^{\delta}\right), & q=1,\\ V(\eta)+(F(t,y))^{\delta}, & q=0.\end{cases} \tag{30}$$

Setting $t=0$ in Lemma 1 relates the q-optimal measure to the distortion solution.

Corollary 2 *The optimizer $Q^* \equiv Q^{(q)}$ in the dual problem*

$$v(\eta) := \inf_{Q \in \mathcal{M}} EV\left(\eta \frac{dQ}{dP}\right), \tag{31}$$

satisfies

$$EV\left(\eta \frac{dQ^{(q)}}{dP}\right) = \begin{cases} F^\delta V\left(\eta / F^\delta\right), & q \le 1, q \ne 0, \\ F^\delta + V(\eta), & q = 0, \end{cases}$$

where $F \equiv F(0, y)$.

Proof Set $t = 0$ in (29) and equate the result with the defining relation (31).

□

Finally, using the specific form (17) of V we are able to recast (30) as a relation between the optimal distance $H_q(Q^{(q)}, P)$ and F^δ.

Proposition 2 *The optimal distance $H_q(Q^{(q)}, P)$ satisfies*

$$H_q(Q^{(q)}, P) = \begin{cases} F^{\delta(1-q)}, & q < 1, q \ne 0, \\ -\log\left(F^\delta\right), & q = 1, \\ F^\delta, & q = 0. \end{cases} \tag{32}$$

Proof Equate (30) at $t = 0$ with (20).

□

Remark 4 *For $q = 1$, using the stochastic representation (24) for F, Proposition 2 may be written in the form*

$$\exp\left(-\bar{\rho}^2 \mathcal{H}(Q^E, P)\right) = E^{Q^M} \exp\left(-\frac{1}{2}\bar{\rho}^2 K_T\right),$$

equivalent to Corollaries 2.1 and 3.1 in Stoikov and Zariphopoulou [27].

3.1 The dual stochastic control problem

The preceding analysis used the Legendre transform representation (19) for the dual value function in conjunction with the distortion solution (2) for the primal problem. In contrast, we now approach the dual problem (18) directly via dynamic programming and the associated HJB equation. This gives to the following representation for the density of the dual optimizer Q^* with respect to P.

Proposition 3 *The dual optimizer $Q^* = Q^{(q)}$ is given by*

$$\frac{dQ^*}{dP} = \mathcal{E}\left(-\lambda \cdot B - \psi^* \cdot Z\right)_T,$$

with

$$\psi_t^* = \begin{cases} -\delta\bar{\rho}b(t, Y_t)\frac{\partial}{\partial y}\log F(t, Y_t), & t \in [0, T], \quad q < 1, q \neq 0 \\ 0 & q = 0, \end{cases} \tag{33}$$

where $F(t, y)$ is the function in the distortion solution (22) and δ is given in (23).

Proof From the definition (18) and the P-dynamics (7) of Y and (9) of M, the Bellman equation for v is

$$\inf_\psi \left[v_t + av_y + \frac{1}{2}\left(\lambda^2 + \psi^2\right)\eta^2 v_{\eta\eta} + \frac{1}{2}b^2 v_{yy} - (\rho\lambda + \bar{\rho}\psi)\, b\eta v_{\eta y} \right] = 0. \tag{34}$$

Performing the minimization gives the optimal value of ψ in feedback form as

$$\psi^*(t, \eta, y) = \frac{\bar{\rho}b(t, y)}{\eta}\frac{v_{\eta y}(t, \eta, y)}{v_{\eta\eta}(t, \eta, y)}. \tag{35}$$

Inserting (35) into (34) converts the Bellman equation to

$$v_t + av_y + \frac{1}{2}\lambda^2\eta^2 v_{\eta\eta} + \frac{1}{2}b^2 v_{yy} - \rho\lambda b\eta v_{\eta y} - \frac{1}{2}\bar{\rho}^2 b^2 \frac{v_{\eta y}^2}{v_{\eta\eta}} = 0, \quad v(T, \eta, y) = V(\eta). \tag{36}$$

Note that (36) is also obtained by applying the Legendre transform (19) to the Bellman equation (27) for u. Moreover, one can easily verify, using the PDE (25) for $F(t, y)$, that (30) solves the HJB equation (36), and the optimality of the proposed solution (30) follows from a verification theorem. See [3] for an example of such a verification for the dual problem with power utility.

