STRONGLY MINIMAL GROUPS IN THE THEORY OF COMPACT COMPLEX SPACES

MATTHIAS ASCHENBRENNER, RAHIM MOOSA, AND THOMAS SCANLON

Abstract. We characterise strongly minimal groups interpretable in elementary extensions of compact complex analytic spaces.

1. Introduction

In [19] Zilber observed that compact complex manifolds may be naturally regarded as structures of finite Morley rank for which the axioms of Zariski-type structures hold. As such, the key to a model theoretic structure theory for sets definable in compact complex manifolds is a description of the interpretable strongly minimal groups. Pillay and Scanlon described these groups in [15] but left open the question of what strongly minimal groups might be definable in elementary extensions of compact complex manifolds. In this paper, we complete the classification.

We regard a compact complex manifold M as a structure in the language having a predicate for each closed analytic subvariety of each Cartesian power of M. It is convenient to consider all compact complex analytic spaces at the same time. To do so, we form the many sorted structure $\mathcal{A}$ having a sort for each (isomorphism class of) compact complex analytic space(s) and having as basic relations on the product of sorts $S_1 \times \cdots \times S_n$ the closed analytic subvarieties. As every point in $\mathcal{A}$ is distinguished by a basic relation, this structure cannot be saturated. More seriously, even if one restricts to a single sort M in $\mathcal{A}$ it can happen that there is no countable reduct for which every definable set in M is parametrically definable in the reduct. That is, not all compact complex manifolds are *essentially saturated* in the sense of [10]. Consequently, to study properties of elementary extensions of $\mathcal{A}$ one cannot work entirely within the standard model.

By definition, elementary properties transfer from $\mathcal{A}$ to its elementary extensions. Much of our work consists of unwinding results for the standard model to find their elementary content. In addition to close readings and reworkings of existing proofs, we use properties of families of analytic spaces. In particular, Grothendieck's relative infinitesimal neighbourhoods play an important role.

The structure $\mathcal{A}$ is interpretable in $\mathbb{R}_{\text{an}}$, the field of real numbers with restricted analytic functions, a well-studied o-minimal structure. We use this observation in only one place in our arguments, but as Peterzil and Starchenko have shown, it could form the basis for a theory of complex analytic spaces over arbitrary o-minimal expansions of real closed fields (see [12, 13]). However, the strongly minimal groups

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definable in the interpreted complex analytic spaces in nonstandard models of the theory of $\mathbb{R}_{\mathrm{an}}$ may have properties not enjoyed by any group in an elementary extension of $\mathcal{A}$. For instance, while every (complex-) one-dimensional group in any elementary extension of $\mathcal{A}$ is algebraic [11], this does not hold in the $\mathbb{R}_{\mathrm{an}}$ world [13]. Thus, while the o-minimal approach may be useful, it must be paired with work internal to $\mathcal{A}$.

Complex algebraic geometry lives in $\mathcal{A}$ in the sense that the field of complex numbers is a definable set (being the complement of the point at infinity on the complex projective line) and the field operations on $\mathbb{C}$ are themselves definable. Moreover, by Chow's theorem, $\mathcal{A}$ induces no additional structure on $\mathbb{C}$. It follows from the classification of locally compact fields that any other field interpretable in $\mathcal{A}$ is definably isomorphic to $\mathbb{C}$. Moosa showed that this conclusion continues to hold in elementary extensions of $\mathcal{A}$ even though the statement of the result is not facially elementary and the Euclidean topology is unavailable in the elementary extensions. Moosa proves in addition a nonstandard version of the Riemann existence theorem from which one may show, using the Zilber trichotomy, that any strongly minimal group interpretable in an elementary extension of $\mathcal{A}$ is either a one-dimensional algebraic group over the (nonstandard) complex numbers or is locally modular.

We direct our attention to the strongly minimal locally modular groups in elementary extensions of $\mathcal{A}$ (though local modularity itself plays no role in our arguments). In $\mathcal{A}$ itself, these groups are the simple nonalgebraic complex tori [15]. We prove a version for elementary extensions.

Theorem 1.1. *Suppose $\mathcal{A}' \succeq \mathcal{A}$ is a saturated elementary extension of $\mathcal{A}$ with $\mathbb{C}'$ its interpretation of $\mathbb{C}$. If G is a strongly minimal group intepretable in $\mathcal{A}'$ then either G is definably isomorphic to an algebraic group over $\mathbb{C}'$ or G is "compact" in either of the following senses:*

(i) *viewing G as a definable manifold by interpreting $\mathcal{A}'$ in an elementary extension of $\mathbb{R}_{\mathrm{an}}$, G is definably compact, or*
(ii) *there exists a Zariski closed set X in $\mathcal{A}'$ and a surjective definable map $\pi\colon X \to G$ which is holomorphic with respect to the natural nonstandard meromorphic manifold structure on G.*

Formulation (i) of "compact" is an easy consequence of (ii) and it is in the sense of the latter that the theorem is proved. See Section 4 for a more precise formulation.

We would say that the group G is a *nonstandard complex torus* if G were of the form T_a where $\nu\colon T \to B$ is a holomorphic map between compact complex manifolds whose general fibres are (uniformly) complex tori and $a \in B(\mathcal{A}')$ is a nonstandard point of B. It may very well be the case that any "compact" group in the sense of Theorem 1.1 must be a nonstandard complex torus, but we were unable to resolve this question as it implicates some subtle issues in the theory of moduli of complex tori.

Remark 1.2. If G lives in a sort that is of Kähler-type (i.e., is a holomorphic image of a compact Kähler manifold) then, using essential saturation (cf. [10]), Theorem 1.1 follows automatically from the characterisation of strongly minimal groups in $\mathcal{A}$ itself [15]. In fact, in that case we can replace "compact" group in the conclusion by "nonstandard complex torus". However, not every strongly minimal group in $\mathcal{A}'$ lives in a sort that is of Kähler-type, see [14] for an example.[1]

This paper is organized as follows. In Section 2 we recall and supply a detailed proof of the main compactification result of [15] that any strongly minimal group G in $\mathcal{A}$ may be embedded as a Zariski open subset of some compact complex manifold. In Section 3 we reformulate the statement of this compactification so as to make its elementary content transparent. In Section 4, we recall some of the theory of elementary extensions of $\mathcal{A}$, show how to transfer part of the compactification theorem and then analyse the case where there is no action of G on its boundary. In Section 5 we analyze the remaining case transposing a theorem of Fujiki to the nonstandard context to complete the proof.

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2. Compactifications in $\mathcal{A}$

In this section we concentrate on proving in detail the following main compactification result from [15].

Proposition 2.1. *Let G be a complex manifold expressible as the disjoint union of an open set U and a finite set F, where U is a non-empty Zariski open subspace of an irreducible compact complex manifold X. There exist:*

- *a compact complex manifold G^* and a holomorphic embedding $\iota\colon G \to G^*$ such that $\iota(G)$ is a Zariski open subset of G^*, and*
- *a holomorphic surjection $\pi\colon X \to G^*$;*

such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & G^* \\ \uparrow & & \uparrow \iota \\ U & \longrightarrow & G \end{array}$$

and π is a biholomorphism off $\iota(F) \subseteq G^*$.

The published proof of this result while correct is, to our taste, incomplete in that the nontrivial verification of the efficacy of the construction is left to the reader. Many of the details of the proof that follows involve routine topological manipulations, but the proof as a whole is remarkably tricky. It should be noted that while we work with the Euclidean topology, the hypotheses and the conclusion of this proposition concern objects definable in the structure $\mathcal{A}$.

We first reduce to the case when F is a singleton. Given (G, U, X, F) we prove the result by induction on $|F|$ with the case of $|F| = 0$ being trivial. If $F = F' \dot\cup \{g\}$, let $G' := G \setminus F'$. Then G' is a manifold expressible as $U \dot\cup \{g\}$. If the result is true for $(G', U, X, \{g\})$, then it follows for (G, U, X, F). Indeed, suppose $\iota'\colon G' \hookrightarrow (G')^*$ and $\pi'\colon X \to (G')^*$ satisfy the conclusion for $(G', U, X, \{g\})$. After replacing

[1] An explicit argument for why this example is not definably isomorphic to a strongly minimal group in a sort of Kähler-type is given in Example 6.1 of [11].

$(G')^*$ by its image under a suitable biholomorphism and composing π' with this biholomorphism, we may assume that $(G')^*$ contains G' as a Zariski open subset and ι' is the natural inclusion $G' \subseteq (G')^*$. Now G may be expressed as $G' \cup F'$, and by induction we find $\iota\colon G \hookrightarrow G^*$ and $\widetilde{\pi}\colon (G')^* \to G^*$ satisfying the conclusion for $(G, \iota(G'), (G')^*, F')$. Take $\pi := \widetilde{\pi} \circ \pi'\colon X \to G^*$.

So it suffices to prove the Proposition in the case that $F = \{g\}$ is a singleton, and for the rest of this section we assume that F has this form.

We begin with some notation: For a subset S of a topological space R we denote the closure of S in R by $\mathrm{cl}_S(R)$ and the interior of S in R by $\mathrm{int}_R(S)$. The boundary of S in R is denoted by $\mathrm{bd}_R(S) := \mathrm{cl}_R(S) \setminus \mathrm{int}_R(S)$, and the frontier of S in R by $\mathrm{fr}_R(S) := \mathrm{cl}_R(S) \setminus S$. When the context is clear we drop the reference to R in this notation. Recall also that a subset S of R is called *regular open* if $S = \mathrm{int}(\mathrm{cl}(S))$. Note that for every $S \subseteq R$, the set $\mathrm{int}(\mathrm{cl}(S))$ is regular open and $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(S))) \subseteq \mathrm{cl}(S)$, with equality if S is open.

Lemma 2.2. *Let $V \subseteq G$ be a regular open co-ordinate neighbourhood of g in G and let D be the frontier of $\mathrm{cl}_G(V) \setminus \{g\}$ in X. That is,*

$$D := \mathrm{cl}_X(\mathrm{cl}_G(V) \setminus \{g\}) \setminus (\mathrm{cl}_G(V) \setminus \{g\}).$$

Then $D \cup (V \setminus \{g\})$ is open in X.

