Elimination of Ramification I: The Generalized Stability Theorem*

Franz-Viktor Kuhlmann

17. 6. 2005

1 Introduction

1.1 The Main Theorem

In this paper, we consider the **defect** (also called **ramification deficiency**) of finite extensions of valued fields. For a valued field (K,v), we will denote its value group by vK and its residue field by $\overline{K}$ or by Kv. An extension of valued fields is written as $(L|K,v)$, meaning that v is a valuation on L and K is equipped with the restriction of this valuation. Every finite extension L of a valued field (K,v) satisfies the **fundamental inequality** (cf. [En], [Z–S]):

$$n \;\geq\; \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \tag{1}$$

where $n=[L:K]$ is the degree of the extension, $v_1,\ldots,v_{\mathrm{g}}$ are the distinct extensions of v from K to L, $\mathrm{e}_i=(v_iL:vK)$ are the respective ramification indices and $\mathrm{f}_i=[Lv_i:Kv]$ are the respective inertia degrees. Note that $\mathrm{g}=1$ if (K,v) is henselian.

We say that (K,v) is **defectless in** L if equality holds in (1). Further, (K,v) is called a **defectless** (or **stable**) field if it is defectless in every finite extension L of K. Finally, (K,v) is called a **separably defectless** field if it is defectless in every finite separable extension, and an **inseparably defectless** field if it is defectless in every finite purely inseparable extension of K. Note that every trivially valued field is a defectless field.

Every valued field of residue characteristic 0 is defectless; this is a consequence of the "Lemma of Ostrowski" (see Section 2.3). So we will always assume in the following that

$$p=\mathrm{char}(\overline{K})>0\;.$$

*I wish to thank Peter Roquette for his invaluable help and support. I also thank F. Delon, B. Green, A. Prestel, F. Pop, and R. Strebel for inspiring discussions and suggestions that helped me prove the Generalized Stability Theorem. During the work on this paper, I was partially supported by a Canadian NSERC grant. For the last two weeks of this work I have also been a member of the Newton Institute in Cambridge; I gratefully acknowledge its support.

Let $(L|K,v)$ be an extension of valued fields of finite transcendence degree. Then the following well known form of the "Abhyankar inequality" holds (it is a consequence of Lemma 10 below):

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{rr}\, vL/vK \;+\; \mathrm{trdeg}\, \overline{L}|\overline{K}\;, \tag{2}$$

where $\mathrm{rr}\, vF/vK := \dim_{\mathbb{Q}}\, (vF/vK)\otimes\mathbb{Q}$ is the **rational rank** of the abelian group vF/vK, i.e., the maximal number of rationally independent elements in vF/vK. We will say that $(L|K,v)$ is **without transcendence defect** if equality holds in (2). This paper is devoted to the proof of the following theorem, which has important applications.

Theorem 1 (Generalized Stability Theorem)
Let $(F|K,v)$ be a valued algebraic function field without transcendence defect. If (K,v) is a defectless field, then (F,v) is a defectless field. The same holds for "inseparably defectless" in the place of "defectless". If vK is cofinal in vF, then it also holds for "separably defectless" in the place of "defectless".

Take a valued field (K,v) and fix an extension of the valuation v to the separable-algebraic closure K^{sep} of K. Then the **absolute ramification field** K^r, the **absolute inertia field** (also called **strict henselization**) K^i, and the **henselization** K^h of (K,v) (with respect to the chose extension of v) are the ramification field, the inertia field and the decomposition field, respectively, of the extension $(K^{\mathrm{sep}}|K,v)$. Since all extensions of the valuation v from K to K^{sep} are conjugate (i.e., are obtained from each other by composing with an automorphism of $K^{\mathrm{sep}}|K$), these fields are unique up to valuation preserving isomorphism. Therefore, we will often speak of "*the* henselization" and work with K^h withour previously fixing an extension of the valuation. If $K^h = K$, then (K,v) is called **henselian**. This holds if and only if the extension of v from K to every algebraic extension field is unique (implying that g$=1$ in (1)), or equivalently, if and only if (K,v) satisfies Hensel's Lemma. An algebraic extension of a henselian field (K,v) is called **purely wild** if it is linearly disjoint from $K^r|K$.

The algebraic closure of a field K will be denoted by $\tilde{K}$. If $F|K$ is an arbitrary field extension and $L|K$ is an algebraic extension, then $L.F$ will denote the field compositum of L and F inside of $\tilde{F}$.

Corollary 2 *Let $(F|K,v)$ be a valued algebraic function field without transcendence defect, and $E|F$ a finite extension. Fix an extension of v from F to $\tilde{K}.F$. Then there is a finite extension $L_0|K$ such that for every algebraic extension L of K containing L_0, $(L.F,v)$ is defectless in $L.E$. If (K,v) is henselian, then $L_0|K$ can be chosen to be purely wild.*

Theorem 1 was stated and proved in [K1]; the proof presented here is an improved version. The theorem is a generalization of a result of Grauert and Remmert [G–R] which is restricted to the case of algebraically closed complete ground fields of rank 1 (i.e., with archimedean ordered value group). A generalization of the Grauert–Remmert

Theorem was given by Gruson [G]. A good presentation of it can be found in the book [B–G–R] of Bosch, Güntzer and Remmert (§5.3.2, Theorem 1). The proof uses methods of nonarchimedean analysis. Further generalizations are due to Matignon and Ohm; see also [G–M–P]. In [O], Ohm arrived independently of [K1] at a quite general version of the Stability Theorem; but like all of its forerunners, it is still restricted to the case $\mathrm{trdeg}(F|K) = \mathrm{trdeg}(\overline{F}|\overline{K})$. The idea of Ohm's proof is to reduce the general case of valuations of arbitrary rank to the case of rank 1 and to give an auxiliary argument for the transition from completions to henselizations. In this way, he is able to deduce his theorem from [B–G–R], Proposition 3, p. 215 (more precisely, from a generalized version of this proposition which is proved but not stated in [B–G–R]).

In contrast to this approach, we give a new proof which replaces the analytic methods of [B–G–R] by valuation theoretical arguments. Such arguments seem to be more adequate for a theorem that is of (Krull) valuation theoretical nature.

Similar to Ohm's approach, our proof of Theorem 1 in Section 5 uses reduction to valuations of rank 1; further reduction leads to Galois extensions of degree p of special henselized algebraic function fields F over algebraically closed ground fields ($p = \mathrm{char}\,\overline{K}$). Such Galois extensions are Artin–Schreier extensions generated by elements ϑ with $\wp(\vartheta) := \vartheta^p - \vartheta = a \in F$ if $\mathrm{char}\,(F) = p > 0$, and Kummer extensions (generated by p–th roots of elements $a \in F$) if $\mathrm{char}(F) = 0$. These extensions are studied in Section 4. We deduce normal forms for a which allow us to read off that the extension is defectless. A normal form for a can also be seen as a result on the structure of the group $F/\wp(F)$ or $F^\times/(F^\times)^p$, respectively. Such results can already be found in the work of Hasse, Whaples, Epp [E] and others, and in Matignon's proof of Theorem 1. For the case of $\mathrm{char}\,(F) = 0$, we have taken over Matignon's approach to replace an earlier proof which worked with extensions of Artin-Schreier form but did not lead to a normal form result for a. But it should be pointed out that even in characteristic 0, the Artin-Schreier form is still related with the Kummer form, as the transformation (5) and its further use will show. While in characteristic $p > 0$ we can make full use of the additivity of the Artin–Schreier polynomial $\wp(X) = X^p - X$, we use in characteristic 0 results on p-th roots of 1-units obtained in Section 2.2. Although not worked out in the present paper, these are connected with a "pseudo-additivity" of Artin–Schreier polynomials over henselian fields which applies only to elements of restricted value.

In the case where $\overline{F}|\overline{K}$ is an algebraic function field of transcendence degree 1, we have to use the existence of a **Frobenius-closed** basis for this function field. This is a basis where every p–th power of a basis element is again a basis element. The existence of such a basis is shown in [K1] and [K5]. It seems that such bases help to replace arguments of algebraic geometry that were used in the earlier approaches; it would certainly be an interesting task to determine the connection between both methods.

1.2 Applications

• **Elimination of ramification.** This is the task of finding a (nice) transcendence basis $\mathcal{T}$ for a given valued algebraic function field $(F|K,v)$ such that for some extension of v to the algebraic closure of F, the extension $(F^h|K(\mathcal{T})^h,v)$ of respective henselizations is **unramified**, that is, the residue fields form a separable extension $\overline{F}|\overline{K(\mathcal{T})}$ of degree equal to $[F^h : K(\mathcal{T})^h]$, which by the fundamental inequality implies that the value groups vF and $vK(\mathcal{T})$ are equal. (Note that passing to the henselization of a valued field does not change value group and residue field.) The above condition is equivalent to (F,v) being a subfield of the absolute inertia field of the extension $(K(\mathcal{T})^i,v)$, for some extension of v to $K(\mathcal{T})^{\text{sep}}$. Since all of these extensions of the valuation are conjugate and hence the respective absolute inertia fields of $(K(\mathcal{T}),v)$ are all isomorphic over F, our condition does not depend on the chosen extension.

A **standard algebraically valuation independent set** in a valued field extension $(F|K,v)$ is a set $\mathcal{T}=\{x_i,y_j \mid i\in I,\, j\in J\}$ where the values vx_i , $i\in I$, are rationally independent values in vF modulo vK, and the residues $\overline{y}_j$, $j\in J$, are algebraically independent over $\overline{K}$. If in addition $\mathcal{T}$ is a transcendence basis of $F|K$, then it is called a **standard valuation transcendence basis** of $(F|K,v)$. We use Theorem 1 to prove:

Theorem 3 *Take a defectless field (K,v) and a valued algebraic function field $(F|K,v)$ without transcendence defect. Assume that $\overline{F}|\overline{K}$ is a separable extension and vF/vK is torsion free. Then $(F|K,v)$ admits elimination of ramification in the following sense: there is a standard valuation transcendence basis $\mathcal{T}=\{x_1,\ldots,x_r,y_1,\ldots,y_s\}$ of $(F|K,v)$ such that*

a) $vF=vK\oplus\mathbb{Z}vx_1\oplus\ldots\oplus\mathbb{Z}vx_r$,

b) $\overline{y}_1,\ldots,\overline{y}_s$ form a separating transcendence basis of $\overline{F}|\overline{K}$.

For each such transcendence basis $\mathcal{T}$ and every extension of v to the algebraic closure of F, $(F^h|K(T)^h,v)$ is unramified.

Corollary 4 *Let $(F|K,v)$ be a valued algebraic function field without transcendence defect. Fix an extension of v to $\tilde{F}$. Then there is a finite extension $L_0|K$ and a standard valuation transcendence basis $\mathcal{T}$ of $(L_0.F|L_0,v)$ such that for every algebraic extension L of K containing L_0 , the extension $((L.F)^h|L(\mathcal{T})^h,v)$ is unramified.*

• **Local uniformization in positive and in mixed characteristic.** Theorem 3 is a crucial ingredient for our proof that Abhyankar places of algebraic function fields (i.e., places for which the corresponding valued algebraic function field is without transcendence defect) admit local uniformization (cf. [K–K1], [K2], [K3]), provided that the corresponding (necessary) separability condition for their residue field extension is satisfied. The proof uses solely valuation theory. The arithmetic case of local uniformization (also proved in [K–K1]) uses Theorem 1 in mixed characteristic. Building on this result, we prove in [K–K2] that all places admit local uniformization after a finite extension of the function field. This needs a second main theorem, which is proved in [K8] using a generalization of a method employed in [Ep] (see also [K4]).

Let us mention at this point that our approach has some similarity with S. Abhyankar's method of using ramification theory in order to reduce the question of resolution of singularities to the study of Galois extensions of degree p and the search for suitable normal forms of Artin–Schreier–like minimal polynomials.

• **Model theory of valued fields.** If $\mathcal{L}$ is an elementary language and $\mathcal{A} \subset \mathcal{B}$ are $\mathcal{L}$-structures, then we will say that $\mathcal{A}$ is **existentially closed in** $\mathcal{B}$ and write $\mathcal{A} \prec_\exists \mathcal{B}$ if every existential sentence with parameters from $\mathcal{A}$ that holds in $\mathcal{B}$ also holds in $\mathcal{A}$. When we talk of fields, then we use the language of rings or fields. When we talk of valued fields, we augment this language by a unary predicate for the valuation ring or a binary predicate for valuation divisibility. When we talk of ordered abelian groups, then we use the language of groups augmented by a binary predicate for the ordering. For the meaning of "existentially closed in" in the setting of valued fields and of ordered abelian groups, see [K–P]. We use Theorem 1 to prove the following Ax–Kochen–Ershov Principle:

Theorem 5 *Take a henselian defectless valued field (K, v) and an extension $(L|K, v)$ of finite transcendence degree without transcendence defect. If vK is existentially closed in vL (in the language of ordered groups) and $\overline{K}$ is existentially closed in $\overline{L}$ (in the language of fields), then (K, v) existentially closed in (L, v) (in the language of valued fields).*

This is used in [K7] to prove Ax–Kochen–Ershov Principles and further model theoretic results for tame valued fields.

With the tools developed in this paper, it is also possible to prove versions of Theorems 1 and 5 that work with infinite standard valuation transcendence bases.

2 Preliminaries

2.1 Facts from general valuation theory

We will use the following facts throughout the paper, often without citing. The first lemma is a consequence of the fundamental inequality (1).

Lemma 6 *Let $(L|K, v)$ be an algebraic extension of valued fields. Then vL/vK is a torsion group and the extension $Lv|Kv$ of residue fields is algebraic. If $L|K$ is finite, then so are vL/vK and $Lv|Kv$. If v is trivial on K (i.e., $vK = \{0\}$), then v is trivial on L.*

For each extension of the valuation v from K to its algebraic closure $\tilde{K}$, the value group of $\tilde{K}$ is divisible and the residue field of $\tilde{K}$ is algebraically closed.

An extension $(L|K, v)$ is called **immediate** if the canonical embeddings of vK in vL and of Kv in Lv are onto. For the next results, and general background in valuation theory, we refer the reader to [En], [R], [W], [Z–S], [K9].

Lemma 7 *The henselization K^h of a valued field (K,v) (which is unique up to valuation preserving isomorphism over K) is an immediate separable-algebraic extension and has the following universal property: if (L,v') is an arbitrary henselian extension field of (K,v), then there is a unique valuation preserving embedding of (K^h,v) in (L,v') over K. Thus, a henselization of (K,v) can be chosen in every henselian valued extension field of (K,v).*

Lemma 8 *An algebraic extension of a henselian valued field, equipped with the unique extension of the valuation, is again henselian.*

If $L|K$ is algebraic, then $(L.K^h,v)$ is the henselization of (L,v) (with respect to the unique extension of v from K^h to $\tilde{K}$).

Take any valued field (K,w) and a valuation $\overline{w}$ on the residue field Kw. Then the **composition** $w\circ\overline{w}$ is the valuation whose valuation ring is the subring of the valuation ring of w consisting of all elements whose w-residues lie in the valuation ring of $\overline{w}$. (Note that we identify equivalent valuations.) While $w\circ\overline{w}$ does actually not mean the composition of w and $\overline{w}$ as mappings, this notation is used because in fact, up to equivalence the place associated with $w\circ\overline{w}$ is indeed the composition of the places associated with w and $\overline{w}$.

For a valued field (K,v), every convex subgroup Γ of vK gives rise to a valuation v_Γ such that $v_\Gamma K$ is isomorphic to vK/Γ and the valuation ring of v is contained in that of v_Γ. Then v induces a valuation $\overline{v}_\Gamma$ on the residue field Kv_Γ (whose valuation ring is the image of the valuation ring of v under the place associated with v_Γ), such that $v=v_\Gamma\circ\overline{v}_\Gamma$. The value group $\overline{v}_\Gamma(Kv_\Gamma)$ of $\overline{v}_\Gamma$ can be identified with Γ and its residue field $(Kv_\Gamma)\overline{v}_\Gamma$ with Kv.

