

# A Generic Identification Theorem for $L^*$ -Groups of Finite Morley Rank

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## 1 Introduction

### 1.1 The Even Type Conjecture: Global Strategy

According to a long-standing and thoroughly unresolved conjecture in model theory due to Zilber and the last author, simple groups of finite Morley rank should be algebraic. The present paper outlines some of the last steps in a series of results which aim at the following more tractable part of this conjecture.

**Even Type Conjecture.** *Let  $G$  be a simple group of finite Morley rank of even type. Then  $G$  is a Chevalley group over an algebraically closed field of characteristic two.*

See [16] for an informal introduction to the subject, [1] for a recent survey of the classification programme, and [17] for general background on groups of finite Morley rank.

An infinite simple group  $G$  of finite Morley rank whose Sylow 2-subgroups are of bounded exponent is said to be of *even type* if its Sylow 2-subgroups are

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infinite, and of *degenerate type* if they are finite. According to the main conjecture, there should be no simple groups of finite Morley rank of degenerate type, but this is the most difficult instance of the conjecture, an analog of the Feit-Thompson Theorem. In attacking the Even Type Conjecture inductively, one encounters the difficulty that relevant sections may be of degenerate type. This can however be overcome, and therefore it is useful to consider a hypothetical minimal counterexample. This allows us to assume that every proper simple definable connected section of  $G$  is *either* a Chevalley group over an algebraically closed field of characteristic two, or a group of degenerate type. We adopt the terminology of [9] and say that  $G$  is an  $L^*$ -group in this case (when degenerate type simple sections are excluded, these groups have been called  $K^*$ -group in keeping with the finite group theoretic terminology).

The hypothesis of even type is less ad hoc than may appear, and is in fact part of a systematic case division. It has been shown in [9] that the Even Type Conjecture implies that in any counterexample to the general algebraicity conjecture, the Sylow 2-subgroups contain a divisible abelian group of finite index. In this case, if the Sylow 2-subgroup is infinite, we speak of groups of *odd type*. This last class has also been extensively investigated, by methods specific to that case.

In the present paper we are concerned with a recognition theorem which comes in to the final stages of the classification of the simple groups of even type. In the  $K^*$ -case, the analysis of groups of even type was undertaken in [2, 26, 3, 4, 7], and this material is being adapted to the  $L^*$ -case, see in particular [9, 10, 11, 12] for the first part of this adaptation, with more to come. Our recognition theorem has also been given in the  $K^*$ -case [15], and must be adapted to the  $L^*$  setting. We also enlarge on some earlier arguments that were given rather sketchily in earlier accounts, and correct some inaccuracies.

Much as in the case of the finite simple groups, our recognition theorem concerns only the “generic” case, corresponding to one of three cases in the global classification strategy for even type. Accordingly, before entering into the proof of the theorem, we describe that strategy and the role played by the present result.

With  $G$  a fixed simple  $L^*$ -group of even type and finite Morley rank, we set up the following notation. Let  $S$  be a 2-Sylow<sup>o</sup> subgroup of  $G$ , that is, the *connected component* of an ordinary Sylow 2-subgroup. Then  $N_G^o(S)$  is solvable by Proposition 1.10 below. Let  $\mathcal{M}$  be the collection of subgroups  $P \leq G$  satisfying the following conditions.

- $P$  is a 2-local<sup>o</sup> subgroup (that is,  $P$  has the form  $N_G^o(U)$  with  $U$  some non-trivial definable connected 2-subgroup),
- $N_G^o(S) < P$ ,
- $P$  is minimal with respect to these properties.

We will call a subgroup of  $G$  *minimal parabolic* if it belongs to the class  $\mathcal{M}$ . This is suggested by the analogy with algebraic groups. Indeed, when  $G$  is a

simple algebraic group over an algebraically closed field of characteristic two, the group  $N^\circ(S)$  is a Borel subgroup, and the subgroups belonging to  $\mathcal{M}$  are the *minimal parabolic subgroups* of  $G$  containing that Borel subgroup (cf. §2.1). However, for us the term is merely a synonym for membership in the class  $\mathcal{M}$ . More generally, one may define a *parabolic* subgroup, relative to  $N^\circ(S)$ , as a 2-local<sup>o</sup> subgroup containing  $N^\circ(S)$ .

One of the goals of the first phase of the analysis of simple  $L^*$ -groups of even type would be to control the structure of these groups, and in the  $K^*$ -case this was completed in [7]. It appears that similar results can be obtained in the  $L^*$ -case after an extensive reworking of a body of  $K^*$ -theory [6]. The structure of the minimal parabolic subgroups  $P$  is as follows. Writing  $O_2(P)$  for the largest connected definable 2-subgroup of  $P$ , which plays the role of the unipotent radical, we will have

$$P/O_2(P) = L * T$$

a central product with definable factors, in which  $L$  is of the form  $\mathrm{SL}_2(K)$  and  $T$  has degenerate type;  $T$  stands for *torus*, but we are a long way from claiming here that it actually is a torus.

A critical point is that the group  $G$  is generated by its minimal parabolic subgroups except when  $G$  is itself of type  $\mathrm{SL}_2(K)$ ; in this case  $\mathcal{M} = \emptyset$ , with our definitions. This is derived from a strong form of the  $C(G, T)$ -theorem which holds in  $L^*$ -groups of finite Morley rank, and is given in the  $K^*$ -case in [4]. Accordingly, in the second phase of the classification, as one turns to the problem of identification of the group in question, one has the following natural division of the problem into three cases, corresponding to groups of Lie rank one, Lie rank two, and higher Lie rank, respectively. Much as in the finite case, the methods used for identification vary widely according to the case in question.

**Thin Groups:**  $|\mathcal{M}| \leq 1$ ;

**Quasi-thin Groups:**  $G$  is generated by two groups belonging to  $\mathcal{M}$ ;

**Generic Groups:**  $|\mathcal{M}| \geq 3$ , and any two groups in  $\mathcal{M}$  generate a proper subgroup of  $G$ .

By a direct application of the  $C(G, T)$ -Theorem, whose  $L^*$  version will be given in [6], it follows that the thin groups are of type  $\mathrm{SL}_2(K)$ . The treatment of the quasithin groups uses the *amalgam method* in the manner of Delgado/Stellmacher. The method used has been summarized in [5], for the quasi-thin  $K^*$  case, which takes notes of certain issues peculiar to the case of finite Morley rank. This is a long argument, most of which is closely parallel to the finite case and has not been fully documented in the literature; it should appear in great detail in [6]. The result as anticipated is that these groups are isomorphic to a Lie rank two group over an algebraically closed field of characteristic two, that is, one of  $\mathrm{PSL}_3(K)$ ,  $\mathrm{PSp}_4(K)$ , or  $G_2(K)$ .

In [14], generic  $K^*$ -groups of even type were shown to be algebraic by constructing a  $BN$ -pair of Tits rank at least 3 and then applying the classification of

$BN$ -pairs of finite Morley rank given in [27]. This undeniably efficient approach has the drawback that in a sense it requires the classification to be done *twice*, as use of the underlying theory of buildings throws away most of the information gained along the way and then solves the same problem from scratch.

The aim of this paper is to give a direct identification theorem for generic  $L^*$ -groups which can be used to complete the proof of the Even Type Conjecture, as will be seen in detail in [6]. This theorem will be given in a self-contained form, together with two more concrete versions which depend on the existing body of knowledge concerning  $L^*$ -groups. Our discussion above serves to place these applications in context, and apart from the introduction of a certain amount of useful notation is not otherwise relevant.

The identification theorem will be based on the analysis of the centralizers of  $p$ -elements for odd primes  $p$ , see Theorems 1.1, 1.2 and 1.3 below. As we are dealing with groups expected to be of characteristic two type, this is a “semisimple” strategy, in keeping with the usual approach in finite group theory. We note however that the bulk of [6] adheres to a “unipotent” strategy, inspired by “third generation” finite group theory, and the switch to a semisimple strategy takes place just at the point where this identification theorem is invoked (and can even be avoided by heavy use of the theory of buildings).

Two points are worth emphasizing here. First, we offer a treatment of the generic case of the Even Type Conjecture which proceeds directly to the desired identification without going through the (admirable, but extensive) classification of buildings of spherical type in Tits rank at least three; we will indicate the point at which a detour into the theory of buildings would be possible. Secondly, our Generic Identification Theorem has a proof which is completely self-contained, but the applications depend on the  $C(G, T)$ -theorem in the  $L^*$ -case, Fact 1.4 below. This is actually the culmination of the first of the two phases of the analysis, and will be found in [6].

## 1.2 The Generic Identification Theorem

Our main theorem is an adaptation of [15] to the context of  $L^*$ -groups of even type. It concerns simple  $L^*$ -groups of even type (see §1.1.4 for general terminology). The genericity assumption in this theorem is expressed in terms of a maximal  $p$ -torus, that is a divisible abelian  $p$ -group, for  $p$  some suitable odd prime. In addition we use the following notation.

**Notation 1.1** *Let  $G$  be a group of finite Morley rank.*

- $O_2(G)$  is the largest normal unipotent 2-subgroup of  $G$ . This is sometimes called  $O_2^o(G)$  elsewhere in the literature.
- $U_2(G)$  is the subgroup of  $G$  generated by all unipotent 2-subgroups of  $G$ .

A word is in order concerning the ideology behind the consideration of  $U_2(G)$ . In a simple group  $G$  of even type,  $U_2(G)$  will be  $G$ . In a degenerate type group  $U_2(G) = 1$ . In the  $L$ -setting  $U_2(G)$  pulls out the “manageable” part of  $G$ , and

in fact if  $G$  is an  $L$ -group then  $U_2(G)$  is a  $K$ -group [9]. So a common theme when adapting  $K^*$ -group material to the wider  $L^*$  setting is the consideration of  $U_2(H)$  for various subgroups  $H$ , as appropriate.

After these preliminaries we can state the main result.

**Theorem 1.1 (Generic Identification)** *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type, and  $p$  an odd prime. Suppose that  $G$  contains a maximal  $p$ -torus  $D$  of Prüfer rank at least 3, and that relative to  $D$  the following generation and reductivity hypotheses are satisfied.*

$$(G) \langle U_2(C_G^\circ(x)) : x \in D, |x| = p \rangle = G,$$

(R) *For every element  $x$  of order  $p$  in  $D$ ,*

$$O_2(C_G^\circ(x)) = 1.$$

*Then  $G$  is a Chevalley group of Lie rank at least three over an algebraically closed field of characteristic two.*

Notice that the only Chevalley groups not covered by the conclusion of Theorem 1.1 are those of types  $A_1$ ,  $A_2$ ,  $B_2$ , and  $G_2$ .