Using (30) and the specific form (17) of $V(\cdot)$ for each utility function, the representation (35) loses all dependence on η and reduces to

$$\psi^*(t, y) = -\delta\bar{\rho}b(t, y)\frac{F_y(t, y)}{F(t, y)}, \tag{37}$$

for power and exponential utility, and to zero for logarithmic utility. The optimal control process $\psi^* := (\psi_t^*)_{0 \le t \le T}$ is given by $\psi_t^* = \psi^*(t, Y_t)$, and the proof is complete.

□

Note that for $\rho^2 = 1$, the dual PDE becomes linear (a trait of complete markets), $\psi_t^* = 0$, and the dual optimizer is the unique martingale measure of the complete market. Also, for deterministic mean-variance trade-off K, $\psi_t^* = 0$ and the dual optimizer is the minimal martingale measure.

The next result is a novel representation for the solution of the dual stochastic control problem, when it is reformulated as that of finding the q-optimal measure and the optimal distance $H_q(Q^{(q)}, P)$. We show that this problem is equivalent to the *stochastic control problem* of minimizing (or maximizing, for $0 < q < 1$) a cost functional of the form

$$E^{P^{q,\psi}} C^{q,\psi} \tag{38}$$

over control processes $\psi \in \Psi$, where $C^{q,\psi}$ is given by

$$C^{q,\psi} := \begin{cases} \left(-\frac{1}{2}\right)^{1+q} \left(K_T + \int_0^T \psi_t^2 dt\right), & q \in \{0,1\}, \\ \exp\left(-\frac{1}{2}q(1-q)\left(K_T + \int_0^T \psi_t^2 dt\right)\right), & q < 1, q \neq 0, \end{cases} \tag{39}$$

and $P^{q,\psi}$ is defined in (11). Note that we do not use the projection $\widehat{P}^{q,\psi}$ in (38) as we do not assume at the outset that $C^{q,\psi}$ is $\mathcal{G}_T$-measurable (but we shall see that the optimal control ψ^* does indeed turn out to be $\mathbb{G}$-adapted).

Theorem 1 *For $q \notin (0,1)$, the dual problem to locate the q-optimal measure has the stochastic control representation*

$$\inf_{Q \in \mathcal{M}} H_q(Q, P) = \inf_{\psi \in \Psi} E^{P^{q,\psi}} C^{q,\psi}, \tag{40}$$

where $C^{q,\psi}$ is defined in (39), and the solution satisfies

$$\Phi^{(q)}\left(\inf_{\psi \in \Psi} E^{P^{q,\psi}} C^{q,\psi}\right) = E^{P^{q,0}} \Phi^{(q)}(C^{q,0}), \tag{41}$$

where $\Phi^{(q)}(\cdot)$ is given by

$$\Phi^{(q)}(x) = \begin{cases} x^{(1-q\rho^2)/(1-q)}, & q < 1, q \neq 0, \\ \exp(-\bar{\rho}^2 x), & q = 1, \\ x, & q = 0, \end{cases}$$

and $P^{q,\psi}$ is the measure defined in (11). For $0 < q < 1$ the result holds with the infima in (40) and (41) replaced by suprema.

Proof We prove the result for $q < 0$, corresponding to the power utility case with HARA parameter $0 < \gamma < 1$. The other cases are established in a similar manner (and the case $q = 0$ is trivial, with $P^{0,\psi} = P$ regardless of ψ).

From (8), for a martingale measure $Q \equiv Q^\psi$,

$$H_q(Q^\psi, P) = E\left[\left(\frac{dQ^\psi}{dP}\right)^q\right] = E\left[(\mathcal{E}(-\lambda \cdot B - \psi \cdot Z)_T)^q\right].$$

Using (11) we convert the expectation to one under $P^{q,\psi}$ to obtain

$$H_q(Q^\psi, P) = E^{P^{q,\psi}} C^{q,\psi}.$$

Minimizing $H_q(Q,P)$ over $Q \in \mathcal{M}$ is equivalent to minimizing $E^{P^{q,\psi}} C^{q,\psi}$ over control processes $\psi \in \Psi$, so the optimal measure $Q^* \equiv Q^{(q)}$ satisfies

$$\inf_{\psi\in\Psi} E^{P^{q,\psi}} C^{q,\psi} = H_q(Q^{(q)}, P).$$

Using Proposition 2 and the stochastic representation (24) for F, we may rewrite the right-hand side to obtain

$$\inf_{\psi\in\Psi} E^{P^{q,\psi}} C^{q,\psi} = \left(E^{\widehat{P}^{q,0}}\left[\left(C^{q,0}\right)^{(1-q\rho^2)/(1-q)}\right]\right)^{(1-q)/(1-q\rho^2)}, \tag{42}$$

where we have used $\widehat{P}^M = \widehat{P}^{q,0}$. Since $C^{q,0}$ is $\mathcal{G}_T$-measurable we can replace $\widehat{P}^{q,0}$ by $P^{q,0}$ in the expectation on the right hand side of (42) and the proof is complete.

