PRICING AND TRADING CREDIT DEFAULT SWAPS UNDER DETERMINISTIC INTENSITY[*]

Tomasz R. Bielecki[†]
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, IL 60616, USA

Monique Jeanblanc[‡]
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France

Marek Rutkowski[§]
School of Mathematics
University of New South Wales
Sydney, NSW 2052, Australia
and
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland

June 28, 2005

Contents

1   Introduction

The topic of this work is a detailed study of credit default swaps within the framework of a reduced-form model of credit risk. We start, in Section 2, by dealing with the valuation and trading of a generic defaultable claim. The presentation in this section, although largely based on Section 2.1 in Bielecki and Rutkowski (2002), is adapted to our current purposes, and the notation is modified accordingly. We believe that it is more convenient to deal with a generic dividend-paying asset, rather than with a specific example of a credit derivative since most basic properties of prices of defaultable assets and related trading strategies are already apparent in a general set-up. In Section 3, we first provide results concerning the valuation and trading of credit default swaps under the assumption that the default intensity is deterministic and the interest rate is zero. Subsequently, we derive a closed-form solution for replicating strategy for an arbitrary non-dividend paying defaultable claim in a market in which a bond and a credit default swap are traded, and we examine the market completeness.

This work was completed during our visit to the Isaac Newton Institute for Mathematical Sciences in Cambridge. We thank the organizers of the programme *Developments in Quantitative Finance* for the kind invitation.

2   Valuation and Trading Defaultable Claims

The goal of this section is to give a brief presentation of general results concerning the valuation and trading of defaultable claims.

2.1   Generic Defaultable Claims

A strictly positive random variable τ, defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, is termed a *random time*. In view of its interpretation, it will be later referred to as a *default time*. We introduce the jump process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ associated with τ, and we denote by $\mathbb{H}$ the filtration generated by this process. We assume that we are given, in addition, some auxiliary filtration $\mathbb{F}$, and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that we have $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$.

Definition 2.1 By a *defaultable claim maturing at T* we mean the quadruple (X, A, Z, τ), where X is an $\mathcal{F}_T$-measurable random variable, A is an $\mathbb{F}$-adapted process of finite variation, Z is an $\mathbb{F}$-predictable process and τ is a random time.

The financial interpretation of the components of a defaultable claim becomes clear from the following definition of the dividend process D, which describes all cash flows associated with a defaultable claim over the lifespan $]0, T]$, that is, after the contract was initiated at time 0. Of course, the choice of 0 as the date of inception is arbitrary.

Definition 2.2 The *dividend process D* of a defaultable claim maturing at T equals, for every $t \in [0, T]$,

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T, \infty[}(t) + \int_{]0,t]} (1 - H_u) \, dA_u + \int_{]0,t]} Z_u \, dH_u.$$

The financial interpretation of the definition above justifies the following terminology: X is the *promised payoff*, A represents the process of *promised dividends*, and the process Z, termed the *recovery process*, specifies the recovery payoff at default. It is worth stressing that, according to our convention, the cash payment (premium) at time 0 is not included in the dividend process D associated with a defaultable claim. When we deal with a credit default swap, the premium at time 0 is typically equal to zero, and the process A represents in fact the premium paid in instalments

up to maturity date or default, whichever comes first. For instance, if $A_t = -\kappa t$ for some constant $\kappa > 0$, then the 'price' of a stylized credit default swap is formally represented by this constant, referred to as the continuously paid *credit default spread* or *premium* (see Section 3.1 for details).

If the other covenants of the contract are known (i.e., the payoffs X and Z are given), the valuation of a swap is equivalent to finding the level of the spread κ that makes the swap worthless at inception. Typically, in a credit default swap we have $X = 0$, and Z is determined in reference to a recovery rate of the reference credit-risky entity. In a more realistic approach, the process A is discontinuous, with jumps occurring at the premium payment dates. In this note, we shall only deal with a stylized CDS with a continuously paid premium.

Let us return to the general set-up. It is clear that the dividend process D follows a process of finite variation on $[0, T]$. Since

$$\int_{]0,t]} (1 - H_u)\, dA_u = \int_{]0,t]} \mathbb{1}_{\{\tau > u\}}\, dA_u = A_{\tau-}\mathbb{1}_{\{\tau \le t\}} + A_t\mathbb{1}_{\{\tau > t\}},$$

it is also apparent that if default occurs at some date t, the 'promised dividend' $A_t - A_{t-}$ that is due to be received or paid at this date is disregarded. If we denote $\tau \wedge t = \min(\tau, t)$ then we have

$$\int_{]0,t]} Z_u\, dH_u = Z_{\tau \wedge t}\mathbb{1}_{\{\tau \le t\}} = Z_\tau\mathbb{1}_{\{\tau \le t\}}.$$

Let us stress that the process $D_u - D_t$, $u \in [t, T]$, represents all cash flows from a defaultable claim received by an investor who purchases it at time t. Of course, the process $D_u - D_t$ may depend on the past behavior of the claim (e.g., through some intrinsic parameters, such as credit spreads) as well as on the history of the market prior to t. The past dividends are not valued by the market, however, so that the current market value at time t of a claim (i.e., the price at which it trades at time t) depends only on future dividends to be paid or received over the time interval $]t, T]$.

Suppose that our underlying financial market model is arbitrage-free, in the sense that there exists a *spot martingale measure* $\mathbb{Q}^*$ (also referred to as a *risk-neutral probability*), meaning that $\mathbb{Q}^*$ is equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$, and the price process of any tradeable security, paying no coupons or dividends, follows a $\mathbb{G}$-martingale under $\mathbb{Q}^*$, when discounted by the *savings account* B, where

$$B_t = \exp\left(\int_0^t r_u\, du\right).$$

2.2   Buy-and-hold Strategy

We write S^i, $i = 1, 2, \ldots, k$ to denote the price processes of k primary securities in an arbitrage-free financial model. We make the standard assumption that the processes S^i, $i = 1, 2, \ldots, k - 1$ follow semimartingales. In addition, we set $S_t^k = B_t$ so that S^k represents the value process of the savings account. The last assumption is not necessary, however. We can assume, for instance, that S^k is the price of a T-maturity risk-free zero-coupon bond, or choose any other strictly positive price process as as numéraire.

For the sake of convenience, we assume that S^i, $i = 1, 2, \ldots, k - 1$ are non-dividend-paying assets, and we introduce the discounted price processes S^{i*} by setting $S_t^{i*} = S_t^i / B_t$. All processes are assumed to be given on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{Q})$, where $\mathbb{Q}$ is interpreted as the real-life (i.e., statistical) probability measure.

Let us now assume that we have an additional traded security that pays dividends during its lifespan, assumed to be the time interval $[0, T]$, according to a process of finite variation D, with $D_0 = 0$. Let S denote a (yet unspecified) price process of this security. In particular, we do not postulate a priori that S follows a semimartingale. It is not necessary to interpret S as a price process of a defaultable claim, though we have here this particular interpretation in mind.

Let a $\mathbb{G}$-predictable, $\mathbb{R}^{k+1}$-valued process $\phi = (\phi^0, \phi^1, \ldots, \phi^k)$ represent a generic trading strategy, where ϕ_t^j represents the number of shares of the jth asset held at time t. We identify here S^0

with S, so that S is the 0th asset. In order to derive a pricing formula for this asset, it suffices to examine a simple trading strategy involving S, namely, the buy-and-hold strategy.