The **rank** of a valued field (K,v) is the order type of the chain of non-trivial convex subgroups of its value group vK. It has rank 1 (i.e., its only convex subgroups are $\{0\}$ and vK) if and only if vK is **archimedean**, that is, embeddable in the ordered additive group of the reals. This holds if and only if v is not the composition of two non-trivial valuations. If (K,v) has finite rank n, then v is the composition $v=v_1\circ\ldots\circ v_n$ of n many rank 1 valuations. The following lemma does in general not hold for valuations of rank >1:

Lemma 9 *If (K,v) is a valued field of rank 1, then K is dense in its henselization. In particular, the completion of (K,v) is henselian.*

For the easy proof of the following lemma, see [B], chapter VI, §10.3, Theorem 1.

Lemma 10 *Let $(L|K,v)$ be an extension of valued fields. Take elements $x_i,y_j\in L$, $i\in I$, $j\in J$, such that the values vx_i, $i\in I$, are rationally independent over vK, and the residues y_jv, $j\in J$, are algebraically independent over Kv. Then the elements x_i,y_j, $i\in I$, $j\in J$, are algebraically independent over K.*

Moreover, if we write

$$f\;=\;\sum_k c_k\prod_{i\in I}x_i^{\mu_{k,i}}\prod_{j\in J}y_j^{\nu_{k,j}}\in K[x_i,y_j\mid i\in I,j\in J]$$

in such a way that for every $k \neq \ell$ there is some i s.t. $\mu_{k,i} \neq \mu_{\ell,i}$ or some j s.t. $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$vf \;=\; \min_k v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k \, vc_k \,+\, \sum_{i\in I} \mu_{k,i} vx_i \;. \tag{3}$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$\begin{aligned} vK(x_i, y_j \mid i\in I, j\in J) &= vK \oplus \bigoplus_{i\in I} \mathbb{Z} vx_i \\ K(x_i, y_j \mid i\in I, j\in J)v &= Kv\,(y_j v \mid j\in J)\;. \end{aligned}$$

Moreover, the valuation v on $K(x_i, y_j \mid i\in I, j\in J)$ is uniquely determined by its restriction to K, the values vx_i and the residues $y_j v$.

Corollary 11 *Take a valued algebraic function field $(F|K,v)$ without transcendence defec and set $r = \mathrm{rr}\, vF/vK$ and $s = \mathrm{trdeg}\, \overline{F}|\overline{K}$. Choose elements $x_1,\ldots,x_r,\ y_1,\ldots,y_s \in F$ such that the values $vx_1,\ldots,vx_r$ are rationally independent over vK and the residues $\overline{y}_1,\ldots,\overline{y}_s$ are algebraically independent over $\overline{K}$. Then $\mathcal{T} = \{x_1,\ldots,x_r,y_1,\ldots,y_s\}$ is a standard valuation transcendence basis $(F|K,v)$ Moreover, the quotient vF/vK and the extension $\overline{F}|\overline{K}$ are finitely generated.*

Proof: By the foregoing theorem, the elements $x_1,\ldots,x_r,y_1,\ldots,y_s$ are algebraically independent over K. Since $\mathrm{trdeg}\, F|K = r+s$ by assumption, these elements form a basis and hence a standard valuation transcendence basis of $(F|K,v)$.

It follows that the extension $F|K(\mathcal{T})$ is finite. By the fundamental inequality (1), this yields that $vF/vK(\mathcal{T})$ and $\overline{F}|\overline{K(\mathcal{T})}$ are finite. Since already $vK(\mathcal{T})/vK$ and $\overline{K(\mathcal{T})}|\overline{K}$ are finitely generated by the foregoing theorem, it follows that also vF/vK and $\overline{F}|\overline{K}$ are finitely generated. □

Since an infinite number of distinct convex subgroups in an ordered abelian group give rise to an infinite number of rationally independent elements in that group, the following is also a consequence of Lemma 10:

Corollary 12 *Every valued field of finite transcendence degree over its prime field has finite rank.*

Lemma 13 *If $(F|K,v)$ is a valued algebraic function field without transcendence defect and $v = w\circ\overline{w}$, then $(F|K,w)$ and $(Fw|Kw,\overline{w})$ are valued function fields without transcendence defect.*

Proof: The value group $\overline{w}(Fw)$ can be considered (in a canonical way) a subgroup of vF and then wF is isomorphic to $vF/\overline{w}(Fw)$; the same holds for K in the place of F.

From this it follows that $\mathrm{rr}\, vF/vK = \mathrm{rr}\, wF/wK + \mathrm{rr}\, \overline{w}(Fw)/\overline{w}(Kw)$. Hence,

$$\begin{aligned} \mathrm{trdeg}\, F|K &= \mathrm{rr}\, vF/vK + \mathrm{trdeg}\, Fv|Kv \\ &= \mathrm{rr}\, wF/wK + \mathrm{rr}\, \overline{w}(Fw)/\overline{w}(Kw) + \mathrm{trdeg}\,(Fw)\overline{w}|(Kw)\overline{w} \\ &\leq \mathrm{rr}\, wF/wK + \mathrm{trdeg}\, Fw|Kw \leq \mathrm{trdeg}\, F|K , \end{aligned}$$

and we find that equality must hold everywhere. The last equality then shows that $(F|K,w)$ is without transcendence defect, which by Corollary 11 implies that $Fw|Kw$ is an algebraic function field. We also obtain that $\mathrm{rr}\, \overline{w}(Fw)/\overline{w}(Kw) + \mathrm{trdeg}\,(Fw)\overline{w}|(Kw)\overline{w} = \mathrm{trdeg}\, Fw|Kw$, which shows that $(Fw|Kw,\overline{w})$ is without transcendence defect. □

2.2 p-th roots of 1-units in henselian fields of mixed characteristic

A well known application of Hensel's Lemma shows that in every henselian field, each 1-unit (an element of the form $1+b$ with $vb > 0$) is an n-th power for every n not divisible by the residue characteristic p. If the latter is not the case, then one considers the "level" of the 1-unit. For our purposes, we need a more precise result for a 1-unit to have a p-th root.

Throughout this paper, we will take C to be an element in the algebraic closure of $\mathbb{Q}$ such that $C^{p-1} = -p$, where $p > 0$ is a prime. Take a henselian field (K,v) of characteristic 0 and residue characteristic p. Extend the valuation v to the algebraic closure of K. Note that

$$C^p = -pC \quad \text{and} \quad vC = \frac{1}{p-1} vp > 0 .$$

Consider the polynomial

$$X^p - (1+b) \tag{4}$$

with $b \in K$. Performing the transformation

$$X = CY + 1 , \tag{5}$$

we obtain the polynomial

$$f(Y) = Y^p + g(Y) - Y - \frac{b}{C^p} \tag{6}$$

with

$$g(Y) = \sum_{i=2}^{p-1} \binom{p}{i} C^{i-p} Y^i \tag{7}$$

a polynomial with coefficients in $K(C)$ of value > 0.

Lemma 14 *Take (K, v) and C as above. Then K contains C if and only if it contains all p-th roots of unity.*

Proof: Since $\operatorname{char} \overline{K} = p$, the restriction of v to $\mathbb{Q} \subset K$ is the p-adic valuation. Since (K, v) is henselian, it contains $(\mathbb{Q}^h, v_p)$. The p-th roots of unity being algebraic over $\mathbb{Q}$, it suffices to prove that $\mathbb{Q}^h(C)$ is the smallest extension of $\mathbb{Q}^h$ which contains all p-th roots of unity. Applying the transformation (5) to the polynomial (4) with $b = 0$, we find that K contains all p-th roots of unity if the polynomial $f(Y) = Y^p + g(Y) - Y$ splits over K. But this holds by Hensel's Lemma because the polynomial $\overline{f}(Y) = Y^p - Y$ splits over $\mathbb{F}_p$. Let $\eta \neq 1$ be a p-th root of unity. Since the non-zero roots of f have nonzero residue and thus value zero, $v(\eta - 1) = vC = vp/(p-1)$. We find that $(v\mathbb{Q}^h(\eta) : v\mathbb{Q}^h) \geq p - 1$. Consequently,

$$[\mathbb{Q}^h(\eta) : \mathbb{Q}^h] \;\geq\; p - 1 \;\geq\; [\mathbb{Q}^h(C) : \mathbb{Q}^h] \;\geq\; [\mathbb{Q}^h(\eta) : \mathbb{Q}^h] \;,$$

showing that equality holds everywhere and that $\mathbb{Q}^h(\eta) = \mathbb{Q}^h(C)$. □

Lemma 15 *Let (K, v) be a henselian field containing all p-th roots of unity. Then for all $b \in K$,*

$$vb > \frac{p}{p-1} vp \;\;\Rightarrow\;\; 1 + b \in (K^\times)^p \;.$$

Proof: Consider the polynomial (6). If $vb > \frac{p}{p-1} vp = vC^p$, then $\overline{f}(Y) = Y^p - Y$, which splits over $\overline{K}$. By Hensel's Lemma, this implies that $f(Y)$ splits over K. Via the transformation (5), it follows that $1 + b$ has a p-th root in K. □

Corollary 16 *Let (K, v) be a henselian field containing all p-th roots of unity. Take any 1-units $1 + b$ and $1 + c$ in K. Then:*

a) $1 + b \in (1 + b + c) \cdot (K^\times)^p$ *if* $vc > \frac{p}{p-1} vp$ *.*

b) $1 + b \in (1 + b + c) \cdot (K^\times)^p$ *if* $1 + c \in (K^\times)^p$ *and* $vbc > \frac{p}{p-1} vp$ *.*

c) $1 + c^p + pc \in (K^\times)^p$ *if* $vc^p > vp$ *.*

d) $1 + b - pc \in (1 + b + c^p) \cdot (K^\times)^p$ *if* $vb \geq \frac{1}{p-1} vp$ *and* $vc^p > vp$ *.*

Proof: a): $1 + b \in (1 + b + c)(K^\times)^p$ is true if the quotient

$$\frac{1 + b + c}{1 + b} \;=\; 1 + \frac{c}{1 + b}$$

is an element of $(K^\times)^p$. By hypothesis we have $vb > 0$ and thus $v\frac{c}{1+b} = vc$. Now our assertion follows from Lemma 15.

b): An application of part a) shows that

$$(1+b+c) \;\in\; (1+b)(1+c)\cdot (K^\times)^p \quad \text{if} \;\; vbc > \tfrac{p}{p-1}vp.$$

The assertion of b) is an immediate consequence of this.
c): If $vc^p > vp$ then for every $i = 2, \ldots, p-1$ we have

$$v\left(\begin{array}{c} p\\ i \end{array}\right)c^i \;\geq\; vp + 2vc \;>\; \frac{p+2}{p}vp \;\geq\; \frac{p}{p-1}vp\;;$$

note that the last inequality holds for every $p \geq 2$. This together with assertion a) yields

$$1 + c^p + pc \in \left(1 + c^p + pc + \sum_{i=2}^{p-1}\left(\begin{array}{c} p\\ i \end{array}\right)c^i\right)(K^\times)^p \;=\; (1+c)^p(K^\times)^p \;=\; (K^\times)^p\;.$$

d): In view of part c), the assertion follows from part b) where b is replaced by $b - pc$ and c is replaced by $c^p + pc$. Note that b) can be applied since $v(b-pc)(c^p+pc) > \frac{1}{p-1}vp + vp = \frac{p}{p-1}vp$. □

2.3 The defect

Assume that $(L|K,v)$ is a finite extension and the extension of v from K to L is unique. Then the Lemma of Ostrowski says that

$$[L:K] \;=\; (vL:vK)\cdot [Lv:Kv]\cdot p^\nu \quad \text{ with } \nu \geq 0 \tag{8}$$

where p is the **characteristic exponent** of Kv, that is, $p = \operatorname{char} Kv$ if this is positive, and $p = 1$ otherwise (cf. [En], [R], [K9]). The factor

$$\mathrm{d}(L|K,v) \;:=\; p^\nu \;=\; \frac{[L:K]}{(vL:vK)[Lv:Kv]}$$

is called the **defect** of the extension $(L|K,v)$. If $\nu > 0$, then we talk of a **non-trivial** defect. If $[L:K] = p$ then $(L|K,v)$ has non-trivial defect if and only if it is immediate, i.e., $(vL:vK) = 1$ and $[Lv:Kv] = 1$. If $\mathrm{d}(L|K,v) = 1$, then we call $(L|K,v)$ a **defectless extension**. Note that $(L|K,v)$ is always defectless if $\operatorname{char} Kv = 0$. Therefore,

Corollary 17 *Every valued field (K,v) of residue characteristic $\operatorname{char} Kv = 0$ is a defectless field.*

The following lemma shows that the defect is multiplicative. This is a consequence of the multiplicativity of the degree of field extensions and of ramification index and inertia degree. We leave the straightforward proof to the reader.

Lemma 18 *Fix an extension of a valuation* v *from* K *to its algebraic closure. If* $L|K$ *and* $M|L$ *finite extensions and extension of* v *from* K *to* M *is unique, then*

$$\mathrm{d}(M|K,v) = \mathrm{d}(M|L,v)\cdot \mathrm{d}(L|K,v) \tag{9}$$

In particular, $(M|K,v)$ *is defectless if and only if* $(M|L,v)$ *and* $(L|K,v)$ *are defectless.*

Theorem 19 *Take a valued field* (K,v) *and fix an extension of* v *to* $\tilde{K}$*. Then* (K,v) *is defectless if and only if its henselization* $(K,v)^h$ *in* $(\tilde{K},v)$ *is defectless. The same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".*

Proof: For "separably defectless", our assertion follows directly from [En], Theorem (18.2). This theorem shows that (K,v) is defectless in a finite separable extension L if and only if $(L_i.K^h|K^h,v)$ is defectless for all fields L_i isomorphic to L over K. Since $K^h|K$ is separable, $L|K$ is separable if and only if all $L_i.K^h|K^h$ are. Hence if K^h is separably defectless, then so is K. For the converse, observe that for every finite extension $L'|K^h$ there is a finite extension $L|K$ such that $L' \subseteq L.K^h$. If $L'|K$ is separable, then $L|K$ can be chosen separable (and if $L'|K$ is purely inseparable, then $L|K$ can be chosen purely inseparable). Hence if (K,v) is separably defectless, then $(L.K^h|K^h,v)$ is defectless, and it follows from Lemma 18 that $(L'|K^h,v)$ is defectless.

A closer look at Endler's proof of Theorem (18.2) reveals that the restriction to separable extensions $L|K$ is not necessary. Since the henselization is a separable extension, the factors g_{λ_i} appearing in (17.3) of [En] are all equal to 1, showing that the crucial equality $[L:K] = \sum_{i=1}^{r} [\overline{L}_i : \overline{K}]$ in Endler's proof remains true for *all* finite extensions $L|K$. From this we obtain our assertion for "defectless" and for "inseparably defectless". □

Since a henselian field has a unique extension of the valuation to every algebraic extension field, we obtain:

Corollary 20 *A valued field* (K,v) *is defectless if and only if* $\mathrm{d}(L|K^h,v) = 1$ *for every finite extension* $L|K^h$.

Using this corollary together with Lemma 18, one easily shows:

Corollary 21 *Every finite extension of a defectless field is again a defectless field.*

Lemma 22 *Let* (K,v) *be a valued field with* $v = w\circ\overline{w}$*. If* (K,w) *and* $(Kw,\overline{w})$ *are defectless fields, then so is* (K,v).

Proof: Take a finite extension $L|K$; we wish to show that equality holds in (1). Let $w_1,\ldots,w_{\mathrm{g}_w}$ be all extensions of w from K to L, and set $\mathrm{e}_i^w = (w_iL : wK)$ and $\mathrm{f}_i^w = [Lw_i : Kw]$ for $1\leq i\leq \mathrm{g}_w$. Further, for $1\leq i\leq \mathrm{g}_w$, let $\overline{w}_1,\ldots,\overline{w}_{\mathrm{g}_i}$ be all extensions of $\overline{w}$ from Kw to Lw_i, and set $\mathrm{e}_{ij} = (\overline{w}_{ij}(Lw_i) : \overline{w}(Kw))$ and $\mathrm{f}_{ij} = [(Lw_i)\overline{w}_{ij} : (Kw)\overline{w}] = [L(w_i\circ \overline{w}_{ij}) : Kv]$ for $1\leq j\leq \mathrm{g}_i$. Since (K,w) is a defectless field, we have $[L:K] = \sum_{i=1}^{\mathrm{g}_w} \mathrm{e}_i^w \mathrm{f}_i^w$.