### 1.3 Applications

We will formulate two increasingly concrete versions of the Generic Identification Theorem, the second of which puts the result in the form needed for the proof of the Even Type Conjecture. These make use of a certain body of material which requires adaptation from the  $K^*$  context to the  $L^*$  context. The additional facts needed will be stated (and proved, below) as auxiliary propositions.

All of this material is conditional on an appropriate version of the  $C(G, T)$ -theorem for  $L^*$ -groups of even type, a version which has in fact been proved but will appear only in the text [6], as it depends on an extensive body of material which has appeared in the literature in its  $K^*$ -formulation, only part of which has been published in its  $L^*$  form.

The result needed is analogous to the version given in [4, Theorem 3.5] in the context of  $K^*$ -groups. There the following definition was made.

**Definition 1.2** *Let  $G$  be a group of finite Morley rank, and  $S$  a Sylow<sup>o</sup> 2-subgroup. Then  $\hat{C}(G, S)$  is the subgroup of  $G$  generated by all groups of the form  $N^\circ(X)$ , with  $X$  varying over nontrivial unipotent subgroups of  $S$  which are normal in  $N^\circ(S)$ .*

This definition is borrowed from finite group theory, and the “ $C$ ” here stands for “characteristic”; in the finite case, one varies  $X$  over all characteristic subgroups of  $S$  (a Sylow 2-subgroup). In our finite Morley rank context, with  $G$  a simple  $L^*$ -group of even type, one can see that  $\hat{C}(G, S)$  is nothing but the group generated by parabolic subgroups of  $G$  containing  $N^\circ(S)$  (these would be

proper parabolic subgroups in the algebraic context, as by definition we consider only 2-local<sup>o</sup> subgroups).

Now the group  $\hat{C}(G, S)$  was called “ $C(G, S)$ ” in the  $K^*$ -context, but actually it would be better to define the latter as follows, and this is the tack taken in [6].

**Definition 1.3** *Let  $G$  be a group of finite Morley rank, and  $S$  a Sylow<sup>o</sup> 2-subgroup. Then  $C(G, S)$  is the subgroup of  $G$  generated by all groups of the form  $U_2(N^\circ(X))$ , with  $X$  varying over nontrivial unipotent subgroups of  $S$  which are normal in  $N^\circ(S)$ .*

This amounts to replacing parabolic subgroups by Levi factors in the generating set, with  $S$  itself thrown in for good measure. The two notions of “ $C(G, S)$ ” are not necessarily equivalent even in the  $K^*$ -case, but the distinction is relatively unimportant there because of the absence of degenerate sections. In fact, the distinction is never *very* important, as we shall see, but for ease of application in the  $L^*$  context the second is to preferred in the statement of the  $C(G, T)$ -theorem, which we now give.

**Fact 1.4** ([6]) *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type,  $S$  a 2-Sylow<sup>o</sup> subgroup of  $G$ . If  $C(G, S) < G$ , then  $G$  is of type  $SL_2$ .*

Note that the name of this theorem is taken from the context of finite group theory, where our  $S$  generally is called  $T$ .

We make some comments on the proof of this theorem. The proof for the  $K^*$  case in [4, Theorem 3.5] works equally well with our narrower definition of  $C(G, S)$ , thereby slightly strengthening the theorem. Furthermore, that proof goes over directly to the  $L^*$ -case once the body of material on which it relies is adapted to the  $L^*$  setting. This consists essentially of two points: a weak embedding theorem and the treatment of groups with “standard components” of type  $SL_2$ . The proof of the weak embedding theorem in the  $L^*$  context is found in [12], and deviates significantly from the  $K^*$  proof. The treatment of standard components of type  $SL_2$  was given in the  $K^*$  case in [3], and the treatment for the  $L^*$  case will be found in [6] in due course; again it deviates from the  $K^*$  case significantly, though not as much as the the proof of the weak embedding theorem. This particular point is the main one missing from the published literature at this stage, and the reader is advised to treat these results as conditional until that process is complete.

Another point is that for all *practical purposes* the two notions of  $C(G, S)$  are equivalent, in view of the following elementary lemma.

**Lemma 1.5** *Let  $G$  be a simple group of finite Morley rank of even type, and  $S$  a Sylow<sup>o</sup> 2-subgroup. Then  $\hat{C}(G, S) < G$  iff  $C(G, S) < G$ .*

**Proof.** We have  $C(G, S) \leq \hat{C}(G, S)$  and it suffices to show that  $\hat{C}(G, S) \leq N(C(G, S))$ .

The group  $\hat{C}(G, S)$  is generated by subgroups of the form  $N^\circ(X)$  where  $X \leq S$  is unipotent and  $X \triangleleft N^\circ(S)$ . Let  $H = U_2(N^\circ(X))$ . By a Frattini

argument we have  $N^\circ(X) \leq H \cdot N(S)$ . But  $H \leq C(G, S)$  by definition, and  $N(S)$  normalizes  $C(G, S)$ , so  $N^\circ(X)$  normalizes  $C(G, S)$ , and thus  $\hat{C}(G, S)$  normalizes  $C(G, S)$ .  $\square$

This is a point of purely academic interest, as the most straightforward approach to the subject (in finite Morley rank) is to ignore  $\hat{C}(G, S)$  and work directly with  $C(G, S)$ .

We can now state the first of our two concrete incarnations of the Generic Identification Theorem.

**Theorem 1.2** *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type. Assume for some odd prime  $p$  that  $G$  contains a  $p$ -torus of Prüfer rank at least 3 which normalizes a Sylow $^\circ$  2-subgroup of  $G$ . Then  $G$  is isomorphic to a Chevalley group over an algebraically closed field of characteristic two.*

This relies on the following two propositions.

**Proposition 1.6** ( *$p$ -Uniqueness Theorem*) *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type,  $S$  a 2-Sylow $^\circ$  subgroup of  $G$  and  $D$  a  $p$ -torus in  $G$  normalizing  $S$  with  $\text{pr}(D) \geq 2$ , where  $p$  is an odd prime. Then*

$$G = \langle U_2(C_G(x)) \mid x \in D, |x| = p \rangle.$$

For the second proposition, we introduce some additional notation.

**Definition 1.7** *Let  $H$  be a group of finite Morley rank. A 2-local $^\circ$  subgroup of  $H$  is a subgroup of the form  $N_G^\circ(U)$  with  $U$  a nontrivial unipotent 2-subgroup.*

**Proposition 1.8 (Reductivity of Centralizers)** *Let  $G$  be an  $L^*$ -group of finite Morley rank and even type with  $O_2(G) = 1$ , and  $p$  an odd prime. Let  $D$  be a  $p$ -torus in  $G$  of Prüfer rank at least 3, normalizing a Sylow $^\circ$  2-subgroup of  $G$ . Then  $O_2(C_G(x)) = 1$  for every element  $x \in D$  of order  $p$ .*

Using Theorem 1.2, one can then treat the generic case of the Even Type Conjecture.

**Theorem 1.3** *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type. Let  $S$  be a 2-Sylow $^\circ$  subgroup of  $G$ , and  $\mathcal{M}$  the set of minimal 2-local $^\circ$  subgroups which contain  $N_G^\circ(S)$  as a proper subgroup. Let  $|\mathcal{M}| \geq 3$ , and assume that*

$$\langle P_1, P_2 \rangle < G$$

*for any two subgroups  $P_1, P_2 \in \mathcal{M}$ . Then  $G$  is a Chevalley group over an algebraically closed field of characteristic two.*

This depends mainly on the following analog of a theorem proved by Niles in the finite case [28].

**Fact 1.9 (Niles' Theorem [14])** *Let  $G$  be a group of finite Morley rank and even type, and  $S$  a Sylow $^\circ$  2-subgroup of  $G$ . Assume that  $G$  contains a set of definable connected subgroups  $P_1, \dots, P_n$  which satisfy the following conditions.*

1.  $G = \langle P_1, \dots, P_n \rangle$ .
2. Each of the groups  $P_i$  contains  $N_G^\circ(S)$ , and  $N_G^\circ(S)$  is solvable.
3. If  $L_i := U_2(P_i)$  then  $\bar{L}_i = L_i/O_2(L_i) \simeq \mathrm{SL}_2(F_i)$ , where  $F_i$  is an algebraically closed field of characteristic 2.
4. If  $L_{ij} := \langle L_i, L_j \rangle$  then  $\bar{L}_{ij} = L_{ij}/O_2^\circ(L_{ij})$  is one of the following groups:  $(P)\mathrm{SL}_3(F_{ij})$ ,  $\mathrm{SL}_2(F_{ij}) \times \mathrm{SL}_2(F'_{ij})$ ,  $\mathrm{Sp}_4(F_{ij})$ ,  $G_2(F_{ij})$ , where  $F_{ij}$  and  $F'_{ij}$  are algebraically closed fields of characteristic two.

*Then  $G_0 := \langle L_1, \dots, L_n \rangle$  is normal in  $G$ , and has a definable spherical BN-pair of Tits rank  $n$ .*

The hypothesis of solvability on  $N^\circ(S)$  turns out to present some difficulties when we apply this theorem in the  $L^*$  context. It is possible to prove the same result without that hypothesis (cutting down  $N^\circ(S)$  to a suitable solvable subgroup) but as the next result suggests, we will be able to apply the theorem in its stated form.

**Proposition 1.10** *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type,  $S$  a 2-Sylow $^\circ$  subgroup of  $G$ . Then  $N_G^\circ(S)$  is solvable.*

## 1.4 Definitions

A general source for notation and terminology is [17]. This is also the source for various elementary remarks made without reference in the remainder of this subsection. We write  $N^\circ(H)$  for  $N_G(H)^\circ$ ,  $C^\circ(H)$  for  $C_G(H)^\circ$ , and so forth.

**Definition 1.11** *Let  $G$  be a group of finite Morley rank.*

- A subgroup  $H$  of  $G$  is unipotent if it is definable, connected, nilpotent, and of bounded exponent. In practice, we only deal with unipotent 2-groups here.
- $G$  is a  $K$ -group if every infinite connected simple definable section of  $G$  is a Chevalley group over an algebraically closed field.
- $G$  is a  $K^*$ -group if every proper connected definable simple section of  $G$  is a Chevalley group over an algebraically closed field.
- A Sylow $^\circ$  2-subgroup of  $G$  is the connected component of a maximal 2-subgroup. The notation ‘‘Sylow $^\circ$ ’’ may be pronounced: ‘‘connected Sylow’’ (this does not mean that the full Sylow is connected!).
- $G$  is of even type if its Sylow $^\circ$  2-subgroup is nontrivial and unipotent.