Indeed, we will show that $D \cup (V \setminus \{g\}) = \mathrm{int}_X(\mathrm{cl}_X(V \setminus \{g\}))$. Note that D is nonempty as $\mathrm{cl}_G(V) \setminus \{g\}$ is not compact, and hence not closed, in X. The proof of Lemma 2.2 is preceded by a series of claims.

Claim 2.3. $\mathrm{int}_G(\mathrm{cl}_G(V) \setminus \{g\}) = V \setminus \{g\}$

Proof. Since $\mathrm{cl}_G(V) \setminus \{g\} = \mathrm{cl}_G(V) \cap (G \setminus \{g\})$, the set $G \setminus \{g\}$ is open in G, and V is regular open in G, we have

$$\mathrm{int}_G(\mathrm{cl}_G(V) \setminus \{g\}) = \mathrm{int}_G(\mathrm{cl}_G(V)) \cap \mathrm{int}_G(G \setminus \{g\}) = V \cap (G \setminus \{g\}),$$

hence $\mathrm{int}_G(\mathrm{cl}_G(V) \setminus \{g\}) = V \setminus \{g\}$ as claimed. □

Claim 2.4. $U \cap \mathrm{cl}_X(V \setminus \{g\}) = \mathrm{cl}_G(V) \setminus \{g\}$

Proof. Let $x \in U$, and suppose first that $x \in \mathrm{cl}_G(V)$. Then any G-neighbourhood of x has a nonempty intersection with V. If $N \subseteq X$ is any sufficiently small X-neighbourhood of x, then $g \notin N$ so that N is also a G-neighbourhood of x; hence $N \cap (V \setminus \{g\}) \neq \varnothing$. This shows $x \in \mathrm{cl}_X(V \setminus \{g\})$. Now suppose $x \notin \mathrm{cl}_G(V)$, so there is an open neighbourhood N of x in G with $N \subseteq U$ and $N \cap V = \varnothing$. Then N is also an X-neighbourhood of x disjoint from $V \setminus \{g\}$, hence $x \notin \mathrm{cl}_X(V \setminus \{g\})$. □

Claim 2.5. $U \cap \mathrm{int}_X(\mathrm{cl}_X(V \setminus \{g\})) = V \setminus \{g\}$

Proof. Since U is an open subspace of both G and X, for every $S \subseteq U$ we have $\mathrm{int}_G\, S = \mathrm{int}_X\, S$. The claim follows from this remark and Claim 2.3 by applying int_X on both sides of the equation in Claim 2.4. □

Claim 2.6. $\mathrm{cl}_X(\mathrm{cl}_G(V) \setminus \{g\}) = \mathrm{cl}_X(V \setminus \{g\})$

Proof. Certainly $\mathrm{cl}_X(\mathrm{cl}_G(V) \setminus \{g\}) \supseteq \mathrm{cl}_X(V \setminus \{g\})$, and the reverse inclusion follows by taking cl_X on both sides of the equation stated in Claim 2.4. □

Note that from Claims 2.4 and 2.6 we get $D \subseteq X \setminus U$.

Claim 2.7. $\mathrm{cl}_X(V \setminus \{g\}) = (V \setminus \{g\}) \cup \mathrm{bd}_G(V) \cup D$

Proof. By the definition of D and Claim 2.6, $D \cup (\mathrm{cl}_G(V) \setminus \{g\}) = \mathrm{cl}_X(V \setminus \{g\})$. On the other hand, $\mathrm{cl}_G(V) \setminus \{g\} = \mathrm{bd}_G(V) \cup (V \setminus \{g\})$. □

By Claim 2.5 we have $\mathrm{bd}_G(V) \cap \mathrm{int}_X(\mathrm{cl}_X(V \setminus \{g\})) = \varnothing$; hence by Claim 2.7 we get $\mathrm{int}_X(\mathrm{cl}_X(V \setminus \{g\})) \subseteq (V \setminus \{g\}) \cup D$. Our aim now is to show the reverse inclusion and thereby prove Lemma 2.2. We first observe:

Claim 2.8. $\mathrm{bd}_G(V) \cup D \supseteq \mathrm{bd}_X(\mathrm{cl}_X(V \setminus \{g\}))$

Proof. Since $V \setminus \{g\} \subseteq \mathrm{int}_X(\mathrm{cl}_X(V \setminus \{g\}))$, Claim 2.7 yields

$$\mathrm{bd}_X(\mathrm{cl}_X(V \setminus \{g\})) = \mathrm{cl}_X(V \setminus \{g\}) \setminus \mathrm{int}_X(\mathrm{cl}_X(V \setminus \{g\})) \subseteq \mathrm{cl}_X(V \setminus \{g\}) \setminus (V \setminus \{g\}) \subseteq \mathrm{bd}_G(V) \cup D.$$

□

In the next two claims we will use the fact that any complex manifold admitting a finite atlas where the model spaces are polydiscs (so, in particular, a compact complex manifold) is interpretable in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$, the real field equipped with restricted analytic functions. The sets we have been dealing with (i.e., V, D, etc.), though not necessarily definable in $\mathcal{A}$, will "live" definably in $\mathbb{R}_{\mathrm{an}}$ and we will thus have recourse to the o-minimal dimension of these sets.

To make this more precise, consider the disjoint union $M := G \amalg X$ of the complex manifolds G and X, that is, the complex manifold whose underlying topological space is the disjoint union of the topological spaces G and X, and whose atlas is the disjoint union of the ones of G and X. Then M, in fact, admits a finite atlas where the model spaces are polydiscs, which we can assume to contain the co-ordinate chart V of G. Hence, M can be identified with a *definable* $\mathbb{C}$*-manifold* in $\mathbb{R}_{\mathrm{an}}$ (as defined in Section 3 of [13]) in such a way that V becomes definable in $\mathbb{R}_{\mathrm{an}}$. Moreover, since U is an $\mathbb{R}_{\mathrm{an}}$-definable open subset of both X and G (in both cases it is the complement of an analytic set), and since the identification of the U of G with the U of X is biholomorphic, this identification is also $\mathbb{R}_{\mathrm{an}}$-definable. It follows that $\overline{V}^G \setminus \{g\}$ as a subset of X is also definable in $\mathbb{R}_{\mathrm{an}}$, and hence so is D.

We will use the following property of o-minimal dimension, which is probably well-known, although we could not locate it in the literature. For this we fix an o-minimal structure R.

Fact 2.9. *Suppose that R expands a real closed field. Let M be a definable manifold of dimension $m > 0$ and S a definable open subset of M. Then the definable set* $\mathrm{bd}_M(\mathrm{cl}_M(S))$ *has local dimension $m-1$ at each point $a \in \mathrm{bd}_M(\mathrm{cl}_M(S))$.*

For the proof, we first reduce to the case that $M = R^m$, by passing to a chart around a and applying the fact that open cells in R^m are definably homeomorphic to R^m, since R expands a real closed field. The next lemma (valid without this extra assumption on R) and the equality $\mathrm{bd}(\mathrm{cl}(S)) = \mathrm{fr}(\mathrm{int}(\mathrm{cl}(S)))$ then immediately yield the claim.

Lemma 2.10. *Let S be a non-empty proper definable regular open subset of R^m, $m > 0$. Then* $\dim\ \mathrm{fr}(S) = m - 1$.

In the proof of the lemma we use the following notations and observations. Let S be a subset of R^m, $m > 0$. For $x \in R$ we put $S_x := \{y \in R^{m-1} : (x, y) \in S\}$. If

S is definable, then each of the following sets is finite:

$$F_S := \{x \in R : \mathrm{fr}(S_x) \neq \mathrm{fr}(S)_x\};$$
$$C_S := \{x \in R : \mathrm{cl}(S_x) \neq \mathrm{cl}(S)_x\};$$
$$I_S := \{x \in R : \mathrm{int}(S_x) \neq \mathrm{int}(S)_x\}.$$

(We have $C_S = F_S$, and finiteness of F_S is shown in [2], Lemma 4.1.7. Using $\mathrm{int}(A) = R^n \setminus \mathrm{cl}(R^n \setminus A)$ for all $n > 0$ and $A \subseteq R^n$ we see that $I_S = C_{R^m \setminus S}$, hence I_S is also finite.) In particular, we obtain:

Lemma 2.11. *Suppose that S is regular open. If S is definable, then S_x is regular open for all but finitely many $x \in R$. Moreover, with $\pi\colon R^m \to R$ denoting the projection onto the first co-ordinate, the set $S' := \{x \in \pi(S) : S_x \neq R^{m-1}\}$ is either empty or infinite.*

Proof. For the first statement, just note that the set of $x \in R$ such that S_x is not regular open is contained in $C_S \cup I_{\mathrm{cl}(S)}$. As for the second statement, if S' is finite, then $\mathrm{cl}(S) = \mathrm{cl}(\pi(S)) \times R^{m-1}$, hence $S = \mathrm{int}(\mathrm{cl}(S)) = \mathrm{int}(\mathrm{cl}(\pi(S))) \times R^{m-1}$, and therefore $S' = \varnothing$. □

(In the first statement of the previous lemma we cannot replace "all but finitely many" by "all", as the example S = the union of the two open discs of radius 1 centered at $(0,1)$ and $(0,-1)$ in R^2 shows.)

Proof of Lemma 2.10. By Theorem 4.1.8 in [2] we have $\dim \mathrm{fr}(S) < \dim S = m$, so it suffices to show that $\dim \mathrm{fr}(S) \geq m-1$. For this, we proceed by induction on m. Note that $\mathrm{fr}(S) \neq \varnothing$, since $S \neq \varnothing$ and $S \neq R^m$. This yields the claim in the case $m = 1$. Suppose $m > 1$ and the claim holds with m replaced by $m-1$. The definable subset $\pi(S)$ of R is open and non-empty, and the definable subset S' of $\pi(S)$ is either empty or has dimension 1. If $S' = \varnothing$, then $S = \pi(S) \times R^{m-1}$, hence $\mathrm{fr}(S) = \mathrm{fr}(\pi(S)) \times R^{m-1}$ and thus $\dim\, \mathrm{fr}(S) \geq m-1$ as required.