Suppose that we have purchased at time 0 one unit of the 0th asset at the initial price S_0, and we hold it until time T. We assume all the proceeds from dividends are re-invested in the savings account B. More specifically, we consider a *buy-and-hold* strategy $\psi = (1, 0, \ldots, 0, \psi^k)$, where ψ^k is a $\mathbb{G}$-predictable process. The associated *wealth process* $U(\psi)$ equals

$$U_t(\psi) = S_t + \psi_t^k B_t, \quad \forall\, t \in [0, T], \tag{1}$$

so that its initial value equals $U_0(\psi) = S_0 + \psi_0^k$.

Definition 2.3 We say that a strategy $\psi = (1, 0, \ldots, 0, \psi^k)$ is *self-financing* if

$$dU_t(\psi) = dS_t + dD_t + \psi_t^k \, dB_t,$$

or more explicitly, for every $t \in [0, T]$,

$$U_t(\psi) - U_0(\psi) = S_t - S_0 + D_t + \int_{]0,t]} \psi_u^k \, dB_u. \tag{2}$$

We assume from now on that the process ψ^k is chosen in such a way (with respect to S, D and B) that a buy-and-hold strategy ψ is self-financing. Also, we make a standing assumption that the random variable $Y = \int_{]0,T]} B_u^{-1} dD_u$ is $\mathbb{Q}^*$-integrable.

Lemma 2.1 *The discounted wealth* $U_t^*(\psi) = B_t^{-1} U_t(\psi)$ *of any self-financing buy-and-hold trading strategy* ψ *satisfies, for every* $t \in [0, T]$,

$$U_t^*(\psi) = U_0^*(\psi) + S_t^* - S_0^* + \int_{]0,t]} B_u^{-1} dD_u. \tag{3}$$

Hence we have, for every $t \in [0, T]$,

$$U_T^*(\psi) - U_t^*(\psi) = S_T^* - S_t^* + \int_{]t,T]} B_u^{-1} \, dD_u. \tag{4}$$

Proof. We define an auxiliary process $\widehat{U}(\psi)$ by setting $\widehat{U}_t(\psi) = U_t(\psi) - S_t = \psi_t^k B_t$ for $t \in [0, T]$. In view of (2), we have

$$\widehat{U}_t(\psi) = \widehat{U}_0(\psi) + D_t + \int_{]0,t]} \psi_u^k \, dB_u,$$

and so the process $\widehat{U}(\psi)$ follows a semimartingale. An application of Itô's product rule yields

$$
\begin{aligned}
d\big(B_t^{-1} \widehat{U}_t(\psi)\big) &= B_t^{-1} d\widehat{U}_t(\psi) + \widehat{U}_t(\psi) \, dB_t^{-1} \\
&= B_t^{-1} dD_t + \psi_t^k B_t^{-1} dB_t + \psi_t^k B_t \, dB_t^{-1} \\
&= B_t^{-1} dD_t,
\end{aligned}
$$

where we have used the obvious identity: $B_t^{-1} dB_t + B_t \, dB_t^{-1} = 0$. Integrating the last equality, we obtain

$$B_t^{-1}\big(U_t(\psi) - S_t\big) = B_0^{-1}\big(U_0(\psi) - S_0\big) + \int_{]0,t]} B_u^{-1} dD_u,$$

and this immediately yields (3). $\qquad\square$

It is worth noting that Lemma 2.1 remains valid if the assumption that S^k represents the savings account B is relaxed. It suffices to assume that the price process S^k is a *numéraire*, that is, a

strictly positive continuous semimartingale. For the sake of brevity, let us write $S^k = \beta$. We say that $\psi = (1, 0, \ldots, 0, \psi^k)$ is self-financing it the wealth process

$$U_t(\psi) = S_t + \psi_t^k \beta_t, \quad \forall\, t \in [0, T],$$

satisfies, for every $t \in [0, T]$,

$$U_t(\psi) - U_0(\psi) = S_t - S_0 + D_t + \int_{]0,t]} \psi_u^k \, d\beta_u.$$

Lemma 2.2 *The relative wealth $U_t^*(\psi) = \beta_t^{-1} U_t(\psi)$ of a self-financing trading strategy ψ satisfies, for every $t \in [0, T]$,*

$$U_t^*(\psi) = U_0^*(\psi) + S_t^* - S_0^* + \int_{]0,t]} \beta_u^{-1} dD_u,$$

where $S^* = \beta_t^{-1} S_t$.

2.3   Spot Martingale Measure

Our next goal is to derive the risk-neutral valuation formula for the ex-dividend price S_t. To this end, we assume that our market model is arbitrage-free, meaning that it admits a (not necessarily unique) martingale measure $\mathbb{Q}^*$, equivalent to $\mathbb{Q}$, which is associated with the choice of B as a numéraire.

Definition 2.4 We say that $\mathbb{Q}^*$ is a *spot martingale measure* if the discounted price S^{i*} of any non-dividend paying traded security follows a $\mathbb{Q}^*$-martingale with respect to $\mathbb{G}$.

It is well known that the discounted wealth process $U^*(\phi)$ of any self-financing trading strategy $\phi = (0, \phi^1, \phi^2, \ldots, \phi^k)$ is a local martingale under $\mathbb{Q}^*$. In what follows, we shall only consider *admissible* trading strategies, that is, strategies for which the discounted wealth process $U^*(\phi)$ is a martingale under $\mathbb{Q}^*$. Then a market model in which only admissible trading strategies are allowed is *arbitrage-free*, in the usual sense.

Following this line of arguments, we postulate that the trading strategy ψ introduced in Section 2.2 is also *admissible*, so that its discounted wealth process $U^*(\psi)$ follows a martingale under $\mathbb{Q}^*$ with respect to $\mathbb{G}$. This assumption is quite natural if we wish to prevent arbitrage opportunities in the extended model of the financial market. Indeed, since we postulate that S is traded, the wealth process $U(\psi)$ can be formally seen as an additional non-dividend paying tradeable security.

To derive a pricing formula for a defaultable claim, we make a natural assumption that the market value at time t of the 0th security comes exclusively from the future dividends stream, that is, from the cash flows occurring in the open interval $]t, T[$. Since the lifespan of S is $[0, T]$, this amounts to postulate that $S_T = S_T^* = 0$. To emphasize this property, we shall refer to S as the *ex-dividend price* of the 0th asset.

Definition 2.5 A process S with $S_T = 0$ is the *ex-dividend price* of the 0th asset if the discounted wealth process $U^*(\psi)$ of any self-financing buy-and-hold strategy ψ follows a $\mathbb{G}$-martingale under $\mathbb{Q}^*$.

As a special case, we obtain the ex-dividend price a defaultable claim with maturity T.

Proposition 2.1 *The ex-dividend price process S associated with the dividend process D satisfies, for every $t \in [0, T]$,*

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big). \tag{5}$$

Proof. The postulated martingale property of the discounted wealth process $U^*(\psi)$ yields, for every $t \in [0, T]$,

$$\mathbb{E}_{\mathbb{Q}^*}\big(U_T^*(\psi) - U_t^*(\psi) \,\big|\, \mathcal{G}_t \big) = 0.$$

Taking into account (4), we thus obtain

$$S_t^* = \mathbb{E}_{\mathbb{Q}^*}\Big(S_T^* + \int_{]t,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t \Big).$$

Since, by virtue of the definition of the ex-dividend price we have $S_T = S_T^* = 0$, the last formula yields (5). □

It is not difficult to show that the ex-dividend price S satisfies, for every $t \in [0, T]$,

$$S_t = \mathbb{1}_{\{t < \tau\}} \widetilde{S}_t, \tag{6}$$

where the process $\widetilde{S}$ represents the *ex-dividend pre-default price* of a defaultable claim.

