Generix never gives up

Eric Jaligot*

Equipe de Logique Mathématique, Université de Paris VII

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Abstract

We prove conjugacy and generic disjointness of generous Carter subgroups in groups of finite Morley rank. We elaborate on groups with a generous Carter subgroup and on a minimal conterexample to the Genericity Conjecture.

1 Introduction

This paper is a contribution to the theory of groups of finite Morley rank as developped in [BN94] and in an extensive body of work aiming at transfering ideas from the classification of finite simple groups to groups of finite Morley rank. This is motivated by the ultimate Algebraicity Conjecture which postulates that infinite simple groups of the latter class are algebraic over algebraically closed fields. Here we rather tend to focus on aspects that might be very useful in case of a failure of this conjecture.

As usual, we say that a definable subset X of a group G of finite Morley rank is generic in G if it has the same rank as G. In this paper we are mostly interested in the weaker property that only the union of the G-conjugates of X is generic in G, that is $\mathrm{rk}\,(X^G) = \mathrm{rk}\,(G)$, and we say in this case that X is *generous* in G.

The main theorem of the present note will be the following.

Theorem 3.1 *In any group of finite Morley rank, generous Carter subgroups are conjugate and generically disjoint.*

Recall from [FJ05] that any group G of finite Morley rank contains a *Carter* subgroup, that is a definable connected nilpotent subgroup of finite index in its normalizer. Carter subgroups in this abstract context are a good approximation of maximal tori in the algebraic context. There are several genericity

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conjectures around these Carter subgroups, inspired by the fact that maximal tori are generous in algebraic groups. The most natural generalization would be the following.

Genericity Conjecture 4.1 *In any group of finite Morley rank, Carter subgroups are generous.*

In [FJ05] it is asked whether the verification of this conjecture implies a conjugacy theorem for Carter subgroups, which would be an abstract version of the conjugacy of maximal tori in the algebraic context. Theorem 3.1 answers this question positively and proves even more, as it does not require the full force of the Genericity Conjecture 4.1.

The *generic disjointness* conclusion in Theorem 3.1 means that, given a generous Carter subgroup C, there is a definable generic subset Y of C such that each element of Y is in no other generous Carter subgroup than C. In fact, this is an important step toward the proof of the conjugacy and we will even prove in this process the following stronger property.

Corollary 3.6 *Let G be a group of finite Morley rank, C a Carter subgroup of G, and y an element in C which is contained in only finitely many conjugates of C. Then C is the unique maximal definable connected nilpotent subgroup of G containing y.*

Our proof of Theorem 3.1 uses the fact that if C is a generous Carter subgroup, then the finiteness assumption of Corollary 3.6 is satisfied for y generic in C. Such a finiteness property is indeed proved in Proposition 2.1 below for any generous definable connected subgroup. The conjugacy of generous Carter subgroups follows then easily from this property and Corollary 3.6.

The paper is organized as follows. In Section 2 we start with a general theory of generous subgroups. We give two proofs of Proposition 2.1, a complicated one and a simpler one noticed by G. Cherlin. The first one is not more useful than the second one here, but we kept it as the kind of analysis done here might be relevant for other purposes. In Section 3 we prove Theorem 3.1, using merely the finiteness result just shown in Section 2 and properties of Carter subgroups via Corollary 3.6. Theorem 3.1 is also applied shortly to elaborate on the theory of groups with a generous Carter subgroup. In Section 4 we give several (and a priori nonequivalent) versions of the Genericity Conjecture 4.1 and develop on rudimentary tools for the analysis of a minimal conterexample to that conjecture. (Theorem 3.1 has originaly been proved like this for that purpose). The conclusion here is that the problem looks very much like the analog problem in the specific case of minimal simple groups.

The operators C and N denote respectively the centralizer and the normalizer in the ambiant group, unless another group is specified.

2 Generous subgroups

In this section we develop a mini theory of definable connected generous subgroups. Hence G will denote throughout the ambiant group of finite Morley rank and the operative assumption for this section will typically be

$(*)$ H is a definable connected generous subgroup of G.

The main goal of this section is to show that under the assumption $(*)$ a generic element of H can only be in a finite number of conjugates of H.