Since $(Kw, \overline{w})$ is a defectless field, we have $\mathrm{f}_i^w = [Lw_i : Kw] = \sum_{j=1}^{\mathrm{g}_i} \mathrm{e}_{ij}\mathrm{f}_{ij}$. Using that $\mathrm{e}_i^w \mathrm{e}_{ij} = ((w_i \circ \overline{w}_{ij})L : vK)$, we obtain:

$$[L : K] \;=\; \sum_{i=1}^{\mathrm{g}_w} \mathrm{e}_i^w \sum_{j=1}^{\mathrm{g}_i} \mathrm{e}_{ij}\mathrm{f}_{ij} \;=\; \sum_{i=1}^{\mathrm{g}_w}\sum_{j=1}^{\mathrm{g}_i} ((w_i \circ \overline{w}_{ij})L : vK)[L(w_i \circ \overline{w}_{ij}) : Kv] \,.$$

As the valuations $w_i \circ \overline{w}_{ij}$, $1 \leq i \leq \mathrm{g}_w$, $1 \leq j \leq \mathrm{g}_i$, are distinct extensions of v from K to L, the fundamental inequality implies that they are in fact all extensions, and we have proved that equality holds in (1). □

In [K6] we have proved the following:

Proposition 23 *Let (K, v) be a henselian field and N an arbitrary algebraic extension of K within K^r. If $L|K$ is a finite extension, then*

$$\mathrm{d}(L|K, v) \;=\; \mathrm{d}(L.N|N, v) \,.$$

In particular, $(L|K, v)$ is defectless if and only if $(L.N|N, v)$ is defectless. This implies: (K, v) is a defectless field if and only if (N, v) is a defectless field, and the same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".

2.4 Valuation disjoint and valuation regular extensions

Let $(L|K, v)$ be an extension of valued fields, and $\mathcal{B} \subset L$. Then $\mathcal{B}$ is said to be a **valuation independent set in** $(L|K, v)$ if for every $n \in \mathbb{N}$ and each choice of distinct elements $b_1, \ldots, b_n \in \mathcal{B}$ and arbitrary elements $c_1, \ldots, c_n \in K$,

$$v \sum_{i=1}^{v} c_i b_i \;=\; \min_{1 \leq i \leq n} v c_i b_i \,.$$

Further, $\mathcal{B}$ is called a **standard valuation independent set** (**in** $(L|K, v)$) if it is of the form $\mathcal{B} = \{b'_i b''_j \mid 1 \leq i \leq k\,,\ 1 \leq j \leq \ell\}$ where the values $vb'_1, \ldots, vb'_k$ lie in distinct cosets of vL modulo vK, and $b''_1, \ldots, b''_\ell$ are elements of value 0 whose residues are $\overline{K}$-linearly independent. Every standard valuation independent set is a valuation independent set.

Let $(\Omega|K, v)$ be an extension of valued fields and $F|K$ and $L|K$ two subextensions of $\Omega|K$. We say that $(F|K, v)$ is **valuation disjoint from** $(L|K, v)$ (**in** (Ω, v)) if every standard valuation independent set $\mathcal{B}$ of $(F|K, v)$ is also a standard valuation independent set of $(L.F|L, v)$. (It is possible that a standard valuation independent set $\mathcal{B}$ of $(F|K, v)$ remains valuation independent over (L, v) without remaining a standard valuation independent set of $(L.F|L, v)$.)

Let G and G' be two subgroups of some group $\mathcal{G}$, and H a common subgroup of G and G'. We will say that the group extension $G|H$ is **disjoint from** the group extension $G'|H$ (**in** $\mathcal{G}$) if every two elements in G that belong to distinct cosets modulo H will also belong to distinct cosets modulo G'. This holds if and only if $G \cap G' = H$. Hence, our notion "disjoint from" is symmetrical, like "linearly disjoint from" in the field case.

Now valuation disjoint extensions can be characterized as follows:

Lemma 24 *Let $(\Omega|K,v)$ be an extension of valued fields and $F|K$ and $L|K$ subextensions of $\Omega|K$. Then $(F|K,v)$ is valuation disjoint from $(L|K,v)$ in (Ω,v) if and only if*
1) $vF|vK$ is disjoint from $vL|vK$ in $v\Omega$, and
2) $\overline{F}|\overline{K}$ is linearly disjoint from $\overline{L}|\overline{K}$ in $\overline{\Omega}$.
Consequently, the notion of "valuation disjoint" is symmetrical.

Proof: Let $\mathcal{B}$ be a standard valuation independent set in $(F|K,v)$ of the form as given in the definition. By condition 1), the values $vb'_1,\ldots,vb'_k$ also lie in distinct cosets of $v(L.F)$ modulo vL. By condition 2), the residues $b''_1,\ldots,vb''_\ell$ remain $\overline{L}$-linearly independent. Hence, $\mathcal{B}$ be a standard valuation independent set in $(L.F|L,v)$.

For the converse, assume that condition 1) or 2) is not satisfied. If 1) is not satisfied, then there are two elements $b,b' \in F$ whose values belong to distinct cosets modulo vK but to the same coset modulo vL. Hence, $\{b,b'\}$ is a standard valuation independent set in $(F|K,v)$, but not in $(L.F|L,v)$. If 2) is not satisfied, then there are elements $b_1,\ldots,b_n \in F$ of value 0 whose residues are $\overline{K}$-linearly independent but not $\overline{L}$-linearly independent. Then $\{b_1,\ldots,b_n\}$ is a standard valuation independent set in $(F|K,v)$, but not in $(L.F|L,v)$. □

Recall that an extension $F|K$ is called **regular** if it is linearly disjoint from $\tilde{K}|K$, or equivalently, if it is separable and K is relatively algebraically closed in F. An extension $(F|K,v)$ of valued fields will be called **valuation regular** if it is valuation disjoint from $(\tilde{K}|K,v)$ in $(\tilde{F},v)$ for some extension of v from F to $\tilde{F}$. Using that $v\tilde{K}$ is the divisible hull of K and $\overline{\tilde{K}}$ is the algebraic closure of $\overline{K}$ (Lemma 6), one deduces the following characterization of valuation regular extensions:

Lemma 25 *An extension $(F|K,v)$ is valuation regular if and only if*
1) vF/vK is torsion free,
2) $\overline{F}|\overline{K}$ is regular.
Consequently, $(F|K,v)$ is valuation regular if and only if it is valuation disjoint from $(\tilde{K}|K,v)$ for every *extension of v from F to $\tilde{F}$. Every valued field extension of an algebraically closed valued field is valuation regular.*

Since the henselization of a valued field is an immediate extension, this lemma yields:

Corollary 26 *If $(F|K,v)$ is valuation regular, then also $(F^h|K^h,v)$ is valuation regular.*

Important examples of valuation regular extensions are the valued field extensions which are generated by standard algebraically valuation independent sets. Indeed, it follows from Lemma 10 that they satisfy the conditions of the above lemma. Using also Lemma 24, we obtain:

Lemma 27 *Let $(\Omega|K,v)$ be an extension of valued fields containing a standard algebraically valuation independent set $\mathcal{T}$. Then $(K(\mathcal{T})|K,v)$ is a valuation regular extension.*

Lemma 28 *Assume that $(F|K,v)$ is a valuation regular subextension of a valued field extension $(\Omega|K,v)$. If $\mathcal{T}$ is a standard algebraically valuation independent set in $(\Omega|F,v)$, then also $(F(\mathcal{T})|K(\mathcal{T}),v)$ is a valuation regular extension.*

Proof: We write $\mathcal{T} = \{x_i\,,\, y_j \mid i\in I\,,\, j\in J\} \subset F$ such that the values $vx_i\,,\ i\in I$, are rationally independent over vF, and that the residues $\overline{y}_i\,,\ j\in J$, are algebraically independent over $\overline{F}$. Since $(F|K,v)$ is assumed to be valuation regular we know from Lemma 25 that vF/vK is torsion free and that $\overline{F}|\overline{K}$ is regular. The former implies that also $vF\oplus\bigoplus_{i\in I}\mathbb{Z}vx_i$ is torsion free modulo $vK\oplus\bigoplus_{i\in I}\mathbb{Z}vx_i$. Similarly, the latter implies that also the extension $\overline{F}(\overline{y}_j\mid j\in J)\,|\,\overline{K}(\overline{y}_j\mid j\in J)$ is regular (this follows from the fact that the elements $\overline{y}_j$ are algebraically independent over $\overline{F}$ by assumption). Again by Lemma 25, we conclude that $(F(\mathcal{T})|K(\mathcal{T}),v)$ is a valuation regular extension. □

Valuation disjoint extensions are particularly interesting in combination with defectless extensions:

Proposition 29 *Take an extension $(F|K,v)$ of henselian fields and a finite algebraic extension $(L|K,v)$, valuation disjoint from $(F|K,v)$ (in $(\tilde{F},v)$). Then*

$$\mathrm{d}(L|K,v)\ \geq\ \mathrm{d}(L.F|F,v)\,. \tag{10}$$

Proof: Since $vL|vK$ is disjoint from $vF|vK$ and $\overline{L}|\overline{K}$ is linearly disjoint from $\overline{F}|\overline{K}$, we find that

$$\left.\begin{array}{rcl}(v(L.F):vF) & \geq & (vL+vF:vF)=(vL:vK)\\ [\overline{L.F}:\overline{F}] & \geq & [\overline{L}.\overline{F}:\overline{F}]=[\overline{L}:\overline{K}]\,.\end{array}\right\} \tag{11}$$

where the inequalities hold because $v(L.F)$ contains both vL and vF, and $\overline{L.F}$ contains both $\overline{L}$ and $\overline{F}$. Now inequality (10) follows from this, the definition of the defect and the fact that $[(L.F)^h:F^h]=[L^h.F^h:K^h.F^h]\leq[L^h:K^h]$. □

Corollary 30 *Take a valuation regular extension $(F|K,v)$, fix an extension of v from F to $\tilde{F}$ and assume that (K,v) and $(\tilde{K}.F,v)$ are defectless fields. Then also (F,v) is a defectless field.*

Proof: From Theorem 19 we know that (K^h,v) and $((\tilde{K}.F)^h,v)$ are defectless fields and that it suffices to prove that (F^h,v) is a defectless field. We will prove that every finite subextension $E|F^h$ of $(\tilde{K}.F)^h|F^h$ is defectless. Since $((\tilde{K}.F)^h,v)$ is a defectless field, it will then follow from Lemma 2.3 of [K6] that (F^h,v) is a defectless field.

Since $(\tilde{K}.F)^h = \tilde{K}.F^h$ by Lemma 8, there is a finite extension $L|K^h$ such that $E \subseteq L.F^h$. Since (K^h, v) is a defectless field, we know that $\mathrm{d}(L|K^h, v) = 1$. Since $(F|K, v)$ is assumed to be valuation regular, the same holds for $(F^h|K^h, v)$ by Corollary 26. Thus it follows from Lemma 29 that $\mathrm{d}(L.F^h|F^h, v) \leq \mathrm{d}(L|K^h, v) = 1$. Using Lemma 18 we conclude that $\mathrm{d}(E|F^h, v) = 1$. □

More properties of valuation disjoint and valuation regular extensions and an introduction to the notion of valuation separable extensions can be found in [K9].

2.5 Henselized function fields

Theorem 19 shows that a valued field is defectless if and only if its henselization is. So instead of working with a valued algebraic function field, we will rather analyze its henselization. Such a henselization will be called a **henselized function field**. Every finite extension of a henselized function field is again a henselized function field. Further, a henselized function field $(F|K, v)$ will be called **henselized rational with generator** x if $F = K(x)^h$. We will say that $(F|K, v)$ is **henselized inertially generated with generator** x if (F, v) is a finite unramified extension of a henselized rational function field with generator x.

We note that $(F|K, v)$ is henselized inertially generated with generator x if and only if there exists some $y \in F$ such that $F = K(x)^h(y)$, $vy = 0$ and $\overline{K(x)}(\overline{y})|\overline{K(x)}$ is a finite separable extension of degree

$$[\overline{K(x)}(\overline{y}) : \overline{K(x)}] \;=\; [K(x, y) : K(x)] \;=\; [K(x)^h(y) : K(x)^h]\,. \tag{12}$$

Indeed, if the latter holds, then $F|K(x)^h$ is unramified. Conversely, if $F|K(x)^h$ is unramified, then $\overline{F}|\overline{K(x)}$ is a finite separable extension, of degree $[F : K(x)^h]$, and we can choose some $\zeta \in \overline{F}$ such that $\overline{F} = \overline{K(x)}(\zeta)$. Take a monic polynomial f with coefficients in the valuation ring of $K(x)$ and such that $\overline{f}$ is the minimal polynomial of ζ over $\overline{K(x)}$. By Hensel's Lemma, f has a root $y \in F$ such that $\overline{y} = \zeta$. Since $\deg(f) = \deg(\overline{f})$, we obtain the first equation in (12). Consequently,

$$[F : K(x)^h] \geq [K(x)^h(y) : K(x)^h] \geq [K(x, y) : K(x)] = [\overline{F} : \overline{K(x)}] = [F : K(x)^h]\,.$$

Hence, equality holds everywhere, showing that $F = K(x)^h(y)$ and that the second equation in (12) holds. Note that $vF = vK(x)$ by Lemma 10.

Take a henselized inertially generated function field $(F|K, v)$ of transcendence degree 1 without transcendence defect. Then either $\mathrm{rr}\, vF/vK = 1$ and there is a **value-transcendental** element $x \in F$, i.e., its value vx is rationally independent over vK, or $\mathrm{trdeg}\, \overline{F}|\overline{K} = 1$ and there is a **residue-transcendental** element $x \in F$, i.e., $vx = 0$ and the residue $\overline{x}$ is transcendental over $\overline{K}$. In both cases, x is transcendental over K by Lemma 10, hence $F|K(x)$ is finite.

Assume in addition that the element x is a generator for the henselized inertially generated function field $(F|K,v)$. Then in the first case, we call x a **value-transcendental generator**, and in the second case a **residue-transcendental generator** for $(F|K,v)$. In both cases, we also call x a **valuation-transcendental generator**. In the first (the "value-transcendental") case, $\overline{F}|\overline{K}$ is finite, and in the second (the "residue-transcendental") case, vF/vK is a finite torsion group (cf. Corollary 11).

If in addition K is algebraically closed, then by Lemma 6, $\overline{F}=\overline{K}$ is algebraically closed and vK is divisible in the value-transcendental case, and $vF=vK$ is divisible and $\overline{K}$ algebraically closed (and hence relatively algebraically closed in $\overline{F}$) in the residue-transcendental case. In particular, we see that a henselized rational function field of transcendence degree 1 with a value-transcendental generator over an algebraically closed field does not admit proper unramified extensions, hence every henselized inertially generated function field of transcendence degree 1 with a value-transcendental generator over an algebraically closed field is actually a henselized rational function field.

Lemma 31 *Let $(F|K,v)$ be a henselized inertially generated function field with value-transcendental (or residue-transcendental) generator over an algebraically closed field K. If E is a finite extension of F within F^r, then $(E|K,v)$ is again a henselized inertially generated function field with value-transcendental (or residue-transcendental, respectively) generator.*

Proof: Case I): $(F|K,v)$ has a value-transcendental generator x. Then by what we have shown above, $F=K(x)^h$. Since $vF=vK(x)=vK\oplus\mathbb{Z}vx$, vK is divisible and $vE|vF$ is finite, we have that $vE=vK\oplus\mathbb{Z}\frac{vx}{n}$ for some $n\geq 1$. Since $E\subset F^r$, n is equal to $[E:F]$ and prime to char $\overline{K}$. As $\overline{F}=\overline{K}$ is algebraically closed and $\overline{E}|\overline{F}$ is finite, we find that $\overline{E}=\overline{F}$. Therefore, Hensel's Lemma can be used to find an element y in the henselian field E such that $y^n=cx$ for some $c\in F$ of value 0. We have that $vy=\frac{vx}{n}$, $K(x)^h\subset K(y)^h$, $\overline{K(y)}=\overline{K(x)}$ and

$$\begin{aligned}[E:K(x)^h] &\geq [K(y)^h:K(x)^h] \geq (vK(y):vK(x))\cdot[\overline{K(y)}:\overline{K(x)}]\\ &= (vK(y):vK(x)) \geq n = [E:K(x)^h]\,.\end{aligned}$$

Hence, equality holds everywhere, showing that $E=K(y)^h$. Since vy is rationally independent over vK, $(E|K,v)$ is a henselized rational function field with value-transcendental generator y.