- $G$  is an  $L$ -group if every connected simple definable section of  $G$  which is of even type is a Chevalley group over an algebraically closed field.
- $G$  is an  $L^*$ -group if every proper connected definable simple section of  $G$  which is of even type is a Chevalley group over an algebraically closed field.
- A group  $H$  of even type is said to be reductive if  $H$  is connected and  $O_2(H) = 1$ .

The only  $L^*$ -groups we consider will be themselves groups of even type. Thus their definable sections are either again of even type, or have trivial Sylow<sup>o</sup> 2-subgroups. The latter are called groups of *degenerate* type.

**Definition 1.12** *Let  $p$  be a prime.*

- A  $p$ -torus  $T$  is a divisible abelian  $p$ -group.
- An abelian divisible  $p$ -group in which every proper subgroup is cyclic is called quasicyclic; such a group is isomorphic to the group of  $p^{\text{th}}$ -power roots of unity in the complex numbers.
- If a  $p$ -torus contains finitely many elements of order  $p$  then it is a direct sum of finitely many quasicyclic subgroups. The number of summands is called the Prüfer  $p$ -rank of  $S$  and is denoted  $\text{pr}(S)$ . It coincides with the dimension of the subgroup  $T[p] = \{t \in T : pt = 0\}$  (additive notation) over  $\mathbb{F}_p$ . This applies in particular when  $T$  is a subgroup of a group of finite Morley rank.

**Notation 1.13** *Let  $H$  be a group of finite Morley rank.*

- $O(H)$  is the maximal normal connected definable subgroup of  $H$  without involutions.
- $O^*(H)$  is the maximal normal connected definable subgroup of  $H$  of degenerate type.

One can define reductivity more broadly, but we are interested only in the even type case, where the operator  $O_2$  supplies a clear notion of unipotent radical. The reductivity hypothesis of the Generic Identification Theorem says that  $C^\circ(x)$  is reductive for each element  $x \in D$  of order  $p$ .

## 2 Background Material

### 2.1 Algebraic Groups

General background on the structure of linear algebraic groups is found in the text [24], and an overview may be found in [29]. We require the connection

with root systems, which is fundamental to the recognition process, and we will make free use of it.

A connected algebraic group  $G$  is usually called *simple* if it has no proper normal *connected and closed* subgroups. Such a group turns out to be quasisimple with finite center, as an abstract group. The classical classification theorem for simple algebraic groups states that simple algebraic groups over algebraically closed fields are Chevalley groups, that is, groups constructed from Chevalley bases in simple complex Lie algebras as described, for example, in [20].

Now fix a maximal torus  $T$  in a connected simple algebraic group  $G$  and denote the corresponding root system by  $\Phi$ . For each  $\alpha \in \Phi$ , denote the corresponding root subgroup by  $X_\alpha$ . The subgroup  $\langle X_\alpha, X_{-\alpha} \rangle$  is known to be isomorphic to  $SL_2$  or  $PSL_2$  (which we indicate briefly by “ $(P)SL_2$ ”) and is called a *root  $SL_2$ -subgroup*. We will tend to drop the notation  $(P)SL_2$  in our main work, as we will be working in characteristic two where the two abstract groups are in any case isomorphic.

Fixing a Borel subgroup  $B$  in  $G$  containing  $T$  corresponds to fixing a set of positive roots  $\Phi^+$  in  $\Phi$ , and a parabolic subgroup containing  $B$  properly corresponds to a subset of  $\Phi$  containing  $\Phi^+$  properly. A minimal parabolic subgroup corresponds to a subset of  $\Phi$  consisting of  $\Phi^+$  together with one negative root. Such a subgroup is a product of  $B$  and the corresponding root  $PSL_2$ -subgroup (see [29]). If  $P$  is a minimal parabolic subgroup and the characteristic of the underlying field is  $p > 0$ , then  $O'_p(P/O_p(P)) \simeq SL_2$ . If  $P$  is a parabolic subgroup containing exactly two proper parabolic subgroups, then  $O'_p(P/O_p(P))$  is a semisimple algebraic group of Lie rank 2; that is either simple or a central product of two copies of  $(P)SL_2$ .

Every parabolic subgroup other than  $G$  itself has nontrivial unipotent radical, and is connected. In our more abstract setting, we build these properties in to the definition of parabolic subgroup.

We mention a few miscellaneous points that are frequently useful.

**Fact 2.1** *Let  $G$  be a quasisimple algebraic group in an expanded language, and suppose that  $G$  has finite Morley rank.*

1. *Any connected definable group of automorphisms of  $G$  induces inner automorphisms on  $G$ .*
2. *Any semisimple element of  $G$  has a reductive centralizer (i.e., the connected component of the centralizer is reductive).*

The first point is found in [17], and the second is a purely algebraic fact, cf. [31, 3.19].

## 2.2 The Curtis-Tits Theorem

The Curtis-Tits theorem may be expressed as follows: a simply connected quasisimple algebraic group is the free amalgam of the system of subgroups and

inclusion maps corresponding to all root  $\mathrm{SL}_2$  subgroups and subgroups generated by pairs of such subgroups, taken relative to a fixed maximal torus. The classical form of the result is somewhat weaker, as noted in [34], but a proof of this version in the finite case, valid in general, is in [23].

Note that the Dynkin diagram can be construed as giving information about the structure of the groups generated by pairs of root  $\mathrm{SL}_2$  subgroups, which captures the local information in the system of groups referred to above. Indeed, a stronger form of the Curtis-Tits theorem, proved by Timmesfeld in [34], says that the group in question is determined, not just by the full system of groups and subgroups, but by the collection of subsystems corresponding to pairs of roots. One approach would be to derive this from the Curtis-Tits theorem by patching together a family of local isomorphisms, adjusting them so as to match on their overlap (a root  $\mathrm{SL}_2$ -subgroup, or trivial). This is possible only because the Dynkin diagrams are simply connected, and in any case requires attention. To illustrate the point, we note that in odd or zero characteristic groups of types  $B_3$  and  $C_3$  have the same local data, from a certain point of view, and are nonisomorphic. However if one formulates the notion of local data carefully (bearing in mind the labelling of root  $\mathrm{SL}_2$ -subgroups by the roots), this “counterexample” disappears. In fact Timmesfeld proves the stronger result directly and derives the Curtis-Tits theorem from it.

Timmesfeld’s theorem goes as follows. It makes use of his notion of *rank one* subgroup, a considerable generalization of the Lie rank one twisted Chevalley groups, discussed at length in the first chapter of his book [33]. In our application these rank one groups will only be of the form  $(\mathrm{P})\mathrm{SL}_2$ . In this context, the notion of *unipotent subgroup* is an abstract one, coinciding with the usual notion in the context of a Chevalley group.

**Fact 2.2 ([34])** *Let  $\Phi$  be an irreducible spherical root system of Tits rank at least 3, with fundamental system  $\Pi$  and Dynkin diagram  $\Delta$ . Let  $G$  be any group generated by rank one groups  $X_r = \langle A_r, A_{-r} \rangle$  for  $r \in \Pi$ , with unipotent subgroups  $A_r, A_{-r}$  satisfying the condition*

$$N_{X_r}(A_r) \cap N_{X_r}(A_{-r}) \leq N(X_s)$$

for all  $r, s \in \Pi$ . Set  $X_{rs} = \langle X_r, X_s \rangle$  for  $r, s \in \Pi$  distinct, and assume the following all hold.

1.  $X_r, X_s$  commute for  $r, s$  not connected in  $\Delta$ .
2. If  $r, s$  are connected in  $\Delta$ , then there is a group  $\bar{X} = \bar{X}_{rs}$  of Lie type with root system  $\bar{\Phi}_{rs}$  (the span of  $r, s$  in  $\Phi$ ), which is generated by subgroups  $\bar{A}_\alpha$  for  $\alpha \in \bar{\Phi}_{rs}$ , and there is a surjective homomorphism  $\phi_{rs} : X_{rs} \rightarrow \bar{X}_{rs}$ , such that:
  - (a)  $\phi_{rs}[A_\alpha] = \bar{A}_\alpha$  for  $\alpha \in \bar{\Phi}_{rs}$ ;
  - (b)  $\ker \phi_{rs} \leq Z(X_{rs})$ ;
  - (c) If  $\bar{X}_{rs}$  is defined over a field of order 2 or 3, or is of the form  $\mathrm{PSL}_3(4)$ , then  $\ker \phi_{rs}$  is a  $2'$ -group or a  $3'$ -group respectively.

Then there is a group  $\bar{G}$  of Lie type  $\mathcal{B}$ , with root system  $\Phi$  and with fundamental system  $\Pi$ , and there is a surjective homomorphism  $\sigma : G \rightarrow \bar{G}$  mapping the groups  $A_{\pm r}$  for  $r \in \Pi$  onto the corresponding fundamental root groups and their opposites in  $\bar{G}$ . Furthermore,  $\ker \sigma \leq Z(G) \cap H$ , where  $H$  is the subgroup generated by the groups  $H_r = N_{X_r}(A_r) \cap N_{X_r}(A_{-r})$  for  $r \in \Pi$ .

The following case is the one which concerns us here.

**Proposition 2.3** *Let  $\Phi$  be an irreducible root system (of spherical type) and rank at least 3, and let  $\Pi$  be a system of fundamental roots for  $\Phi$ . Let  $X$  a group generated by subgroups  $X_r$  for  $r \in \Pi$ , Set  $X_{rs} = \langle X_r, X_s \rangle$ . Suppose that  $X_{rs}$  is a group of Lie type  $\Phi_{rs}$  over an infinite field, with  $X_r$  and  $X_s$  corresponding root  $\mathrm{SL}_2$ -subgroups with respect to some maximal torus of  $X_{rs}$ . Then  $X/Z(X)$  is isomorphic to a group of Lie type via a map carrying the subgroups  $X_r$  to root  $\mathrm{SL}_2$ -subgroups.*

Note that if  $X$  is, in addition, a group of finite Morley rank, then it follows from the theory of central extensions [8] that  $X$  is itself a Chevalley group.

## 2.3 $L$ -groups and signalizer functors

**Notation 2.4** *Let  $H$  be a group of finite Morley rank.*

- A quasisimple component of  $H$  is a quasisimple subnormal subgroup of  $H$ . If  $H$  is connected then its quasisimple components are connected, and are normal in  $H$ .
- $E(H)$  is the subgroup of  $H$  generated by its connected quasisimple components. Note that  $E(H) = E(H^\circ)$ .