Proposition 2.1 *Let H be a definable connected subgroup of a group G of finite Morley rank. If H is generous in G, then there is a definable generic subset Y of H such that each element of Y belongs to only finitely many conjugates of H.*

We are going to give two proofs of Proposition 2.1. The first one is rather complicated and involves elementary considerations on conjugacy classes of elements, more specifically centralizers and fusion in H of generic elements of H. All this is done assuming H connected, and this detailed analysis of conjugacy classes might be relevant for other purposes. The second one is a very short and conceptual proof noticed by G. Cherlin, which consists in looking at a geometry naturally associated to the problem.

We start with general lemmas implied by assumption $(*)$.

2.1 Generic lemmas

Our two first lemmas are true independantly of the connectedness of H.

Lemma 2.2 *Assume $(*)$, but H not necessarily connected. Then $N(H)/H$ is finite.*

Proof.

A general rank computation gives $\mathrm{rk}\,(H^G) \leq \mathrm{rk}\,(H) + \mathrm{rk}\,(G/N(H)) = \mathrm{rk}\,(G) - \mathrm{rk}\,(N(H)/H)$, and thus the result follows. □

Lemma 2.3 *Assume $(*)$, but H not necessarily connected. Let X be a definable subset of H. If X is generous in G, then X is generous in H.*

Proof.

By Lemma 2.2, $N(H)/H$ is finite. Assume toward a contradiction that $Y := X^H$ is not generic in H. Then, as $Z := Y^{N(H)}$ is the union of finitely many conjugates of Y, Z is thus also nongeneric in H. Now the same rank computation as in the preceding lemma gives $\mathrm{rk}\,(Z^G) \leq \mathrm{rk}\,(Z) + \mathrm{rk}\,(G/N(H)) = \mathrm{rk}\,(Y) + \mathrm{rk}\,(G) - \mathrm{rk}\,(H) < \mathrm{rk}\,(G)$. But $Z^G = (X^{N(H)})^G = X^G$ must be generic in G, a contradiction. □

We also mention the following consequence of Lemma 2.3 when H is connected.

Lemma 2.4 *Assume* $(*)$*. Then any definable generic subset* X *of* H *is generous in* G*.*

Proof.

Assume toward a contradiction that X in not generous in G. Let $Z = H \setminus X$. As $H^G = X^G \cup Z^G$ is generic in G, Z must be generous in G. Let $Z_1 = Z \cap X^G$ and $Z_2 = Z \setminus Z_1$. Now Z_1 is not generous in G, as otherwise X would be. As $Z^G = Z_1^G \cup Z_2^G$ is generic in G, it follows that Z_2 is generous in G. By Lemma 2.3, Z_2 must be generous in H as well. Then Z_2^H and X are two definable generic subsets of H. By connectedness of H, they have a nontrivial intersection [BN94, Theorem 5.12], a contradiction to the definition of Z_2. □

2.2 Fusion in generous connected subgroups

In this subsection we give our first proof of Proposition 2.1. As the next subsection will give a shorter and more conceptual proof, the reader may skip this one.

We start by looking at properties equivalent to assumption $(*)$. For an element $x \in H$ we consider the two following properties concerning respectively its centralizer in G and its fusion in H:

(I) $C^\circ(x) \leq H$,

(II) $\mathrm{rk}\,(x^G \cap H) = \mathrm{rk}\,(x^{N(H)})$.

The next lemma relates property $(*)$ with properties (I) and (II).

Lemma 2.5 *Let* G *be a group of finite Morley rank and* H *a definable subgroup. If* H *connected and generous in* G*, then properties* (I) *and* (II) *hold generically in* H*. Conversely, if* H *has finite index in its normalizer (but is not necessarily connected) and satisfies properties* (I) *and* (II) *generically, then* H *is generous in* G*.*

Proof.

Assume first H connected and generous in G. Consider the map

$$\begin{array}{rccc} \Psi : & H \times G & \longrightarrow & G \\ & (x, g) & \longmapsto & x^g. \end{array}$$

We first claim that the fiber of an element $\Psi(x, g)$, with $(x, g) \in H \times G$, has a rank equal to $\mathrm{rk}\,(x^G \cap H) + \mathrm{rk}\,(C(x))$. When solving an equation $x^g = x'^{g'}$, with $(x', g') \in H \times G$, then x' varies freely in $x^G \cap H$, and then g' varies freely in a coset $C(x')g'_0$, where g'_0 is a fixed element conjugating x' to x^g. Hence our rank equality follows.