Case II): $(F|K,v)$ has a residue-transcendental generator x. Then by what we have shown earlier, $vF=vK$ is divisible, and since vE/vF is finite, we find that $vE=vF$. Hence $(E|F,v)$ is unramified and $(E|K,v)$ is again a henselized inertially generated function field with residue-transcendental generator x. □

3 Inseparably defectless fields

In this section, we shall give the proof of the "inseparably defectless" version of the Generalized Stability Theorem:

Proposition 32 *Let $(F|K,v)$ be a valued algebraic function field without transcendence defect. If (K,v) is an inseparably defectless field, then (F,v) is an inseparably defectless field.*

Proof: We choose a standard valuation transcendence basis $\mathcal{T}$ of $(F|K,v)$ as in Corollary 11. Then $F|K(\mathcal{T})$ is a finite extension. By Lemma 4.15 of [K6], every finite extension of an inseparably defectless field is again an inseparably defectless field. Hence it suffices to prove that $(K(\mathcal{T}),v)$ is an inseparably defectless field.

Every finite purely inseparable extension L of $K(\mathcal{T})$ is contained in an extension $E = K'(\mathcal{T}^{1/p^m}) = K'(t^{1/p^m} \mid t \in \mathcal{T})$ for a suitable $m \in \mathbb{N}$ and some finite purely inseparable extension K' of K. Since $K'|K$ is algebraic, we know from Lemma 6 that vK'/vK is a torsion group and $\overline{K'}|\overline{K}$ is algebraic. Consequently, the values $vx_i^{1/p^m} = \frac{vx_i}{p^m}$, $1 \leq i \leq r$, are still rationally independent over vK', and the residues $\overline{y^{1/p^m}}_i = \overline{y}_i^{1/p^m}$, $1 \leq j \leq s$, are still algebraically independent over $\overline{K'}$. This proves that $\mathcal{T}^{1/p^m}$ is a standard valuation transcendence basis of $(E|K',v)$. Now Lemma 10 shows that

$$vE = vK' \oplus \mathbb{Z}vx_1^{1/p^m} \oplus \ldots \oplus \mathbb{Z}vx_r^{1/p^m} = vK' \oplus \mathbb{Z}\frac{vx_1}{p^m} \oplus \ldots \oplus \mathbb{Z}\frac{vx_r}{p^m}$$

and that

$$\overline{E} = \overline{K'}\left(\overline{y_1^{1/p^m}}, \ldots, \overline{y_s^{1/p^m}}\right) = \overline{K'}(\overline{y_1}^{1/p^m}, \ldots, \overline{y_s}^{1/p^m}) \,,$$

whence

$$\begin{aligned}
[E : K(\mathcal{T})] &= [K'(\mathcal{T}^{1/p^m}) : K'(\mathcal{T})] \cdot [K'(\mathcal{T}) : K(\mathcal{T})] = p^{m(r+s)} \cdot [K' : K] \\
&= p^{mr} \cdot p^{ms} \cdot (vK' : vK) \cdot [\overline{K'} : \overline{K}] = p^{mr} \cdot (vK' : vK) \cdot p^{ms} \cdot [\overline{K'} : \overline{K}] \\
&= (vE : vK(\mathcal{T})) \cdot \left[\overline{E} : \overline{K(\mathcal{T})}\right]
\end{aligned}$$

since $(K'|K,v)$ is defectless by hypothesis. As the extension of the valuation is unique in every purely inseparable algebraic extension, this equation shows that $(E|K(\mathcal{T}),v)$ is defectless. Then by Lemma 18, also its subextension $(L|K(\mathcal{T}),v)$ is defectless. □

4 Galois extensions of degree p

In this section, we will consider the structure of Galois extensions $E|F$ of degree p of a henselized inertially generated function field $(F|K,v)$ of rank 1 and of transcendence degree 1 with a valuation-transcendental generator x. Throughout, we will assume that

$$\left.\begin{array}{l}(K,v) \text{ is henselian and } p=\operatorname{char}\overline{K}>0\,,\\ \text{and } K \text{ is closed under } p\text{-th roots.}\end{array}\right\} \tag{13}$$

If $\operatorname{char} K = p$, then the latter condition is equivalent to K being perfect. If $\operatorname{char} K = 0$, then it implies that K contains all p-th roots of unity.

In Section 2.5 we have shown that if K is algebraically closed, then we may assume that F has the following structure:

a) The value-transcendental case: $(F|K,v)$ is henselized rational, so we have that

$$F = K(x)^h \text{ is of rank 1 and } x \text{ is value-transcendental over } K. \tag{14}$$

In this case, $\overline{F} = \overline{K}$ and $vF = vK \oplus \mathbb{Z}vx$.

b) The residue-transcendental case: here we have

$$\left.\begin{array}{l}F = K(x)^h(y) \text{ is of rank 1, where } x \text{ is residue-transcendental over } K,\ vy=0,\\ \overline{F} = \overline{K}(\overline{x},\overline{y}) \text{ with } \overline{K} \text{ relatively algebraically closed in } \overline{F}, \text{ and}\\ \overline{K}(\overline{x},\overline{y})|\overline{K}(\overline{x}) \text{ separable with } [\overline{K}(\overline{x},\overline{y}):\overline{K}(\overline{x})] = [K(x,y):K(x)]\,.\end{array}\right\} \tag{15}$$

In this case, $vF = vK$.

In what follows we will *not* assume that K is algebraically closed. Instead, we will assume (13) together with (14) or (15), respectively.

If $\operatorname{char} K = p$, then the Galois extension $E|F$ of degree p is an Artin-Schreier extension (cf. [L], Theorem 6.4). That is, the extension is of the form

$$E = F(\vartheta) \quad \text{where} \quad a := \vartheta^p - \vartheta \in F\,. \tag{16}$$

By the additivity of the Artin-Schreier polynomial $\wp(X) = X^p - X$, for every $d \in F$ we have:

$$E = F(\vartheta - d), \quad \wp(\vartheta - d) = \vartheta^p - d^p - \vartheta + d = a - d^p + d \in a + \wp(F)\,. \tag{17}$$

This shows that we can replace a by any other element of $a + \wp(F)$ without changing the Artin-Schreier extension. Note that by Hensel's Lemma, $X^p - X - a$ has a root in the henselian field F whenever $va > 0$. Denoting the valuation ideal of F by $\mathcal{M}_F$, we thus have

$$\mathcal{M}_F \subset \wp(F)\,. \tag{18}$$

If $\operatorname{char} K = 0$ and K contains all p-th roots of unity, then $E|F$ is a Kummer extension (cf. [L], Theorem 6.2). That is, the extension is of the form

$$E = F(\vartheta) \quad \text{where} \quad a := \vartheta^p \in F\,. \tag{19}$$

For every $d \in F^\times$ we have

$$E = F(\vartheta d), \quad (\vartheta d)^p = ad^p \in a(F^\times)^p , \tag{20}$$

showing that we can replace a by any other element of $a(F^\times)^p$ without changing the extension $E|F$.

The main goal of this section is the proof of the following result:

Proposition 33 *Assume that (K, v) satisfies condition (13), Further, let (F, v) be of the form (14) or (15), and $(E|F, v)$ a Galois extension of degree p. Then either $(E|F, v)$ is defectless or there is a Galois extension $L|K$ of degree p with non-trivial defect such that $(L.E|L.F, v)$ is defectless.*

Corollary 34 *Let $(F|K, v)$ be a henselized inertially generated function field of transcendence degree 1 and rank 1. If K is algebraically closed, then every Galois extension $(E|F, v)$ of degree p is defectless.*

Remark 35 The normal forms we will derive will show that if we drop the condition that K be closed under p-th roots, then the assertion of Proposition 33 remains true if we replace K by a finite extension L (which depends on F), F by $L.F$ and E by $L.E$. This extension $L|K$ can be chosen to be purely inseparable if $\mathrm{char}\, K = p$, and to be generated by successive adjunction of p-th roots if $\mathrm{char}\, K = 0$. Indeed, it suffices to choose K' large enough to contain all of the finitely many coefficients that are needed in the respective normal forms.

4.1 The value-transcendental case

We will first discuss the value-transcendental case of Proposition 33. In order to find a normal form for the element a suitable to show that the extension is defectless, we will consider the ring

$$R = K[x, x^{-1}]$$

which consists of all finite Laurent series

$$\varphi(x) = \sum_{i \in I} c_i x^i , \quad c_i \in K , \quad I \subset \mathbb{Z} \text{ finite.} \tag{21}$$

Lemma 36 *Let $F = K(x)^h$ where $(K(x), v)$ is of rank 1 and x is value-transcendental over K. Then R is dense in F and thus,*

$$F = R + \wp(F) .$$

Proof: Since (F,v) and hence also $(K(x),v)$ is of rank 1, we know from Lemma 9 that $K(x)$ is dense in its henselization $K(x)^h$. Therefore, and in view of (18),

$$F = K(x) + \mathcal{M}_F = K(x) + \wp(F) .$$

Now it suffices to prove that

$$K(x) \subset R + \mathcal{M}_F , \tag{22}$$

i.e., that *R is dense in its quotient field* $K(x)$. For this, it is enough to show that for every $0 \neq \varphi(x) \in R$ and every $\alpha \in vK(x)$ there is some $\tilde{\varphi}(x) \in R$ such that

$$v\left(\frac{1}{\varphi(x)} - \tilde{\varphi}(x)\right) > \alpha .$$

Using the notation of (21) we have that

$$v\varphi(x) = \min_{i\in I} vc_i x^i = vc_k x^k \tag{23}$$

for a unique $k \in I$ since x is value-transcendental over K. We write

$$\frac{1}{\varphi(x)} = \frac{c_k^{-1}x^{-k}}{1-\psi(x)} \quad \text{with} \quad \psi(x) = 1 - c_k^{-1}x^{-k}\varphi(x) \in R$$

Since $v\psi(x) > 0$ and vK is archimedean by hypothesis, there is some $n \in \mathbb{N}$ such that $(n+1)v\psi(x) > \alpha + vc_k x^k$. Then by the geometric expansion, the element

$$\tilde{\varphi}(x) := c_k^{-1}x^{-k}\sum_{i=0}^{n}\psi(x)^n \in R$$

satisfies

$$v\left(\frac{1}{\varphi(x)} - \tilde{\varphi}(x)\right) = v\left(\frac{1}{1-\psi(x)} - \sum_{i=0}^{n}\psi(x)^n\right) - vc_k x^k = (n+1)v\psi(x) - vc_k x^k > \alpha .$$

□

• **The equal characteristic case.**

We deduce the following normal form, which proves the equal characteristic value-transcendental case of Proposition 33:

Proposition 37 *Let K, F and E be as in the value-transcendental case of Proposition 33, and assume that* $\mathrm{char}\, K = p$. *Then either*

$$E \;=\; F(\vartheta) \quad \textit{where} \;\; \vartheta^p - \vartheta \in K , \tag{24}$$

or

$$E = F(\vartheta) \quad where \;\; \vartheta^p - \vartheta = c_0 + \sum_{i\in I} c_i x^i\,, \;\; c_i \in K$$

with finite non-empty $I \subset p\mathbb{Z} \setminus \mathbb{Z}$ and such that

$$\forall i \in I: \; vc_i x^i < 0$$

and $\forall i \in I: vc_i x^i < vc_0 \leq 0$ if $c_0 \neq 0$.

If (24) does not hold, then $(vE : vF) = p$ and the extension $(E|F,v)$ is defectless. If (24) holds then for $L = K(\vartheta)$ we have that the extension $(L.E|L.F,v)$ is trivial and hence defectless. In this case, $(E|F,v)$ is defectless if and only $(L|K,v)$ is defectless.

Proof: We can assume that $E|F$ is of the form (16). By Lemma 36 and (17) together with (18), we can also assume that a is a finite Laurent series $\varphi(x)$ of the form (21), and only containing summands of value ≤ 0. Again by (17), we can replace a summand $c_{jp}x^{jp}$ of $\varphi(x)$ by a summand $c'_j x^j$ with $c'_j = c_{jp}^{1/p} \in K$ (as K is assumed to be perfect). After a finite repetition of this procedure we arrive at a finite Laurent series $c_0 + \sum_{i\in I} c_i x^i$ with coefficients $c_i \in K$ such that $c_i = 0$ whenever $i \neq 0$ is divisible by p. That is, we can assume that $I \subset \mathbb{Z} \setminus p\mathbb{Z}$. Observe that by the replacement procedure, all summands remain of value ≤ 0 and that the original coefficient $c_0 \in K$ remains unchanged. Note that for all $i \in I$, $vc_i x^i < 0$ because $i \neq 0$ and thus $ivx \notin vK$.

Assume now that (24) does not hold. Then $I \neq \emptyset$. We may assume that $vc_0 > vc_i x^i$ for all $i \in I$. Indeed, if this is not the case then using (17) ν times, we can replace c_0 by c_0^{1/p^ν}. For large enough ν, we obtain that $0 \geq vc_0^{1/p^\nu} = (vc_0)/p^\nu > vc_i x^i$ for all $i \in I$, because the value group $vK(x)$ is archimedean by hypothesis.

Since the elements x^i, $i \in \mathbb{Z}$, are valuation independent over K, there is $j \in I$ such that $v(c_0 + \sum_{i\in I} c_i x^i) = vc_j x^j < 0$. Therefore, we must have that $v\vartheta < 0$. It follows that $v\vartheta^p = pv\vartheta < v\vartheta$ and consequently, $pv\vartheta = v\vartheta^p = vc_j x^j$ by the ultrametric triangle law. As $j \notin p\mathbb{Z}$, this value is not in pvF. Hence $(vE : vF) \geq p$. From the fundamental inequality it then follows that $(vE : vF) = p$.

Finally, assume that (24) holds and set $L = K(\vartheta)$. Since vF/vK is torsion free by assumption and $\overline{F} = \overline{K}$, we know from Lemma 25 that $(F|K,v)$ is valuation regular. Therefore, the algebraic extension $(L|K,v)$ is valuation disjoint from $(F|K,v)$ and Lemma 29 tells us that $\mathrm{d}(L|K,v) \geq \mathrm{d}(L.F|F,v) = \mathrm{d}(E|F,v)$. Hence if $(E|F,v)$ has non-trivial defect, then so has $(L|K,v)$. □

- **The mixed characteristic case.**

The following normal form proves the mixed characteristic value-transcendental case of Proposition 33:

Proposition 38 *Let K, F and E be as in the value-transcendental case of Proposition 33, and assume that* $\mathrm{char}\, K = 0$. *Then*

$$E \;=\; F(\vartheta) \quad \textit{where} \;\; \vartheta^p = x^m u \;,$$

with $m \in \{0, \ldots, p-1\}$ and $u \in K[x, x^{-1}]$ a 1-unit of the form

$$u \;=\; 1 + \sum_{i \in I} c_i x^i \;, \qquad c_i \in K,$$

with finite index set $I \subset \mathbb{Z} \setminus \{0\}$ and

$$\forall i \in I: \;\; 0 < v c_i x^i \leq \frac{p}{p-1} vp \;\; \textit{and} \;\; (i \in p\mathbb{Z} \;\Rightarrow\; v c_i x^i > vp) \;. \tag{25}$$

If $v c_i x^i \geq \frac{1}{p-1} vp$ for all $i \in I$, then it may in addition be assumed that $I \subset \mathbb{Z} \setminus p\mathbb{Z}$. Hence if $I \neq \emptyset$ then $v \sum_{i \in I} c_i x^i = \min_{i \in I} v c_i x^i$ is not divisible by p in vF.

Further, $m \neq 0$ or $I \neq \emptyset$. In both cases, $(vE : vF) = p$.