It is not hard to see that a group  $H$  of finite Morley rank has finitely many quasisimple components, and that  $E(H)$  is the central product of the connected quasisimple components of  $H$ . In a  $K$ -group  $H$ , the quasisimple components of  $E(H)$  are algebraic (this requires the central extension theory of [8]), but in an  $L$ -group  $E(H)$  may also have some factors of degenerate type, in principle.

**Lemma 2.5** *Let  $H$  be an  $L$ -group of even type and finite Morley rank. Then  $U_2(H)$  is a  $K$ -group. If, in addition, the group  $H$  is reductive, then*

$$U_2(H) \triangleleft E(H) \text{ and } H = U_2(H)O^*(H)$$

**Proof.** By [9, 3.4.1],  $U_2(H)$  is a  $K$ -group. Assume now that  $H$  is reductive. By [9, 3.7]  $U_2(H) = E(U_2(H))$ , so  $U_2(H) \triangleleft E(H)$ . The factors of  $E(U_2(H))$  are algebraic and hence  $H = U_2(H) * C_H(U_2(H))$ , with intersection central in  $U_2(H)$  and thus finite. So  $C_H(U_2(H)) = O^*(H)$ .  $\square$

**Lemma 2.6** *Let  $H$  be an  $L$ -group of even type, and  $x \in H$  an element of odd order. Then  $O_2(C_H(x)) \leq O_2(H)$ .*

**Proof.** Since  $O_2(C_H(x)) \leq U_2(H)$  and the latter is a  $K$ -group, we may suppose that  $H = U_2(H)$ , and in particular  $H$  is a  $K$ -group. Consider  $\bar{H} = H/O_2(H)$ . By [2, Prop. 2.43], the centralizer of  $\bar{x}$  in  $\bar{H}$  is covered by  $C_H(x)$ . Hence the image of  $O_2(C_H(x))$  is contained in  $O_2(C_{\bar{H}}(\bar{x}))$ , and it suffices to prove that the latter is trivial. In other words, we may suppose  $O_2(H) = 1$ .

In this case, by the preceding lemma, since  $H = U_2(H)$  we have  $H = E(H)$  is a central product of quasisimple algebraic groups over algebraically closed fields of characteristic two. Hence we easily reduce to the case in which  $H$  is itself a quasisimple algebraic group.

Now  $x$  is a semisimple element of  $H$ , so its centralizer is reductive (in the algebraic sense) and thus  $O_2(C(x)) = 1$ .  $\square$

The foregoing lemma is often referred to as a “balance” property, in the parlance of finite simple group theory. It is exploited via *signalizer functor theory*. We recall the definition in the form given in [18].

**Definition 2.7** *Let  $G$  be a group of finite Morley rank, let  $p$  be a prime, and let  $E \leq G$  be an elementary abelian  $p$ -group. An  $E$ -signalizer functor in  $G$  is a family  $\{\theta(x)\}_{x \in E^\#}$  of definable  $p^\perp$ -subgroups of  $G$  satisfying:*

1.  $\theta(x)^g = \theta(x^g)$  for all  $x \in E^*$  and  $g \in G$ .
2.  $\theta(x) \cap C_G(y) \leq \theta(y)$  for any  $x, y \in E^\#$ .

We rephrase the foregoing lemma in this language.

**Corollary 2.8** *Let  $G$  be a simple  $L^*$ -group of even type,  $p$  an odd prime, and  $E \leq H$  an elementary abelian  $p$ -group. Then the function  $\theta$  defined on  $E^\#$  by*

$$\theta(x) = O_2(C_G(x))$$

*is an  $E$ -signalizer functor in  $G$ .*

**Proof.** We have to check the balance condition

$$O_2(C_G(x)) \cap C_G(y) \leq O_2(C(y)) \text{ for any } x, y \in E^\#$$

Indeed,  $O_2(C_G(x)) \cap C_G(y) \leq O_2(C_{C_G(y)}(x)) \leq O_2(C_G(y))$  by the lemma.  $\square$

This is applied via the nilpotent signalizer functor theorem, stated in [16] in the finite Morley rank case (with a discussion tailored to the tame odd type case), with a detailed proof given in the appendix to [18]. This concerns signalizer functors which are connected and nilpotent (that is, they take their values among connected, nilpotent, and of course definable subgroups of  $G$ —including the trivial subgroup, possibly). The main result runs as follows.

**Fact 2.9** *Let  $G$  be a group of finite Morley rank,  $p$  a prime, and  $E \leq G$  a finite elementary abelian  $p$ -group of  $p$ -rank at least 3. Then the group  $\theta(E)$  defined as*

$$\langle \theta(x) : x \in E^\# \rangle$$

is a nilpotent  $p^\perp$ -group, and

$$\theta(x) = C_{\theta(E)}(x)$$

for all  $x \in E^\#$ .

Of course, under the hypotheses of our Corollary, the function  $O_2(C(x))$  is a connected nilpotent signalizer functor, and since  $\theta(E)$  is then a nilpotent group generated by unipotent 2-subgroups, the group  $\theta(E)$  is itself a unipotent 2-group.

## 2.4 Tate modules

There is a duality theory correlating  $p$ -tori of finite Prüfer rank with free  $\mathbb{Z}_p$ -modules of finite rank, where  $\mathbb{Z}_p$  is the ring of  $p$ -adic integers, called *Tate modules* with reference to their use in connection with the study of Galois actions on the torsion points of elliptic curves.

If  $T$  is a  $p$ -torus of finite Prüfer rank, written additively, and  $T_i = \Omega_i(T)$  is the subgroup defined by

$$p^i x = 0$$

then  $T$  is the direct limit  $\varinjlim T_i$  with respect to inclusions. There is also an inverse system  $(T_i)$  with connecting maps  $T_i \rightarrow T_j$  given by multiplication by  $p^{i-j}$ , and the corresponding inverse limit  $\hat{T} = \varprojlim T_i$  is called the *Tate module* associated with  $T$ . The process is reversible: given a free  $\mathbb{Z}_p$ -module  $M$  of finite rank one considers the quotients  $M_i = M/p^i M$ , and there is a natural embedding  $M_i \rightarrow M_j$  for  $i \leq j$  induced by multiplication by  $p^{i-j}$ , so that  $\hat{M} = \varprojlim M_i$  may be defined.

In particular  $T$  and  $\hat{T}$  have the same endomorphism ring, and, in particular, when  $T$  is embedded in a larger ambient group  $G$  then the Tate module  $\hat{T}$  affords a representation of the group  $N_G(T)/C_G(T)$  of automorphisms of  $T$  induced by  $G$ .

See also [14, §3.3].

## 3 Complex reflection groups

Our identification theorem for simple algebraic groups will make use of two identification theorems for Coxeter groups, one based on the classification of complex reflection groups, and the other due to Goldschmidt and incorporated into the proof of our version of Niles' theorem (as well as the original version). In this section, we give a detailed account of the first of these identification theorems, combining [13] and [19].

A linear transformation on a finite dimensional vector space is a (generalized) *reflection* if it is diagonalizable and has a fixed space of codimension exactly one. A *real* or *ordinary* reflection is a complex reflection of order two. Note that the identity is not considered to be a reflection.

The finite groups generated by reflections were originally classified by Shephard and Todd [30], and their numbering is referred to as the *Shephard-Todd* numbering. The table at the end of this section gives some of the properties of “sporadic” finite irreducible complex reflection groups in dimension at least two, organized according to the following scheme: *Shephard-Todd number; dimension of the representation; Coxeter label (if applicable); group order; order of the center; orders of reflections*, where the last item refers to the orders of the complex reflections occurring in the group. In groups defined over the real field these reflections must have order 2. There are also three infinite families: the first contains the standard representation of the symmetric group (Coxeter type  $A_n$ ), the third consists of dihedral groups acting in dimension 2 and the second is a series  $G(m, l, n)$  to which we will return below.

It will be observed that four of the groups listed are crystallographic Coxeter groups associated with exceptional Dynkin diagrams. Other than that, the most interesting group is probably the one with number 12, which crops up in various contexts such as singularity theory.

Series #2 in the Shephard-Todd classification is a family of groups denoted  $G(m, l, n)$ , where  $n$  is the dimension of the associated vector space, and  $m, l$  are parameters with  $l$  a divisor of  $m$ , which for  $m = 2$  correspond to the Coxeter groups  $B_n$  (or  $C_n$ ) and  $D_n$ . The groups  $G(m, l, n)$  may be described explicitly as follows [21, p. 386]. Let  $A(m, l, n)$  be the group of diagonal matrices  $D$  for which  $D^m = 1$  and  $\det(D)^{m/l} = 1$ . Then  $G(m, l, n)$  is the semidirect product  $A(m, l, n) \rtimes \Pi_n$  with  $\Pi_n$  the group of permutation matrices.

We use the foregoing information to derive a criterion for a finite group to be isomorphic to an irreducible Coxeter group. A very similar statement was given in [13], but the full proof of this important tool has not appeared previously.

**Proposition 3.1** *Let  $W$  be a finite group,  $I \subseteq W$  a subset, and  $n$  an integer, satisfying the following conditions.*

1. *The set  $I$  generates  $W$ , consists of involutions, and is closed under conjugation in  $W$ ;*
2. *The graph  $\Delta_I$  with vertices  $I$  and edges  $(i, j)$  for noncommuting pairs  $i, j \in I$  is connected;*
3. *For all sufficiently large prime numbers  $\ell$ ,  $W$  has a faithful representation  $V_\ell$  over the finite field  $\mathbb{F}_\ell$  in which the elements of  $I$  operate as complex reflections, with no common fixed vectors.*

*Then one of the following occurs.*

- (a)  *$W$  is a dihedral group acting in dimension  $n = 2$ , or cyclic of order two.*
- (b)  *$W$  is isomorphic to an irreducible crystallographic Coxeter group, that is,  $A_n, B_n, C_n, D_n$  ( $n \geq 3$ ),  $E_n$  ( $n = 6, 7, \text{ or } 8$ ), or  $F_4$  ( $n = 4$ ),*
- (c)  *$W$  is a semidirect product of a quaternion group of order 8 with the symmetric group  $\text{Sym}_3$ , acting naturally, represented in dimension 2.*

If, in addition, over some field,  $W$  has an irreducible representation of dimension at least 3, in which the elements of  $I$  act as reflections, then case (b) applies.

**Proof.** Note that as  $W$  is generated by finitely many reflections, the dimensions of the representations  $V_\ell$  are bounded. Let  $V$  be a nonprincipal ultraproduct of these representations, which is a representation of  $W$  over the field  $F$  obtained as the corresponding ultraproduct of the finite fields  $\mathbb{F}_\ell$ . Then the field  $F$  has characteristic zero and cardinality  $2^{\aleph_0}$ , and can be identified with a subfield of the complex field  $\mathbb{C}$ . Let  $\tilde{V}$  be the complexification of  $V$ ; we consider  $W$  with its complex representation  $\tilde{V}$ .