Let H_1 be the unique definable generic subset of H on which $\mathrm{rk}\,(C(x))$, $\mathrm{rk}\,(C_H(x))$, and $\mathrm{rk}\,(x^G \cap H)$ are constant. Assuming H^G generic, then H_1^G has to be generic in G by Lemma 2.4. By additivity of rank we get $\mathrm{rk}\,(H_1) + \mathrm{rk}\,(G) =$

$\mathrm{rk}\,(H_1^G) + \mathrm{rk}\,(x^G \cap H) + \mathrm{rk}\,(C(x))$ where the two last terms are constant not depending on x, as the latter varies in H_1. Hence $\mathrm{rk}\,(H_1) = \mathrm{rk}\,(x^G \cap H) + \mathrm{rk}\,(C(x))$ and we indeed get

$$(\dagger) \qquad \mathrm{rk}\,(H) = \mathrm{rk}\,(x^G \cap H) + \mathrm{rk}\,(C(x)).$$

As $x^{N(H)} \subseteq x^G \cap H$, we have on H_1 that (recall that H has finite index in its normalizer by Lemma 2.2) $\mathrm{rk}\,(x^G \cap H) \geq \mathrm{rk}\,(x^{N(H)}) = \mathrm{rk}\,(x^H)$, that is $\mathrm{rk}\,(x^G \cap H) \geq \mathrm{rk}\,(H) - \mathrm{rk}\,(C_H(x))$. Replacing $\mathrm{rk}\,(H)$ in this inequality by its expression in ($\dagger$) gives

$$\mathrm{rk}\,(C_H(x)) \geq \mathrm{rk}\,(C(x)),$$

which means that $C^\circ(x) = C_H^\circ(x)$. Now, ($\dagger$) also gives $\mathrm{rk}\,(x^G \cap H) = \mathrm{rk}\,(H) - \mathrm{rk}\,(C(x)) = \mathrm{rk}\,(H) - \mathrm{rk}\,(C_H^\circ(x)) = \mathrm{rk}\,(x^H) = \mathrm{rk}\,(x^{N(H)})$. Properties (I) and (II) are shown for $x \in H_1$.

The converse is just a converse: assuming that (I) and (II) hold generically in H, consider the set H_1 where they hold generically, and the same map Ψ as above. Then the same computation $\mathrm{rk}\,(H_1) + \mathrm{rk}\,(G) = \mathrm{rk}\,(H_1^G) + \mathrm{rk}\,(x^G \cap H) + \mathrm{rk}\,(C(x))$ gives, with properties (I) and (II) holding on H_1, the genericity of H_1^G and, a fortiori, that of H^G. □

From now on we assume that H is a definable connected subgroup satisfying ($*$). We will study the fusion in H of generic elements of H in the following series of three lemmas, leading eventually to our first proof of Proposition 2.1.

Lemma 2.6 *Assume ($*$). Then there is an integer $\alpha \geq 1$, an integer $\beta \geq 0$, and a definable generic $N(H)$-invariant subset X of H such that, for every $x \in X$:*

(I) $C^\circ(x) \leq H$ *and* $\mathrm{rk}\,(C^\circ(x)) = \alpha$,

(II) $\mathrm{rk}\,(x^G \cap H) = \mathrm{rk}\,(H) - \alpha$,

(III) $x^G \cap X$ *is the union of exactly β distinct $N(H)$-conjugacy classes.*

Proof.

By Lemma 2.5, there is an integer $\alpha \geq 0$ such that the definable subset X of H defined by:

(I) $C^\circ(x) \leq H$ and $\mathrm{rk}\,(C^\circ(x)) = \alpha$,

(II) $\mathrm{rk}\,(x^G \cap H) = \mathrm{rk}\,(H) - \alpha$,

is generic in H. Note also that X, defined as such, is $N(H)$-invariant. By (I), each $N(H)$-conjugacy class in X has rank $\mathrm{rk}\,(H) - \alpha$. Now each G-conjugacy class in X has the same rank: it has rank at least $\mathrm{rk}\,(H) - \alpha$ by the preceding, and at most $\mathrm{rk}\,(H) - \alpha$ by property (II). Hence each G-conjugacy class in X of an element $x \in X$ is a finite union of $N(H)$-conjugacy classes. This gives a definable map from the set of $N(H)$-conjugacy classes in X to the set of G-conjugacy classes in X which has finite fibers. By the "elimination of infinite quantifier" Borovik-Poizat Axiom (cf. [BN94], p. 57), there is a uniform bound on the size of these fibers. Now X is definably partitioned into finitely many sets depending on the size of these fibers, and there is a $\beta \geq 1$ such that, for x generic in X, $x^G \cap X$ is the union of exactly β distinct $N(H)$-conjugacy classes. By imposing this new condition to X, shrinking it by a nongeneric subset if necessary, we can furthermore impose that

(III) $x^G \cap X$ is the union of exactly β distinct $N(H)$-conjugacy classes.