Proof: We can assume that $E|F$ is of the form (19). Since $vF = vK \oplus \mathbb{Z} vx$ and $\overline{F} = \overline{K}$, we can write $a = c x^k u$ where $k \in \mathbb{Z}$, $c \in K$ and $u \in F$ a 1-unit. Using (20) and our assumption that K is closed under p-th roots, we may replace a by $x^m u$ where $m = k \bmod p$.

By Lemma 36 and part a) of Corollary 16, we may assume that

$$u \;=\; 1 + \sum_{i \in I} c_i x^i \in R \quad \text{with} \quad c_i \in K \,,\; v c_i x^i \;\leq\; \frac{p}{p-1} vp \;\; \text{and} \;\; I \subset \mathbb{Z} \;\; \text{finite.}$$

Since u is a 1-unit, we have $v \sum c_i x^i > 0$, and since the elements x^i, $i \in \mathbb{Z}$, are valuation independent, we find that $v c_i x^i > 0$ for each $i \in I$.

Since K is closed under p-th roots, a monomial $c_i x^i$ is a p^ν-th power in F if and only if $i \in p^\nu \mathbb{Z}$; indeed, if $i \notin p^\nu \mathbb{Z}$, then $v(c_i x^i)^{1/p^\nu} = v c_i^{1/p^\nu} + \frac{i}{p^\nu} vx \notin vK \oplus \mathbb{Z} vx = vF$. In a first step, we will eliminate all p-th powers $\neq c_0$ of value $\leq vp$ in the above sum. Assuming that there are any, let ν be the largest positive integer such that p^ν-th powers $\neq c_0$ appear. Suppose that there are n many, and write them as $z_1^{p^\nu}, \ldots, z_n^{p^\nu}$. Denote by y the sum of all other summands, so that $u = 1 + y + \sum_{j=1}^n z_j^{p^\nu}$. Using (20), we replace u by

$$u' \;:=\; \frac{u}{(1 + \sum_{j=1}^n z_j)^{p^\nu}} \;=\; \frac{1 + \sum_{j=1}^n z_j^{p^\nu}}{(1 + \sum_{j=1}^n z_j)^{p^\nu}} + \frac{y}{(1 + \sum_{j=1}^n z_j)^{p^\nu}} \;\;.$$

Using the geometrical series expansion as in the proof of Lemma 36, we find that the first quotient on the right hand side is equivalent to 1 modulo $p\mathcal{M}_F$. The second quotient on the right hand side is equivalent modulo $p\mathcal{M}_F$ to a finite sum of products of the summands in y with the p^ν-th powers $z_1^{p^\nu}, \ldots, z_n^{p^\nu}$. Since the summands $\neq c_0$ in y aren't p^ν-th powers, the products aren't either. By the geometrical series expansion we also find that there are only finitely many summands $c_i' x^i$ in u' of value $\leq \frac{p}{p-1} vp$. By part a) of Corollary 16 we can thus replace u' by the sum of these finitely many elements, which is a 1-unit. We will call it again u. Iterating this process until we finish with $\nu = 1$, we obtain a 1-unit u in which no p-th powers $c_i x^i \neq c_0$ of value $\leq vp$ appear:

$$u \;=\; 1 + \sum_{i\in I} c_i x^i \quad \text{with finite } I \subset \mathbb{Z}\,, c_i x^i \in \mathcal{M}_F\,, c_i x^i \in p\mathcal{M}_F \;\text{ if }\; 0 \neq i \in p\mathbb{Z}\,. \qquad (26)$$

We note that

$$\frac{u}{1+c_0} \;=\; \frac{1+c_0}{1+c_0} + \sum_{i\in I\setminus\{0\}} \frac{c_i}{1+c_0} x^i \;=\; 1 + \sum_{i\in I\setminus\{0\}} \frac{c_i}{1+c_0} x^i$$

where the values of the summands have not changed. Since K is closed under p-th roots, $1+c_0$ is a p-th power in K. We may thus replace u by $\frac{u}{1+c_0}$ and I by $I \setminus \{0\}$ so that we may assume that $I \subset \mathbb{Z} \setminus \{0\}$.

Suppose that all summands $c_i x^i$ have value $\geq \frac{1}{p-1} vp$. Then part d) of Corollary 16 shows that we may replace every monomial of the form $c_i x^i$ with $i \in p\mathbb{Z}$ by the monomial $-p\tilde{c}_i x^{i/p}$ where $\tilde{c}_i \in K$ is a p-th root of c_i. After an iterated application we may then assume that $I \cap p\mathbb{Z} = \emptyset$.

Again by part a) of Corollary 16, we delete every monomial $c_i x^i$ of value $> \frac{p}{p-1} vp$.

It remains to prove the last assertions of the lemma. If $m \neq 0$, then $v\vartheta = \frac{m}{p} vx \notin vF$. So assume that $m = 0$. Then $I \neq \emptyset$ because the extension $E|F$ is assumed to be non-trivial. Hence there is some $j \in I$ such that $v\sum_{i\in I} c_i x^i = vc_j x^j$. We have that $j \notin p\mathbb{Z}$: if $vc_i x^i \geq \frac{1}{p-1} vp$ for all $i \in I$ then this follows from $I \subset \mathbb{Z} \setminus p\mathbb{Z}$, and otherwise it follows from the second assertion of (25). By Lemma 14 we know that $C \in K$. Performing the transformation (5), we find that $\eta := \frac{\vartheta - 1}{C}$ is a root of the polynomial (6), with $b = \sum_{i\in I} c_i x^i$. Since $vb = vc_j x^j < \frac{p}{p-1} vp$ (since $j \neq 0$, equality cannot hold), we have that $v\frac{b}{C^p} < 0$. As was shown in the proof of Proposition 37 for the element ϑ, it follows that $pv\eta = v\eta^p = vbC^{-p}$. This yields:

$$v\eta \;=\; \frac{1}{p}\, vbC^{-p} \;=\; \frac{1}{p}\, vc_j - vC + \frac{j}{p}\, vx \;\notin\; vK \oplus \mathbb{Z}vx \;=\; vF\,.$$

In all cases, we find as in the proof of Proposition 37 that $(vE : vF) = p$. □

4.2 Frobenius-closed bases

The ring $R = K[x, x^{-1}]$ that we used above has the following crucial properties:

– R contains K and its quotient field $K(x)$ is dense in $F = K(x)^h$,

– R admits a valuation basis $\mathcal{B} = \{x^i \mid i \in I\}$ over K, containing the element 1, such that the values vx^i, $i \in I$, form a system of representatives of $vK(x)$ modulo vK,

– The basis $\mathcal{B}$ is **Frobenius-closed**, i.e., the p-th power of every element in $\mathcal{B}$ lies again in $\mathcal{B}$.

We will need an analogue of such rings also in the residue-transcendental case. But there, we will have to work with a henselized inertially generated function field. The main goal of this section is the proof of the following lemma:

Lemma 39 *Let $(F|K,v)$ be a henselized inertially generated function field of rank 1 and transcendence degree 1 with a residue-transcendental generator. Further, assume that $\overline{K}$ is perfect of characteristic $p>0$ and relatively algebraically closed in $\overline{F}$, and that K is of arbitrary characteristic and closed under p-th roots. Then F contains a subring R which satisfies:*

(LFC1) *R contains K and its quotient field Quot(R) is dense in F,*

(LFC2) *R admits a valuation basis $\mathcal{B}=\{u_j \mid j\in J\}$ over K of elements of value 0 and containing the element 1, whose residues $\overline{u_j}$, $j\in J$, form a basis of $\overline{F}|\overline{K}$,*

(LFC3) *$\mathcal{B}$ is Frobenius-closed.*

The valuation basis $\mathcal{B}$ of this lemma may be called a **lifting of a Frobenius-closed basis (LFC)** in view of part a) of the following lemma. Part b) is the crucial property of Frobenius-closed bases of algebraic function fields that we are going to exploit.

Lemma 40 *Assume that Properties (LFC2) and (LFC3) hold. Then:*
a) The basis $\overline{\mathcal{B}}$ of $\overline{F}|\overline{K}$ consisting of all $\overline{u}_j$, $j\in J$, is also Frobenius-closed. If $\overline{u}_m=\overline{u}_n^p$ then $u_m=u_n^p$.
b) If the sum

$$\sum_{i\in I}\overline{c}_i\overline{u}_i\,,\ \ \overline{c}_i\in\overline{K}\,,\ I\subset J \text{ finite}$$

is a p-th power, then for every $i\in I$ with $\overline{c}_i\neq 0$, the basis element $\overline{u}_i$ is a p-th power of a basis element.
c) If

$$0\;\neq\;\vartheta^p-\vartheta\;=\;\sum_{i\in I}\overline{c}_i\overline{u}_i\,,\ \ \overline{c}_i\in\overline{K}\,,\ I\subset J \text{ finite}$$

then there is some $i\in I$ with $\overline{c}_i\neq 0$ such that the basis element $\overline{u}_i$ is a p-th power of a basis element.

Proof: a): Since $\mathcal{B}$ is Frobenius-closed, every u_j^p is an element of $\mathcal{B}$. Hence $\overline{u}_j^p=\overline{u_j^p}\in\overline{\mathcal{B}}$ which shows that $\overline{\mathcal{B}}$ is Frobenius-closed. If $\overline{u}_m=\overline{u}_n^p$ then $v(u_m-u_n^p)>0=vu_m$ which is only possible if $u_m=u_n^p$ since $\mathcal{B}$ is assumed to be a valuation basis.
b): Assume that the sum is equal to

$$\left(\sum_{j\in J_0}\overline{d}_j\overline{u}_j\right)^p\,,\ \ \overline{d}_j\in\overline{K}$$

where $J_0 \subset J$ is a finite index set. Then

$$\sum_{i\in I} \overline{c}_i \overline{u}_i \;=\; \sum_{j\in J_0} \overline{d}_j^p \overline{u}_j^p$$

where the elements $\overline{u}_j^p$ are also basis elements by part a), which shows that every $\overline{u}_i$ which appears on the left hand side (i.e., $\overline{c}_i \neq 0$) equals a p-th power $\overline{u}_j^p$ appearing on the right hand side.
c): Similar to b), hence left to the reader. □

We also deduce the following analogue of Lemma 36.

Lemma 41 *Let F be henselian of rank 1. Then the properties (LFC1) and (LFC2) imply that R is dense in F and hence again,*

$$F = R + \wp(F)\;.$$

Proof: In view of (LFC1), we have to show that R is dense in its quotient field, for which it suffices to show that for every $r \in R$ and each $\alpha \in vF$ there is $r' \in R$ such that $v(\frac{1}{r} - r') > \alpha$. Assume that there exists an element $s \in R^\times$ with $v(rs-1) > 0$ and write

$$\frac{1}{r} = \frac{s}{1-(1-rs)}\;.$$

Note that $1 - rs \in R$ and proceed as in the proof of Lemma 36. It remains to show the existence of s. Now the condition $K \subset R$ which is part of property (LFC1), together with property (LFC2) implies that $vR = vF$ and $\overline{R} = \overline{F}$. This shows that there is some $s_1 \in R$ such that $vrs_1 = 0$ and that the residue of the element $rs_1 \in R$ has an inverse in $\overline{R}$, say $\overline{s_2}$ for suitable $s_2 \in R$. Then the element $s = s_1 s_2$ has the desired property since $vrs_1s_2 = vrs_1 = 0$ and $\overline{rs_1s_2} = 1$. □

We will now prove Lemma 39. Since $\overline{K}$ is assumed to be perfect and $\overline{F}|\overline{K}$ an algebraic function field of transcendence degree 1 with $\overline{K}$ relatively algebraically closed in $\overline{F}$, Theorem 10 of [K5] shows that there exists a Frobenius-closed basis of $\overline{F}|\overline{K}$. We have to lift this basis to a Frobenius-closed basis of F over K.

• **The case of** $\mathrm{char}\, K = p$**.**

Since K is assumed to be henselian and perfect, it contains a field of representatives for the residue field $\overline{K}$ (we leave the easy proof to the reader). We identify this field with $\overline{K}$, so that we can write

$$\overline{K} \subset K \quad \text{with } \overline{a} = a \text{ for all } a \in \overline{K}\;. \tag{27}$$

This embedding can be extended to an embedding of $\overline{F}$ in F as follows. Let $f(x,y) = 0$ be the irreducible equation for x, y over K, normed such that f has coefficients of value

≥ 0 with $\overline{f}(X,Y) \neq 0$. By (15), the polynomials $f(X,Y)$ and $\overline{f}(X,Y)$ have the same degree in Y, and

$$\frac{\partial \overline{f}}{\partial Y}(\overline{x},\overline{y}) \neq 0 \,. \tag{28}$$

By our embedding of the residue field, we can view the polynomial $\overline{f}(x,Y)$ as a polynomial over $K(x) \subset F$; from (28) it follows by Hensel's Lemma that this polynomial has exactly one zero $y' \in F$ with $\overline{y'} = \overline{y}$. We have $K(x)^h(y') \subset K(x)^h(y)$. Again from (28) it follows that the polynomial $f(x,Y)$ has exactly one root in $K(x)^h(y')$ whose residue is equal to $\overline{y'} = \overline{y}$. This root must be y, hence $y \in K(x)^h(y')$ and we have shown

$$K(x)^h(y') \;=\; K(x)^h(y) \;=\; F \,. \tag{29}$$

The residue map induces on $\overline{K}$ the identity and an isomorphism

$$\overline{K}(x) \longrightarrow \overline{K}(\overline{x})$$

since both x and $\overline{x}$ are transcendental over $\overline{K}$. It leaves the coefficients of the irreducible polynomial $\overline{f}(X,Y)$ fixed and sends the zero (x,y') of $\overline{f}(X,Y)$ to the zero $(\overline{x},\overline{y})$, hence it induces an isomorphism

$$\overline{K}(x,y') \longrightarrow \overline{K}(\overline{x},\overline{y}) = \overline{F} \,.$$

By this isomorphism we identify

$$x = \overline{x} \;, \quad y' = \overline{y}$$

such that

$$\overline{F} \subset F \;, \quad F = (K.\overline{F})^h \;, \tag{30}$$

the latter being a consequence of (29).

By construction, K and $\overline{F}$ are linearly disjoint over $\overline{K}$. We form the subring generated by both fields in F:

$$R = K \otimes_{\overline{K}} \overline{F} \subset F \,.$$

By (30), F is the henselization of the quotient field of R. Since the rank of F is 1, the field $\mathrm{Quot}(R)$ is dense in its henselization by Lemma 9, hence R satisfies property (LFC1).

Every $\overline{K}$-basis of $\overline{F}$ is at the same time a K-basis of R. As the residue map induces the identity on $\overline{F}$, every such basis is a valuation basis of R over K. Thus, R satisfies (LFC2). If we choose, as indicated above, a Frobenius-closed basis of $\overline{F}|\overline{K}$ then R together with this basis satisfies also property (LFC3).

We summarize what we have proved:

Lemma 42 *In the case of* $\mathrm{char}\, K = p$ *there exists an embedding of the residue field* $\overline{F}$ *in* F *respecting the residue map such that* $\overline{K} = K \cap \overline{F}$, *that* K *is linearly disjoint from* $\overline{F}$ *over* $\overline{K}$ *and that* $F = (K.\overline{F})^h$. *The ring* $R = K \otimes_{\overline{K}} \overline{F} \subset F$ *satisfies properties (LFC1), (LFC2) and (LFC3).*

• **The case of** $\mathrm{char}\, K = 0$**.**

In this case, K contains $\mathbb{Q}$ and its valuation induces the p-adic valuation v_p on $\mathbb{Q}$. Since K is algebraically closed, it contains a valued extension field (K_0, v) of $(\mathbb{Q}, v)$ such that $\overline{K_0} = \overline{K}$ and $vK_0 = v_p\mathbb{Q} = \mathbb{Z}vp$; this field can be constructed as follows:
Take $\mathcal{T}$ to be a set of preimages for a transcendence basis $\overline{\mathcal{T}}$ of $\overline{K}|\mathbb{F}_p$; then $\mathcal{T}$ is an algebraically valuation independent set and we have $\overline{\mathbb{Q}(\mathcal{T})} = \mathbb{F}_p(\overline{\mathcal{T}})$ and $v\mathbb{Q}(\mathcal{T}) = v\mathbb{Q}$ by Lemma 10. Now $\overline{K}\,|\,\overline{\mathbb{Q}(\mathcal{T})}$ is an algebraic extension which can be viewed as a (transfinite) tower of finite separable and finite purely inseparable extensions. By induction we succesively lift all of these extensions, preserving their degrees. The separable extensions are lifted by Hensel's Lemma, using our assumption that (K, v) is henselian. The purely inseparable extensions can be lifted using our assumption that K is closed under p-th roots. We obtain a tower of finite extensions, starting from the field $\mathbb{Q}(\mathcal{T})$; since all of them have the same degree as the corresponding extensions of their residue fields, the fundamental inequality shows that these extensions will all have the same value group as $\mathbb{Q}(\mathcal{T})$, which is $\mathbb{Z}vp$. The union over this tower is the desired field K_0.