□

Remark 5 *Writing (42) explicitly gives*

$$\inf_{\psi\in\Psi} E^{P^{q,\psi}}\left[e^{\theta\Delta\left(K_T+\int_0^T \psi_t^2 dt\right)}\right] = \left(E^{P^{q,0}}\left[e^{\theta K_T}\right]\right)^{\Delta}, \tag{43}$$

where $\theta = -\frac{1}{2}q(1-q\rho^2)$ *and* $\Delta = (1-q)/(1-q\rho^2)$. *The infimum in (43) is an example of a* risk-sensitive control *problem [7]. The integral in the* exponential *cost functional depends on the control* ψ, *as does the measure* $P^{q,\psi}$. *The novel feature of (43) is that the solution of the control problem is obtained by setting* $\psi = 0$, *taking the exponential integral to the power of* $1/\Delta$, *and taking the resultant expectation to the power* Δ.

3.2 The Hobson representation equation

For $q \geq 1$ Hobson [13] derives the following representation equation for the q-optimal measure. Let there be previsible processes ν^*, ψ^* and a finite constant c such that

$$\frac{1}{2}qK_T = N_T + \frac{1}{2}(1-q)[N]_T + L_T + \frac{1}{2}[L]_T + c, \tag{44}$$

where

$$\begin{aligned} L_t &:= \int_0^t \psi_u^* dZ_u, \\ N_t &:= \int_0^t \nu_u^*(dB_u + q\lambda(u,Y_u)du). \end{aligned}$$

If a solution to (44) can be found, in the form of the triple (ν^*, ψ^*, c), then this defines a candidate for the q-optimal measure via $dQ^{(q)}/dP = \mathcal{E}(-\lambda\cdot B - \psi^*\cdot Z)_T$, and that measure is indeed optimal if certain regularity conditions are met. We show that (44) holds for $q \leq 1$ by arguing from the properties of the process $F(t, Y_t)$, related to the optimal dual control ψ_t^* by (33), so that (44) does indeed give a representation for the dual optimizer in the setting of this paper.

Proposition 4 *The representation equation (44) holds with*

$$\nu_t^* = \frac{\rho}{\bar{\rho}}\psi_t^*, \tag{45}$$

where ψ^* *is given by (33):*

$$\psi_t^* = -\delta\bar{\rho}b(t, Y_t)\frac{\partial}{\partial y}\log F(t, Y_t), \tag{46}$$

and $c = -\delta \log F$*, for* $q \neq 0$*. For* $q = 0$*,* $c = \psi^* = 0$*.*

Proof For $q = 0$ the result is trivial.

For $q \leq 1$, define the function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ by

$$f(t, y) := -\delta \log F(t, y).$$

From the PDE (25) for F, $f(t, y)$ solves the semi-linear PDE

$$f_t + (a - qb\rho\lambda)f_y + \frac{1}{2}b^2 f_{yy} - \frac{1}{2}\frac{b^2}{\delta}f_y^2 + \frac{1}{2}q\lambda^2 = 0, \quad f(T, y) = 0. \tag{47}$$

From (46) we have

$$\psi_t^* = \bar{\rho}b(t, Y_t)f_y(t, Y_t),$$

so that N, L are given by

$$\begin{aligned} N_t &= \rho\int_0^t b(u, Y_u)f_y(u, Y_u)(dB_u + \lambda(u, Y_u)du), \\ L_t &= \bar{\rho}\int_0^t b(u, Y_u)f_y(u, Y_u)dZ_u. \end{aligned}$$

A straightforward computation establishes that

$$\begin{aligned} & N_T + \frac{1}{2}(1-q)[N]_T + L_T + \frac{1}{2}[L]_T \\ = & \int_0^T \left(q\rho\lambda b f_y + \frac{1}{2}(1 - q\rho^2)b^2 f_y^2\right) dt + \int_0^T b f_y dW_t \\ = & \int_0^T \left(f_t + a f_y + \frac{1}{2}b^2 f_{yy}\right) dt + \int_0^T b f_y dW_t + \frac{1}{2}qK_T \\ = & \int_0^T df(t, Y_t) + \frac{1}{2}qK_T \\ = & = -f(0, y) + \frac{1}{2}qK_T \\ = & \delta \log F(0, y) + \frac{1}{2}qK_T \\ = & -c + \frac{1}{2}qK_T, \end{aligned}$$

where we have used the PDE (47) satisfied by f and the Itô formula.