Then  $V$  and  $\tilde{V}$  are finite dimensional as well, over their respective fields, and the elements of  $I$  operate as (ordinary) reflections on  $V$  and hence on  $\tilde{V}$ . We claim that the action of  $W$  on  $\tilde{V}$  is irreducible. The action is completely reducible since  $W$  is finite and the characteristic is zero. If  $\tilde{V}$  is reducible then it factors as  $V_1 \oplus V_2$  with  $V_1, V_2$  nontrivial invariant subspaces. Then setting  $I_i = \{w \in I : [w, \tilde{v}] \leq V_i\}$ , it follows that  $(I_1, I_2)$  is a partition of  $I$  into commuting subsets, one of which must be empty. So we may suppose  $[I, \tilde{V}] \subseteq V_1$ , so  $[I, V] < V$ ; as  $V$  is an ultraproduct this yields  $[I, V_\ell] < V_\ell$  for infinitely many  $\ell$ , a contradiction.

We remark that the same argument shows that for  $\ell$  not dividing the order of  $W$ , if the elements of  $I$  act as complex reflections on a vector space over  $\mathbb{F}_\ell$  and have no common fixed vectors there, then the representation in question is irreducible.

Now returning to our complex representation, the classification of the irreducible complex reflection groups applies. Leaving aside the Coxeter groups, we have to deal with the groups numbered 4–27 or 29–34, as well as those of the form  $G(m, l, n)$  with  $m > 2$ .

By a slight variation of Schur's lemma, we claim that the center of  $W$  acts via scalar matrices in every representation  $V_0$  in which the generating set  $I$  acts via reflections. Take  $z \in Z(W)$  and take  $i \in I$ . Then  $z$  preserves the one-dimensional space  $[i, V_0]$  and hence has an eigenvalue  $\alpha$  on this space. The  $\alpha$ -eigenspace for  $z$  is  $W$ -invariant and hence equal to  $V$ .

Accordingly, the order of the center of  $W$  divides  $\ell - 1$  for all sufficiently large primes  $\ell$ . By Dirichlet's theorem, there are arbitrarily large primes congruent to  $-1$  modulo  $|Z(W)|$ , and hence  $|Z(W)|$  divides 2. But after leaving aside the crystallographic Coxeter groups,  $|Z(W)| > 2$  with the exception of the groups numbered

$$4, 12, 23, 24, 30, 33$$

in the table following. As the last column in the table shows, group #4 contains no ordinary reflections, and may be excluded. Group #12 is referred to in case (c).

We claim that  $W$  cannot occur twice on our list. If  $W \simeq G(m, \ell, n)$ , then in any representation over  $\mathbb{C}$ ,  $A(m, \ell, n)$  is diagonalizable and its eigenspaces are permuted by  $W$ , so the representation is imprimitive. But the individually listed groups are primitive. So there is no overlap between the family  $G(m, \ell, n)$



and the groups listed. As the Fitting subgroup of  $G(m, \ell, n)$  is  $A(m, \ell, n)$ , it is easy to recover both  $m$  and  $n$  from the group  $G(m, \ell, n)$ ; so any group  $G(m, \ell, n)$  occurs at most once. The remaining groups on our list are of distinct orders. So the dimension  $n$  of the representation  $V$  is independent of the nonprincipal ultrafilter chosen, and hence all but finitely many of the representations  $V_\ell$  have dimension  $n$ .

For the groups numbered 23, 24, 30, 33 one works with the order, which must divide the order of  $\mathrm{GL}_n(\ell)$  for almost all primes  $\ell$ . We use the fact that the orders shown are divisible by the values 5, 7,  $5^2$ , and  $3^4$  respectively, in dimensions 3, 4, 5, 5 respectively. For example in case 33 we may take  $\ell$  congruent to 2 mod  $3^4$ , so that  $|\mathrm{GL}_5(\ell)|$  is congruent to  $2^{10}(2^5 - 1)(2^4 - 1)(2^3 - 1)(2^2 - 1)(2 - 1)$ , and the only factors divisible by 3 here are  $2^4 - 1, 2^2 - 1$  giving a factor of  $3^2$  but not  $3^4$ , a contradiction.

It remains to consider the groups  $G(m, l, n)$  with  $m > 2$ . We will work with particular elements of  $G(m, l, n)$ . Let  $\zeta$  be a primitive  $m$ -th root of unity and let  $D_1, D_2$  be the following diagonal matrices, considered as elements of  $W$ :

$$\mathrm{diag}(\zeta, \zeta^{-1}, \dots); \quad \mathrm{diag}(\zeta, \zeta, \zeta^{-2}, \dots)$$

where diagonal entries not shown all equal 1. The coefficients are not necessarily in the base field  $F$ ; this is the representation after complexification. However the traces  $\tau_1 = \zeta + \zeta^{-1}$  and  $\tau_2 = 2\zeta + \zeta^{-2}$  are in the base field, and as this is an ultraproduct, with respect to whatever ultrafilter we like, it follows that we have similar elements  $\tau_1, \tau_2$  in any field prime  $\mathbb{F}_\ell$  with  $\ell$  sufficiently large; that is, there is a primitive  $m^{\mathrm{th}}$  root of unity  $\zeta_\ell$  in an extension of  $\mathbb{F}_\ell$  for which the corresponding formulas hold.

Now one finds that  $(\tau_1 - 2)\zeta = \tau_2 - \tau_1^2 + 1$ , and over  $\mathbb{F}_\ell$  this implies that either  $\tau_1 = 2$  or  $\zeta \in \mathbb{F}_\ell$ . But when  $\tau_1 = 2$  the equation  $\zeta + \zeta^{-1} = 2$  yields  $\zeta = 1$ , and hence in any case  $\zeta \in \mathbb{F}_\ell$ . This means that  $m$  divides  $\ell - 1$  for almost all  $\ell$ , and hence  $m \leq 2$ , which corresponds to a Coxeter group.

This exhausts the treatment of all cases and proves that one of cases (a-c) occurs.

Turning to the final point, if  $W$  has a faithful representation in which the elements of  $I$  act as reflections, in dimension  $d \geq 3$ , then it is certainly not dihedral. As far as the group listed as #12 is concerned (case (c)), this is generated by three reflections and hence has no suitable representation in dimension 4 or more. In dimension 3, since the commutator subgroup of  $W$  is the extension of a quaternion group  $Q$  by a cyclic group of order 3, and the center of  $Q$  is central in  $W$ , we find first that the central involution of  $Q$  is scalar, and secondly that it has no square root in  $\mathrm{SL}_3$ , hence none in  $Q$ , and this is a contradiction.  $\square$

## 4 Proof of Theorem 1.1

We recall the result to be proved.

Number	Dim.	Name	$ W $	$ Z(W) $	$ r $ (possible)
4	2	#4	$2^3 * 3$	2	[ 3 ]
5	2	#5	$2^3 * 3^2$	6	[ 3 ]
6	2	#6	$2^4 * 3$	4	[ 2, 3 ]
7	2	#7	$2^4 * 3^2$	12	[ 2, 3 ]
8	2	#8	$2^5 * 3$	4	[ 4 ]
9	2	#9	$2^6 * 3$	8	[ 2, 4 ]
10	2	#10	$2^5 * 3^2$	12	[ 3, 4 ]
11	2	#11	$2^6 * 3^2$	24	[ 2, 3, 4 ]
12	2	#12	$2^4 * 3$	2	[ 2 ]
13	2	#13	$2^5 * 3$	4	[ 2 ]
14	2	#14	$2^4 * 3^2$	6	[ 2, 3 ]
15	2	#15	$2^5 * 3^2$	12	[ 2, 3 ]
16	2	#16	$2^3 * 3 * 5^2$	10	[ 5 ]
17	2	#17	$2^4 * 3 * 5^2$	20	[ 2, 5 ]
18	2	#18	$2^3 * 3^2 * 5^2$	30	[ 3, 5 ]
19	2	#19	$2^4 * 3^2 * 5^2$	60	[ 2, 3, 5 ]
20	2	#20	$2^3 * 3^2 * 5$	6	[ 3 ]
21	2	#21	$2^4 * 3^2 * 5$	12	[ 2, 3 ]
22	2	#22	$2^4 * 3 * 5$	4	[ 2 ]
23	3	$H_3$	$2^3 * 3 * 5$	2	[ 2 ]
24	3	#24	$2^4 * 3 * 7$	2	[ 2 ]
25	3	#25	$2^3 * 3^4$	3	[ 3 ]
26	3	#26	$2^4 * 3^4$	6	[ 2, 3 ]
27	3	#27	$2^4 * 3^3 * 5$	6	[ 2 ]
28	4	$F_4$	$2^7 * 3^2$	2	[ 2 ]
29	4	#29	$2^9 * 3 * 5$	4	[ 2 ]
30	4	$H_4$	$2^6 * 3^2 * 5^2$	2	[ 2 ]
31	4	#31	$2^{10} * 3^2 * 5$	4	[ 2 ]
32	4	#32	$2^7 * 3^5 * 5$	6	[ 3 ]
33	5	#33	$2^7 * 3^4 * 5$	2	[ 2 ]
34	6	#34	$2^9 * 3^7 * 5 * 7$	6	[ 2 ]
35	6	$E_6$	$2^7 * 3^4 * 5$	1	[ 2 ]
36	7	$E_7$	$2^{10} * 3^4 * 5 * 7$	2	[ 2 ]
37	8	$E_8$	$2^{14} * 3^5 * 5^2 * 7$	2	[ 2 ]

Table 1: Sporadic complex reflection groups

**Theorem 1.1** *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type, and  $p$  an odd prime. Suppose that  $G$  contains a maximal  $p$ -torus  $D$  of Prüfer rank at least 3, and that relative to  $D$  the following generation and reductivity hypotheses are satisfied.*

(G)  $\langle U_2(C_G^\circ(x)) : x \in D, |x| = p \rangle = G$ ,

(R) For every element  $x$  of order  $p$  in  $D$ ,

$$O_2(C_G^\circ(x)) = 1.$$

Then  $G$  is a Chevalley group of Lie rank at least three over an algebraically closed field of characteristic two.

The proof will follow the line of [15]. We retain the hypotheses and notation of this theorem throughout the present section.