The *cum-dividend price* process $\bar{S}$ associated with the dividend process D is a $\mathbb{G}$-martingale under $\mathbb{Q}^*$, given by the formula, for every $t \in [0, T]$,

$$\bar{S}_t = B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]0,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t \Big). \tag{7}$$

The savings account B can be replaced by an arbitrary numéraire β. The corresponding valuation formula becomes, for every $t \in [0, T]$,

$$S_t = \beta_t\, \mathbb{E}_{\mathbb{Q}^\beta}\Big(\int_{]t,T]} \beta_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t \Big), \tag{8}$$

where $\mathbb{Q}^\beta$ is a martingale measure on $(\Omega, \mathcal{G}_T)$ associated with a numéraire β, that is, a probability measure on $(\Omega, \mathcal{G}_T)$ given by the formula

$$\frac{d\mathbb{Q}^\beta}{d\mathbb{Q}^*} = \frac{\beta_T}{\beta_0 B_T}, \quad \mathbb{Q}^*\text{-a.s.}$$

2.4 Self-Financing Trading Strategies

Let us now examine a general trading strategy $\phi = (\phi^0, \phi^1, \ldots, \phi^k)$ with $\mathbb{G}$-predictable components. The associated *wealth process* $U(\phi)$ equals $U_t(\phi) = \sum_{i=0}^k \phi_t^i S_t^i$, where, as before $S^0 = S$. A strategy ϕ is said to be *self-financing* if $U_t(\phi) = U_0(\phi) + G_t(\phi)$ for every $t \in [0, T]$, where the *gains process* $G(\phi)$ is defined as follows:

$$G_t(\phi) = \int_{]0,t]} \phi_u^0\, dD_u + \sum_{i=0}^k \int_{]0,t]} \phi_u^i\, dS_u^i.$$

Corollary 2.1 *Let $S^k = B$. Then for any self-financing trading strategy ϕ, the discounted wealth process $U^*(\phi) = B_t^{-1} U_t(\phi)$ follows a martingale under $\mathbb{Q}^*$.*

Proof. Since B is a continuous process of finite variation, Itô's product rule gives

$$dS_t^{i*} = S_t^i\, dB_t^{-1} + B_t^{-1} dS_t^i$$

for $i = 0, 1, \ldots, k$, and so

$$dU_t^*(\phi) = U_t(\phi)\, dB_t^{-1} + B_t^{-1} dU_t(\phi)$$

$$
\begin{aligned}
&= U_t(\phi)\, dB_t^{-1} + B_t^{-1}\Big(\sum_{i=0}^{k} \phi_t^i\, dS_t^i + \phi_t^0\, dD_t\Big) \\[2mm]
&= \sum_{i=0}^{k} \phi_t^i\big(S_t^i\, dB_t^{-1} + B_t^{-1} dS_t^i\big) + \phi_t^0 B_t^{-1} dD_t \\[2mm]
&= \sum_{i=1}^{k-1} \phi_t^i\, dS_t^{i*} + \phi_t^0\big(dS_t^* + B_t^{-1} dD_t\big) = \sum_{i=1}^{k-1} \phi_t^i\, dS_t^{i*} + \phi_t^0\, d\widehat{S}_t,
\end{aligned}
$$

where the auxiliary process $\widehat{S}$ is given by the following expression:

$$
\widehat{S}_t = S_t^* + \int_{]0,t]} B_u^{-1}\, dD_u.
$$

To conclude, it suffices to observe that in view of (5) the process $\widehat{S}$ satisfies

$$
\widehat{S}_t = \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]0,T]} B_u^{-1}\, dD_u\,\Big|\, \mathcal{G}_t\Big), \tag{9}
$$

and thus it follows a martingale under $\mathbb{Q}^*$. $\square$

It is worth noting that $\widehat{S}_t$, given by formula (9), represents the discounted *cum-dividend price* at time t of the 0th asset, that is, the arbitrage price at time t of all past and future dividends associated with the 0th asset over its lifespan. To check this, let us consider a buy-and-hold strategy such that $\psi_0^k = 0$. Then, in view of (4), the terminal wealth at time T of this strategy equals

$$
U_T(\psi) = B_T \int_{]0,T]} B_u^{-1}\, dD_u. \tag{10}
$$

It is clear that $U_T(\psi)$ represents all dividends from S in the form of a single payoff at time T. The arbitrage price $\pi_t(\widehat{Y})$ at time $t < T$ of a claim $\widehat{Y} = U_T(\psi)$ equals (under the assumption that this claim is attainable)

$$
\pi_t(\widehat{Y}) = B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]0,T]} B_u^{-1}\, dD_u\,\Big|\, \mathcal{G}_t\Big)
$$

and thus $\widehat{S}_t = \pi_t(\widehat{Y})/B_t$. It is clear that discounted cum-dividend price follows a martingale under $\mathbb{Q}^*$ (under the standard integrability assumption).

Remarks. (i) Under the assumption of uniqueness of a spot martingale measure $\mathbb{Q}^*$, any $\mathbb{Q}^*$-integrable contingent claim is attainable, and the valuation formula established above can be justified by means of replication.

(ii) Otherwise – that is, when a martingale probability measure $\mathbb{Q}^*$ is not uniquely determined by the model $(S^1, S^2, \ldots, S^k)$ – the right-hand side of (5) may depend on the choice of a particular martingale probability, in general. In this case, a process defined by (5) for an arbitrarily chosen spot martingale measure $\mathbb{Q}^*$ can be taken as the no-arbitrage price process of a defaultable claim. In some cases, a market model can be completed by postulating that S is also a traded asset.

2.5   Martingale Properties of Prices of a Defaultable Claim

In the next result, we summarize the martingale properties of prices of a generic defaultable claim.

Corollary 2.2 *The discounted cum-dividend price $\widehat{S}_t$, $t \in [0,T]$, of a defaultable claim is a $\mathbb{Q}^*$-martingale with respect to $\mathbb{G}$. The discounted ex-dividend price S_t^*, $t \in [0,T]$, satisfies*

$$
S_t^* = \widehat{S}_t - \int_{]0,t]} B_u^{-1}\, dD_u, \quad \forall\, t \in [0,T],
$$

and thus it follows a supermartingale under $\mathbb{Q}^*$ if and only if the dividend process D is increasing.

In an application considered in Section 3, the finite variation process A is interpreted as the positive premium paid in instalments by the claimholder to the counterparty in exchange for a positive recovery (received by the claimholder either at maturity or at default). It is thus natural to assume that A is a decreasing process, and all other components of the dividend process are increasing processes (that is, we postulate that $X \geq 0$, and $Z \geq 0$). It is rather clear that, under these assumptions, the discounted ex-dividend price S^* is neither a super- or submartingale under $\mathbb{Q}^*$, in general.

Assume now that $A \equiv 0$, so that the premium for a defaultable claim is paid upfront at time 0, and it is not accounted for in the dividend process D. We postulate, as before, that $X \geq 0$, and $Z \geq 0$. In this case, the dividend process D is manifestly increasing, and thus the discounted ex-dividend price S^* is a supermartingale under $\mathbb{Q}^*$. This feature is quite natural since the discounted expected value of future dividends decreases when time elapses.