By the remarks above, the set F consisting of all $x \in S'$ such that $\mathrm{fr}(S_x) \neq (\mathrm{fr}\, S)_x$ or S_x is not regular open is finite, and if $x \in S' \setminus F$ then $\dim\, \mathrm{fr}(S_x) \geq m-2$ by inductive hypothesis. Hence if $\dim S' = 1$, then also $\dim(S' \setminus F) = 1$, and

$$\mathrm{fr}(S) \cap \pi^{-1}(S' \setminus F) = \bigcup_{x \in S' \setminus F} \{x\} \times \mathrm{fr}(S_x)$$

yields $\dim \mathrm{fr}(S) \geq \dim\big(\mathrm{fr}(S) \cap \pi^{-1}(S' \setminus F)\big) \geq m-1$ by [2], Proposition 4.1.5. □

Claim 2.12. *If $D \cap \mathrm{bd}_X(\mathrm{cl}_X(V \setminus \{g\})) \neq \varnothing$, then the o-minimal dimension of D is at least $2n-1$ where n is the complex dimension of X.*

Proof. Suppose $x \in D \cap \mathrm{bd}_X(\mathrm{cl}_X(V \setminus \{g\}))$. Since $D \cap U = \varnothing$, we have $x \notin \mathrm{bd}_G(V)$. As $\mathrm{bd}_G(V)$ is also closed in X there exists an open neighbourhood P of x in X such that $P \cap \mathrm{bd}_G(V) = \varnothing$. In $\mathbb{R}_{\mathrm{an}}$, o-minimal dimension for complex analytic sets is twice the complex dimension. Hence the o-minimal dimension of X is $2n$, and the set $V \setminus \{g\}$ is open in X and definable in $\mathbb{R}_{\mathrm{an}}$. By Fact 2.9, $\mathrm{bd}_X(\mathrm{cl}_X(V \setminus \{g\}))$ has o-minimal dimension $2n-1$ everywhere, in particular, at x. So, the o-minimal dimension of $P \cap \mathrm{bd}_X(\mathrm{cl}_X(V \setminus \{g\}))$ is $2n-1$. Moreover, by 2.8 and our choice of P, we get $P \cap \mathrm{bd}_X(\mathrm{cl}_X(V \setminus \{g\})) \subseteq D$. □

Claim 2.13. $D \subseteq \mathrm{int}_X(\mathrm{cl}_X(V \setminus \{g\}))$

Proof. Since $X \setminus U$ is a proper analytic subset of the irreducible space X, the complex dimension of $X \setminus U$ is at most $n-1$, and hence its o-minimal dimension is at most $2n-2$. By $D \cap U = \varnothing$ and Claim 2.12, $D \cap \mathrm{bd}_X(\mathrm{cl}_X(V \setminus \{g\})) = \varnothing$. As $D \subseteq \mathrm{cl}_X(V \setminus \{g\})$, we conclude that $D \subseteq \mathrm{int}_X(\mathrm{cl}_X(V \setminus \{g\}))$. □

Proof of Lemma 2.2. We already know that $\mathrm{int}_X(\mathrm{cl}_X(V \setminus \{g\})) \subseteq (V \setminus \{g\}) \cup D$. The reverse inclusion also holds, by Claim 2.13. So the set $D \cup (V \setminus \{g\}) = \mathrm{int}_X(\mathrm{cl}_X(V \setminus \{g\}))$ is open, as desired. □

Lemma 2.14. *As in 2.2, suppose $V \subseteq G$ is a regular open co-ordinate neighbourhood of g in G and D is the frontier of $\mathrm{cl}_G(V) \setminus \{g\}$ in X. Suppose $N \subseteq V$ is another regular open co-ordinate neighbourhood of g in G and let D_N be the frontier of $\mathrm{cl}_G(N) \setminus \{g\}$ in X.*

Then $D_N = D$.

Proof. By Claims 2.5 and 2.6 we have $D = \mathrm{cl}_X(V \setminus \{g\}) \cap (X \setminus U)$, and similarly $D_N = \mathrm{cl}_X(N \setminus \{g\}) \cap (X \setminus U)$ by applying these claims to N in place of V. Now

$$\begin{aligned} D &= \mathrm{cl}_X(V \setminus \{g\}) \cap (X \setminus U) \\ &= \mathrm{cl}_X((V \setminus N) \cup (N \setminus \{g\})) \cap (X \setminus U) \\ &= \left[\mathrm{cl}_X(V \setminus N) \cup \mathrm{cl}_X(N \setminus \{g\})\right] \cap (X \setminus U) \\ &= \left[\mathrm{cl}_X(V \setminus N) \cap (X \setminus U)\right] \cup \left[\mathrm{cl}_X(N \setminus \{g\}) \cap (X \setminus U)\right] \end{aligned}$$

Since $V \setminus N$ is bounded away from g, we have $\mathrm{cl}_X(V \setminus N) = \mathrm{cl}_G(V \setminus N) \subseteq U$, hence $D = \mathrm{cl}_X(N \setminus \{g\}) \cap (X \setminus U) = D_N$.

□

Proof of Proposition 2.1. As mentioned before, it suffices to prove the Proposition in the case that $|F| = 1$. Let $F = \{g\}$, and set V and D to be as in Lemma 2.2. Let $W \supseteq V$ be a larger co-ordinate neighbourhood around g in G for which $\mathrm{cl}_G(V) \subseteq W$. As a set, we define $G^* := (X \setminus D) \dot\cup \{g\}$ and the function $\pi\colon X \to G^*$ by $\pi(x) := g$ if $x \in D$ and $\pi(x) := x$ otherwise. That is, G^* is formed by collapsing D to the point g, and π is this collapsing map.

We give G^* a complex manifold structure by specifying a system of local co-ordinate neighborhoods about each point in G^*. In what follows we write Δ for the standard $\dim G$-unit polydisc. Around g, take $O_g := V$ with its co-ordinate function $\phi_g\colon \Delta \to O_g$ as the chart. For $x \in X \setminus D$ find a chart $O_x \subseteq X \setminus D$ in the sense of X with co-ordinate function $\psi_x\colon \Delta \to O_x$. If $x \in W$, then we may (and do) choose O_x so that it is contained in $W \subseteq G$ and, so, ψ_x is also a chart in the sense of G as well. If $x \notin W$, then, in particular, $x \notin \mathrm{cl}_X(V \setminus \{g\})$. So, we may (and do) choose O_x so that $O_x \cap V = \varnothing$.

Consider two distinct points $x \neq y \in G^*$. We must show that

$$\vartheta := \phi_y^{-1} \circ \phi_x \restriction \phi_x^{-1}(O_x \cap O_y)$$

is holomorphic. If neither x nor y is equal to g, then because $X \setminus D$ is a complex manifold, the transition map ϑ is a transition map in the sense of X and is therefore holomorphic. So, we may suppose that $x = g \neq y$. If $y \notin W$, then $O_y \cap O_g = \varnothing$ so that there is nothing to check. If $y \in W$, then ϑ is a transition map in the sense of G and is holomorphic.

Next we verify that π is continuous. Let $a \in X$. If $a \notin D$, then there is a neighbourhood about a on which π is the identity, so π is clearly continuous at

a. If $a \in D$, and $N \subseteq V \subseteq G^*$ is a regular open co-ordinate neighbourhood of $g = \pi(a)$, then

$$\pi^{-1}(N) = D \cup (N \setminus \{g\}) = D_N \cup (N \setminus \{g\})$$

by Lemma 2.14. The latter is open in X by Lemma 2.2 (applied to N instead of V). That is, π is continuous at a. So π is continuous everywhere.

The map π is the identity on $X \setminus D$, and from the choice of co-ordinate neighbourhoods it is clear that it is biholomorphic there. Hence to show that π is holomorphic on X it suffices to consider $a \in D$ and N a co-ordinate neighbourhood about a in X, and show that $f := \pi \upharpoonright N$ is holomorphic.

Since $D \subseteq X \setminus U$, $A := N \cap D$ is *thin* in N; it is contained in the analytic set $N \cap (X \setminus U)$. Also, f is holomorphic on $N \setminus A$ and locally bounded everywhere (as it is continuous on N). By the Riemann extension theorem, f is holomorphic on N.

Finally, let $\iota \colon G \to G^*$ be the inclusion map on the underlying sets (recalling that $U \subseteq X \setminus D$). It follows from the definitions that ι is an embedding and that π agrees with ι on U. It remains to show that $\iota(G)$ is Zariski open in G^*. Note that $G^* \setminus U = \pi(X \setminus U)$ and hence U is Zariski open in G^* (by Remmert's Proper Mapping Theorem). It follows that $\iota(G) = U \cup \{g\}$ is constructible and it suffices to show that $U \cup \{g\}$ is open in G^*. As π is a biholomorphism on U we need only consider the point g. But V is an open neighbourhood of g in G^* contained in $U \cup \{g\}$. □

3. Standard strongly minimal groups

Suppose (G, μ) is a group definable in $\mathcal{A}$. We will mostly deal with strongly minimal, hence abelian, (G, μ). Therefore, we sometimes also write the group operation $(g, h) \mapsto \mu(g, h) \colon G \times G \to G$ of G as $\mu(g, h) = g + h$, even if (G, μ) is not abelian. Suppose, moreover, that we can write $G = U \cup F$ where U is a non-empty Zariski open subset of an irreducible compact complex manifold X and F is a finite set of points disjoint from U. For example, when G is strongly minimal we can always write G in this way (by quantifier elimination).

Every group interpretable in $\mathcal{A}$ has the structure of a complex Lie group making it into a connected *meromorphic group* in the sense of [15]. Indeed, the proof given in [1] of the Weil-Hrushovski theorem for groups interpretable in algebraically closed fields generalises immediately to groups interpretable in $\mathcal{A}$.

So (G, μ) has the structure of a meromorphic group. In particular, Proposition 2.1 applies to the complex manifold G, and for now we identify G with $\iota(G) \subseteq G^*$. Moreover, μ is holomorphic with respect to this manifold structure and extends to a meromorphic map $\mu^* \colon G^* \times G^* \to G^*$. Let $S := G^* \setminus G$ and let $\Gamma(\mu^*) \subseteq (G^*)^3$ denote the graph of μ^*.