Let  $\Sigma$  be the set of all definable subgroups of  $G$  isomorphic to  $\mathrm{SL}_2$  (since we work in even type, we do not need to distinguish  $\mathrm{SL}_2$  and  $\mathrm{PSL}_2$ ), and normalized by  $D$ . We will refer to these (optimistically) as “root  $\mathrm{SL}_2$ -subgroups” for  $G$ . We aim to show that with a suitable labelling, these root  $\mathrm{SL}_2$ -subgroups will satisfy the hypotheses of Proposition 2.3.

**Lemma 4.1**  $G$  is generated by the groups  $L$  for  $L \in \Sigma$ .

**Proof.** Let  $G_0 = \langle L : L \in \Sigma \rangle$ . Let  $\hat{D} = C_G(D)$ . We make no special claim about the structure of the group  $\hat{D}$ .

We have by hypothesis  $G = \langle U_2(C^\circ(x)) : x \in D \text{ of order } p \rangle$ . We claim for  $x \in D$  of order  $p$  we have the following.

$$(*) \quad U_2(C^\circ(x)) \leq G_0 \hat{D}$$

This follows from our reductivity hypothesis on  $C^\circ(x)$ . We have  $U_2(C^\circ(x)) = E(U_2(C^\circ(x)))$ . It is easy to see that  $D$  acts on each quasisimple component of  $U_2(C^\circ(x))$  like the  $p$ -torsion in a maximal torus. As Chevalley groups are generated by root  $\mathrm{SL}_2$ -subgroups relative to a maximal torus, it follows that  $E(U_2(C^\circ(x))) \leq G_0$ . So  $(*)$  holds.

Now applying our generation hypothesis we conclude that  $G = G_0 \hat{D}$ . On the other hand  $\hat{D}$  normalizes  $G_0$  and hence  $G_0 \triangleleft G$ ,  $G_0 = G$ , as claimed.  $\square$

The next point is to get some control over the subgroups generated by pairs of root  $\mathrm{SL}_2$ -subgroups, using the inductive (i.e.,  $L^*$ ) hypothesis and the assumption that the Prüfer rank of  $D$  is large. The key point here is that our “root  $\mathrm{SL}_2$ -subgroups” actually do turn out to be conventional root  $\mathrm{SL}_2$ -subgroups of certain definable subgroups of  $G$ .

**Lemma 4.2** For  $L \in \Sigma$ ,  $L = U_2(C_G(C_D(L)))$ . In particular, if  $L$  is contained in a  $D$ -invariant definable subgroup  $H$  of  $G$  which is itself isomorphic to a quasisimple algebraic group, then under this isomorphism  $L$  corresponds to a root  $\mathrm{SL}_2$ -subgroup with respect to  $C_H(D)$ .

**Proof.** Set  $L^\perp = C_D(L)$ . This is nontrivial, and evidently  $L \leq U_2(C_G(L^\perp))$ . Set  $\hat{L} = U_2(C_G(L^\perp))$ .

Fix  $x \in L^\perp$  of order  $p$  and let  $H = C_G^\circ(x)$ . By hypothesis  $O_2(H) = 1$ . Now applying Lemma 2.6 repeatedly, we find that  $O_2(C_G(L^\perp)) = 1$  and hence

$O_2(\hat{L}) = 1$ . It follows that  $\hat{L}$  is a central product of quasisimple algebraic groups. Now if  $D$  has Prüfer rank  $n$ , then the Prüfer rank of  $L^\perp$  is  $n - 1$ . As  $D \leq \hat{L} \cdot L^\perp$  and  $D$  is a maximal  $p$ -torus, the Prüfer rank of  $\hat{L}$  is 1 and  $\hat{L}$  is of type  $SL_2$  as well. As  $L \leq \hat{L}$  it follows easily that  $L = \hat{L}$ .

The second statement follows easily from the first, taking into account the structure of quasisimple algebraic groups, since  $C_H(C_C(L))$  is easily seen to be (or rather, to correspond to) a Zariski closed subgroup.  $\square$

**Lemma 4.3** *For  $K, L \in \Sigma$ , if  $K$  and  $L$  do not commute then the subgroup  $\langle K, L \rangle$  in  $G$  is a Lie rank two Chevalley group.*

Note that if  $K$  and  $L$  do commute there is no reason, at this stage, to suppose that they are over the same field.

The proof here goes as in [15], and we just sketch it. Let  $H = \langle K, L \rangle$ . One shows first that  $C_D(H) > 1$ , using the fact that  $D$  has Prüfer rank at least three and acts on  $K$  and  $L$  by inner automorphisms; for the latter, consider the action of the definable closure of  $D$ .

So for some  $x \in D$  of order  $p$ , we have  $H \leq U_2(C(x)) = U_2(E(C(x)))$ . Here  $U_2(C(x))$  is  $D$ -invariant and thus its quasisimple components are  $D$ -invariant. As  $K$  and  $L$  do not commute it follows that they lie in the same quasisimple component and the preceding lemma can be invoked; they can be viewed as root  $SL_2$ -subgroups with respect to the same maximal torus.

At this stage one may easily derive the following, which is convenient though not essential for the argument.

**Lemma 4.4** *The base fields of the groups  $L \in \Sigma$  are definably isomorphic.*

For noncommuting pairs this is a consequence of the preceding lemma. The general case follows from this, as otherwise  $\Sigma$  would split into pairwise commuting subfamilies and  $G$  would acquire nontrivial proper definable normal subgroups.

We have everything we need to apply Proposition 2.3 apart from a suitable labelling of  $\Sigma$  by a root system, and this is in essence the problem of constructing and identifying the Weyl group associated to  $G$ , which is nontrivial. The criterion for this is provided by Proposition 3.1.

According to that criterion, it suffices to show the following points, where the group in question is  $W_0$ ,  $I$  is the distinguished set of involutions generating  $W_0$ , and  $n$  is the Prüfer  $p$ -rank of  $T$

1. The set  $I$  is closed under conjugation in  $W_0$ .
2. The graph  $\Delta_I$  on the vertex set  $I$  in which edges correspond to noncommuting pairs of involutions is connected.
3. For all sufficiently large prime numbers  $\ell$ ,  $W_0$  has a faithful irreducible representation over  $\mathbb{F}_\ell$  in which the elements of  $I$  act as generalized reflections.

4. In the action of  $W_0$  on  $D$ , the elements of  $I$  act as reflections of order two, and have no common fixed points.
5.  $W$  has an irreducible representation of dimension at least three over some field.
6.  $W_0$  is finite.

Now  $W_0$  acts on the set  $\Sigma$  of distinguished “root  $SL_2$ ” subgroups, hence preserves  $I$ . In view of the structure of the groups  $\langle K, L \rangle$  for  $K, L \in \Sigma$ , if  $K$  and  $L$  do not commute then  $w_K$  and  $w_L$  do not commute, so the graph  $\Delta_I$  is connected. This disposes of the first two points. For the rest, we must examine the action of  $W_0$  on  $T$ , and specifically on the subgroup  $T_\ell = T[\ell]$  consisting of the torsion of exponent  $\ell$ . We claim that all of these  $W_0$ -modules are faithful and irreducible, with the generators  $r_L$  acting as reflections of order two. As the Prüfer  $p$ -rank is at least three, the module  $T_p$  is at least three dimensional over  $\mathbb{F}_p$ . Furthermore, as these representations are finite, if they are faithful then  $W_0$  is finite. So this will suffice.

As far as the action of  $r_L$  on  $T_p$  is concerned, we  $T = T_L C_T(L)$  and thus  $r_L$  acts as a reflection of order two.

For the irreducibility, since the representations are generated by reflections and the graph  $\Delta$  is connected, it suffices to check that the  $r_L$  have no common centralizer in  $T$ . But an element of  $T$  which centralizes  $r_L$  must centralize  $L$  and hence  $C_T(W_0)$  centralizes the subgroup generated by all  $L \in \Sigma$ , which is  $G$  (Lemma 4.1).

So it remains only to check that these representations are faithful. Let  $N = N_G(T)$ . We claim that more generally the action of  $N/C_G(T)$  on each  $T_\ell$  is faithful (for  $\ell$  odd and not equal to the characteristic of the base field), or in other words that  $C_N(T_\ell)$  centralizes  $T$ .

So consider  $x \in N(T)$  centralizing  $T_\ell$  for some prime  $\ell$ . Then  $x$  acts on the set  $\Sigma$ . If  $L \in \Sigma$  then  $L \cap L^x$  contains a  $T_p \cap L$ . If  $L \neq L^x$  then  $|L \cap L^x| \leq Z(L)$  has order at most two, a contradiction. So for each  $L \in \Sigma$ ,  $x$  acts on  $L$  and centralizes  $T_p \cap L$ . As  $x$  normalizes  $T \cap L$  and acts as an inner automorphism of  $L$ , it either inverts or centralizes  $T \cap L$ ; since it centralizes  $T_p \cap L$ ,  $x$  centralizes  $T_L$ . Since this holds for all  $L$ ,  $x$  centralizes  $T$ .

So we have

$W_0$  is a crystallographic Coxeter group

By the proof of that result, the generators  $r_L$  correspond to reflections in  $W_0$  (that is, elements of  $W_0$  which act as reflections in the usual real representation of  $W_0$ ). We claim

All reflections of  $W_0$  are of the form  $r_L$  ( $L \in \Sigma$ )

Since the set of generators  $r_L$  is closed under conjugation, and since reflections corresponding to roots of fixed length are conjugate, there are only two possibilities: either the reflections  $r_L$  exhaust all reflections in  $W_0$ , or else there are two root lengths, and the  $r_L$  vary over roots of one length. But in the latter

case the group generated by the  $r_L$  is associated to the root system consisting of roots of that fixed length, and is not the group  $W_0$ . So this proves our claim.

Now the group  $W_0$  largely determines the associated Dynkin diagram, apart from an indication of root lengths. So let  $I_0$  be the Dynkin diagram without the root length information, correlated with a set  $r_i$  ( $i \in I_0$ ) of reflections generating  $W_0$ . Here  $r_i = r_{L_i}$  for some  $L_i \in \Sigma_0$ .

We claim the group  $L_i$  ( $i \in I_0$ ) generate  $G$ . Let  $L \in \Sigma$ . We claim that  $\langle L_i : i \in I_0 \rangle$  contains  $L$ . Since  $r_L$  is conjugate under the action of  $W_0$ , and  $W_0$  is generated by the  $r_i$ , with  $r_i \in L_i$ , we may suppose that  $r_L = r_i$  for some  $i$ . On the other hand if  $L$  and  $L_i$  are distinct, we have already determined the structure of  $\langle L, L_i \rangle$ , a Chevalley group of Lie rank two, and if  $L \neq L_i$  it follows that  $r \neq r_i$ . So we have  $L = L_i$  in this case, and our claim holds.