By Lemma 10,

$$\overline{K_0(x)} \;=\; \overline{K_0}(\overline{x}) \;=\; \overline{K}(\overline{x})\;. \tag{31}$$

We choose f with $f(x, y) = 0$ as in the positive characteristic case. Again, (28) holds. By (31), we may choose a polynomial $g(X, Y) \in K_0[X, Y]$ with coefficients of value ≥ 0 such that g has the same degree in Y as f and

$$\overline{g}(X, Y) \;=\; \overline{f}(X, Y)\;.$$

From (28) it follows by Hensel's Lemma that $g(x, Y)$ has exactly one zero $y' \in F$ with $\overline{y'} = \overline{y}$. We have $K(x)^h(y') \subset K(x)^h(y)$. Again from (28) it follows that the polynomial $f(x, Y)$ has exactly one root in $K(x)^h(y')$ whose residue is equal to $\overline{y'} = \overline{y}$. This root must be y, hence $y \in K(x)^h(y')$ and we have shown

$$K(x)^h(y') \;=\; K(x)^h(y) \;=\; F\;.$$

Hence we assume from now on that y is algebraic over $K_0(x)$ with

$$[K_0(x, y) : K_0(x)] = [\overline{K_0}(\overline{x}, \overline{y}) : \overline{K_0}(\overline{x})] = [\overline{K}(\overline{x}, \overline{y}) : \overline{K}(\overline{x})] = [K(x, y) : K(x)]\;.$$

In particular, this shows that the function field $F_0 = K_0(x, y)$ is linearly disjoint from K over K_0. Moreover, we have

$$\overline{F_0} = \overline{K_0}(\overline{x}, \overline{y}) = \overline{F}$$

and, in view of Lemma 8,

$$F = K(x)^h(y) = K(x, y)^h = (K.K_0(x, y))^h = (K.F_0)^h\;. \tag{32}$$

Now we lift the Frobenius-closed basis $\overline{\mathcal{B}}$ of $\overline{F}|\overline{K}$ to F_0. First we observe that every basis element $\overline{u} \neq 1$ in $\overline{\mathcal{B}}$ is an element of $\overline{F} \setminus \overline{K}$, which implies that there exists an integer $\nu = \nu(\overline{u})$ such that $\overline{u} \notin \overline{F}^{p^\nu}$. This shows that

$$\overline{\mathcal{B}} = \{1\} \cup \{\overline{u}^{p^n} \mid n \in \mathbb{N} \text{ and } \overline{u} \in \overline{\mathcal{B}} \setminus \overline{F}^p\} .$$

For every $\overline{u} \in \overline{\mathcal{B}} \setminus \overline{F}^p$ we choose an element $u \in F_0$ with residue $\overline{u}$. Let $\mathcal{B}'$ be the collection of all these elements u. Then

$$\mathcal{B} = \{1\} \cup \{u^{p^n} \mid n \in \mathbb{N} \text{ and } u \in \mathcal{B}'\}$$

is a valuation independent set and a set of representatives for $\mathcal{B}$. Let

$$R_0 = K_0[\mathcal{B}]$$

be the subring of F_0 generated over K_0 by the elements from $\mathcal{B}$. Since

$$vR_0 = v_p\mathbb{Q} = vF_0 \quad \text{and} \quad \overline{R_0} = \overline{F} = \overline{F_0}$$

and since the value group $v_p\mathbb{Q}$ is isomorphic to $\mathbb{Z}$, we conclude that R_0 is dense in F_0.

Recall that K and F_0 are linearly disjoint over K_0. We form the subring generated by K and R_0 in F:

$$R = K \otimes_{K_0} R_0 \subset F .$$

Since R_0 is dense in F_0 and K is of rank 1 by assumption, R is dense in the ring

$$R' = K \otimes_{K_0} F_0 \subset F .$$

From (32) it follows that F is the henselization of the quotient field of R'. Since the rank of F is 1, the field $\mathrm{Quot}(R')$ is dense in its henselization by Lemma 9, and the fact that R is dense in R' implies that $\mathrm{Quot}(R)$ is dense in $\mathrm{Quot}(R')$. Hence R satisfies property (LFC1).

By construction, $\mathcal{B}$ is a valuation basis of R over K, containing 1, closed under p-th powers and such that $\overline{\mathcal{B}}$ is a Frobenius-closed basis of $\overline{F} \mid \overline{K}$. Hence R satisfies properties (LFC2) and (LFC3).

We summarize what we have proved:

Lemma 43 *In the case of* $\mathrm{char}\, K = 0$ *there exists a subfield* K_0 *of* K *and an algebraic function field* $F_0|K_0$ *linearly disjoint from* $K|K_0$ *such that* $vF_0 = vK_0$ *is discrete,* $\overline{K_0} = \overline{K}$ *and* $\overline{F_0} = \overline{F}$*. The field* F_0 *contains a valuation independent set* $\mathcal{B}$ *including 1 and closed under p-th powers such that* $\overline{\mathcal{B}}$ *is a Frobenius-closed basis of* $\overline{F}$ *over* $\overline{K}$*. The ring* R_0 *generated by* $\mathcal{B}$ *over* K_0 *is dense in* F_0*, and the ring* R *generated by* K *and* R_0 *in* F *satisfies properties (LFC1), (LFC2) and (LFC3).*

4.3 The residue-transcendental case

We will now discuss the residue-transcendental case of Proposition 33.

- **The equal characteristic case.**

Using Lemma 42, we derive the following normal form, which proves the equal characteristic residue-transcendental case of Proposition 33:

Proposition 44 *Let K, F and E be as in the residue-transcendental case of Proposition 33, and assume that* $\mathrm{char}\, K = p$. *Choose a ring R with Frobenius-closed basis $\mathcal{B}$ in $F|K$ as in Lemma 42. Then either*

$$E \;=\; F(\vartheta) \quad \textit{where } \; \vartheta^p - \vartheta \in K\;, \tag{33}$$

or

$$E = F(\vartheta) \quad \textit{where } \; \vartheta^p - \vartheta = \sum_{i\in I} c_i u_i\;, \qquad 0 \neq c_i \in K,\; u_i \in \mathcal{B}$$

with I a finite index set, and such that no element $u_i \neq 1$ is a p-th power in $\mathcal{B}$, and

$$\forall i \in I:\; vc_i u_i = vc_i \leq 0\;.$$

If $vc_j u_j < 0$ for some $j \in I$ with $u_j \neq 1$, then $\overline{E}|\overline{F}$ is purely inseparable of degree p. If $vc_i u_i = 0$ for all $i \in I$, then $\overline{E}|\overline{F}$ is separable of degree p. In both cases, $(E|F, v)$ is defectless.

Suppose that $u_\ell = 1$ for some $\ell \in I$ and $c_\ell u_\ell$ is the only summand of value < 0. If $I = \{\ell\}$ then (33) holds and the extension $(L.E|L.F, v)$ is trivial. Otherwise, upon taking L to be the Galois extension of K generated by a root of the polynomial $X^p - X - c_\ell$, we have that $\overline{L.E}|\overline{L.F}$ is purely inseparable of degree p. In both cases, $(L.E|L.F, v)$ is defectless and if $(E|F, v)$ has non-trivial defect, then $(L|K, v)$ has non-trivial defect.

Proof: We can assume that $E|F$ is of the form (16). According to Lemma 41, and by use of (17) together with (18), we can also assume that

$$a \;=\; \sum_{i\in I} c_i u_i \quad \text{where } \; c_i \in K,\; u_i \in \mathcal{B}$$

with I a finite index set and such that $vc_i u_i \leq 0$ for all $i \in I$. Note that

$$va = \min_{i\in I} vc_i u_i \leq 0 \tag{34}$$

since property (LFC2) of R says that the elements u_i, $i \in I$, form a valuation basis of R over K.

After a replacement procedure similar to the one used in the proof of Proposition 37, we can assume that no element $u_i \neq 1$ is a p-th power of another element in $\mathcal{B}$. Note that if $u_i \neq 1$, then $u_i \in F \setminus K$, which shows that there exists an integer $\nu = \nu(u_i)$ such that $u_i \notin F^{p^\nu}$, and thus also an integer $\mu = \mu(u_i) \leq \nu$ such that $u_i \notin \mathcal{B}^{p^\mu}$.

Assume that $vc_ju_j < 0$ for some $j \in I$ with $u_j \neq 1$, and choose j such that

$$vc_ju_j \;=\; \min_{i\in I\,,\,u_i\neq 1} vc_iu_i < 0\;.$$

If $u_\ell = 1$ for some $\ell \in I$, then as we did for the element c_0 in the proof of Proposition 37, we can replace $c_\ell u_\ell = c_\ell \in K$ by a suitable p^ν-th root in K whose value is larger than vc_ju_j. We can thus assume that $vc_ju_j = \min_{i\in I} vc_iu_i$. As in the proof of Proposition 37 we find that $v\vartheta = \frac{1}{p}vc_ju_j = \frac{1}{p}vc_j$. Set $d = c_j^{-1/p} \in K$ and $\chi := d\vartheta$. Then $v\chi = 0$ and $vd > 0$. We obtain:

$$\chi^p - d^{p-1}\chi \;=\; d^p(\vartheta^p - \vartheta) \;=\; c_j^{-1}\sum_{i\in I} c_iu_i\;.$$

Set $d_i = c_i/c_j$; then $vd_i \geq 0$ and $d_j = 1$. Since $vd^{p-1}\chi > 0$, we see that

$$\overline{\chi}^p \;=\; \overline{\sum_{i\in I} d_iu_i} \;=\; \sum_{i\in I}\overline{d}_i\overline{u}_i$$

Since $\overline{d}_j = 1$ and $\overline{u}_j \neq 1$ is not a p-th power in $\overline{\mathcal{B}}$ by construction, we can infer from part b) of Lemma 40 that $\overline{\chi} \notin \overline{F}$. Thus, $[\overline{E} : \overline{F}] \geq p$. By the fundamental inequality (1), equality holds. Since the extension is generated by $\overline{\vartheta}$, it is purely inseparable.

Assume that all summands c_iu_i have value 0. Then also ϑ has value 0, and we obtain that

$$\overline{\vartheta}^p - \overline{\vartheta} \;=\; \sum_{i\in I}\overline{c}_i\overline{u}_i\;.$$

If the polynomial $X^p - X - \sum\overline{c}_i\overline{u}_i$ were reducible, then Hensel's Lemma would yield that $[E : F] < p$ in contradiction to our assumption. Hence $[\overline{E} : \overline{F}] \geq p$, and from the fundamental inequality (1) it follows that equality holds. As the extension is generated by $\overline{\vartheta}$, it is separable.

Assume that $u_\ell = 1$ for some $\ell \in I$ and $c_\ell u_\ell$ is the only summand of value < 0. Choose $\vartheta_0 \in \tilde{K}$ such that $\vartheta_0^p - \vartheta_0 = c_\ell$, and set $L = K(\vartheta_0)$. Then for $\vartheta_1 := \vartheta - \vartheta_0 \in L.E$, we obtain from (17) that

$$L.E = L.F(\vartheta_1) \quad\text{where}\quad \vartheta_1^p - \vartheta_1 = \sum_{i\in I\setminus\{\ell\}} c_iu_i\;.$$

If the polynomial $X^p - X - \sum_{\ell\neq i\in I}\overline{c}_i\overline{u}_i$ is reducible, then it splits completely. But by part c) of Lemma 40, this is only possible if the sum is empty, that is, $I = \{\ell\}$. In this case, (33) holds with $\vartheta^p - \vartheta = c_\ell$. If the polynomial is irreducible, then $[\overline{L.E} : \overline{L.F}] \geq p$, and again, equality holds. As the extension $\overline{L.E}|\overline{L.F}$ is generated by $\overline{\vartheta}_1$, it is separable.

Finally, it remains to prove the very last assertion of the lemma. Since $\overline{K}$ is assumed to be perfect and relatively algebraically closed in $\overline{F}$, the extension $\overline{F}|\overline{K}$ is regular. Since also

$vF = vK$, we know from Lemma 25 that $(F|K, v)$ is valuation regular. As in the proof of Proposition 37 it follows that if $(L.F|F, v)$ has non-trivial defect, then $(L|K, v)$ has non-trivial defect. Suppose that $(E|F, v)$ has non-trivial defect. Since $E|F$ is a subextension of $L.E|F$ it then follows from Lemma 18 that $(L.E|F, v)$ has non-trivial defect. Since $(L.E|L.F, v)$ is defectless, it follows again from Lemma 18 that $(L.F|F, v)$ and hence also $(L|K, v)$ has non-trivial defect. □

- **The mixed characteristic case.**

Using Lemma 43, we derive the following normal form, which proves the mixed characteristic residue-transcendental case of Proposition 33:

Proposition 45 *Let K, F and E be as in the residue-transcendental case of Proposition 33, and assume that* $\operatorname{char} K = 0$. *Choose a ring R with Frobenius-closed basis $\mathcal{B}$ in $F|K$ as in Lemma 43. Then*

$$E = F(\vartheta) \quad \textit{where} \;\; \vartheta^p \;=\; ru$$

such that $r \in R$ has value 0 and either $r = 1$ or $\overline{r} \notin \overline{F}^p$, and such that $u \in R$ is a 1*-unit of the form*

$$u \;=\; 1 + \sum_{i\in I} c_i u_i \;, \qquad c_i \in K,\; 1 \neq u_i \in \mathcal{B}$$

where I is a finite index set and

$$\forall i \in I:\; 0 < vc_iu_i = vc_i \leq \frac{p}{p-1} vp \;\; \textit{and} \;\; (u_i \in \mathcal{B}^p \;\Rightarrow\; vc_i > vp)\;.$$

If $vc_i \geq \frac{1}{p-1} vp$ for all $i \in I$, then it may be assumed that no u_i at all appearing in the sum is a p-th power in $\mathcal{B}$.

In all cases, $[\overline{E} : \overline{F}] = p$, with the extension being separable if and only if $r = 1$ and $vc_i = \frac{p}{p-1} vp$ for all $i \in I$.

Proof: We can assume that $E|F$ is of the form (19). Since $vF = vK$, we can write $a = cb$ where $c \in K$ with $vc = va$. Using (20) and our assumption that K is closed under p-th roots, we may replace a by b. Now $vb = 0$, so $\overline{b} \neq 0$. If $\overline{b}$ is a p-th power in $\overline{F}$, then we may choose $b_0 \in F$ with $\overline{b}_0 = \overline{b}^{1/p}$ and write $b = b_0^p u$ with u a 1-unit. As before, we may then replace b by u. Now suppose that $\overline{b}$ is not a p-th power in $\overline{F}$. By (LCF2) we can choose $r \in R$ such that $\overline{r} = \overline{b}$ and write $b = ru$ with u a 1-unit. In summary, we have shown that we can assume $\vartheta^p = ru$ with $r \in R$ of value 0 and either $r = 1$ or $\overline{r} \notin \overline{F}^p$, and $u \in F$ a 1-unit.

In view of Lemma 41 and part a) of Corollary 16, we may further assume that

$$u \;=\; 1 + \sum_{i\in I} c_i u_i \quad \text{with} \quad c_i \in K,\; u_i \in \mathcal{B}\,,\; vc_iu_i \;\leq\; \frac{p}{p-1}\, vp \;\; \text{and} \;\; I \subset \mathbb{Z} \;\; \text{finite.}$$

Since u is a 1-unit, we have $v\sum c_i u_i > 0$, and since the elements u_i form a valuation basis of R over K by (LFC2), we find that $vc_iu_i > 0$ for each $i \in I$.