The bottom line is that the martingale properties of the price of a defaultable claim depend on the specification of a claim and conventions regarding the prices (ex-dividend price or cum-dividend price). This point will be illustrated below by means of a detailed analysis of prices of credit default swaps.

3   Valuation and Trading Credit Default Swaps

We are now in the position to apply the general theory to the case of a particular class contracts, specifically, credit default swaps. We work throughout under a spot martingale measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{G}_T)$. In the first step, we shall work under additional assumptions that the auxiliary filtration $\mathbb{F}$ is trivial, so that $\mathbb{G} = \mathbb{H}$ and the interest rate $r = 0$. Subsequently, these restrictions will be relaxed.

3.1   Valuation of a Credit Default Swap

A stylized credit default swap is formally introduced through the following definition.

Definition 3.1 A *credit default swap* with a constant spread κ and *recovery at default* is a defaultable claim $(0, A, Z, \tau)$, where $Z_t \equiv \delta(t)$ and $A_t = -\kappa t$ for every $t \in [0, T]$. An RCLL function $\delta : [0, T] \to \mathbb{R}$ represents the *protection payment* and a constant $\kappa \in \mathbb{R}$ is termed the *spread* (or the *premium*) of a CDS.

We shall first analyze the valuation and trading credit default swaps in a simple model of default risk with the filtration $\mathbb{G} = \mathbb{H}$ generated by the process $H_t = \mathbb{1}_{\{\tau \leq t\}}$. We denote by F the cumulative distribution function of the default time τ under $\mathbb{Q}^*$, and we assume that F is a continuous function, with $F(0) = 0$ and $F(T) < 1$ for some fixed date $T > 0$. Also, we write $G = 1 - F$ to denote the *survival probability function* of τ, so that $G(t) > 0$ for every $t \in [0, T]$. For simplicity, we assume that the interest rate $r = 0$, so that the price of a savings account $B_t = 1$ for every t. Our results can be easily extended to the case of a constant r.

Note that we have only one tradeable asset in our model (a savings account), and we wish to value a defaultable claim within this model. It is clear that any probability measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{H}_T)$, equivalent to $\mathbb{Q}$, can be chosen as a spot martingale measure for our model. The choice of $\mathbb{Q}^*$ is reflected in the cumulative distribution function F (in particular, in the default intensity if F is absolutely continuous).

3.1.1   Ex-dividend Price of a CDS

Consider a CDS with the spread κ, which was initiated at time 0 (or indeed at any date prior to the current date t). Its market value at time t does not depend on the past otherwise than through the level of the spread κ. For the moment, we assume that κ is an arbitrary constant. Unless stated

otherwise, we assume that the recovery (also known here as the *protection payment*) is received at the time of default and it is equal $\delta(t)$ if default occurs at time t.

In view of (5), the ex-dividend price of a CDS maturing at T with spread κ is given by the formula

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}^*}\Big(\delta(\tau)\mathbb{1}_{\{t<\tau\leq T\}} - \mathbb{1}_{\{t<\tau\}}\kappa\big((\tau \wedge T) - t\big)\,\Big|\,\mathcal{H}_t\Big). \tag{11}$$

Note that in Lemma 3.1, we do not need to specify the inception date s of a CDS. We only assume that the maturity date T, the spread κ, and the protection payment δ are given.

Lemma 3.1 *The ex-dividend price at time $t \in [s,T]$ of a credit default swap with spread κ and recovery at default equals*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\frac{1}{G(t)}\left(-\int_t^T \delta(u)\,dG(u) - \kappa\int_t^T G(u)\,du\right). \tag{12}$$

Proof. We have, on the set $\{t < \tau\}$,

$$
\begin{aligned}
S_t(\kappa) &= -\frac{\int_t^T \delta(u)\,dG(u)}{G(t)} - \kappa\left(\frac{-\int_t^T u\,dG(u) + TG(T)}{G(t)} - t\right) \\
&= \frac{1}{G(t))}\left(-\int_t^T \delta(u)\,dG(u) - \kappa\Big(TG(T) - tG(t) - \int_t^T u\,dG(u)\Big)\right).
\end{aligned}
$$

Since

$$\int_t^T G(u)\,du = TG(T) - tG(t) - \int_t^T u\,dG(u),$$

we conclude that (12) holds.   □

The ex-dividend price of a CDS can also be represented as follows (see (6))

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\widetilde{S}_t(\kappa), \quad \forall\, t \in [0,T], \tag{13}$$

where $\widetilde{S}_t(\kappa)$ stands for the *ex-dividend pre-default price* of a CDS. It is useful to note that formula (12) yields an explicit expression for $\widetilde{S}_t(\kappa)$ and that it follows a continuous function provided that G is continuous.

3.1.2   Market CDS Spreads

Assume now that a CDS was initiated at some date $s \leq t$ and its initial price was equal to zero. Since a CDS with this property plays an important role, we introduce a formal definition. In Definition 3.2, it is implicitly assumed that a recovery function δ is given.

Definition 3.2 A *market CDS started at s* is a CDS initiated at time s whose initial value is equal to zero. A T-maturity *CDS market spread* at time s is the level of the spread $\kappa = \kappa(s,T)$ that makes a T-maturity CDS started at s worthless at its inception. A CDS market spread at time s is thus determined by the equation $S_s(\kappa(s,T)) = 0$, where S is defined by (12).

In our set-up, by virtue of Lemma 3.1, the T-maturity market spread $\kappa(s,T)$ is a solution to the equation

$$\int_s^T \delta(u)\,dG(u) + \kappa(s,T)\int_s^T G(u)\,du = 0,$$

and thus we have, for every $s \in [0,T]$,

$$\kappa(s,T) = -\frac{\int_s^T \delta(u)\,dG(u)}{\int_s^T G(u)\,du}. \tag{14}$$

Remarks. Let us comment briefly on a model calibration. Suppose that at time 0 the market gives the premium of a CDS for any maturity T. In this way, the market chooses the risk-neutral probability measure $\mathbb{Q}^*$. Specifically, if $\kappa(0,T)$ is the T-maturity market CDS spread for a given recovery function δ then we have

$$\kappa(0,T) = -\frac{\int_0^T \delta(u)\,dG(u)}{\int_0^T G(u)\,du}.$$

Hence, if credit default swaps with the same recovery function δ and varying maturities are traded at time 0, it is possible to find the implied risk-neutral c.d.f. F (and thus the default intensity γ under $\mathbb{Q}^*$) from the *term structure of CDS spreads* $\kappa(0,T)$ by solving an ordinary differential equation.

Standing assumptions. We fix the maturity date T, and we write briefly $\kappa(s)$ instead of $\kappa(s,T)$. In addition, we assume that all credit default swaps have a common recovery function δ.

Note that the ex-dividend pre-default value at time $t \in [0,T]$ of a CDS with any fixed spread κ can be related to the market spread $\kappa(t)$. We have the following result, in which the quantity $\nu(t,s) = \kappa(t) - \kappa(s)$ represents the *calendar CDS market spread* (for a given maturity T).