At this point we may apply Proposition 2.3.

## 5 Theorem 1.2

We will deal first with the auxiliary issues which come into the proof of Theorem 1.2.

### 5.1 $p$ -Uniqueness

We begin with two generation lemmas. The first is a consequence of [18, 3.4,3.7].

**Fact 5.1** *Let  $H$  be a solvable  $p^\perp$ -group of finite Morley rank. Let  $E$  be a finite elementary abelian  $p$ -group of rank at least two acting definably on  $H$ . Then  $H = \langle C_H^\circ(x) : x \in E^\# \rangle$ .*

**Lemma 5.2** *Let  $H$  be a connected  $K$ -group of finite Morley rank and of even type with  $H = U_2(H)$ , and  $E$  an elementary abelian  $p$ -group with  $m(E) \geq 2$ . Suppose that  $E$  is contained in a  $p$ -torus  $D$  which acts definably on  $H$  (that is, as a subgroup of a definable group of automorphisms of  $H$ ). Then*

$$H = \langle U_2(C_H(x)) : x \in E^\# \rangle$$

**Proof.**

Let  $H_0 = \langle U_2(C_H(x)) : x \in E^\# \rangle$ . We have

$$O_2(H) = \langle C_{O_2(H)}^\circ(x) : x \in E^\# \rangle$$

by Fact 5.1. Thus  $O_2(H) \leq H_0$ . As in the proof of Lemma 2.6, we may therefore reduce to the case  $O_2(H) = 1$ . Then  $H$  is a product of simple algebraic groups over algebraically closed fields.

Now we invoke the  $D$ -invariance. The definable closure of  $D$  in its ambient group is a connected group and hence normalizes each quasisimple component of  $H$ , and acts by inner automorphisms on  $H$  [17]. We may assume the action is faithful as our claim trivializes otherwise. Thus it suffices to treat the case in which  $H$  is a quasisimple algebraic group and  $D$  is a subgroup of  $H$ , which is

then contained in a maximal torus of  $H$ . Now considering the action of  $D$  on the unipotent radical of each Borel subgroup containing  $D$ , applying Fact 5.1 we find that these unipotent radicals all lie in  $H_0$ . But it is easy to see that these groups generate  $H$  (in fact it suffices to consider one Borel subgroup containing  $D$  together with the opposite Borel subgroup).  $\square$

The following extension to the  $L^*$  case, which we call the  $p$ -Uniqueness Theorem, is conditional on the  $C(G, T)$ -theorem.

**Proposition 1.6 ( $p$ -Uniqueness)** *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type,  $S$  a 2-Sylow<sup>o</sup> subgroup of  $G$ , and  $D$  a  $p$ -torus in  $G$  normalizing  $S$  of Prüfer rank at least two, where  $p$  is an odd prime. Then*

$$G = \langle U_2(C_G(x)) : x \in D, x \text{ of order } p \rangle$$

**Proof.** In view of the condition on the Prüfer rank of  $D$ ,  $G$  cannot be of the form  $\mathrm{SL}_2$ . Invoking the  $C(G, T)$ -theorem we conclude

$$G = C(G, S)$$

It will suffice to show that

$$C(G, S) \leq \langle U_2(C_G(x)) : x \in D, x \text{ of order } p \rangle$$

Fix  $X \leq S$  unipotent and normal in  $N^\circ(S)$ , and let  $H = U_2(N^\circ(X))$ , a  $K$ -group. By our generation lemma we have

$$H = \langle U_2(C_H(x)) : x \in D, x \text{ of order } p \rangle$$

and our claim follows.  $\square$

This result can be generalized—it is not necessary to assume that  $E$  is contained in a  $p$ -torus—but the proof is more complicated and involves reduction to the case just treated [19].

## 5.2 Reductivity of centralizers

The next result depends on the  $C(G, T)$ -theorem via the  $p$ -Uniqueness Theorem.

**Proposition 1.8 (Reductivity of Centralizers)** *Let  $G$  be an  $L^*$ -group of finite Morley rank and even type with  $O_2(G) = 1$ , and  $p$  an odd prime. Let  $D$  be a  $p$ -torus in  $G$  of Prüfer rank at least 3, normalizing a Sylow<sup>o</sup> 2-subgroup of  $G$ . Then  $O_2(C_G(x)) = 1$  for every element  $x \in D$  of order  $p$ .*

**Proof.** Let  $E \leq \Omega_1(D)$ , be an elementary abelian  $p$ -group of rank 3. For  $a \in E^\#$  let  $\theta(a) = O_2(C_G(a))$ . By Corollary 2.8,  $\theta$  is a nilpotent  $E$ -signalizer functor on  $G$ . Let  $Q = \theta(E)$ , that is

$$\langle \theta(x) : x \in E^\# \rangle$$

Then by Fact 2.9  $Q$  is nilpotent and is therefore a unipotent 2-subgroup of  $G$ .

We will show that  $U_2(C(x)) \leq N(Q)$  for all  $x \in E^\#$ , and apply the  $p$ -Uniqueness Theorem.

We claim that for  $A \leq E$  elementary abelian of rank two, we have  $N(A) \leq N(Q)$ . Indeed,  $Q = \langle C_Q^\circ(a) : a \in A^\# \rangle$  by Fact 5.1, so  $N(A)$  normalizes  $Q$ .

Now for  $x \in E^\#$ , choose  $E_0 \leq E$  elementary abelian of rank two and not containing  $x$ . Let  $H = U_2(C(x))$ . Then by Lemma 5.2 we have

$$H = \langle C_H(y) : y \in E_0^\# \rangle \leq \langle C_G(\langle x, y \rangle) : y \in E_0^\# \rangle$$

and  $C_G(\langle x, y \rangle) \leq N(Q)$  since each such group  $\langle x, y \rangle$  is elementary abelian of rank two.

So  $U_2(C(x)) \leq N(Q)$  for all  $x \in E^\#$ , and by the  $p$ -Uniqueness Theorem we have  $N(Q) = G$ . Since  $O_2(G) = 1$  we conclude  $Q = 1$ .  $\square$

### 5.3 Proof of Theorem 1.2

**Theorem 1.2** *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type. Assume for some odd prime  $p$  that  $G$  contains a  $p$ -torus of Prüfer rank at least 3 which normalizes a Sylow $^\circ 2$ -subgroup of  $G$ . Then  $G$  is isomorphic to a Chevalley group over an algebraically closed field of characteristic two.*

**Proof.** Let  $D$  be a maximal  $p$ -torus of Prüfer rank at least 3 contained in  $N_G^\circ(S)$  for some 2-Sylow $^\circ$  subgroup  $S$  in  $G$ . Then the hypotheses  $(G, R)$  of Theorem 1.1 are satisfied, by Propositions 1.6 and 1.8, respectively.  $\square$

## 6 Theorem 1.3

### 6.1 Solvability of $N^\circ(S)$

The next result occurs as a hypothesis in the version of Niles' Theorem given in [15] and quoted above. It presents no difficulty in the  $K^*$  context, but does require attention in the  $L^*$  context, being a point where degenerate sections could intervene strongly, in principle. This result also depends on the  $C(G, T)$ -Theorem for the  $L^*$  context.

The following notion will be useful.

**Definition 6.1** *Let  $G$  be a group of finite Morley rank. Let  $O_{K^\perp}(G)$  be the smallest connected normal definable subgroup  $H$  of  $G$  such that  $G/H$  is a  $K$ -group. It is easy to see that this group exists, and is unique.*

Before our main result we insert a small lemma which generally goes unremarked in connection with operators of this type, but deserves mention here.

**Lemma 6.2** *Let  $G$  be a group of finite Morley rank, and  $H$  a definable subgroup. The operator  $O_{K^\perp}$  is idempotent, that is  $O_{K^\perp}(O_{K^\perp}(G)) = O_{K^\perp}(G)$ . In particular, if  $O_{K^\perp}(G) \leq H$  then  $O_{K^\perp}(H) = O_{K^\perp}(G)$ .*



**Proof.** The first point is an expression of the fact that the class of  $K$ -groups is closed under extension (short exact sequences).

For the second, if  $H \leq G$  then  $O_{K^\perp}(H) \leq O_{K^\perp}(G)$ , and applying the same principle to the inclusion  $O_{K^\perp}(G) \leq H$  yields the reverse inclusion.  $\square$

**Proposition 1.10** *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type,  $S$  a 2-Sylow<sup>o</sup> subgroup of  $G$ . Then  $N_G^\circ(S)$  is solvable.*

**Proof.** It suffices to show that  $B$  is a  $K$ -group, since  $B/S$  is a group of degenerate type, and hence solvable in that case. In other words, we aim to show that  $O_{K^\perp}(B) = 1$ , and to show this, we will argue that  $C(G, S) \leq N((O_{K^\perp}(B)))$ , and invoke the  $C(G, T)$ -Theorem.

Fix  $X \leq S$  unipotent and normal in  $B$ , and let  $H = N^\circ(X)$ . We must show that  $U_2(H)$  normalizes  $O_{K^\perp}(B)$ . In fact we claim that

$$(*) \quad O_{K^\perp}(B) = O_{K^\perp}(H)$$

and hence  $H$  itself normalizes this group.

Let  $\bar{H} = H/O_2(H)$ . Then  $\bar{H} = E(\bar{H}) = \bar{H}_D * \bar{H}_K$  where  $\bar{H}_D$  is the product of the quasisimple components of degenerate type, and  $\bar{H}_K$  is the product of the algebraic quasisimple components. Let  $H_D, H_K$  be the preimage of  $\bar{H}_D$  and  $\bar{H}_K$ , respectively, in  $H$ . Then  $S \leq H_K$  and it follows easily that  $H_D$  normalizes  $S$ , that is  $H_D \leq B$ .

At the same time clearly  $O_{K^\perp}(H) \leq H_D$ , so  $O_{K^\perp}(H) \leq B$ . Hence  $O_{K^\perp}(B) = O_{K^\perp}(H)$  by the preceding lemma, and in particular  $H$  normalizes  $O_{K^\perp}(B)$ . Varying  $X$ , it follows that  $C(G, S)$  normalizes  $O_{K^\perp}(B)$  and thus by the  $C(G, T)$ -theorem either  $O_{K^\perp}(B) = 1$ , as claimed, or else  $G$  is of type  $SL_2$  in characteristic two, and  $B$  is a Borel subgroup in this case.  $\square$

Taking into account the structure of solvable groups of even type, we may phrase this result as follows.

**Corollary 6.3** *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type,  $S$  a 2-Sylow<sup>o</sup> subgroup of  $G$ . Then  $N_G^\circ(S)$  is a Borel subgroup of  $G$ .*

For this reason, the group  $N^\circ(S)$  is often called a *standard Borel subgroup* of  $G$  (allowing for the existence of other maximal connected solvable definable subgroups, also called Borel subgroups).