All further properties stated for the summands c_iu_i in the proposition are now obtained by a replacdement procedure as in the proof of Proposition 38.

It remains to prove the last assertions of the lemma. If $\overline{r} \notin \overline{F}^p$ then $\overline{\vartheta} = \overline{r}^{1/p} \notin \overline{F}$.

Now assume that $r = 1$. By Lemma 14 we know that $C \in K$. Performing the transformation (5), we find that

$$\eta := \frac{\vartheta - 1}{C}$$

is a root of the polynomial (6), where $b = \sum_{i\in I} c_iu_i$. Suppose first that $vc_iu_i = \frac{p}{p-1}vp$ for all $i \in I$. Then $vbC^{-p} = 0$, and the residue polynomial $Z^p - Z - \overline{bC^{-p}}$ does not admit a zero in $\overline{F}$ since otherwise $\eta \in F$ by Hensel's Lemma and $E|F$ would be trivial, contrary to our assumption that its degree is p.

Now suppose that $vc_iu_i < \frac{p}{p-1}vp$ for some $i \in I$. Then for a suitable $j \in I$,

$$vb = \min_{i\in I} vc_iu_i = vc_j < \frac{p}{p-1}vp \tag{35}$$

and $vbC^{-p} < 0$. As in the proof of Proposition 38 we find that

$$v\eta = \frac{1}{p}vbC^{-p} = \frac{1}{p}vc_j - vC < 0 .$$

By our assumption on K there is some $d \in K$ such that $d^p = c_j^{-1}$. We have that $vdC\eta = 0$ and $vdC > 0$. Hence for $\chi := dC\eta$ we obtain:

$$\chi^p - (dC)^{p-1}\chi = (dC)^p(\eta^p - \eta) = (dC)^p \frac{b}{C^p} = c_j^{-1}\sum_{i\in I} c_iu_i .$$

As in Proposition 44 we see that $\overline{\chi} \notin \overline{F}$.

In all cases, we find that $[\overline{E} : \overline{F}] \geq p$ and hence, $[\overline{E} : \overline{F}] = p$ by the fundamental inequality. It follows that $\overline{E}|\overline{F}$ is a separable extension (generated by a root of an Artin-Schreier polynomial) if $r = 1$ and $vc_i = \frac{p}{p-1}vp$ for all $i \in I$, and that it is a purely inseparable extension in the remaining cases. □

5 Proof of the Generalized Stability Theorem

We prove Theorem 1 by a stepwise reduction to the analysis of Galois extensions of degree p of certain valued algebraic function fields without transcendence defect, followed by an application of the results of Section 4. Since every valued field (K, v) of residue characteristic $\operatorname{char} Kv = 0$ is a defectless field, we will during this proof always assume that $\operatorname{char} Kv = p > 0$.

- **Reduction to transcendence degree 1.**

Lemma 46 *To prove Theorem 1, it suffices to prove*

(R1) *Every valued algebraic function field of transcendence degree 1 without transcendence defect over a defectless ground field is a defectless field.*

Proof: By induction on the transcendence degree of the function field. The case of transcendence degree 1 is covered by **(R1)** which we assume to be true. Assume that $(F|K, v)$ is a valued algebraic function field of transcendence degree > 1 without transcendence defect. Choose any subfunction field $F_0|K$ in $F|K$ such that $0 < \operatorname{trdeg} F_0|K < \operatorname{trdeg} F|K$. By the additivity of transcendence degree and rational rank, both $(F_0|K, v)$ and $(F|F_0, v)$ are valued algebraic function fields without transcendence defect. Hence if (K, v) is a defectless field, then by induction hypothesis, (F_0, v) and consequently also (F, v) is a defectless field. □

• Reduction to algebraically closed ground fields.

Lemma 47 *To prove* **(R1)***, it suffices to prove*

(R2) *Every valued algebraic function field of transcendence degree 1 without transcendence defect over an algebraically closed field is a defectless field.*

Proof: Assume that $(F|K, v)$ is a valued algebraic function field without transcendence defect of transcendence degree 1 over the defectless field (K, v). We pick a valuation-transcendental element $t \in F$. By Lemma 27, $(K(t)|K, v)$ is a valuation regular extension, and so is $(\tilde{K}(t)|\tilde{K}, v)$ under every extension of v to $\tilde{K}(t)$. Hence by **(R2)**, $(\tilde{K}(t), v)$ is a defectless field. From Corollary 30 we infer that the same holds for $(K(t), v)$. Since $F|K(t)$ is finite, it follows from Corollary 21 that also (F, v) is a defectless field. □

• Reduction to finite rank.

Lemma 48 *To prove* **(R2)***, it suffices to prove*

(R3) *Every valued algebraic function field without transcendence defect of transcendence degree 1 over an algebraically closed ground field of finite rank is a defectless field.*

Proof: Assume that assertion **(R3)** is true. Let $(F|K, v)$ be a valued algebraic function field without transcendence defect of transcendence degree 1 over the algebraically closed field K. We wish to show that (F, v) is a defectless field. As in the preceding proof, we choose a valuation transcendental element t, and by Corollary 21 we only have to show that $(K(t), v)$ is a defectless field. By Theorem 19 we may as well show that $(K(t)^h, v)$ is a defectless field.

Let $(K(t)^h(a_1, \ldots, a_n)|K(t)^h, v)$ be an arbitrary finite extension; we have to show that it is defectless. There exists a finitely generated extension k_1 of the prime field of K such that $a_1, \ldots, a_n$ are already algebraic over $k_1(t)$. We take k to be the algebraic closure of k_1 inside K. Since k_1 is finitely generated over its prime field, the rank of (k_1, v) and thus also of its algebraic closure (k,v) must be finite by Lemma 12. By construction, $(k(t)|k,v)$ is a valued algebraic function field without transcendence defect of transcendence degree 1 over the algebraically closed ground field (k,v) of finite rank. So **(R3)** together with Theorem 19 implies that $(k(t)^h(a_1,\ldots,a_n)|k(t)^h,v)$ is a defectless extension.

Since k is algebraically closed, the extension $(K|k,v)$ is valuation regular (Lemma 25). By Lemma 28 and Corollary 26, also $(K(t)|k(t),v)$ and $(K(t)^h|k(t)^h,v)$ are valuation regular. Hence the algebraic extension $(k(t)^h(a_1,\ldots,a_n)|k(t)^h,v)$ is valuation disjoint from $(K(t)^h|k(t)^h,v)$. Now we apply Lemma 29 to obtain that

$$\mathrm{d}(K(t)^h(a_1,\ldots,a_n)|K(t)^h,v) \;\leq\; \mathrm{d}(k(t)^h(a_1,\ldots,a_n)|k(t)^h,v) \;=\; 1\;.$$

□

- **Reduction to rank 1.**

Lemma 49 *To prove* **(R3)***, it suffices to prove*

(R4) *Every valued algebraic function field of rank 1 without transcendence defect of transcendence degree 1 over an algebraically closed ground field is a defectless field.*

Proof: Assume that assertion **(R4)** is true. Let (F,v) be a valued algebraic function field of transcendence degree 1 over an algebraically closed ground field K of finite rank. Then (F,v) must also have finite rank (cf. (2) and note that the rank of vF cannot exceed the rank of vK plus $\mathrm{rr}\, vL/vK$). Let $v=w_1\circ\ldots\circ w_n$ be the decomposition of v into valuations w_i of rank 1. By Lemma 6, every $Kw_1\circ\ldots\circ w_i$ is an algebraically closed field. By a repeated application of Lemma 13, $(F|K,w_1)$ and every $(Fw_1\circ\ldots\circ w_i|Kw_1\circ\ldots\circ w_i,w_{i+1})$ are valued algebraic function fields without transcendence defect. Hence (R4) yields that every $(Fw_1\circ\ldots\circ w_i,w_{i+1})$ is a defectless field. Now a repeated application of Lemma 22 shows that (F,v) itself is a defectless field. □

To complete our proof, we show that (R4) is true. Let $(F|K,v)$ satisfy the conditions of **(R4)**. Then there is a valuation-transcendental element $x\in F$ and $F|K(x)$ is finite. Since every finite extension of a defectless field is again defectless by Corollary 21, it suffices to prove (R4) under the additional assumption that $F|K$ is rational with a valuation-transcendental generator. In view of Theorem 19, we can replace (F,v) by its henselization. More generally, we will now prove our assertion under the assumption that $(F|K,v)$ is a henselized inertially generated function field of rank 1 and transcendence degree 1 with a valuation-transcendental generator over the algebraically closed field K. Given an arbitrary finite extension $(E|F,v)$, we have to show that it is defectless. If $E|F$ is purely inseparable, then it is defectless by Proposition 32. So we will assume now that $E|F$ contains a non-trivial separable subextension.

Since the ramification group is a p-group (cf. [En]), $F^{\mathrm{sep}}|F^r$ is a p-extension. It follows from the general theory of p-groups (cf. [H], Chapter III, §7, Satz 7.2 and the following remark) via Galois correspondence that the maximal separable subextension of $E.F^r|F^r$ is a finite tower of Galois extensions of degree p. But then there is already a finite subextension $N|F$ of $F^r|F$ such that $E.N|N$ is such a tower. Lemma 31 shows that N, being a finite subextension inside F^r, is again a henselized inertially generated function field of transcendence degree 1 with a valuation-transcendental generator over K. Also, it is again of rank 1. By Proposition 23 we have $\mathrm{d}(E|F,v) = \mathrm{d}(E.N|N,v)$, hence it suffices to prove that $E.N|N$ is defectless. Since this is trivial if $E.N = N$, we assume that $E.N \neq N$. Then the first extension in the tower is defectless by Corollary 34. This yields $\mathrm{d}(E|F,v) = \mathrm{d}(E.N|N,v) < [E.N : N] \leq [E : F]$, that is, $(E|F,v)$ cannot be immediate. We have proved:

Henselized inertially generated function fields of rank 1 and of transcendence degree 1 with a valuation-transcendental generator over an algebraically closed ground field do not admit proper immediate algebraic extensions.

We use this fact to prove:

Lemma 50 *Every henselized algebraic function field (F,v) of rank 1 and of transcendence degree 1 without transcendence defect over an algebraically closed ground field K is a henselized inertially generated function field with a valuation-transcendental generator.*

Proof: Case I) Suppose F contains a value-transcendental element x. Since $\overline{K(x)} = \overline{K}$ is algebraically closed and $\overline{F}|\overline{K(x)}$ is finite, we have $\overline{F} = \overline{K(x)}$. Further, vF is a finite extension of $vK(x) = vK \oplus \mathbb{Z}vx$. Since vK is divisible, we have that $vF = vK \oplus \mathbb{Z}\alpha$ for some $\alpha \in vF$. Choose $x' \in F$ such that $vx' = \alpha$. Then $vF = vK \oplus \mathbb{Z}vx' = vK(x')$ by Lemma 10, and $\overline{F} = \overline{K} = \overline{K(x')}$. Now the henselian field F contains the henselization $K(x')^h$, and we have just shown that $F|K(x')^h$ is an immediate extension. But the henselized inertially generated function field $K(x')^h$ of rank 1 and of transcendence degree 1 with value-transcendental generator x' over K does not admit any proper immediate algebraic extension. This yields $F = K(x')^h$.

Case II) Suppose F contains a residue-transcendental element x. Since $vK(x) = vK$ is divisible and $vF/vK(x)$ is finite, we have $vF = vK(x)$. Since $\overline{F}|\overline{K(x)}$ is finite and $\overline{K(x)} = \overline{K}(\overline{x})$ is an algebraic function field of transcendence degree 1 over $\overline{K}$, the same holds for $\overline{F}$. Since $\overline{K}$ is algebraically closed, there is a separating transcendence basis $\{\xi\}$ of $\overline{F}|\overline{K}$. We choose $x' \in F$ such that $\overline{x'} = \xi$. The henselian field F contains the henselization $K(x')^h$. Since $\overline{F}|\overline{K}(\xi)$ is a finite separable extension, Hensel's Lemma shows that we can lift ξ to an element $y' \in F$ such that the finite subextension $K(x')^h(y')|K(x')^h$ of $F|K(x')^h$ has the same degree as $\overline{F}|\overline{K}(\xi)$ and such that $\overline{K(x')^h(y')} = \overline{F}$. It follows that $K(x')^h(y')|K(x')^h$ is unramified and that $(K(x')^h(y')|K,v)$ is a henselized inertially generated function field. Since $vK(x')^h(y') = vF$, we find that $F|K(x')^h(y')$ is immediate. But as proved above, the henselized inertially generated function field $K(x')^h(y')^h$ of rank 1 and of transcendence degree 1 with residue-transcendental generator over K does not admit any proper immediate algebraic extension. This yields $F = K(x')^h(y')$. □

The proof that $E.N|N$ is defectless is now done by induction on the number of extensions appearing in the tower. If there is only one, we are done. Otherwise there exists a normal subextension $E'|N$ of $E.N|N$ of degree p. We have already shown that this extension is defectless, and from the preceding lemma we infer that E' is again a henselized inertially generated function fields of rank 1 and transcendence degree 1 with a valuation-transcendental generator over K (again, its rank is 1 since it is an algebraic extension of N). By induction hypothesis, $(E|E',v)$ is also defectless since it has a smaller degree than $E|N$. Hence by Lemma 18, $(E|N,v)$ is defectless. This completes the proof of the first assertion of Theorem 1.

The "inseparably defectless" version of Theorem 1 has already been proved in Section 3 (see Proposition 32). So it remains to prove the "separably defectless" version. Assume that (K,v) is a separably defectless field, $(F|K,v)$ is a valued algebraic function field without transcendence defect, and that vK is cofinal in vF. Then the completion F^c of (F,v) contains the completion K^c of (K,v). A valued field is separably defectless if and only if its completion is defectless ([K6], Theorem 5.2). Hence, K^c is a defectless field. We consider the subfield $F.K^c \subset F^c$ which is an algebraic function field over K^c. Since $(K^c|K,v)$ and $(F^c|F,v)$ are immediate extensions and $\mathrm{trdeg}\, F.K^c|K^c \leq \mathrm{trdeg}\, F|K$, $(F.K^c|K^c,v)$ is again a valued algebraic function field without transcendence defect. By the "defectless" version of Theorem 1 it follows that $(F.K^c,v)$ is a defectless field. Hence also its completion (F^c,v) is defectless. By the above cited theorem it follows that (F,v) is defectless. □

• **Proof of Corollary 2.** By Theorem 1, $(\tilde{K}.F,v)$ is a defectless field for every extension of the valuation v. Therefore, $(\tilde{K}.F,v)$ is defectless in $\tilde{K}.E$. Let $v_1,\ldots,v_{\mathrm{g}}$ be all extensions of v from $\tilde{K}.F$ to $\tilde{K}.E$. Then there exists a finite extension $L_0|K$ such that:

1) $[L_0.E:L_0.F]=[\tilde{K}.E:\tilde{K}.F]$,
2) the restrictions of $v_1,\ldots,v_{\mathrm{g}}$ to $L_0.E$ are all distinct,
3) $(v_iL_0.E:v_iL_0.F)\geq(v_i\tilde{K}.E:v_i\tilde{K}.F)$ and $[L_0.Ev_i:L_0.Fv_i]\geq[\tilde{K}.Ev_i:\tilde{K}.Fv_i]$ for $1\leq i\leq \mathrm{g}$. (In order to obtain the first inequality, choose representatives of the distinct cosets of $v_i\tilde{K}.E/v_i\tilde{K}.F$ and choose L_0 so large that they appear in $v_iL_0.E$. For the second inequality, choose a basis of $\tilde{K}.Ev_i|\tilde{K}.Fv_i$ and choose L_0 so large that it is contained in $L_0.Ev_i$.)

These conditions will remain true whenever L_0 is replaced by any algebraic extension L of K which contains L_0. Since equality holds in the fundamental inequality for the extension $\tilde{K}.E$ of $(\tilde{K}.F,v)$, the same is then true for the extension $L.E$ of $(L.F,v)$.