Proposition 3.1 *The ex-dividend price of a market CDS started at s with recovery δ at default and maturity T equals, for every $t \in [s,T]$,*

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}}\,(\kappa(t) - \kappa(s))\,\frac{\int_t^T G(u)\,du}{G(t)} = \mathbb{1}_{\{t<\tau\}}\,\nu(t,s)\,\frac{\int_t^T G(u)\,du}{G(t)}, \tag{15}$$

or more explicitly,

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}}\,\frac{\int_t^T G(u)\,du}{G(t)}\left(\frac{\int_s^T \delta(u)\,dG(u)}{\int_s^T G(u)\,du} - \frac{\int_t^T \delta(u)\,dG(u)}{\int_t^T G(u)\,du}\right). \tag{16}$$

Proof. To establish equality (16), it suffices to observe that $S_t(\kappa(s)) = S_t(\kappa(s)) - S_t(\kappa(t))$, and to use (12) and (14). $\qquad\square$

Remarks. A representation of the value of a swap in terms of market swap rates is well known to hold for default-free interest rate swaps. It is especially useful if the calendar spread follows a stochastic process; in particular, it leads to the Black swaption formula within the framework of Jamshidian's (1997) model of co-terminal forward swap rates.

3.1.3 Case of a Constant Default Intensity

Assume that $\delta(t) = \delta$ is independent of t, and $F(t) = 1 - e^{-\gamma t}$ for a constant default intensity $\gamma > 0$ under $\mathbb{Q}^*$. In this case, the valuation formulae for a CDS can be further simplified. In view of Lemma 3.1, the ex-dividend price of a CDS with spread κ equals, for every $t \in [0,T]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}(\delta\gamma - \kappa)\gamma^{-1}\left(1 - e^{-\gamma(T-t)}\right).$$

The last formula (or the general formula (14)) yields $\kappa(s) = \delta\gamma$ for every $s < T$, so that the market spread $\kappa(s)$ is independent of s. As a consequence, the ex-dividend price of a market CDS started at s equals zero not only at the inception date s, but indeed at any time $t \in [s,T]$, both prior to and after default). Hence, this process follows a trivial martingale under $\mathbb{Q}^*$. As we shall see in what follows, this martingale property the ex-dividend price of a market CDS is an exception, rather than a rule.

3.2   Price Dynamics of a CDS

In what follows, we assume that

$$G(t) = \mathbb{Q}^*(\tau > t) = \exp\left(-\int_0^t \gamma(u)\, du\right)$$

where the default intensity $\gamma(t)$ under $\mathbb{Q}^*$ is deterministic. We first focus on the dynamics of the ex-dividend price of a CDS with spread κ started at some date $s < T$.

Lemma 3.2 *The dynamics of the ex-dividend price $S_t(\kappa)$ on $[s, T]$ are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt, \tag{17}$$

where the $\mathbb{H}$-martingale M under $\mathbb{Q}^*$ is given by the formula

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u)\, du, \quad \forall\, t \in \mathbb{R}_+. \tag{18}$$

Hence, the process $\bar{S}_t(\kappa)$, $t \in [s, T]$, given by the expression

$$\bar{S}_t(\kappa) = S_t(\kappa) + \int_s^t \delta(u)\, dH_u - \kappa \int_s^t (1 - H_u)\, du \tag{19}$$

is a $\mathbb{Q}^*$-martingale for $t \in [s, T]$.

Proof. It suffices to recall that

$$S_t(\kappa) = \mathbb{1}_{\{t < \tau\}} \widetilde{S}_t(\kappa) = (1 - H_t)\widetilde{S}_t(\kappa)$$

so that

$$dS_t(\kappa) = (1 - H_t)\, d\widetilde{S}_t(\kappa) - \widetilde{S}_{t-}(\kappa)\, dH_t.$$

Using formula (12), we find easily that we have

$$d\widetilde{S}_t(\kappa) = \gamma(t)\widetilde{S}_t(\kappa)\, dt + (\kappa(s) - \delta(t)\gamma(t))\, dt.$$

In view of (18), the proof of (17) is complete. To prove the second statement, it suffices to observe that the process N given by

$$N_t = S_t(\kappa) - \int_s^t (1 - H_u)(\kappa - \delta(u)\gamma(u))\, du = -\int_s^t S_{u-}(\kappa)\, dM_u$$

is an $\mathbb{H}$-martingale under $\mathbb{Q}^*$. But for every $t \in [s, T]$

$$\bar{S}_t(\kappa) = N_t + \int_s^t \delta(u)\, M_u,$$

so that $\bar{S}(\kappa)$ also follows an $\mathbb{H}$-martingale under $\mathbb{Q}^*$. Note that the process $\bar{S}(\kappa)$ given by (19) represents the cum-dividend price of a CDS, so that the martingale property $\bar{S}(\kappa)$ is expected. $\square$

Equality (17) emphasizes the fact that a single cash flow of $\delta(\tau)$ occurring at time τ can be formally treated as a dividend stream at the rate $\delta(t)\gamma(t)$ paid continuously prior to default. It is clear that we also have

$$dS_t(\kappa) = -\widetilde{S}_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt. \tag{20}$$

It can be useful to reformulate the dynamics of a market CDS in terms of market observables, such as CDS spreads.

Corollary 3.1 *The dynamics of the ex-dividend price $S_t(\kappa(s))$ on $[s, T]$ are also given as*

$$dS_t(\kappa(s)) = -S_{t-}(\kappa(s))\,dM_t + (1 - H_t)\left(\frac{\int_t^T G(u)\,du}{G(t)}\,d_t\nu(t, s) - \nu(t, s)\,dt\right). \tag{21}$$

Proof. In the present set-up, for any fixed s, the calendar spread $\nu(t, s)$, $t \in [s, T]$ is a continuous function of bounded variation. In view of (17), it suffices to check that

$$\frac{\int_t^T G(u)\,du}{G(t)}\,d_t\nu(t, s) - \nu(t, s)\,dt = (\kappa(s) - \delta(t)\gamma(t))\,dt, \tag{22}$$

where $d_t\nu(t, s) = d_t(\kappa(t) - \kappa(s)) = d\kappa(t)$. Equality (22) follows by elementary computations.   □

3.2.1   Trading Strategies with a CDS

We shall show that in the present set-up, in order to replicate an arbitrary contingent claim Y settling at time T and satisfying the usual integrability condition, it suffices to deal with two traded assets: a CDS with maturity $U \geq T$ and a constant savings account $B = 1$. Since one can always work with discounted values, the last assumption is not restrictive.

According to Section 2.4, a strategy $\phi_t = (\phi_t^0, \phi_t^1)$, $t \in [0, T]$, is self-financing if the wealth process $U(\phi)$, defined as

$$U_t(\phi) = \phi_t^0 S_t(\kappa) + \phi_t^1, \tag{23}$$

satisfies

$$dU_t(\phi) = \phi_t^0\,dS_t(\kappa) + \phi_t^0\,dD_t, \tag{24}$$

where $S(\kappa)$ is the ex-dividend price of a CDS with the dividend stream D. As usual, we say that a strategy ϕ replicates a contingent claim Y if $U_T(\phi) = Y$. On the set $\{\tau \leq t \leq T\}$ the ex-dividend price $S(\kappa)$ equals zero, and thus the total wealth is necessarily invested in B, so that it is constant. This means that ϕ replicates Y if and only if $U_{\tau \wedge T}(\phi) = Y$.