Fact 3.1 (Lemma 3.3 of [15])**.** *Suppose $S \neq \varnothing$. Then every component of S has codimension* 1 *in G^* and μ^* restricts to a meromorphic map $\mu^*_S \colon G^* \times S \to S$. Moreover, for each component C of S, μ^* induces a generic action of (G, μ) on C.*

More precisely, for each $g \in G$, $\Gamma(\mu^*)_g \cap (C \times C)$ is the graph of a bimeromorphic map $\mu^*_C(g, -) \colon C \to C$; and for $g, h \in G$,

$$\mu^*_C(g, -) \circ \mu^*_C(h, -) = \mu^*_C(g + h, -)$$

on a non-empty Zariski open subset of C.

In order to transfer the above fact to elementary extensions we would like to remove the reference to the compactification G^* (whose existence we cannot *a priori* establish in the nonstandard case). To this end we formulate the following corollary of Proposition 2.1 and Fact 3.1:

Let $\Gamma(\mu) \subseteq G^3$ be the graph of μ, and consider the Zariski closure $\overline{\Gamma(\mu) \cap U^3}$ of $\Gamma(\mu) \cap U^3$ in X^3. Since the general fibres over X^2 under the map $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ are singletons, $\overline{\Gamma(\mu) \cap U^3}$ has a unique irreducible component $\Gamma \subseteq X^3$ which projects onto X^2.

Corollary 3.2. *Either G is compact — and hence is a complex torus — or Γ induces a generic action of (G, μ) on some component of $X \setminus U$.*

The latter case can be stated more precisely as: for some component C' of $X \setminus U$, $\Gamma \cap (X \times C' \times C')$ is the graph of a meromorphic map $\mu' \colon X \times C' \to C'$; for each $g \in U$, $\mu'(g, -) \colon C' \to C'$ is bimeromorphic; and for $g, h \in U$ such that $g + h \in U$,

$$\mu'(g, -) \circ \mu'(h, -) = \mu'(g + h, -)$$

on a non-empty Zariski open subset of C'.

Proof. If $G = G^*$ (that is, if $S = \varnothing$) then G is a complex torus. So we assume that $S \neq \varnothing$. Let $\pi \colon X \to G^*$ be as in Proposition 2.1. Then

$$X \setminus U = \pi^{-1}(S) \cup \pi^{-1}(F)$$

and $\pi^{-1}(S)$ is just a copy of S in X. In particular, every component of $\pi^{-1}(S)$ has codimension 1 in X and hence is a component of $X \setminus U$. Let $C' = \pi^{-1}(C)$ be one such component. By Fact 3.1 μ^* induces a generic action of (G, μ) on C. Note that $\pi^3(\Gamma)$ agrees with $\Gamma(\mu^*)$ on a non-empty Zariski open subset of $(G^*)^3$ and hence $\pi^3(\Gamma) = \Gamma(\mu^*)$. Since π^3 is biholomorphic over $(G^* \setminus F)^3$, it lifts a generic action to a generic action. That is, Γ induces a generic action of (G, μ) on C'. □

Remark 3.3. The components of $X \setminus U$ split up into those coming from the boundary of G in G^* and those that collapse to the finite set F. It is on the components of the former type that we have a generic action.

When G is strongly minimal, the latter case of Corollary 3.2 implies that G is a linear algebraic group. Indeed, using the fact that C' is of lower dimension one shows that G acts holomorphically on C' as the identity (see the proof of Lemma 3.5 in [15]). It then follows from an argument of Fujiki's in [5] that G is linear algebraic. We will mimic this argument in elementary extensions.

But first we deal with the former case of Corollary 3.2 in elementary extensions.

4. Elementary extensions and the case of no boundary action

Let $\mathcal{A}'$ be a sufficiently saturated elementary extension of $\mathcal{A}$. Definable sets and maps in $\mathcal{A}'$ can be understood in terms of uniformly definable families of sets and maps in the standard model $\mathcal{A}$. A systematic discussion of this correspondence is given in section 2 of [11], and we restrict ourselves to only a few remarks here.

First, given an irreducible Zariski closed set F in $\mathcal{A}$ we use $F(\mathcal{A}')$ to denote its interpretation in the nonstandard model. By a *generic point* of F we will mean a point $a \in F(\mathcal{A}')$ that is not contained in $H(\mathcal{A}')$ for any proper Zariski closed subset $H \subseteq F$. By saturation, every irreducible Zariski closed set has generic points. Conversely, given a sort X of $\mathcal{A}$ and a point $a \in X(\mathcal{A}')$, the *locus* of a is the smallest Zariski closed set $F \subseteq X$ such that $a \in F(\mathcal{A}')$. By noetherianity of the Zariski topology, such a Zariski closed set exists (and is irreducible). Note that a Zariski closed subset F of X is the locus of an element $a \in F(\mathcal{A}')$ if and only if a is generic in F.

If P is a $\varnothing$-definable property of points in F, then P holds in some non-empty Zariski open subset of F (we say that *P holds for general $x \in F$*) if and only if it holds for a generic point.

By a *Zariski closed set in $\mathcal{A}'$ over s* we mean a set of the form

$$G(\mathcal{A}')_s := \{x \in X(\mathcal{A}') : (x, s) \in G(\mathcal{A}')\}$$

where X, Y are sorts of $\mathcal{A}$, $G \subseteq X \times Y$ is a Zariski closed subset, and $s \in Y(\mathcal{A}')$. Taking $S \subseteq Y$ to be the locus of s and restricting the second co-ordinate projection map to $F := G \cap (X \times S)$, we see that $G(\mathcal{A}')_s$ is the fibre of $F \to S$ over s. That is, every Zariski closed set in $\mathcal{A}'$ is the fibre of a holomorphic map from a compact complex analytic space to an irreducible compact complex analytic space, over a generic point. This description of Zariski closed sets in $\mathcal{A}'$ is more canonical and behaves well with respect to parameters: working over additional parameters in $\mathcal{A}'$ corresponds to base change in $\mathcal{A}$.

The Zariski closed sets in $\mathcal{A}'$ form the closed sets of a noetherian topology on each sort, and every such set can be written uniquely as an irredundant union of finitely many *absolutely irreducible* Zariski closed sets — sets that cannot be written as a union of two proper Zariski closed subsets. Absolutely irreducible Zariski closed sets are exactly the generic fibres of *fibre spaces*, i.e., holomorphic maps between irreducible compact complex spaces whose general fibres are irreducible (see Lemma 2.7 of [11]).

Suppose A and B are absolutely irreducible Zariski closed sets in $\mathcal{A}'$, where A is the fibre of $F \to S$ over some generic point $s \in S(\mathcal{A}')$ and B is the fibre of $G \to T$ over some generic point $t \in T(\mathcal{A}')$. A map $f \colon A \to B$ is *holomorphic* if there exist

- common base extensions $F_Z := F \times_S Z \to Z$ and $G_Z := G \times_T Z \to Z$;
- an irreducible Zariski closed subset Γ of the fibre product $F_Z \times_Z G_Z$; and,
- a point $z \in Z$ which maps to s and t respectively,

such that the graph of f is the fibre of $\Gamma \to Z$ over z. In a similar manner, one can define *meromorphic* maps between absolutely irreducible Zariski closed sets in $\mathcal{A}'$.

One advantage of working in a saturated model, besides the existence of generic points, is homogeneity. In particular, if F is a definable set in $\mathcal{A}'$ and P is a set of parameters in $\mathcal{A}'$, then F is definable over P if and only if every automorphism of $\mathcal{A}'$ that fixes P pointwise fixes F setwise. We use this fact often and abbreviate it to the phrase "by automorphisms".

We now extend the notion of meromorphic group as given in [15] to $\mathcal{A}'$:

Definition 4.1. A *meromorphic group in $\mathcal{A}'$* (or a *nonstandard meromorphic group*) is given by the following data: a definable group $(\mathcal{G}, +)$ in $\mathcal{A}'$ with a finite covering by definable sets $W_1, \dots, W_n$; and definable bijections $\phi_i \colon W_i \to U_i$ where U_i is a non-empty Zariski open subset of an absolutely irreducible Zariski closed set X_i, for $i = 1, \dots, n$; such that

(i) for each $i \neq j$, the subset $\phi_i(W_i \cap W_j)$ of X_i is Zariski open, and the induced bijection from $\phi_i(W_i \cap W_j)$ to $\phi_j(W_i \cap W_j)$ is a biholomorphic map that extends to a meromorphic map from X_i to X_j; and

(ii) for each i, j, k, the set

$$\big\{(x,y) \in U_i \times U_j : \phi_i^{-1}(x) + \phi_j^{-1}(y) \in W_k\big\}$$

is Zariski open in $X_i \times X_j$ and the induced map from this set to U_k given by $(x,y) \longmapsto \phi_k\big(\phi_i^{-1}(x) + \phi_j^{-1}(y)\big)$ is holomorphic and extends to a meromorphic map from $X_i \times X_j$ to X_k.

We say the meromorphic group $(\mathcal{G}, +)$ is defined over a set of parameters P if all the definable sets and maps involved in the above data are over P.

Lemma 4.2. *Every group interpretable in $\mathcal{A}'$ has the structure of a nonstandard meromorphic group.*

Proof. By elimination of imaginaries for $\mathrm{Th}(\mathcal{A})$ (see the appendix of [10]) we need only consider definable groups. As in the standard case, the result now follows by the proof of the Weil-Hrushovski theorem (see [1]). □

For the rest of this paper, we let $(\mathcal{G}, +)$ be a *strongly minimal group* definable in $\mathcal{A}'$. There exists an irreducible compact complex space Y, a uniformly definable family $(G \to Y, \mu\colon G \times_Y G \to G)$ of groups over Y (in the standard model), and a generic point $a \in Y(\mathcal{A}')$, such that $(\mathcal{G}, +) = (G_a, \mu_a)$. Note that the set G may be merely a definable set (not necessarily Zariski closed), and the maps $G \to Y$ and μ merely definable functions (not necessarily holomorphic). By strong minimality of $\mathcal{G}$ and quantifier elimination, there exist

- an irreducible compact complex space over Y, $f\colon X \to Y$, whose general fibres are smooth and irreducible;
- a non-empty Zariski open set U of X; and
- a compact complex space F over Y, whose general fibres are finite;

such that $G_y = U_y \cup F_y$ for general $y \in Y$. In particular, G_y is a (standard) definable group of the form discussed in the previous section. Note that $\mathcal{G} = U_a \cup F_a$.