## 6.2 Proof of Theorem 1.3

The following is conditional on the  $C(G, T)$  theorem for the  $L^*$  case, which it depends on both directly and via a reduction to Theorem 1.2.

**Theorem 1.3** *Let  $G$  be a simple  $L^*$ -group of finite Morley rank and even type. Let  $S$  be a 2-Sylow<sup>o</sup> subgroup of  $G$ , and  $\mathcal{M}$  the set of minimal 2-local<sup>o</sup> subgroups which contain  $N_G^\circ(S)$  as a proper subgroup. Let  $|\mathcal{M}| \geq 3$ , and assume that*

$$\langle P_1, P_2 \rangle < G$$

for any two subgroups  $P_1, P_2 \in \mathcal{M}$ . Then  $G$  is a Chevalley group over an algebraically closed field of characteristic two.

We will retain the notation and hypotheses of this theorem through the end of this section.

The first part of our analysis aims at showing that Niles' Theorem in the form of Fact 1.9 is applicable.

Recall that  $\mathcal{M}$  consists of all definable subgroups  $P \leq G$  satisfying the following conditions.

- $P$  is a 2-local<sup>o</sup> subgroup ( $P = N_G^o(U)$  with  $U$  2-unipotent);
- $N_G^o(S) < P$ ;
- $P$  is minimal with respect to these properties.

The following lemma has much the same content as the  $C(G, T)$ -Theorem, and will be derived from the latter.

**Lemma 6.4** *The group  $G$  is generated by the groups  $U_2(P)$  for  $P \in \mathcal{M}$ .*

**Proof.** We set

$$G_0 = \langle U_2(P) : P \in \mathcal{M} \rangle$$

and we invoke the  $C(G, T)$ -Theorem. So it suffices to show that for any non-trivial unipotent subgroup  $X$  of  $S$  which is normal in  $N^o(S)$ , the group  $H = U_2(N^o(X))$  is contained in  $G_0$ . We remark that  $S \leq G_0$  in any case, simply because  $\mathcal{M}$  is nonempty.

Let  $\hat{X} = O_2(H)$  and  $\hat{H} = N^o(Q)$ . Then  $X \leq \hat{X} \leq H \leq \hat{H}$ . Furthermore  $O_2(\hat{H}) \leq S \leq H$  and hence  $O_2(\hat{H}) \leq \hat{H}$ . So  $O_2(\hat{H}) = \hat{X}$ . Replacing  $H$  and  $X$  by  $\hat{H}$  and  $\hat{X}$ , we may therefore suppose that  $X = O_2(H)$ .

The group  $H$  may be solvable, in which case  $U_2(H) = S \leq G_0$  and we have nothing to prove. Assume therefore that it is nonsolvable.

Now  $U_2(H)/X$  is a central product of quasisimple algebraic groups over algebraically closed fields of characteristic two, and  $N^o(S)/X$  is the central product of Borel subgroups there. So  $U_2(H)$  is generated by subgroups  $P$  containing  $N^o(S)$  such that  $P/X$  is a minimal parabolic subgroup in one of the quasisimple components of  $U_2(H)/X$ .

Let  $Y = O_2(P)$  and consider  $\hat{P} = N^o(Y)$ . Notice that  $P \leq U_2(\hat{P})$ , and so it will suffice to show that  $\hat{P} \in \mathcal{M}$ .

Arguing as above it follows that  $O_2(\hat{P}) = Y$ , and thus  $\hat{P}$  is a 2-local<sup>o</sup> subgroup of  $G$ . Since  $N^o(S) < \hat{P}$ , it remains to check only that  $\hat{P}$  is minimal among 2-local<sup>o</sup> subgroups properly containing  $N^o(S)$ .

Now  $U_2(\hat{P})/Y$  is a central product of quasisimple algebraic groups over algebraically closed fields of characteristic two, and contains  $P/Y$ . Here  $P/Y$  contains a Sylow<sup>o</sup> 2-subgroup of  $\hat{P}$ . Thus  $P = U_2(\hat{P})$ . By a Frattini argument  $\hat{P} = P \cdot N^o(S)$ . As  $P/Y$  is a group of type  $SL_2$ , there is no group intermediate between  $N^o(S)$  and  $\hat{P}$ .

Thus  $\hat{P} \in \mathcal{M}$  and  $P \leq G_0$ , and we conclude. □

**Notation 6.5** *As the groups  $U_2(P)$  for  $P \in \mathcal{M}$  are connected and generate  $G$ , finitely many of them suffice. Let  $P_1, \dots, P_n$  be any subset of  $\mathcal{M}$  which generates  $G$ . By hypothesis,  $n \geq 3$ . Set  $L_i = U_2(P_i)$ .*

We have now obtained clauses (1, 2) from the hypotheses of Niles' Theorem. Clause (3) is a consequence of (and equivalent to) the minimality hypothesis on members of  $\mathcal{M}$ . This is not immediately clear, but the argument can be extracted from the proof of the last lemma. The fourth and last clause depends mainly on the hypothesis that  $n > 2$ , so that the subgroup  $L_{ij} = \langle L_i, L_j \rangle$  is always proper. Then as  $L_{ij} = U_2(L_{ij})$ , this group is a  $K$ -group. Thus  $\bar{L}_{ij} = L_{ij}/O_2(L_{ij})$  is a central product of quasisimple algebraic groups, and if  $L_i$  and  $L_j$  do not commute then they lie in a single quasisimple component of  $\bar{L}_{ij}$ , and thus  $\bar{L}_{ij}$  is itself a quasisimple algebraic group generated by two minimal parabolic subgroups, as one sees easily by decoding the definition of  $\mathcal{M}$  inside  $\bar{L}_{ij}$ .

Thus all the hypotheses of Niles' Theorem are satisfied, and the group  $G$  has a definable spherical  $(B, N)$ -pair of Tits rank  $n$ .

Now as noted in [14] one could call on the classification of buildings of spherical type and Tits rank at least three at this point, and pass to the desired conclusion directly, using [27]. But Theorem 1.2 offers a "low-cost" alternative, which we now pursue.

We must return to the proof of Fact 1.9 as given in [28] and examine the  $(B, N)$ -pair actually constructed there. As a point of notation, we remark that the operator  $O^{2'}$  as defined in that article is the operator we denote by  $U_2$  here.

**Notation 6.6** *Fix a complement  $H$  to  $S$  in  $N^\circ(S)$ , which exists as  $N^\circ(S)$  is solvable. Let  $B_i = N_{P_i}(S)$  and  $N_i = N_{L_i}(H)$ . let  $B = \langle B_i : 1 \leq i \leq n \rangle$  and  $N = \langle N_i : 1 \leq i \leq n \rangle$ .*

Of course, these groups  $B$  and  $N$  are shown to furnish a  $(B, N)$  pair for the group generated by the  $P_i$  (in our current case, the full group  $G$ ). In particular the Weyl group  $W$  is  $B/(B \cap N)$ . The group  $B \cap N$  is shown to be  $H \times C_S(H)$  in the course of the proof of [28, 3.2]. We will need to pull out a subgroup of  $B \cap N$  whose structure we can control.

**Notation 6.7** *Let  $H_i = H \cap L_i$ , and let  $T = \langle H_1, \dots, H_n \rangle$ .*

Observe that the  $H_i$  and  $T$  are definable divisible abelian groups.

As  $H$  is a definable connected group acting on  $L_i/O_2(L_i)$ , it induces a group of inner automorphisms, and as  $H$  is a  $2^\perp$ -group it acts (modulo the kernel) as a subgroup of a maximal torus of the Borel subgroup  $N^\circ(S) \cap L_i$ . Furthermore as  $H$  is a complement to  $S$  in  $N^\circ(S)$ ,  $H_i$  is itself a maximal torus of  $N_{L_i}(S)$ . As  $H$  acts on  $L_i$  like  $H_i$ , in particular  $H$  commutes with  $H_i$ . It follows that  $T$  is central in  $B \cap N$ . For the further analysis of this action one should bear in mind the decomposition  $T = H_i \times C_T(L_i)$  implicit in the analysis just undertaken, with  $H_i$  the covering an algebraic torus of  $L_i/O_2(L_i)$ .

We claim that  $W$  acts naturally on  $T$ . Since  $T$  is central in  $B \cap N$ , this reduces to the claim that  $N$  normalizes  $T$ . This holds in view of the particular definition of  $N$ . We remark further that as one would expect, the generators  $w_i$  of the quotients  $N_i/(B \cap N)$  are the distinguished involutions which turn out to be the distinguished generators of the Coxeter group  $W$ ; again, this is in [28, 3.2], specifically in the paragraph devoted to the treatment of the condition (BN4) there. Observe that  $w_i$  acts by inversion on  $H_i$  and centralizes the complement  $C_H(L_i)$ .

In view of the conditions on the  $L_{ij}$ , for each odd prime  $p$  the  $p$ -primary torsion subgroup  $T_p$  of  $T$  has Prüfer rank at least 2. What we need to know is that for at least one odd prime  $p$ , this Prüfer rank is at least 3; then Theorem 1.2 applies. This is our final result.

**Lemma 6.8** *For all odd primes  $p$ , the  $p$ -primary torsion subgroup  $T_p$  of  $T$  has Prüfer rank at least 3.*

**Proof.** Let  $\hat{T}_p$  be the associated Tate module, as in §2.4. By our remarks above, in the action of  $W$  on  $\hat{T}_p$ , the distinguished generators  $w_i$  act as reflections. In other words, we have a reflection representation, not necessarily faithful, of the indecomposable Coxeter group  $W$  on  $T_p$ . We can now invoke Proposition 3.1, which is perhaps an odd thing to do since we already know that  $W$  is a Coxeter group.

In any case, the possibilities  $(a, c)$  listed there both fall away as  $W$  is a Coxeter group on at least 3 generators, and  $W$  is a crystallographic Coxeter group. Furthermore, as  $W$  is a reflection representation and  $T$  is generated by subgroups inverted by various reflections in  $W$ , it is easy to see that the action of  $W$  on  $\hat{T}_p$  is faithful, and as  $W$  is indecomposable it is also irreducible. Then from the information about complex reflection groups used in the proof of Proposition 3.1, we see that  $\hat{T}_p$  affords the standard representation of  $W$ , and in particular  $\hat{T}_p$  is of rank at least three, so that  $T_p$  has Prüfer rank at least three.  $\square$

Now Theorem 1.2 applies and completes the proof of Theorem 1.3.

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