Lemma 3.3 *For any self-financing strategy ϕ we have, on the set $\{\tau \leq T\}$,*

$$\Delta_\tau U(\phi) := U_\tau(\phi) - U_{\tau-}(\phi) = \phi_\tau^0(\delta(\tau) - \widetilde{S}_\tau(\kappa)). \tag{25}$$

Proof. In general, the process ϕ^0 is $\mathbb{G}$-predictable. In our model, ϕ^0 is assumed to be an RCLL function. The jump of the wealth process $U(\phi)$ at time τ equals, on the set $\{\tau \leq T\}$,

$$\Delta_\tau U(\phi) = \phi_\tau^0 \Delta_\tau S + \phi_\tau^1 \Delta_\tau D$$

where $\Delta_\tau S(\kappa) = S_\tau(\kappa) - S_{\tau-}(\kappa) = -\widetilde{S}_\tau(\kappa)$ (recall that the ex-dividend price $S(\kappa)$ drops to zero at default time) and manifestly $\Delta_\tau D = \delta(\tau)$.   □

3.3   Hedging of a Contingent Claim in the CDS Market

An $\mathcal{H}_T$-measurable random variable can be represented as follows

$$Y = \mathbb{1}_{\{T \geq \tau\}}h(\tau) + \mathbb{1}_{\{T < \tau\}}c(T), \tag{26}$$

where $h : [0, T] \to \mathbb{R}$ is a Borel function, and $c(T)$ is a constant. For concreteness, we shall deal with claims Y such that h is an RCLL function, but this restriction is not essential.

We first recall a suitable version of the predictable representation theorem. Subsequently, we derive closed-form solution for the replicating strategy for a claim Y given by (26) and settling at time T. As tradeable assets, we shall use a CDS started at time 0 and maturing at T, and a savings account.

3.3.1 Representation Theorem

For any RCLL function $\widehat{h} : \mathbb{R}_+ \to \mathbb{R}$ such that the random variable $\widehat{h}(\tau)$ is integrable, we set $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(\widehat{h}(\tau) \,|\, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. It is clear that $\widehat{M}$ is an $\mathbb{H}$-martingale under $\mathbb{Q}^*$. The following version of the martingale representation theorem is well known (see, for instance, Blanchet-Scalliet and Jeanblanc (2004), Jeanblanc and Rutkowski (2002) or Proposition 4.3.2 in Bielecki and Rutkowski (2002)).

Proposition 3.2 *Assume that G is continuous and $\widehat{h}$ is an RCLL function such that the random variable $\widehat{h}(\tau)$ is $\mathbb{Q}^*$-integrable. Then the $\mathbb{H}$-martingale $\widehat{M}$ admits the following integral representation*

$$\widehat{M}_t = \widehat{M}_0 + \int_{]0,t]} \left(\widehat{h}(u) - \widehat{g}(u) \right) dM_u, \tag{27}$$

where the continuous function $\widehat{g} : \mathbb{R}_+ \to \mathbb{R}$ is given by the formula

$$\widehat{g}(t) = \frac{1}{G(t)} \mathbb{E}_{\mathbb{Q}^*} \left(\mathbb{1}_{\{\tau > t\}} \widehat{h}(\tau) \right) = -\frac{1}{G(t)} \int_t^\infty \widehat{h}(u) \, dG(u). \tag{28}$$

Remark. It is easily seen that on the set $\{t \leq \tau\}$ we have $\widehat{g}(t) = \widehat{M}_{t-}$. Therefore, formula (27) can also be rewritten as follows

$$\widehat{M}_t = \widehat{M}_0 + \int_{]0,t]} \left(\widehat{h}(u) - \widehat{M}_{u-} \right) dM_u. \tag{29}$$

3.3.2 Replication of a Defaultable Claim

Assume now that a random variable Y given (26) represents a contingent claim settling at T. Formally, we deal with a defaultable claim of the form $(X, 0, Z, \tau)$, where $X = c(T)$ and $Z_t = h(t)$.

To deal with such a claim, we shall apply Proposition 3.2 to the function $\widehat{h}$, where $\widehat{h}(t) = h(t)$ for $t < T$ and $\widehat{h}(t) = c(T)$ for $t \geq T$ (recall that $\mathbb{Q}^*(\tau = T) = 0$). In this case, we obtain

$$\widehat{g}(t) = \frac{1}{G(t)} \left(-\int_t^T h(u) \, dG(u) + c(T)G(T) \right), \tag{30}$$

and thus for the process $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y \,|\, \mathcal{H}_t)$, $t \in [0, T]$, we have

$$\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y) + \int_{]0,t]} (h(u) - \widehat{g}(u)) \, dM_u \tag{31}$$

with $\widehat{g}$ given by (30). Recall that $\widetilde{S}(\kappa)$ is the pre-default ex-dividend price process of a CDS with spread κ and maturity T. We know that $\widetilde{S}(\kappa)$ is a continuous function of t if G is continuous.

Proposition 3.3 *Assume that the inequality $\widetilde{S}_t(\kappa) \neq \delta(t)$ holds for every $t \in [0, T]$. Let ϕ^0 be an RCLL function given by the formula*

$$\phi_t^0 = \frac{h(t) - \widehat{g}(t)}{\delta(t) - \widetilde{S}_t(\kappa)}, \tag{32}$$

and let $\phi_t^1 = U_t(\phi) - \phi_t^0 S_t(\kappa)$, where the process $U(\phi)$ is given by (24) with the initial condition $U_0(\phi) = \mathbb{E}_{\mathbb{Q}^*}(Y)$, where Y is given by (26). Then the self-financing trading strategy $\phi = (\phi^0, \phi^1)$ is admissible and it s a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$, where $X = c(T)$ and $Z_t = h(t)$.

Proof. The idea of the proof is based on the observation that it is enough to concentrate on the formula for trading strategy prior to default. In view of Lemma 3.2, the dynamics of the price $S(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt.$$

and thus we have, on the set $\{\tau > t\}$,

$$d\widetilde{S}_t(\kappa) = dS_t(\kappa) = \big(\gamma(t)\widetilde{S}_t(\kappa) + \kappa - \delta(t)\gamma(t)\big)dt.$$

From Corollary 2.1, we know that the wealth $U(\phi)$ of any admissible self-financing strategy is a $\mathbb{H}$-martingale under $\mathbb{Q}^*$. Since under the present assumptions $dB_t = 0$, for the wealth process $U_t(\phi)$ we obtain, on the set $\{\tau > t\}$,

$$dU_t(\phi) = \phi_t^0\, (d\widetilde{S}_t(\kappa) - \kappa\, dt) = -\phi_t^0\gamma(t)\big(\delta(t) - \widetilde{S}_t(\kappa)\big)\, dt. \tag{33}$$

For the martingale $\widehat{M} = \mathbb{E}_{\mathbb{Q}^*}(Y \,|\, \mathcal{H}_t)$ associated with Y, in view of (31) we obtain, on the set $\{\tau > t\}$,

$$d\widehat{M}_t = -\gamma(t)(h(t) - \widehat{g}(t))\, dt. \tag{34}$$

We wish to find ϕ^0 such that $U_t(\phi) = \widehat{M}_t$ for every $t \in [0, T]$. To this end, we first focus on the equality $\mathbb{1}_{\{t<\tau\}}U_t(\phi) = \mathbb{1}_{\{t<\tau\}}\widehat{M}_t$ for pre-default values. A comparison of (33) with (34) yields

$$\phi_t^0 = \frac{h(t) - \widehat{g}(t)}{\delta(t) - \widetilde{S}_t(\kappa)}, \quad \forall\, t \in [0, T]. \tag{35}$$

We thus see that if $U_0(\phi) = \widehat{M}_0$ then also $\mathbb{1}_{\{t<\tau\}}U_t(\phi) = \mathbb{1}_{\{t<\tau\}}\widehat{M}_t$ for every $t \in [0, T]$. As usual, the second component of a self-financing strategy ϕ is given by (23), that is, $\phi_t^1 = U_t(\phi) - \phi_t^0 S_t(\kappa)$, where $U(\phi)$ is given by (24) with the initial condition $U_0(\phi) = \mathbb{E}_{\mathbb{Q}^*}(Y)$. In particular, we have that $\phi_0^1 = \mathbb{E}_{\mathbb{Q}^*}(Y) - \phi_0^0 S_0(\kappa)$.