□

An immediate corollary of (44) is the following representation of the likelihood ratio $dQ^{(q)}/dP$, given in [13] for $q \geq 1$ and extended here to $q \leq 1$.

Corollary 3 *The dual optimizer has Radon-Nikodym derivative given in terms of the traded asset price S by*

$$\log \frac{dQ^{(q)}}{dP} = c - \int_0^T \theta_t dS_t + \frac{1}{2}(1-q)\int_0^T \theta_t^2 d[S]_t,$$

where

$$\theta_t := \frac{\lambda(t,Y_t) - \nu_t^*}{\sigma(t,Y_t)S_t}, \quad 0 \leq t \leq T, \tag{48}$$

$c = -\delta \log F$*, and* $\nu^* = (\rho/\bar{\rho})\psi^*$ *is defined in (45).*

Proof Since $dQ^{(q)}/dP = \mathcal{E}(-\lambda \cdot B - L)_T$ we have

$$\log \frac{dQ^{(q)}}{dP} = -(\lambda \cdot B)_T - \frac{1}{2}K_T - L_T - \frac{1}{2}[L]_T.$$

Eliminate the terms involving $L_T, [L]_T$ using (44) and use $dB_t = (dS_t/\sigma(t,Y_t)S_t) - \lambda(t,Y_t)dt$ to get

$$\log \frac{dQ^{(q)}}{dP} = c - \int_0^T \theta_t dS_t + \frac{1}{2}(1-q)\int_0^T (\lambda(t,Y_t) - \nu_t^*)^2 dt,$$

which is the required result, since $d[S]_t = \sigma^2(t,Y_t)S_t^2 dt$.

□

3.3 Relation between optimal portfolio and dual optimizer

The process θ in Corollary 3 is essentially the optimal trading strategy, as the next result confirms. For a utility function $U(x)$, denote the Arrow-Pratt measure of risk aversion by

$$A(x) := -\frac{U''(x)}{U'(x)}.$$

Then we have the following relation between the primal and dual feedback control functions.

Proposition 5 *The optimal portfolio feedback control $\pi^*(t,x,y)$ in (26) is related to the optimal dual control $\psi^*(t,y)$ in (37) by*

$$\pi^\star(t,x,y) = -\frac{1}{\sigma(t,y)A(x)}\left(\lambda(t,y) - \left(\frac{\rho}{\bar{\rho}}\right)\psi^*(t,y)\right), \tag{49}$$

and the optimal portfolio process $\pi_t^* = \pi^*(t,X_t^*,Y_t)$ is given by

$$\pi_t^* = \frac{\theta_t S_t}{A(X_t^*)}, \tag{50}$$

where θ_t is the process in (48), and X_t^* is the optimal wealth process (13) with $\pi_t = \pi_t^*$.

Proof We give the proof for $q \neq 0$. Use the distortion solution (22) in (26) to give

$$\pi^*(t,x,y) = \frac{1}{\sigma(t,y)A(x)} \left(\lambda(t,y) + \delta\rho b(t,y) \frac{F_y(t,y)}{F(t,y)} \right).$$

Then (49) follows from (37), and (50) follows from $\pi_t^* = \pi^*(t, X_t^*, Y_t)$ and the expression (48) for θ.

□

Proposition 5 is an explicit example of the classical relation between the dual optimizer and the optimal terminal wealth:

$$X_T^* = I\left(\eta \frac{dQ^*}{dP}\right), \tag{51}$$

where $I(\cdot) \equiv -V(\cdot)$ is the inverse of the gradient of the utility function, $I \equiv (U')^{-1}$, and η is a constant related to the initial wealth x by $\eta = u'(x)$, or equivalently $x = -v'(\eta)$. In other words, we have:

Corollary 4 *The relation (50) implies (51).*

Proof We demonstrate the equivalence for power utility, $q < 1, q \neq 0$. The proofs for other preferences follow the same lines.