By Lemma 4.2, $(\mathcal{G}, +)$ is a nonstandard meromorphic group. We can choose Y and $a \in Y(\mathcal{A}')$ such that this meromorphic group structure is also over a. It follows that there is a uniformly definable meromorphic group structure on (G_y, μ_y) for general $y \in Y$.

Proposition 4.3. *Suppose that $+$ does not extend to a generic action of $\mathcal{G}$ on any absolutely irreducible components of $X_a \setminus U_a$. Then for general $y \in Y(\mathcal{A})$, G_y is definably isomorphic to a complex torus.*

Moreover, possibly after base extension, there is a definable map $\pi\colon X \to G$ over Y such that for general $y \in Y$, $\pi_y\colon X_y \to G_y$ is a holomorphic surjection with respect to the complex manifold structure on G_y viewed as a meromorphic group.

That is, $\pi_a\colon X_a \to \mathcal{G}$ is a holomorphic surjection *with respect to the nonstandard meromorphic manifold structure on $\mathcal{G}$.*

Proof. Let Γ_a be the unique absolutely irreducible component of $\overline{\Gamma(+) \cap U_a^3} \subseteq X_a^3$ that projects onto the first two factors. By automorphisms, Γ_a is defined over a and hence Γ_a is the generic fibre over Y of an irreducible analytic subset $\Gamma \subseteq X^3$.

Claim 4.4. *For generic $g \in X_a$, $\Gamma_a(g) := \Gamma_a \cap \big(\{g\} \times X_a^2\big)$ is absolutely irreducible.*

Proof of claim. In the standard model the general fibres of of the first projection $\Gamma \to X$ have a unique maximal dimensional irreducible component that projects onto X (viewing the fibres as subsets of $X \times_Y X$ and taking the first projection).

This is because they are generically one-to-one over X. These distinguished components are uniformly definable over X (by automorphisms). So there is a constructible $\Gamma' \subseteq \Gamma$ whose general fibres over X are these distinguished irreducible components of the fibres of Γ. Counting dimension and using the irreducibility of Γ we see that the Zariski closure of Γ' must be Γ. As the general fibres of Γ' are Zariski closed, $\overline{\Gamma'}$ and Γ' have the same general fibres. So the general fibres of Γ are irreducible. So the generic fibre is absolutely irreducible. □

Possibly after base extension, we may assume that the absolutely irreducible components of $X_a \setminus U_a$ are defined over a — write them as $(C_1)_a, \dots, (C_n)_a$ where $C_1, \dots, C_n$ are irreducible analytic subsets of X. Now fix $i \leq n$. To say that "$+$ extends to a generic action of $\mathcal{G}$ on $(C_i)_a$" is to say that

(i) over some Zariski open subset of $X_a \times (C_i)_a$, $\Gamma_a \cap \big(X_a \times (C_i)_a \times (C_i)_a\big)$ is the graph of a well-defined function to $(C_i)_a$; and
(ii) for generic $g \in X_a$, $\Gamma_a^i(g) := \Gamma_a \cap \big(\{g\} \times (C_i)_a^2\big)$ induces a well-defined bijection between Zariski open subsets of $(C_i)_a$; and
(iii) for generic $g, h \in X_a$, $\Gamma_a^i(g+h)$ agrees with

$$\big\{(x,y) \in (C_i)_a^2 : \exists z\ (x,z) \in \Gamma_a^i(h) \text{ and } (z,y) \in \Gamma_a^i(g)\big\}.$$

Both (i) and (ii) are definable properties of a: (i) is expressed by stating that the co-ordinate projection

$$\Gamma_a \cap \big(X_a \times (C_i)_a \times (C_i)_a\big) \to X_a \times (C_i)_a$$

is surjective with generic fibre a singleton; and (ii) is expressed by saying that for generic $g \in X_a$, both co-ordinate projections from $\Gamma_a^i(g) := \Gamma_a \cap (\{g\} \times (C_i)_a^2)$ to $(C_i)_a$ are surjective with generic fibres singletons. (To say that the generic fibre has a property P is equivalent to saying that the set of point in the base over which the fibre has property P is of the same dimension as the base — and hence is definable if P is.) Note that (iii) is always true since $\Gamma_a(g+h)$ agrees with

$$\big\{(x,y) \in X_a^2 : \exists z\ (x,z) \in \Gamma_a(h) \text{ and } (z,y) \in \Gamma_a(h)\big\}$$

on the nonempty Zariski open subset of X_a^2 where Γ agrees with $+$; and hence on all of X_a^2.

It follows that "$+$ does not extend to a generic action of $\mathcal{G}$ on any absolutely irreducible components of $X_a \setminus U_a$" is a definable property of a. Hence for general $y \in Y$, μ_y does not extend to a generic action of G_y on any irreducible component of $X_y \setminus U_y$. By Corollary 3.2 this implies that G_y is definably isomorphic to a complex torus, as desired.

For the "moreover" clause, fix a sufficiently general $y \in Y$. Let $\iota\colon G_y \to G_y^*$ be the compactification of G_y obtained in Proposition 2.1. In this case $\iota(G_y) = G_y^*$, and we set $\pi_y\colon X_y \to G_y$ to be the composition of the associated holomorphic surjection $X_y \to G_y^*$ with ι^{-1}. As ι is an isomorphism on the complex manifold G_y viewed as a meromorphic group, π_y is a holomorphic surjection with respect to this structure. Despite the optimistic notation, we have yet to verify that π_y is uniformly definable in y (we do not claim that ι is). We know from Proposition 2.1 that π_y is the identity on U_y, and so it remains to consider π_y on $X_y \setminus U_y$. Note that $\pi_y(X_y \setminus U_y)$ is equal to the finite set F_y. It follows that π_y is constant on each of the irreducible components of $X_y \setminus U_y$ and takes values in F_y. Since F_y and the components of $X_y \setminus U_y$ are uniformly definable in y, for each possible behavior of

π_y on $X_y \setminus U_y$, the set of y having this behavior is definable. As there are only finitely many possibilities, one of them holds on a nonempty Zariski open subset of Y, as desired. □

Remark 4.5. Proposition 4.3 says that if $+$ does not extend to a generic action on the boundary then $\mathcal{G}$ is "compact" in the sense of Theorem 1.1.

Question 4.6. Does it follow from the conclusion of Proposition 4.3 that (possibly after base extension) there exists a compact complex space G^* over Y and a definable map $\iota\colon G \to G^*$ over Y such that for general $y \in Y$, ι_y is a definable isomorphism between G_y and G^*_y? If so it would follow that $\mathcal{G}$ is a *nonstandard complex torus.*

5. The case of a boundary action

In this final section we complete the proof of Theorem 1.1 by dealing with the case when the hypothesis of Proposition 4.3 fails — that is, when $+$ *does* extend to a generic action of $\mathcal{G}$ on some absolutely irreducible component of $X_a \setminus U_a$. By making an argument of Fujiki's (Proposition 2.7 of [5]) uniform in parameters, we will show that $(\mathcal{G}, +)$ is definably isomorphic to a linear algebraic group over $\mathbb{C}'$. We begin by reviewing some notions from complex geometry.

5.1. **Douady spaces.** For any first-order structure M in a language $\mathcal{L}$ one may list all of the (parametrically) definable subsets of M by considering all $\mathcal{L}$-formulae in $1+n$ variables (as n varies) and all instances of these formulae with tuples from M substituted for the parameter variables. Of course, this method of listing the definable sets may be redundant as two different formulae may define the same subset of the model. In some cases it is possible to achieve a correspondence between syntax and semantics. That is, there may exist some subset $\mathcal{S}$ of all the $\mathcal{L}$-formulae such that, perhaps, allowing for imaginary parameters, every definable subset of M is defined by $\{x \in M \mid M \models \phi(x; \boldsymbol{m})\}$ for a unique $\phi \in \mathcal{S}$ and unique parameter $\boldsymbol{m}$. In the cases that M carries a good definable topology, one might even hope that some topology on the parameters for the definable sets reflects the way in which the definable sets lie in M.

The compact analytic subspaces of a complex analytic space have a particularly nice parametrisation called the Douady space, which we describe below. For certain compact complex manifolds M, in particular, Kähler manifolds, the Douady spaces may be used to produce canonical formulae for the definable subsets of M. Even outside this setting, the Douady spaces provide a canonical *analytic*, though possibly non-definable, parameterisation of analytic subspaces. A more complete discussion of the model-theoretic relevance of Douady spaces can be found in [10].

The theory of Douady spaces may be applied in a very general setting, and we shall require it for non-reduced complex analytic spaces. For a modern treatment of complex analytic spaces, including the non-reduced case, the reader is advised to consult [17].

For any complex analytic space X (possibly non-compact and non-reduced), Douady [3] constructed a universal family for the compact analytic subspaces of X. That is, there exists a complex analytic space $D = D(X)$ and a closed analytic subspace $Z = Z(X) \subseteq D \times X$ such that:

(a) The projection $Z \to D$ is a flat and proper surjection.

(b) If S is a complex analytic space and G is an analytic subspace of $S \times X$ that is flat and proper over S, then there exists a unique holomorphic map $g\colon S \to D$ such that $G \simeq S \times_D Z$ canonically.

$D(X)$ is called the *Douady Space of* X, $Z(X)$ is called the *universal family of* X, and $g\colon S \to D(X)$ as in (b) is called the *Douady map associated to* $G \to S$.

Condition (*b*) says that *every flat family of compact analytic subspaces of* X *is witnessed uniquely by a subfamily of* $Z(X)$ *over* $D(X)$. The condition of flatness, while technically necessary, may seem somewhat ill-motivated from the model theoretic point of view. We can, however, avoid considerations of flatness as follows: if G and S are as in (*b*) with S now reduced but $G \to S$ not necessarily flat, then we can always find, by a theorem of Frisch, a non-empty Zariski-open subset $U \subseteq S$ over which G is flat. Hence, we have a Douady map $g : U \to D(X)$. Moreover, by Hironaka's Flattening theorem, g extends to a meromorphic map $S \to D(X)$. Hence, in the non-flat case we still have a Douady map, however, it is only meromorphic and not necessarily holomorphic.