To complete the proof, that is, to show that $U_t(\phi) = \widehat{M}_t$ for every $t \in [0, T]$, it suffices to compare the jumps of both processes at time τ (both martingales are stopped at τ). It is clear from (31) that the jump of $\widehat{M}$ equals $\Delta_\tau \widehat{M} = h(\tau) - \widehat{g}(\tau)$. Using (25), we get for the jump of the wealth process

$$\Delta_\tau U(\phi) = \phi_\tau^0(\delta(\tau) - \widetilde{S}_\tau(\kappa)) = h(\tau) - \widehat{g}(\tau),$$

and thus we conclude that $U_t(\phi) = \widehat{M}_t$ for every $t \in [0, T]$. In particular, ϕ is admissible and $U_T(\phi) = U_{\tau \wedge T}(\phi) = h(\tau \wedge T) = Y$, so that ϕ replicates a claim Y.   □

Note that if $\kappa = \kappa(0)$ then $S_0(\kappa(0)) = 0$, so that $\phi_0^1 = U_0(\phi) = \mathbb{E}_{\mathbb{Q}^*}(Y)$.

Let us now analyze the condition $\widetilde{S}_t(\kappa) \neq \delta(t)$ for every $t \in [0, T]$. It ensures, in particular, that the wealth process $U(\phi)$ has a non-zero jump at default time for any the self-financing trading strategy such that $\phi_t^0 \neq 0$ for every $t \in [0, T]$. It appears that this condition is not restrictive, since it is satisfied under mild assumptions.

Indeed, if $\kappa > 0$ and δ is a non-increasing function then the inequality $\widetilde{S}_t(\kappa) < \delta(t)$ is valid for every $t \in [0, T]$ (this follows easily from (11)). For instance, if $\gamma(t) > 0$ and the protection payment $\delta > 0$ is constant then it is clear from (14) that the market spread $\kappa(0)$ is strictly positive. Consequently, formula (11) implies that $\widetilde{S}_t(\kappa(0)) < \delta$ for every $t \in [0, T]$, as was required. To summarize, when a tradeable asset is a market CDS with a constant $\delta > 0$ and the default intensity is strictly positive then the inequality holds.

Let us finally observe that if the default intensity vanishes on some set then we do not need to impose the inequality $\widetilde{S}_t(\kappa) \neq \delta(t)$ on this set in order to equate (33) with (34), since the requested equality will hold anyway.

The method of proof is based on the following observation.

Lemma 3.4 *Let M^1 and M^2 be arbitrary two $\mathbb{H}$-martingales under $\mathbb{Q}^*$. If for every $t \in [0,T]$ we have $\mathbb{1}_{\{t<\tau\}}M_t^1 = \mathbb{1}_{\{t<\tau\}}M_t^2$ then $M_t^1 = M_t^2$ for every $t \in [0,T]$.*

Proof. We have $M_t^i = \mathbb{E}_{\mathbb{Q}^*}(h_i(\tau) \,|\, \mathcal{H}_t)$ for some functions $h_i : \mathbb{R}_+ \to \mathbb{R}$ such that $h_i(\tau)$ is $\mathbb{Q}^*$-integrable. Using the well known formula for the conditional expectation

$$\mathbb{E}_{\mathbb{Q}^*}(h_i(\tau) \,|\, \mathcal{H}_t) = \mathbb{1}_{\{t\geq\tau\}}h_i(\tau) - \mathbb{1}_{\{t<\tau\}}\frac{1}{G(t)}\int_t^\infty h_i(u)\,dG(u) = \mathbb{1}_{\{t\geq\tau\}}h_i(\tau) + \mathbb{1}_{\{t<\tau\}}\widehat{g}_i(t),$$

and the assumption that $\mathbb{1}_{\{t<\tau\}}M_t^1 = \mathbb{1}_{\{t<\tau\}}M_t^2$, we obtain the equality $\widehat{g}_1(t) = \widehat{g}_2(t)$ for every $t \in [0,T]$ (recall that $\mathbb{Q}^*(\tau > t) > 0$ for every $t \in [0,T]$). Therefore, we have

$$\int_t^\infty h_1(u)\,dG(u) = \int_t^\infty h_2(u)\,dG(u), \quad \forall\, t \in [0,T].$$

This immediately implies that $h_1(t) = h_2(t)$ on $[0,T]$, almost everywhere with respect to the distribution of τ, and thus we have $h_1(\tau) = h_2(\tau)$, $\mathbb{Q}^*$-a.s. Consequently, $M_t^1 = M_t^2$ for every $t \in [0,T]$. $\square$

In our case, Lemma 3.4 can be applied to the following $\mathbb{H}$-martingales under $\mathbb{Q}^*$: $M^1 = U(\phi)$ is the wealth process of an admissible self-financing strategy ϕ and $M^2 = \widehat{M}$ is the conjectured price of a claim Y, as given by the risk-neutral valuation formula.

Let us note that the method presented above can be extended to replicate a contingent claim defaultable claim (X, A, Z, τ), where $X = c(T)$, $A_t = \int_0^t a(u)\,du$ and $Z_t = h(t)$ for some RCLL functions a and h. In this case, it is natural to expect that the cum-dividend price process π_t associated with a defaultable claim (X, A, Z, τ), is given by the formula, for every $t \in [0,T]$,

$$\pi_t = \widehat{M}_t + \mathbb{1}_{\{t\leq\tau\}}\int_0^\tau a(u)\,du + \mathbb{1}_{\{t<\tau\}}\frac{1}{G(t)}\int_t^T a(u)G(u)\,du, \tag{36}$$

where $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y \,|\, \mathcal{H}_t)$, where Y is given by (26). Hence, the pre-default dynamics of this process are

$$d\pi_t = d\widehat{M}_t + \gamma(t)\widehat{a}(t)\,dt = -\gamma(t)\big(h(t) - \widehat{g}(t) - \widehat{a}(t)\big)\,dt,$$

where we set $\widehat{a}(t) = (G(t))^{-1}\int_t^T a(u)G(u)\,du$. Note that $\widehat{a}(t)$ represents the pre-default value of the future promised dividends associated with A.

Therefore, arguing as in the proof of Proposition 3.3, we find the following formula for a replicating strategy ϕ

$$\phi_t^0 = \frac{h(t) - \widehat{g}(t) - \widehat{a}(t)}{\delta(t) - \widetilde{S}_t(\kappa)}, \quad \forall\, t \in [0,T]. \tag{37}$$

It is easy to see that the jump condition at time τ, mentioned in the second part of the proof of Proposition 3.3, is satisfied in this case as well.

Remark. Of course, if we take as (X, A, Z, τ) a CDS with spread κ and recovery function δ, then we will get $h(t) = \delta(t)$ and $\widehat{g}(t) + \widehat{a}(t) = \widetilde{S}_t(\kappa)$, so that clearly $\phi_t^0 = 1$ for every $t \in [0,T]$.

The following immediate corollary to Proposition 3.3 is worth stating (let us stress once again that the assumption that a claim is represented by an RCLL function, as opposed to a Borel measurable function, is not essential).