Using (50) in (13), the optimal wealth process follows

$$dX_t^* = \frac{\theta_t}{A(X_t^*)} dS_t,$$

where we have used $dS_t/S_t = \sigma(t,Y_t)(dB_t + \lambda(t,Y_t)dt)$. For power utility, $A(x) = 1/\left(x(1-q)\right)$. This and Itô's formula give the logarithm of optimal terminal wealth, with $X_0 = x$, as

$$\log X_T^* = \log x + (1-q)\left(\int_0^T \theta_t dS_t - \frac{1}{2}(1-q) \int_0^T \theta_t^2 d[S]_t \right).$$

Using Corollary 3 this becomes

$$X_T^* = x\left(F^\delta \frac{dQ^*}{dP}\right)^{-(1-q)}. \tag{52}$$

Since the primal and dual value functions $u(x), v(\eta)$ are conjugate we have $x = -v'(\eta)$. Using (30) at $t = 0$ and $V(\eta) = -\eta^q/q$ for power utility we write x as

$$x = -v'(\eta) = \eta^{-(1-q)} F^{\delta(1-q)}.$$

Using this in (52) gives

$$X_T^* = \left(\eta \frac{dQ^*}{dP}\right)^{-(1-q)},$$

which is (51) since, for power utility, $I(\eta) = \eta^{-(1-q)}$.

□

4 Application to exponential hedging

In this section we specialize to exponential utility, $U(x) = -\exp(-\alpha x)$, so $q = 1$. The agent's primal problem is altered to allow for the sale of a claim on Y by incorporating a random terminal endowment $-G$, where $G \equiv G(Y_T)$ represents the payoff of a European claim on Y. We assume the function $G(y)$ is bounded below. This model has been analyzed by Davis [4], Henderson [11] and Monoyios [16] in the case where both assets S, Y are lognormal diffusions, and Zariphopoulou and co-authors [18, 26, 27] in a setting similar to ours. Our focus is again on using the distortion solution to make inferences about the dual minimizer and to derive representations for indifference prices. In a general semimartingale setting Delbaen *et al* [5] derived the fundamental duality linking exponential hedging to entropy minimization. Similar duality results have appeared in [8, 19, 22], and Becherer [1] analyses exponential indifference valuation in a general semimartingale setting.

In what follows, many of the proofs follow the same reasoning as for the problem with no random endowment, and are sketched briefly or omitted altogether, leaving details as an exercise for the reader.

The objective in the primal problem is to maximize

$$J_G(t, x, y; \pi) = \mathbb{E}[U(X_T - G(Y_T))|X_t = x, Y_t = y].$$

The agent's primal value function is

$$u_G(t, x, y) := \sup_{\pi \in \mathcal{A}_G} J_G(t, x, y; \pi), \tag{53}$$

with $u_G(T, x, y) = U(x - G(y))$. The class of admissible strategies $\mathcal{A}_G$ is defined in a similar manner to (16) as follows.

$$\begin{aligned}
\mathcal{A}_b &= \{\pi \in \mathcal{A}_0 : X_t \geq a \in \mathbb{R} \text{ a.s. } \forall t \in [0, T]\}, \\
\mathcal{U}_G^b &= \left\{\Gamma \in L^0(\Omega, \mathcal{F}_T, P) : \Gamma \leq X_T - G(Y_T) \text{ for } \pi \in \mathcal{A}_b \text{ and } E|U(\Gamma)| < \infty\right\} \\
\mathcal{U}_G &= \left\{U(\Gamma) : \Gamma \in \mathcal{U}_G^b\right\}^c \\
\mathcal{A}_G &= \{\pi \in \mathcal{A}_0 : U(X_T) \in \mathcal{U}_G\},
\end{aligned}$$

Using the same methods as earlier, it is straightforward to establish the following distortion solution for u_G. The proof follows the same reasoning as the proof of Proposition 1 and is left as an exercise. The value function u_G satisfies the same PDE as u, with modified terminal boundary condition.

Proposition 6 *With exponential utility, $U(x) = -\exp(-\alpha x)$, the value function (53) is given by*

$$u_G(t, x, y) = U(x) \left(F_G(t, y)\right)^\delta, \tag{54}$$

where $\delta = 1/\bar{\rho}^2$ and $F_G : [0, T] \times \mathbb{R} \to \mathbb{R}^+$ has the stochastic representation

$$F_G(t, y) = E^{\widehat{Q}^M}\left[\exp\left(-\bar{\rho}^2\left(\frac{1}{2}(K_T - K_t) - \alpha G(Y_T)\right)\right)\middle| Y_t = y\right], \tag{55}$$

and where $\widehat{Q}^M$ is the projection of the minimal martingale measure Q^M onto the sigma field $\mathcal{G}_T$.