Now assume in addition that (K,v) is henselian. Then by a result of Pank (cf. [K–P–R]) there exists a field complement W to its absolute ramification field K^r, that is, $W|K$ is linearly disjoint from $K^r|K$ (and hence purely wild), and $K^r.W=\tilde{K}$ (cf. [K–P–R]). Since $W^r=\tilde{K}$, (W,v) is a defectless field. Therefore, we can replace $\tilde{K}$ by W in the above, and obtain that $L_0|K$ as a subextension of $W|K$ is purely wild. □

6 Proofs of the applications

• **Proof of Theorem 3.** By Corollary 11, the factor group vF/vK and the residue field extension $\overline{F}|\overline{K}$ are finitely generated. Since vF/vK is torsion free by assumption, it is thus of the form $\mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_r$ where r is its rational rank. We choose $x_1, \ldots, x_r \in F$ such that $\alpha_i = vx_i + vK$ for $1 \leq i \leq r$. Then $vF = vK \oplus \mathbb{Z}vx_1 \oplus \ldots \oplus \mathbb{Z}vx_r$.

Since $\overline{F}|\overline{K}$ is separable by assumption and finitely generated, it is separably generated. Therefore, we can choose $y_1, \ldots, y_s \in F$ such that $\overline{F}|\overline{K}(\overline{y}_1, \ldots, \overline{y}_s)$ is separable-algebraic, where $s = \operatorname{trdeg} \overline{F}|\overline{K}$. We have thus proved the existence of a valuation transcendence basis $\mathcal{T} := \{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ with properties a) and b).

Now let $\mathcal{T}$ be any such basis and set $F_0 := K(\mathcal{T})$. Then we can choose some $\zeta \in \overline{F}$ such that $\overline{F} = \overline{K}(\overline{y}_1, \ldots, \overline{y}_s, \zeta)$. Since ζ is separable-algebraic over $\overline{K}(\overline{y}_1, \ldots, \overline{y}_s)$, by Hensel's Lemma there exists an element z in the henselization of (F, v) such that $\overline{z} = \zeta$ and that the reduction of the minimal polynomial of z over F_0 is the minimal polynomial of ζ over $\overline{K}(\overline{y}_1, \ldots, \overline{y}_s)$. We obtain that $[\overline{F_0(z)} : \overline{F_0}] \geq [\overline{K}(\overline{y}_1, \ldots, \overline{y}_s, \zeta) : \overline{K}(\overline{y}_1, \ldots, \overline{y}_s)] = [F_0(z) : F_0] \geq [\overline{F_0(z)} : \overline{F_0}]$, so equality must hold. We also obtain that $Fv = F_0(z)v$ and $vF = vF_0(z)$ and $F_0(z) \subset F^h$. As henselizations are immediate extensions and the henselization $F_0(z)^h$ of $F_0(z)$ can be chosen inside of F^h, we obtain an immediate algebraic extension $(F^h|F_0(z)^h, v)$. On the other hand, (K, v) is assumed to be a defectless field. By construction, $(F_0|K, v)$ is without transcendence defect, and the same is true for $(F_0(z)|K, v)$ since this property is preserved by algebraic extensions. Hence we know from Theorem 1 and Theorem 19 that $(F_0(z), v)$ and $(F_0(z)^h, v)$ are defectless fields. Thus, the immediate extension $F^h|F_0(z)^h$ must be trivial, i.e., $F^h = F_0(z)^h$.

On the other hand, $F_0(z)^h = F_0^h(z)$ by Lemma 8. We find that $[F_0(z)^h : F_0^h] = [F_0^h(z) : F_0^h] \leq [F_0(z) : F_0] = [\overline{F_0(z)} : \overline{F_0}] = [\overline{F_0(z)^h} : \overline{F_0^h}] \leq [F_0(z)^h : F_0^h]$, which shows that $[F_0(z)^h : F_0^h] = [\overline{F_0(z)} : \overline{F_0}]$. That is, the extension $(F^h|K(\mathcal{T})^h, v)$, which is equal to the extension $(F_0(z)^h|F_0^h, v)$, is unramified. □

• **Proof of Corollary 4.** The valued algebraic function field $(\tilde{K}.F|\tilde{K}, v)$ satisfies the assumptions of Theorem 3, so we may choose a standard valuation transcendence basis $\mathcal{T}$ of $(\tilde{K}.F|\tilde{K}, v)$ such that $v\tilde{K}.F = v\tilde{K} \oplus \mathbb{Z}vx_1 \oplus \ldots \oplus \mathbb{Z}vx_r$ and that $\overline{y}_1, \ldots, \overline{y}_s$ form a separating transcendence basis of $\overline{\tilde{K}.F}|\overline{\tilde{K}}$ such that $[(\tilde{K}.F)^h : \tilde{K}(\mathcal{T})^h] = [\overline{\tilde{K}.F} : \overline{\tilde{K}}(\overline{y}_1, \ldots, \overline{y}_s)]$. Pick $\zeta \in \overline{\tilde{K}.F}$ such that $\overline{\tilde{K}.F} = \overline{\tilde{K}}(\overline{y}_1, \ldots, \overline{y}_s, \zeta)$ and take $\overline{f}$ to be the separable minimal polynomial of ζ over $\overline{\tilde{K}}(\overline{y}_1, \ldots, \overline{y}_s)$. Then there exists a finite extension $L_0|K$ such that:

1) $\mathcal{T} \subseteq L_0.F$,
2) $[(L_0.F)^h : L_0(\mathcal{T})^h] = [(\tilde{K}.F)^h : \tilde{K}(\mathcal{T})^h]$,
3) $\overline{L_0}(\overline{y}_1, \ldots, \overline{y}_s)$ contains all coefficients of $\overline{f}$.

These conditions will remain true whenever L_0 is replaced by any algebraic extension L of K which contains L_0. For such L, we have that $\mathcal{T} \subset L.F$ remains a valuation transcendence basis of $(L.F|L(\mathcal{T}), v)$ since $L|K$ is algebraic, which yields that $\overline{L(\mathcal{T})} =$

$\overline{L}(\overline{y}_1,\ldots,\overline{y}_s)$ by Lemma 10. Now we compute:

$$\begin{aligned}[(L.F)^h : L(\mathcal{T})^h] \;&\geq\; [\overline{L.F} : \overline{L(\mathcal{T})}] \;\geq\; [\overline{L}(\overline{y}_1,\ldots,\overline{y}_s,\zeta) : \overline{L}(\overline{y}_1,\ldots,\overline{y}_s)] \;=\; \deg \overline{f}\\ &=\; [\overline{\tilde{K}.F} : \overline{\tilde{K}(\mathcal{T})}] \;=\; [(\tilde{K}.F)^h : \tilde{K}(\mathcal{T})^h] \;=\; [(L_0.F)^h : L_0(\mathcal{T})^h]\,.\end{aligned}$$

We see that equality must hold everywhere. In particular, the first equality shows that the extension $((L.F)^h|L(\mathcal{T})^h, v)$ is unramified. □

Remark 51 In addition to the assumptions of Corollary 4, assume that (K,v) is henselian, and choose W as in the proof of Corollary 2. Then $\overline{W}$ is perfect, so $\overline{W.F}|\overline{W}$ is separable. We also know that vW is p-divisible. Hence if $vW.F/vW$ has no torsion prime to p, then $vW.F/vW$ is torsion free. Then we can replace $\tilde{K}$ by W in the above proof, and obtain as in the proof of Corollary 2 that $(L_0|K,v)$ can be chosen purely wild.

• **Proof of Theorem 5.** We follow the spirit of the proof of the Ax–Kochen–Ershov principle for henselian fields with residue characteristic 0 given in the appendix of [K–P]. For the basic model theoretic background, we refer the reader to [C–K], [Pr] or [Po].

Let (K,v) be a henselian defectless field and $(L|K,v)$ an extension without transcendence defect. Assume that the "side conditions" $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$ hold. We choose (K^*,v^*) to be an $|L|^+$-saturated elementary extension of (K,v). Since "henselian" is an elementary property, (K^*,v^*) is henselian like (K,v). Since elementary sentences about the value group or the residue field of a valued field can be encoded in the valued field itself, and since $|vL|\leq|L|$ and $|Lv|\leq|L|$, it follows that v^*K^* is a $|vL|^+$-saturated elementary extension of vK and that K^*v^* is an $|Lv|^+$-saturated elementary extension of Kv. Further, if $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$-structures such that $\mathcal{A}\prec_\exists\mathcal{B}$, then $\mathcal{B}$ can be embedded over $\mathcal{A}$ in every $|\mathcal{B}|^+$-saturated extension of $\mathcal{A}$ (cf. [Pr]; this can also be a proved by a slight modification of the proof of Lemma 6.27 in [Po]). Hence there exist embeddings

$$\rho:\; vL \longrightarrow v^*K^* \qquad\text{and}\qquad \tau:\; Lv \longrightarrow K^*v^*$$

over vK and over Kv, respectively.

Since an existential sentence only talks about finitely many elements, we see that (K,v) is existentially closed in (L,v) if and only if it is existentially closed in every finitely generated subextension (F,v). By the additivity of transcendence degree and rational rank, $(F|K,v)$ is again without transcendence defect. It suffices now to show that (K,v) is existentially closed in (F,v), and we will do this by constructing a valuation preserving embedding ι of F in K^* over K. Then every existential sentence in the language of valued fields which holds in (F,v) will also hold in $(\iota(F),v^*)$, hence also in (K^*,v^*), and then also in (K,v) because (K^*,v^*) is an elementary extension of (K,v).

Our side conditions $vK\prec_\exists vL$ and $Kv\prec_\exists Lv$ imply that $vK\prec_\exists vF$ and $Kv\prec_\exists Fv$ because $vF\subseteq vL$ and $Fv\subseteq Lv$. These conditions in turn imply that vF/vK is torsion free and that $Fv|Kv$ is separable (we leave the proof of these well known facts to the reader). Hence we can choose a valuation transcendence basis $\mathcal{T}$ for $(F|K,v)$ and the element ζ as in the proof of Theorem 3.

We choose elements $x_1^*,\ldots,x_r^*,y_1^*,\ldots,y_s^*\in K^*$ such that $v^*x_i^*=\rho(vx_i)$, $1\leq i\leq r$, and $y_j^*v^*=\tau(y_jv)$, $1\leq j\leq s$. Since $x_1,\ldots,x_r,y_1,\ldots,y_s$ are algebraically independent over K, the assignment $x_i\mapsto x_i^*$, $y_j\mapsto y_j^*$ defines an embedding of $K(\mathcal{T})$ in K^* over K. It follows from Lemma 10 that this embedding is valuation preserving. By the universal property of henselizations (Lemma 7), it extends to a valuation preserving embedding of any given henselization $(K(\mathcal{T})^h,v)$ in the henselian field (K^*,v^*). Via this embedding, we identify $K(\mathcal{T})^h$ with its image in K^*.

We lift the separable minimal polynomial of ζ over $\overline{K}(\overline{y}_1,\ldots,\overline{y}_s)$ to a monic polynomial f with coefficients in the valuation ring of $(K(\mathcal{T}),v)$. Since ζ is a simple root of the reduction of f, Hensel's Lemma shows that F^h contains a root z of f with residue ζ. Similarly, also $\tau\zeta\in K^*v^*$ is a simple root of the reduction of f and we find that K^* contains a root z^* of f with residue $\tau\zeta$. Since $\deg f=[\overline{K}(\overline{y}_1,\ldots,\overline{y}_s,\zeta):\overline{K}(\overline{y}_1,\ldots,\overline{y}_s)]=[K(\mathcal{T})^h(z):K(\mathcal{T})^h]$ (where the last equality follows as in the proof of Theorem 3), f must be irreducible over $K(\mathcal{T})^h$. Therefore, sending z to z^* induces an embedding of $K(\mathcal{T})^h(z)$ in K^* over $K(\mathcal{T})^h$. This embedding must be valuation preserving since the extension of v from $K(\mathcal{T})^h$ to $K(\mathcal{T})^h(z)$ is unique. As in the proof of Theorem 3) we find that $K(\mathcal{T})^h(z)=F^h$, and so the embedding we have constructed also induces a valuation preserving embedding of (F,v). □

References

[B] Bourbaki, N.: *Commutative algebra*, Paris (1972)

[B–G–R] Bosch, S. – Güntzer, U. – Remmert, R.: *Non-Archimedean Analysis*, Berlin (1984)

[C–K] Chang, C. C. – Keisler, H. J.: *Model Theory*, Amsterdam – London (1973)

[En] Endler, O.: *Valuation theory*, Springer, Berlin (1972)

[Ep] Epp, Helmut H. P.: *Eliminating Wild Ramification*, Inventiones Math. **19** (1973), 235–249

[G] Gruson, L.: *Fibrés vectoriels sur un polydisque ultramétrique*, Ann. Sci. Ec. Super., IV. Ser., **177** (1968), 45–89

[G–M–P] Green, B. – Matignon, M. – Pop, F.: *On valued function fields I*, manuscripta math. **65** (1989), 357–376

[G–R] Grauert, H. – Remmert, R.: *Über die Methode der diskret bewerteten Ringe in der nicht archimedischen Analysis*, Invent. Math. **2** (1966), 87–133

[H] Huppert, B.: *Endliche Gruppen I*, Springer, Berlin (1967)

[K–K1] Knaf, H. – Kuhlmann, F.-V.: *Abhyankar places admit local uniformization in any characteristic*, to appear

[K–K2] Knaf, H. – Kuhlmann, F.-V.: *Every place admits local uniformization in a finite extension of the function field*, in preparation

[K1] Kuhlmann, F.-V.: *Henselian function fields and tame fields*, extended version of Ph.D. thesis, Heidelberg (1990)

[K2] Kuhlmann, F.-V.: *On local uniformization in arbitrary characteristic*, The Fields Institute Preprint Series, Toronto (1997)

[K3] Kuhlmann, F.-V.: *Valuation theoretic and model theoretic aspects of local uniformization*, in: Resolution of Singularities — A Research Textbook in Tribute to Oscar Zariski. Herwig Hauser, Joseph Lipman, Frans Oort, Adolfo Quiros (eds.), Progress in Mathematics Vol. **181**, Birkhäuser Verlag Basel (2000), 381–456

[K4] Kuhlmann, F.-V.: *A correction to Epp's paper "Elimination of wild ramification"*, Inventiones Math. **153** (2003), 679–681

[K5] Kuhlmann, F.-V.: *Additive Polynomials and Their Role in the Model Theory of Valued Fields*, to appear in: Proceedings of the Workshop and Conference on Logic, Algebra, and Arithmetic, Tehran, Iran 2003

[K6] Kuhlmann, F.-V.: *A classification of Artin Schreier defect extensions and a characterization of defectless fields*, submitted

[K7] Kuhlmann, F.-V.: *The model theory of tame valued fields*, in preparation

[K8] Kuhlmann, F.-V.: *Elimination of Ramification in Valued Function Fields II: Henselian Rationality*, in preparation

[K9] Kuhlmann, F.-V.: Book in preparation. Preliminary versions of several chapters available at:
`http://math.usask.ca/~fvk/Fvkbook.htm`

[K–P] Kuhlmann, F.–V. – Prestel, A.: *On places of algebraic function fields*, J. reine angew. Math. **353** (1984), 181–195

[K–P–R] Kuhlmann, F.-V. – Pank, M. – Roquette, P.: *Immediate and purely wild extensions of valued fields*, Manuscripta Math. **55** (1986), 39–67

[L] Lang, S.: *Algebra*, Springer, New York, 3rd. ed. (2002)

[O] Ohm, J.: *The henselian defect for valued function fields*, Proc. Amer. Math. Soc. **107** (1989)

[Po] Poizat, B.: *A Course in Model Theory*, Springer Universitext, New York (2000)

[Pr] Prestel, A.: *Einführung in die mathematische Logik und Modelltheorie*, Vieweg, Braunschweig (1986)

[R] Ribenboim, P.: *Théorie des valuations*, Les Presses de l'Université de Montréal, Montréal, 2nd ed. (1968)

[W] Warner, S.: *Topological fields*, Mathematics studies **157**, North Holland, Amsterdam (1989)

[Z–S] Zariski, O. – Samuel, P.: *Commutative Algebra*, Vol. II, New York–Heidelberg–Berlin (1960)

Mathematical Sciences Group, University of Saskatchewan,
106 Wiggins Road, Saskatoon, Saskatchewan, Canada S7N 5E6
email: fvk@math.usask.ca — home page: http://math.usask.ca/~fvk/index.html