Corollary 3.2 *Assume that $\widetilde{S}_t(\kappa) \neq \delta(t)$ for every $t \in [0,T]$. Then the market is complete, in the sense, that any defaultable claim (X, A, Z, τ), where $X = c(T)$, $A_t = \int_0^t a(u)\,du$ and $Z_t = h(t)$ for some constant $c(T)$ and RCLL functions a and h, is attainable through continuous trading in a bond and a CDS. The arbitrage price π_t of a defaultable claim satisfies, for every $t \in [0,T]$,*

$$\pi_t = U_t(\phi) = \widehat{M}_t = \pi_0 + \int_{]0,t]} (h(u) - \pi_{u-})\,dM_u,$$

where

$$\pi_0 = \mathbb{E}_{\mathbb{Q}^*}(Y) + \int_0^T a(t) G(t) \, dt,$$

and $\widetilde{\pi}_t = \widehat{g}(t) + \widehat{a}(t) + A_t$ is its pre-default price, so that we have, for every $t \in [0, T]$

$$\pi_t = \mathbb{1}_{\{t < \tau\}} (\widehat{g}(t) + \widehat{a}(t) + A_t) + \mathbb{1}_{\{t \geq \tau\}} (h(\tau) + A_\tau) = \mathbb{1}_{\{t < \tau\}} \widetilde{\pi}_{t-} + \mathbb{1}_{\{t \geq \tau\}} \pi_\tau.$$

3.3.3   Case of a Constant Default Intensity

As a partial check of the calculations above, we shall consider once again the case of constant default intensity and constant protection payment. In this case, $\kappa(0) = \delta\gamma$ and $S_t(\kappa(0)) = 0$ for every $t \in [0, T]$, so that

$$dU_t(\phi) = -\phi_t^0 \delta\gamma \, dt = -\phi_t^0 \kappa(0) \, dt. \tag{38}$$

Furthermore, for any RCLL function h, formula (35) yields

$$\phi_t^0 = \delta^{-1} \left(h(t) + e^{\gamma t} \int_t^T h(u) \, d(e^{-\gamma u}) - c(T) e^{-\gamma T} \right). \tag{39}$$

Assume, for instance, that $h(t) = \delta$ for $t \in [0, T[$ and $c(T) = 0$. Then (39) gives $\phi_t^0 = e^{-\gamma(T-t)}$. Since $S_0(\kappa(0)) = 0$, we have $\phi_0^1 = \pi_0(Y) = U_0(\phi) = \delta(1 - e^{-\gamma T})$. In view of (38), the gains/losses from positions in market CDSs over the time interval $[0, t]$ equal, on the set $\{\tau > t\}$,

$$U_t(\phi) - U_0(\phi) = -\delta\gamma \int_0^t \phi_u^0 \, du = -\delta\gamma \int_0^t e^{-\gamma(T-u)} \, du = -\delta e^{-\gamma T} (e^{\gamma t} - 1) < 0.$$

Suppose that default occurs at some date $t \in [0, T]$. Then the protection payments is collected, and the wealth at time t becomes

$$U_t(\phi) = U_{t-}(\phi) + \phi_t^0 \delta = \delta(1 - e^{-\gamma T}) - \delta e^{-\gamma T} (e^{\gamma t} - 1) + \delta e^{-\gamma(T-t)} = \delta.$$

The last equality shows that the strategy is indeed replicating on the set $\{\tau \leq T\}$. On the set $\{\tau > T\}$, the wealth at time T equals

$$U_T(\phi) = \delta(1 - e^{-\gamma T}) - \delta e^{-\gamma T} (e^{\gamma T} - 1) = 0.$$

Since $S_t(\kappa(0)) = 0$ for every $t \in [0, T]$, we have that $\phi_t^1 = U_t(\phi)$ for every $t \in [0, T]$.

3.3.4   Short Sale of a CDS

As usual, we assume that the maturity T of a CDS is fixed and we consider the situation where the default has not yet occurred.

1. Long position. We say that an agent has a long position at time t in a CDS if he owns at time t a CDS contract that had been created (initiated) at time s_0 by some two parties and was sold to the agent (by means of assignment for example) at time s. If $s_0 = s$ then the agent is an original counter-party to the contract, that is the agent owns the contract from initiation. If an agent owns a CDS contract, the agent is entitled to receive the protection payment for which the agent pays the premium. The long position in a contract may be liquidated at any time $s < t < T$ by means of assignment or offsetting.

2. Short position. We stress that the short position, namely, selling a CDS contract to a dealer, can only be created for a newly initiated contract. It is not possible to sell to a dealer at time t a CDS contract initiated at time $s_0 < t$.

3. Offsetting a long position. If an agent has purchased at time $s_0 \leq s < T$ a CDS contract initiated at s_0, he can offset his long position by creating a short position at time t. A new contract

is initiated at time t, with the initial price $S_t(\kappa(s_0))$, possibly with a new dealer. This short position offsets the long position outstanding, so that the agent effectively has a zero position in the contract at time t and thereafter.

4. Market constraints. The above taxonomy of positions may have some bearing on portfolios involving short positions in CDS contracts. It should be stressed that not all trades involving a CDS are feasible in practice. Let us consider the CDS contract initiated at time t_0 and maturing at time T. Recall that the ex-dividend price of this contract for any $t \in [t_0, \tau \wedge T[$ is $S_t(\kappa(t_0))$. This is the theoretical price at which the contract should trade so to avoid arbitrage. This price also provides substance for the P&L analysis as it really marks-to-market positions in the CDS contract.

Let us denote the time-t position in the CDS contract of an agent as ϕ_t^0, where $t \in [t_0, \tau \wedge T]$. The strategy is subject to the following constraints:

$$\phi_t^0 \geq 0 \quad \text{if} \quad \phi_{t_0}^0 \geq 0$$

and

$$\phi_t^0 \geq \phi_{t_0}^0 \quad \text{if} \quad \phi_{t_0}^0 \leq 0.$$

It is clear that both restrictions are related to short sale of a CDS. The next simple result shows that under some assumptions a replicating strategy for a claim Y does not require a short sale of a CDS.

Corollary 3.3 *Assume that $\widetilde{S}_t(\kappa) < \delta(t)$ for every $t \in [0, T]$. Let h be a non-increasing function and let $c(T) \leq h(T)$. Then $\phi_t^0 \geq 0$ for every $t \in [0, T]$.*

Proof. It is enough to observe that if h be a non-increasing function and $c(T) \leq h(T)$ then it follows easily from the first equality in (28) that for the function $\widehat{g}$ given by (30) we have that $h(t) \geq \widehat{g}(t)$ for every $t \in [0, T]$. In view of (32), this shows that $\phi_t^0 \geq 0$ for every $t \in [0, T]$. $\square$

References

T.R. Bielecki and M. Rutkowski (2002a) *Credit Risk: Modeling, Valuation and Hedging.* Springer-Verlag, Berlin Heidelberg New York.

C. Blanchet-Scalliet and M. Jeanblanc (2004) Hazard rate for credit risk and hedging defaultable contingent claims. *Finance and Stochastics* 8, 145–159.

F. Jamshidian (1997) LIBOR and swap market models and measures. *Finance and Stochastics* 1, 293–330.

M. Jeanblanc and M. Rutkowski (2002) Default risk and hazard processes. In: *Mathematical Finance – Bachelier Congress 2000*, H. Geman, D. Madan, S.R. Pliska and T. Vorst, editors, Springer-Verlag, Berlin Heidelberg New York, pp. 281–312.