Proof This follows the same reasoning as the proof of Proposition 1

□

The dual value function $v_G(t,\eta,y)$ is given by the Legendre transform

$$v_G(t,\eta,y) = \sup_{x\in\mathbb{R}} \left(u_G(t,x,y) - x\eta\right).$$

The distortion representation for u_G then implies that v_G is given by

$$v_G(t,\eta,y) = V(\eta) - \frac{\eta}{\alpha}\log\left((F_G(t,y))^{\delta}\right), \tag{56}$$

where $V(\eta) = (\eta/\alpha)(\log(\eta/\alpha) - 1)$ is the convex conjugate of the exponential utility function. We recognise (56) as the analogue of (29) or (30) for the optimization problem with random endowment.

The dual value function has the fundamental definition

$$v_G(t,\eta,y) := \inf_{Q\in\mathcal{M}} E\left[V\left(\eta\frac{M_T}{M_t}\right) - \eta\frac{M_T}{M_t}G(Y_T)\middle| Y_t = y\right].$$

Setting $t = 0$ and writing $v_G(\eta) \equiv v_G(0,\eta,y)$ this becomes

$$v_G(\eta) = V(\eta) + \frac{\eta}{\alpha}\inf_{Q\in\mathcal{M}}\left[\mathcal{H}(Q,P) - \alpha E^Q G(Y_T)\right], \tag{57}$$

where we have used the specific form of $V(\cdot)$ for exponential utility, and $\mathcal{H}(Q,P) = H_1(Q,P)$ is the relative entropy between $Q \in \mathcal{M}$ and P. Equating (56) at $t = 0$ with (57) gives the following result, the analogue of Proposition 2 for the problem with random endowment. We write $F_G \equiv F_G(0,y)$ and $G \equiv G(Y_T)$.

Proposition 7 *The dual minimizer in (57) is related to F_G by*

$$-\inf_{Q\in\mathcal{M}}\left[\mathcal{H}(Q,P) - E^Q(\alpha G)\right] = \delta\log F_G. \tag{58}$$

Let Q_G denote the minimizer in (57) and (58). We have the following immediate corollary linking Q_G to an expectation under the minimal martingale measure.

Corollary 5 *The dual minimizer Q_G in (57) satisfies*

$$\exp\left(-\bar{\rho}^2\inf_{Q\in\mathcal{M}}(\mathcal{H}(Q,P) - \alpha E^Q G)\right) = E^{\widehat{Q}^M}\exp\left(-\bar{\rho}^2\left(\frac{1}{2}K_T - \alpha G\right)\right). \tag{59}$$

Proof Use the stochastic representation (55) for F_G (at $t = 0$) in (58).

□

Remark 6 *Note that (59) reduces to the statement of Remark 4 for $G = 0$.*

As remarked earlier, Delbaen *et al* [5] have derived fundamental duality results for exponential hedging in a semimartingale setting, so it is not surprising that we can recover a version of their result from Proposition 7. Using (58) in $u_G(x) = U(x)F_G^\delta$ gives the maximum utility starting at time 0 as

$$\begin{aligned} u_G(x) &= \sup_{\pi \in \mathcal{A}_G} E\left[\exp\left(-\alpha(X_T - G)\right)\right] \\ &= -\exp\left(-\alpha x - \inf_{Q \in \mathcal{M}} (\mathcal{H}(Q, P) - \alpha E^Q G)\right), \end{aligned}$$

which is the fundamental duality in [5].

4.1 The dual control problem

We now treat the dual problem with random endowment as a stochastic control problem over control processes $\psi \in \Psi$, as we did for the problem without random endowment. The dual minimizer Q_G is connected by the Girsanov exponential $dQ_G/dP = \mathcal{E}(-\lambda \cdot B - \psi^G \cdot Z)_T$ to the optimal control process $\psi^G \in \Psi$. Using the same methods as before, ψ^G is given by the analogue of (33), derived from the PDE for v_G. The Bellman PDE for v_G has the same form as (36) for the problem without random endowment, but with terminal boundary condition $v_G(t, \eta, y) = V(\eta) - \eta G(y)$. We therefore obtain:

Proposition 8 *The dual minimizer Q_G is given by*

$$\frac{dQ_G}{dP} = \mathcal{E}\left(-\lambda \cdot B - \psi^G \cdot Z\right)_T,$$

with

$$\psi_t^G = -\frac{b(t, Y_t)}{\bar{\rho}} \frac{\partial}{\partial y} \log F_G(t, Y_t), \quad t \in [0, T], \tag{60}$$

and $F_G(t, y)$ as in (55).