It is instructive to note that, in particular, for every compact analytic subspace A of X there is a unique point $[A] \in D$ such that A is the fibre of Z over $[A]$. Indeed, applying (*b*) to the 0-dimensional variety $S := \{s\}$ and $G := S \times A$, we obtain a holomorphic map $g : \{s\} \to D$ such that A is the fibre of $Z \to D$ above $g(s)$. More precisely, $\{s\} \times_D Z$, which is the sheaf-theoretic fibre of $Z \to D$ over $g(s)$, is isomorphic under the projection $\{s\} \times_D Z \to X$ to A. That $[A] := g(s)$ is the unique such point in D follows from the uniqueness of the Douady map. We call $[A]$ the *Douady point of* A.

In the case that X is a projective variety, $D(X)$ is the Hilbert scheme of X and hence is a countable union of projective varieties.

There is also a relative version of the Douady space constructed by Pourcin [16]: Let X and S be complex spaces and $f : X \to S$ a holomorphic map. Then there exists a complex space $D(X/S)$ with a holomorphic map to S, and a closed analytic subspace $Z(X/S) \subset D(X/S) \times_S X$ such that $Z(X/S) \to D(X/S)$ is proper and flat and such that for any complex space $Y \to S$ and any complex subspace $G \subset Y \times_S X$ that is flat and proper over Y, there is a unique holomorphic map $Y \to D(X/S)$ over S such that $G \simeq Y \times_{D(X/S)} Z(X/S)$ canonically. Loosely speaking, the Douady space of X over S parametrises all flat families of compact subspaces of X whose fibres live in the fibres of f. In particular, $D(X/S)_s = D(X_s)$ and $Z(X/S)_s = Z(X_s)$ for all $s \in S$. So the relative Douady space bundles together the Douady spaces of all the fibres of f in a uniform manner.

The components of the Douady spaces are not necessarily compact even when X and S are. Hence the Douady spaces are not *a priori* definable in $\mathcal{A}$. However, the following fact will play an important role for us: if X and S are compact and $f : X \to S$ is projective[2] then the irreducible components of $D(X/S)$ are compact and projective over S (cf. Theorem 5.2 of [6]).

5.2. **Automorphism groups.** One of the first applications of Douady spaces was to the group of automorphisms of a complex analytic space X. Let $\mathrm{Aut}(X)$ denote the set of biholomorphic maps from X to X. Identifying an automorphism with the Douady point of its graph, we can view $\mathrm{Aut}(X)$ as a subset of $D(X \times X)$.

[2]Recall that $f : X \to S$ is *projective* if X is biholomorphic over S to an analytic subspace of some projective bundle over S.

Douady [3] showed that $\mathrm{Aut}(X)$ is an open subset of $D(X \times X)$ and, equipped with the inherited complex structure, is a complex Lie group acting biholomorphically on X. Fujiki observes that in fact $\mathrm{Aut}(X)$ is a Zariski open subset of $D(X \times X)$ (cf. Lemma 1 of [5] and Lemma 5.5 of [4]).

We follow Fujiki [7] in describing the relative version of automorphism groups: Suppose $f\colon X \to S$ is a proper surjective morphism of irreducible complex analytic spaces (not necessarily reduced or compact). Suppose for the moment, that f is flat. Then there exists a Zariski open subset $\mathrm{Aut}_S(X) \subset D(X \times_S X/S)$ such that for all $s \in S$, $\mathrm{Aut}_S(X)_s = \mathrm{Aut}(X_s)$ (cf. Schuster [18]). That is, $\mathrm{Aut}_S(X)$ bundles together the automorphism groups of the fibres of f in a uniform manner. The inherited complex structure on $\mathrm{Aut}_S(X)$ makes it into a complex Lie group over S acting biholomorphically on X over S.

Now suppose that f is not flat but S is reduced. Then as discussed above, there is a non-empty Zariski-open subset $U \subseteq S$ over which f is flat. We may consider $\mathrm{Aut}_U(X_U)$, which is a Zariski open subset of $D(X_U \times_U X_U/U) = \pi^{-1}(U)$ where $\pi : D(X \times_S X/S) \to S$. Following Fujiki [7], by the *essential closure* of $\mathrm{Aut}_U(X_U)$ in $D(X \times_S X/S)$ we mean the union of those irreducible components of the Zariski closure of $\mathrm{Aut}_U(X_U)$ in $D(X \times_S X/S)$ that project onto S. Note that the essential closure does not depend on U: if $V \subseteq U$ is another Zariski open subset then the essential closure of $\mathrm{Aut}_V(X_V)$ coincides with that of $\mathrm{Aut}_U(X_U)$. The essential closure of $\mathrm{Aut}_U(X_U)$ in $D(X \times_S X/S)$ is denoted by $\mathrm{Aut}^*_S(X)$. Shrinking U if necessary, we have that for all $s \in U$, $\mathrm{Aut}^*_S(X)_s$ is the Zariski closure of $\mathrm{Aut}(X_s)$ in $D(X_s \times X_s)$.

The following fact summarises the relevant properties of relative automorphism groups in the case we will be considering:

Fact 5.1. *Suppose S is a reduced and irreducible compact complex space, X is an irreducible compact complex space, and $f : X \to S$ is a finite surjective morphism. Then the irreducible components of $\mathrm{Aut}^*_S(X)$ are compact and projective over S. Moreover, there exists a non-empty Zariski open subset $U \subseteq S$ over which f is flat and the following hold:*

(i) *For all $s \in U$, $\mathrm{Aut}(X_s)$ is definably isomorphic to a linear algebraic group.*
(ii) *The relative group multiplication $\nu : \mathrm{Aut}_U(X_U) \times_U \mathrm{Aut}_U(X_U) \to \mathrm{Aut}_U(X_U)$ over U, the relative inversion $\iota : \mathrm{Aut}_U(X_U) \to \mathrm{Aut}_U(X_U)$ over U, and the identity section $e : U \to \mathrm{Aut}_U(X_U)$, all extend to meromorphic maps $\nu^* : \mathrm{Aut}^*_S(X) \times_S \mathrm{Aut}^*_S(X) \to \mathrm{Aut}^*_S(X)$ over S, $\iota^* : \mathrm{Aut}^*_S(X) \to \mathrm{Aut}^*_S(X)$ over S, and $e^* : S \to \mathrm{Aut}^*_S(X)$.*

In particular, if $\mathcal{A}'$ is a saturated elementary extension of $\mathcal{A}$ with $\mathbb{C}'$ its interpretaion of $\mathbb{C}$, then any generic fibre of $\mathrm{Aut}_U(X_U) \to U$ in $\mathcal{A}'$ is definably isomorphic to a linear algebraic group over $\mathbb{C}'$.

Remark 5.2. (i) It is important here that we do not require X to be reduced. Indeed, in the case we will be considering the underlying set of X will be S itself and the map f will be the identity on the underlying space, so all the information will live in the non-reduced structure of X and the action of f on the structure sheaf. However, while X itself may not be accessible to us model-theoretically, $\mathrm{Aut}^*_S(X)$ will be a definable object.
(ii) If in Fact 5.1 we only wish to conclude that the generic fibres are algebraic groups then we can weaken the hypothesis on f from being finite to being projective. Moreover, weakening the hypothesis further to f being of

Kähler-type, we still obtain that the components of $\mathrm{Aut}^*_S(X)$ are compact over S and that for general $s \in S$ the fibres form a uniformly definable family of groups (and hence the generic fibres are definable groups).

Proof. Since f is finite, it is projective (this follows from the Finite Mapping Theorem, see [17]). The components of $D(X \times_S X/S)$ are therefore compact and projective over S (cf. Section 5.1 above). The same is thus true of $\mathrm{Aut}^*_S(X)$.

As f is finite, the automorphisms of a fibre of f are just the linear transformations of the structure sheaf of the fibre, which is a finite dimensional complex vector space. Hence each $\mathrm{Aut}(X_s)$ with its inherited complex Lie group structure is isomorphic to a linear algebraic group. To see that it is definably so we need only find a definable embedding of $\mathrm{Aut}(X_s)$ into projective space. The projectivity of $\mathrm{Aut}^*_S(X) \to S$ implies that each fibre is biholomorphic (and hence definably isomorphic) to a projective variety. As $\mathrm{Aut}(X_s)$ is a definable subset of $\mathrm{Aut}^*_S(X)_s$ for $s \in U$, the restriction gives the required definable embedding.

Part (ii) is stated in general as a remark in Section 1 of [7] and proved for the absolute case (when S is a point) in Proposition 2.2 of [5]. It is straightforward to see that these latter arguments extend to the relative case.

Now for the "in particular" clause. To show that the generic fibres are definably isomorphic to nonstandard linear algebraic groups we need to show that the standard fibres are *uniformly* definably isomorphic (over possibly additional parameters) to linear algebraic groups. Since $\mathrm{Aut}^*_S(X) \to S$ is projective, after base change to some compact T, $\mathrm{Aut}^*_S(X) \times_S T \to T$ embeds into $T \times \mathbb{P}_r(\mathbb{C})$ over T (cf. Lemma 3.3 of [11]). By the stable embeddability of the projective sort, this implies that the fibres of $\mathrm{Aut}_U(X_U) \to U$ form a uniformly definable family living entirely in the projective sort of $\mathcal{A}$; that is, the family can be written as a uniformly definable family of subsets of a cartesian power of $\mathbb{P}(\mathbb{C})$, parametrised by a definable subset of a cartesian power of $\mathbb{P}(\mathbb{C})$. Moreover, this is a uniformly definably family of definable groups by part (ii), and each member is definably isomorphic to a linear algebraic group by part (i). Now Chow's theorem (that the analytic subsets of projective space are algebraic) implies that the structure induced on the sort $\mathbb{P}(\mathbb{C})$ by $\mathcal{A}$ is definably bi-interpretable with the complex field. It follows by saturation of $(\mathbb{C}, +, \times)$, that the fibres of $\mathrm{Aut}_U(X_U) \to U$ are uniformly definably isomorphic to linear algebraic groups. □

5.3. **Infinitesimal neighbourhoods.** We review here Grothendieck's theory of infinitesimal neighbourhoods for complex analytic spaces from [8]. An exposition of this material emphasising model-theoretic relevance can also be found in [9].