We have the following representation for the solution of the dual stochastic control problem, the analogue of Theorem 1 for the problem with random endowment.

Theorem 2 *Define*

$$C^{\psi, G} := \frac{1}{2}\left(K_T + \int_0^T \psi_t^2 dt\right) - \alpha G.$$

Then the dual problem with random endowment has the stochastic control representation

$$\inf_{Q\in\mathcal{M}} \left(\mathcal{H}(Q,P) - \alpha E^Q G\right) = \inf_{\psi\in\Psi} E^{Q^\psi} C^{\psi,G}, \tag{61}$$

and the solution satisfies

$$\exp\left(-\bar{\rho}^2 \inf_{\psi\in\Psi} E^{Q^\psi} C^{\psi,G}\right) = E^{Q^0} \exp\left(-\bar{\rho}^2 C^{0,G}\right), \tag{62}$$

where $Q^0 = Q^M$ is the minimal martingale measure.

Proof Given a martingale measure $Q^\psi \in \mathcal{M}$, with $dQ^\psi/dP = \mathcal{E}(-\lambda\cdot B - \psi\cdot Z)_T$, we have

$$\begin{aligned}\mathcal{H}(Q^\psi,P) - \alpha E^{Q^\psi} G &= E^{Q^\psi}\left[\log\left(\frac{dQ^\psi}{dP}\right) - \alpha G\right]\\ &= E^{Q^\psi}\left[\frac{1}{2}\left(K_T + \int_0^T \psi_t^2 dt\right) - \alpha G\right]\\ &= E^{Q^\psi} C^{\psi,G},\end{aligned}$$

which establishes (61). The result then follows from (59).

□

Note that (62) is of the form

$$\Phi^{(1)}\left(\inf_{\psi\in\Psi} E^{Q^\psi} C^{\psi,G}\right) = E^{Q^0}\Phi^{(1)}\left(C^{0,G}\right),$$

where $\Phi^{(1)}(x) = \exp(-\bar{\rho}^2 x)$, as in Theorem 1.

4.2 A representation equation for the dual minimizer

We derive a martingale representation identity for the dual minimizer Q_G, extending (44) for $q=1$ to the problem with random endowment, by considering the integral

$$\int_0^T d(\log F_G(t,Y_T)).$$

Theorem 3 *Define*

$$\begin{aligned}L_t^G &:= \int_0^t \psi_u^G dZ_u,\\ N_t^G &:= \int_0^t \left(\frac{\rho}{\bar{\rho}}\right)\psi_u^G (dB_u + \lambda(u,Y_u)du),\end{aligned}$$

where ψ^G is given by (60). Then N_T^G, L_T^G satisfy

$$\frac{1}{2}K_T - \alpha G = N_T^G + L_T^G + \frac{1}{2}[L^G]_T + \mathcal{H}(Q_G,P) - \alpha E^{Q_G} G.$$

Proof Define $f^G : [0,T] \times \mathbb{R} \to \mathbb{R}$ by $f^G(t,y) = -\delta \log F_G(t,y) = -(1/\bar{\rho}^2) \log F_G(t,y)$. Since $F_G(t,y)$ solves (25) with terminal condition $F_G(T,y) = \exp(\bar{\rho}^2 \alpha G(y))$, $f^G(t,y)$ solves the semi-linear PDE (47) with $q = 1$ and terminal condition $f^G(T,y) = -\alpha G(y)$. Consider

$$\int_0^T df^G(t,Y_t) = f^G(T,Y_T) - f^G(0,y) =: -\alpha G(Y_T) - c_G,$$

where, by (58), $c_G = \mathcal{H}(Q_G, P) - \alpha E^{Q_G} G$. Using Itô's formula and the PDE satisfied by f^G we compute

$$\begin{aligned} -\alpha G - c_G &= \int_0^T df^G(t,Y_t) \\ &= \int_0^T \left(f_t^G + a f_y^G + \frac{1}{2} b^2 f_{yy}^G \right) dt + \int_0^T b f_y^G (\rho dB_t + \bar{\rho} dZ_t) \\ &= \int_0^T \left(\rho \lambda b f_y^G + \frac{1}{2} \bar{\rho}^2 b^2 (f_y^G)^2 - \frac{1}{2} \lambda^2 \right) dt + \int_0^T b f_y^G (\rho dB_t + \bar{\rho} dZ_t) \\ &= \int_0^T \left(\rho b f_y^G (dB_t + \lambda dt) + \bar{\rho} b f^G dZ_t + \frac{1}{2} \bar{\rho}^2 b^2 (f_y^G)^2 dt \right) - \frac{1}{2} K_T, \end{aligned}$$

and the result follows on recognising that

$$\begin{aligned} L_T^G &= \int_0^T \bar{\rho} f_y^G (t,Y_t) dZ_t, \\ N_T^G &= \int_0^T \rho f_y^G (t,Y_t)(dB_t + \lambda(t,Y_t)dt). \end{aligned}$$

□

4.3 Indifference pricing

Define the indifference selling price (at time t, given $Y_t = y$) of the claim G, $P_G(t,y)$, by

$$u_G(t, x + P_G(t,y), y) = u(t,x,y), \tag{63}$$

where u is the value function (15) when no claim is present, given by (22). Note that we have anticipated the well-known property of indifference prices under exponential preferences, namely that they do not depend on the initial cash endowment x. Using (22) and (54) in (63) gives the following representation for the indifference price.

Lemma 2

$$\exp\left(\alpha P_G(t,y)\right) = \left(\frac{F_G(t,y)}{F(t,y)} \right)^\delta. \tag{64}$$

Remark 7 *For $t = 0$, writing $P_G \equiv P_G(0,y)$ and using (32) and (58) in the right hand side of (64), we obtain*

$$\alpha P_G = - \inf_{\mathbb{Q} \in \mathcal{M}} \left[\mathcal{H}(\mathbb{Q}, \mathbb{P}) - \alpha \mathbb{E}^{\mathbb{Q}} G \right] + \mathcal{H}(Q^E, P),$$

or

$$P_G = \sup_{\mathbb{Q}\in\mathcal{M}} \left[E^Q G - \frac{1}{\alpha}\left(\mathcal{H}(Q,P) - \mathcal{H}(Q^E,P)\right)\right], \tag{65}$$

which is the representation found in [5], and from which well-known limits $\lim_{\alpha\to\infty} P_G = \sup_{Q\in\mathcal{M}} E^Q G$ *and* $\lim_{\alpha\to 0} P_G = E^{Q^E} G$ *easily follow.*

In contrast to (65), the next result recasts (64) into a representation for P_G involving the the minimal martingale measure.

Proposition 9 *The indifference price* P_G *has the representation*

$$\exp(\bar{\rho}^2 \alpha P_G) = \frac{E^{\widehat{Q}^M} \exp\left(-\bar{\rho}^2(\frac{1}{2}K_T - \alpha G)\right)}{E^{\widehat{Q}^M} \exp\left(-\bar{\rho}^2 K_T/2\right)}.$$

Proof Use the stochastic representations (24) and (55) for F, F_G, at $t=0$, in (64).

□

The above representation is used in Monoyios [17] to motivate an Esscher transform relation between Q^M and Q^E, which is made rigorous using the Hobson representation equation (44).

Remark 8 *Note that for deterministic* K *Proposition 9 reduces to*

$$\exp(\bar{\rho}^2 \alpha P_G) = E^{\widehat{Q}^M} \exp(\bar{\rho}^2 \alpha G),$$

which is the representation found in [11, 16, 18].

5 Conclusions

The distortion technique has been used to derive a number of representations for the optimal martingale measure of the dual problem. There is a direct link between the optimal distance $H_q(Q^{(q)}, P)$ and the distortion function F^δ, and between the derivative of $\log F^\delta$ and the optimal integrand ψ^* in the Girsanov density $dQ^*/dP = \mathcal{E}(-\lambda \cdot B - \psi^* \cdot Z)_T$. The distortion solution leads to new representations for the solution to the dual stochastic control problem and for utility indifference prices. We are able to treat different preferences in a unified framework using the parameter q in the q-optimal measure.

A topic for future research is to investigate the extent to which the results here can be generalized to non-Markovian model dynamics. Tehranchi [28] has established the distortion solution for general Itô process price dynamics satisfying Assumption 1, providing a platform from which to embark on such an investigation.

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