Let X be a complex space and x a point in X. The *nth infinitesimal neighbourhood of x in X* is the complex subspace $\Delta^{(n)}_{X,x} := (x, \mathcal{O}_{X,x}/\mathfrak{m}^{n+1}_{X,x})$ whose underlying set is $\{x\}$ and whose structure sheaf is the quotient of the local ring $\mathcal{O}_{X,x}$ of X at x by the $(n+1)$st power of the maximal ideal $\mathfrak{m}_{X,x}$ of X at x. These infinitesimal neighbourhoods can be witnessed as the fibres of a complex space over X as follows: Let $\Delta^{(n)}_X$ be the complex subspace of $X \times X$ whose underlying set is the diagonal in $X \times X$ and whose structure sheaf is $\mathcal{O}_{X \times X}/\mathcal{I}^{(n+1)}$ where $\mathcal{I}$ is the ideal sheaf of the diagonal. The first co-ordinate projection induces a finite surjective morphism $\Delta^{(n)}_X \to X$ whose fibres are canonically isomorphic to $\Delta^{(n)}_{X,x}$.

The above construction extends to the relative case: If $f : X \to S$ is a morphism of complex spaces, then $\Delta^{(n)}_{X/S}$ is the complex subspace of $X \times_S X$ whose underlying set is the diagonal in $X \times_S X$ and whose structure sheaf is $\mathcal{O}_{X\times_S X}/\mathcal{I}^{(n+1)}$ where $\mathcal{I}$ is the ideal sheaf of the diagonal. The first projection map induces a finite surjective morphism $\Delta^{(n)}_{X/S} \to X$ over S whose fibre over $x \in X$ is $\Delta^{(n)}_{X_{f(x)},x}$, the nth infinitesimal neighbourhood of x in $X_{f(x)}$ (cf. Corollaire 2.5 of [8]).

5.4. **Proof of Theorem 1.1.** We recover the notation of Section 4 and review our set-up: $\mathcal{A}'$ is a saturated elementary extension of $\mathcal{A}$ and $(\mathcal{G}, +)$ is a strongly minimal group definable in $\mathcal{A}'$. The definable group $(\mathcal{G}, +)$ appears as a generic fibre (G_a, μ_a) of a uniformly definable family of definable groups $(G \to Y, \mu : G \times_Y G \to G)$ in $\mathcal{A}$ where Y is a reduced and irreducible compact complex space and $a \in Y(\mathcal{A}')$ is generic. As $\mathcal{G}$ is strongly minimal it is the union of a nonstandard Zariski open subset of an absolutely irreducible Zariski closed set in $\mathcal{A}'$, together with finitely many additional points. In terms of the standard model, we can choose Y such that there exists a reduced and irreducible compact complex space X with a holomorphic surjection $f : X \to Y$ whose general fibres are smooth and irreducible, a non-empty Zariski open subset $U \subseteq X$, and a reduced compact complex space $F \to Y$ whose general fibres are finite, such that for general $y \in Y$, $G_y = U_y \cup F_y$. In particular, $\mathcal{G} = U_a \cup F_a$.

Let Γ_a be (as in the proof of Proposition 4.3) the unique absolutely irreducible component of the Zariski closure of $\Gamma(+) \cap U_a^3$ in X_a^3 that projects onto X_a^2 via the first two factors. Here $\Gamma(+) \subset \mathcal{G}^3$ is the graph of the group operation. Then, as the notation suggests, Γ_a is the fibre over a of an irreducible Zariski closed set $\Gamma \subset X \times_Y X \times_Y X$. For general $y \in Y$ and general $g \in X_y$, $\Gamma_y(g) := \Gamma_y \cap (\{g\} \times X_y^2)$ (viewed as a subset of X_y^2) is the graph of a meromorphic map $\tau_g \colon X_y \to X_y$ which agrees with translation by g on a non-empty Zariski open set contained in U_y.

In Section 4 we proved that if $+$ does not extend to a generic action on any absolutely irreducible component of $X_a \setminus U_a$, then $\mathcal{G}$ is compact in the sense that there is a holomorphic surjection from X_a to $\mathcal{G}$ with respect to the nonstandard meromorphic group structure on $\mathcal{G}$. Hence to prove Theorem 1.1 it remains to show that if $+$ does extend to a generic action on some components of $X_a \setminus U_a$, then $(\mathcal{G}, +)$ is definably isomorphic to a linear algebraic group over $\mathbb{C}'$ (and hence to the additive or multiplicative group by strong minimality).

We therefore assume that $+$ does extend to a generic action on some components of $X_a \setminus U_a$. In terms of the standard model, this means that there exists an irreducible Zariski closed subset C of $X \setminus U$ whose general fibres over Y are irreducible such that for general $y \in Y$ and $g \in X_y$, τ_g extends to a bimeromorphism $C_y \to C_y$ such that $\tau_g \circ \tau_h = \tau_{g+h}$ on a non-empty Zariski open subset of C_y. The graph of this action is given by the unique irreducible component of $\Gamma_y \cap (X_y \times C_y^2)$ that projects onto the first two co-ordinates – let us call it Γ_y^C. In fact, we have:

Lemma 5.3. *For general $y \in Y$ and general $g \in X_y$, τ_g is the identity on C_y.*

Proof. This is just as in Lemma 3.5 of [15]. Note that the generic types of $\mathcal{G}$ and C_a (over a) are orthogonal. Indeed, by strong minimality of $\mathcal{G}$ any nonorthogonality would be witnessed by (model-theoretic) algebraicity. But $\dim(\mathcal{G}) = \dim X_a > \dim C_a$. Let g, h be generic independent elements of $\mathcal{G}$ and let $x \in C_a$ be generic over $\{g, h\}$. Then $\tau_a(g, x) \in C_a$ is generic in C_a over x. Hence, each of g and h is independent from $\tau_a(g, x)$ over x by orthogonality. As $\mathcal{G}$ has a unique generic type over $\{x, \tau_a(g, x)\}$, we have $\tau_a(g, x) = \tau_a(h, x)$ and thus $\tau_a(g - h, x) = x$. Since $(g - h, x)$ is a generic of $X_a \times C_a$, we have shown that Γ_a^C agrees with the product of X_a with the diagonal in C_a^2 on a non-empty Zariski open subset of $X_a \times C_a^2$ — and hence everywhere by absolute irreducibility. □

In particular, there exists a point $p \in C_a$ such that for general $g \in X_a$, τ_g is defined on a Zariski open subset of X_a containing p, and $\tau_g(p) = p$. We fix from now on this point p.

The idea for the rest of the proof, loosely speaking, is as follows: translation by a general element of $\mathcal{G}$ is defined at and fixes p, and, hence, induces an automorphism of the nonstandard "nth infinitesimal neighbourhoods of p in X_a", for each n. For n sufficiently large, this action will separate points of $\mathcal{G}$. That is, we obtain a generic embedding of $\mathcal{G}$ into the automorphism group of the nth infinitesimal neighbourhood of p in X_a. Since the latter is a linear algebraic group over $\mathbb{C}'$, we obtain the desired conclusion. Since we do not have a natural interpretation of infinitesimal neighbourhoods in $\mathcal{A}'$, in order for this argument to make sense in $\mathcal{A}'$ we need to carry it out uniformly in the standard model.

First of all, we may assume (by taking a base extension if necessary) that $p = \rho(a)$, where $\rho\colon Y \to X$ is a holomorphic section to $f\colon X \to Y$. Let $\Delta^{(n)}_{X/Y} \subset X \times_Y X$ be as in Section 5.2; for $y \in Y$ and $x \in X_y$, the fibre of the first projection map $\Delta^{(n)}_{X/Y} \to X$ over x is canonically isomorphic to the nth infinitesimal neighbourhood of x in X_y. Let $D^{(n)}$ be the restriction of $\Delta^{(n)}_{X/Y}$ to $\rho(Y) \subseteq X$. So $D^{(n)}$ is a closed analytic subspace of $X \times_Y X$ whose support is $\rho(Y) \times_Y \rho(Y)$ and the induced surjective morphism $D^{(n)} \to Y$ is a finite map such that for $y \in Y$ the fibre $D^{(n)}_y$ of $D^{(n)} \to Y$ over y is the nth infinitesimal neighbourhood of $\rho(y)$ in X_y. Fact 5.1 applies and so for $W \subseteq Y$ a sufficiently small non-empty Zariski open subset, $\mathrm{Aut}_W(D^{(n)}_W) \to W$ is a definable group over W with compactification $\mathrm{Aut}^*_Y(D^{(n)}) \to Y$. Moreover, $\mathrm{Aut}_W(D^{(n)}_W)_a$ is definably isomorphic to a linear algebraic group over $\mathcal{A}'$.

Via the diagonal map on the second and third co-ordinates, we can and do identify $X \times_Y X \times_Y \times X$ with an irreducible Zariski closed subset of $X \times_Y (X \times_Y X) \times_Y (X \times_Y X)$. With this identification in mind, let

$$\Gamma^{[n]} := \Gamma \cap (X \times_Y D^{(n)} \times_Y D^{(n)}).$$

We mean here of course the sheaf-theoretic intersection. That is, $\Gamma^{[n]}$ is the (non-reduced) closed subspace of $X \times_Y D^{(n)} \times_Y D^{(n)}$ obtained as the inverse image of Γ under the closed embedding of $X \times_Y D^{(n)} \times_Y D^{(n)}$ in $X \times_Y (X \times_Y X) \times_Y (X \times_Y X)$. For each $y \in Y$ we view

$$\Gamma^{[n]}_y = \Gamma_y \cap \big(X_y \times D^{(n)}_y \times D^{(n)}_y\big)$$

as a family of analytic subspaces of $D^{(n)}_y \times D^{(n)}_y$ via the first projection $\Gamma^{[n]}_y \to X_y$.

Remark 5.4. At this point one could, after taking n sufficiently large, prove that $\Gamma^{[n]} \to X$ is a canonical family and then apply a theorem of Fujiki [6] to argue that the projection $\Gamma^{[n]} \to D^{(n)} \times_Y D^{(n)}$ is Moishezon. One could then deduce that $(\mathcal{G}, +)$ is definably isomorphic to an algebraic group. This would, however, be a rather perverse way to proceed as it would obscure the cause of algebraicity and not shorten the proof considerably. Instead we will prove the linear algebraicity of the general fibres of $X \to Y$ by constructing a meromorphic embedding over

Y, $\gamma_n : X \to \mathrm{Aut}^*_Y(D^{(n)})$ for n sufficiently large (as is done by Fujiki [5] for the absolute case).

We claim that for general $y \in Y$ and general $g \in X_y$, the fibre $\Gamma^{[n]}_y(g)$ of $\Gamma^{[n]}_y \to X_y$ over g is the graph of an automorphism of $D^{(n)}_y$. To make this precise, recall that the *graph* of a holomorphic map $\phi\colon A \to B$ between (possibly non-reduced) complex spaces is the fibre product of $\phi\colon A \to B$ and $\mathrm{id}_B\colon B \to B$. We use the following Lemma.

Lemma 5.5. *Suppose A and B are complex analytic spaces, $\Phi \subseteq A \times B$ is an analytic subspace, and $a \in A$ is such that Φ defines a holomorphic map ϕ on a neighbourhood of a. Let $b = \phi(a)$ and $\Phi^{[n]}_{(a,b)} := \Phi \cap (\Delta^{(n)}_{A,a} \times \Delta^{(n)}_{B,b})$. Then $\Phi^{[n]}_{(a,b)}$ is the graph of a holomorphic map from $\Delta^{(n)}_{A,a}$ to $\Delta^{(n)}_{B,b}$. Indeed, it is the graph of the restriction of ϕ to $\Delta^{(n)}_{A,a}$.*

Proof. Note that if V is an open neighbourhood of a in A then the nth infinitesimal neighbourhood of a in V coincides with the nth infinitesimal neighbourhood of a in A. That is, the entire question is local and we may assume that Φ defines a holomorphic map ϕ on all of A.

By functoriality of the infinitesimal neighbourhoods (section 1 of [8]), there is a unique map

$$\phi^{(n)}_{(a,b)}\colon \Delta^{(n)}_{A,a} \to \Delta^{(n)}_{B,b}$$

which commutes with the inclusions $\Delta^{(n)}_{A,a} \subseteq A$ and $\Delta^{(n)}_{B,b} \subseteq B$ via $\phi\colon A \to B$. So $\phi^{(n)}_{(a.b)}$ is the restriction of ϕ to $\Delta^{(n)}_{A,a}$. The graph of $\phi^{(n)}_{(a,b)}$ is the inverse image of the graph of ϕ under the closed embedding of $\Delta^{(n)}_{A,a} \times \Delta^{(n)}_{B,b}$ in $A \times B$. That is, the graph of $\phi^{(n)}_{(a,b)}$ is $\Phi^{[n]}_{(a,b)}$. □

Corollary 5.6. *For general $y \in Y$ and general $g \in X_y$, $\Gamma^{[n]}_y(g)$ is the graph of the automorphism of $D^{(n)}_y$ induced by τ_g.*

Proof. Recall that $\Gamma_y(g) \subseteq X^2_y$ is the graph of the meromorphic map $\tau_g\colon X_y \to X_y$. Moreover, τ_g is holomorphic on a Zariski-open subset of X_y containing $\rho(y)$, and $\tau_g(\rho(y)) = \rho(y)$.

By Lemma 5.5, $\big(\Gamma_y(g)\big)^{[n]}_{(\rho(y),\rho(y))}$ is the graph of $(\tau_g)^{(n)}_{(\rho(y),\rho(y))}$ which is the holomorphic map from $D^{(n)}_y = \Delta^{(n)}_{X_y,\rho(y)}$ to itself induced by τ_g. On the other hand, as τ_g agrees with translation by g on a non-empty Zariski open subset of X_y, τ_g has τ_{-g} as an inverse on a non-empty Zariski open set. Hence τ_g and τ_{-g} are inverses to each other everywhere where they define a holomorphic map — including at $\rho(y)$. By functoriality, $(\tau_g)^{(n)}_{(\rho(y),\rho(y))}$ and $(\tau_{-g})^{(n)}_{(\rho(y),\rho(y))}$ are inverses, and so $\big(\Gamma_y(g)\big)^{[n]}_{(\rho(y),\rho(y))}$ is the graph of an automorphism of $D^{(n)}_y$. Finally, observe that

$$\big(\Gamma_y(g)\big)^{[n]}_{(\rho(y),\rho(y))} = \Gamma_y(g) \cap (D^{(n)}_y \times D^{(n)}_y) = \Gamma^{[n]}_y(g).$$

□

We are in the following situation:

$$\Gamma^{[n]} \subseteq X \times_Y (D^{(n)} \times_Y D^{(n)})$$

defines a family of analytic subspaces of $D^{(n)} \times_Y D^{(n)}$ parameterised by X over Y. By the universal property of relative Douady spaces (cf. Section 5.1) there is a meromorphic map γ_n from X to the relative Douady space of $D^{(n)} \times_Y D^{(n)}$ over Y, which for general $y \in Y$ and general $g \in X_y$, takes g to the Douady point of $\Gamma_y^{[n]}(g)$. By Corollary 5.6, we have $\gamma_n \colon X \to \mathrm{Aut}^*_Y(D^{(n)})$. Moreover this map is generically a group homomorphism:

Lemma 5.7. *For general $y \in Y$ and general $g, h \in X_y$, $\gamma_n(g+h) = \gamma_n(g) \circ \gamma_n(h)$.*

Proof. As we have a generic action, we know that $\tau_{g+h} = \tau_g \circ \tau_h$ as meromorphic maps from X_y to itself. But by (the proof of) Corollary 5.6, $\gamma(g)$ is the automorphism $(\tau_g)^{(n)}_{(\rho(y),\rho(y))}$ of $D^{(n)}_y$ — and similarly for h and $g+h$. By functoriality of the infinitesimal neighbourhoods,

$$(\tau_g)^{(n)}_{(\rho(y),\rho(y))} \circ (\tau_h)^{(n)}_{(\rho(y),\rho(y))} = (\tau_{g+h})^{(n)}_{(\rho(y),\rho(y))}.$$

□

Lemma 5.8. *Suppose $y \in Y$ and $g, h \in X_y$ are very general. If $\gamma_n(g) = \gamma_n(h)$ for all $n \in \mathbb{N}$, then $g = h$.*

Proof. By 'very general' we will mean outside a countable union of proper Zariski closed sets. Choose y and g, h sufficiently general so as to ensure that for each n, $\gamma_n(g)$ and $\gamma_n(h)$ are (the Douady points of the graphs of) $(\tau_g)^{(n)}_{(\rho(y),\rho(y))}$ and $(\tau_h)^{(n)}_{(\rho(y),\rho(y))}$ respectively. So $(\tau_g)^{(n)}_{(\rho(y),\rho(y))} = (\tau_h)^{(n)}_{(\rho(y),\rho(y))}$ for all n. That is, for each n, the restrictions of τ_g and τ_h coincide on $D^{(n)}_y$, the nth infinitesimal neighbourhood of $\rho(y)$ in X_y. Recalling that $D^{(n)}_y$ is canonically isomorphic to $(\{\rho(y)\}, \mathcal{O}_{X_y,\rho(y)}/\mathfrak{m}^{n+1}_{X_y,\rho(y)})$, and that $\bigcap_n \mathfrak{m}^{n+1}_{X_y,\rho(y)} = 0$, we conclude that $\tau_g = \tau_h$ on some local open neighbourhood of $\rho(y)$. As they are meromorphic maps, this means that $\tau_g = \tau_h$ everywhere where they are defined. Since they agree with translation by g and h (respectively) on a non-empty Zariski open set, $g = h$. □

Corollary 5.9. *For N sufficiently large, $\gamma_N : X \to \mathrm{Aut}^*_Y(D^{(N)})$ is a bimeromorphism with its image.*

Proof. We will show that for some N, γ_N is injective off a countable union of proper Zariski closed sets (this will suffice). Moreover, it suffices to do this fibrewise over Y; so fix a sufficently general $y \in Y$. Now let $\Sigma(g, h)$ be the partial type that says g and h are outside the appropriate countable union of Zariski closed sets that makes Lemma 5.8 work, that $\gamma_n(g) = \gamma_n(h)$ for all $n \in \mathbb{N}$, and that $g \neq h$. By Lemma 5.8, this type is not realised. By ω_1-compactness, as Σ is countable, some finite fragment of Σ is not realised. Let N be the maximum of the n that appear in such a fragment. This N works (noting that if $m \leq n$ and $\gamma_n(g) = \gamma_n(h)$ then $\gamma_m(g) = \gamma_m(h)$). □

Passing to the elementary extension, Lemma 5.7 and Corollary 5.9 imply that $(\gamma_N)_a \colon X_a \to \mathrm{Aut}^*_Y(D^{(n)})_a$ is a definable map which is generically an injective homomorphism from $\mathcal{G}$ to $\mathrm{Aut}(D^{(n)}|_W/W)_a$. Using the Hrushovski-Weil theorem on group chunks in $\mathcal{A}'$, we see that this generic homomorphism extends to a definable group embedding of $\mathcal{G}$ into $\mathrm{Aut}(D^{(n)}|_W/W)_a$. By Fact 5.1, $\mathrm{Aut}(D^{(n)}|_W/W)_a$ is a linear algebraic group over $\mathbb{C}'$. We obtain:

Proposition 5.10. *Suppose that + does extend to a generic action of $\mathcal{G}$ on some absolutely irreducible component of $X_a \setminus U_a$. Then $\mathcal{G}$ is definably isomorphic to the multiplicative or additive group of $\mathbb{C}'$.* □

Propositions 4.3 and 5.10 prove Theorem 1.1. □

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E-mail address: `maschenb@math.uic.edu`

University of Illinois, Chicago, Department of Mathematics, Statistics, and Computer Science, 851 S. Morgan St. (M/C 249), Chicago, IL 60607-7045, USA

E-mail address: `rmoosa@math.uwaterloo.ca`

University of Waterloo, Department of Pure Mathematics, 200 University Avenue West, Waterloo, Ontario N2L 3G1, Canada

E-mail address: `scanlon@math.berkeley.edu`

University of California, Berkeley, Department of Mathematics, Evans Hall, Berkeley, CA 94720-3480, USA