It remains just to notice that X, defined as such, is still $N(H)$-invariant. □

Lemma 2.7 *Assume* $(*)$. *Then there is a definable generic* $N(H)$*-invariant subset* Y *of* X *such that* $y^G \cap H \subseteq Y$ *for every* $y \in Y$.

Proof.

It suffices to prove that $Y := \{x \in X \mid x^G \cap H \subseteq X\}$ is generic in X, as this definable set is clearly $N(H)$-invariant. So assume toward a contradiction that Y is not generic in X. Then $X \setminus Y$ meets α G-conjugacy classes (as each G-conjugacy class in X meets X in $\mathrm{rk}\,(H) - \alpha$ elements).

For every $z \in X \setminus Y$, there exists a G-conjugate z' of z in $H \setminus X$. As $z \in X$, it satisfies property (II), that is $\mathrm{rk}\,(z^G \cap H) = \mathrm{rk}\,(z^H)$. In particular, as $z'^H \subseteq z^G \cap H$, $\mathrm{rk}\,(z'^H) \leq \mathrm{rk}\,(z^H)$; which proves that $\mathrm{rk}\,(C^\circ_H(z)) \leq \mathrm{rk}\,(C^\circ_H(z'))$; i.e. that $C^\circ(z') \leq H$ as $C^\circ(z) = C^\circ_H(z)$ (property (I) of z in H) and z and z' are G-conjugate. In particular $\mathrm{rk}\,(C^\circ_H(z')) = \mathrm{rk}\,(C^\circ_H(z)) = \alpha$ and $\mathrm{rk}\,(z'^H) = \mathrm{rk}\,(H) - \alpha$.

As z was varying in α distinct G-conjugacy classes, we find α distinct H-conjugacy classes of the form z'^H, each of rank equal to $\mathrm{rk}\,(H) - \alpha$. Thus the union of these H-conjugacy classes has rank equal to $\mathrm{rk}\,(H)$. Furthermore, as X is $N(H)$-invariant, an element z' as above satisfies $z'^H \cap X = \emptyset$, and the union of these H-conjugacy classes also meets X trivially. This gives a subset of $H \setminus X$ which is generic in H, and contradicts the connectedness of H as X is already generic in H. □

Lemma 2.8 *Assume* $(*)$. *Then any element* $y \in Y$ *can only be in a finite number of conjugates of* H.

Proof.

Assume $y \in H^g$ for some $g \in G$. Our claim is that g can vary in only finitely many right cosets of $N(H)$.

Fix β representatives y_1, ..., y_β in Y of the β distinct $N(H)$-conjugacy classes of Y belonging to y^G. Fix also β elements g_1, ..., $g_\beta \in G$ such that $y^{g_i} = y_i$. We have $y \in (y^G \cap H)^g$, thus $y \in Y^g$ by definition of Y. Hence

$$y^{g^{-1}} = y_i^\gamma \text{ for some } i \in \{1, \cdots, \beta\} \text{ and some } \gamma \in N(H).$$

As $y^{g_i} = y_i$, it follows that $y_i^{g_i^{-1}g^{-1}} = y_i^\gamma$, i.e. $\gamma g g_i \in C(y_i)$, i.e. $g \in \gamma^{-1}C(y_i)g_i^{-1}$. We have shown that

$$g \in \bigcup_{1 \leq i \leq \beta} N(H)C(y_i)g_i^{-1}$$

and as $C^\circ(y_i) \leq H$ for each i (recall that each y_i is in Y, thus in X, and thus satisfies property (I) in H), this latter set is composed of finitely many right cosets of $N(H)$, as desired. □

Now Lemmas 2.6, 2.7, and 2.8 give our first proof of Proposition 2.1. □

2.3 A geometric proof

This subsection is devoted to our second proof of Proposition 2.1. This much more conceptual proof, mentionned by G. Cherlin, consists in the examination of the geometry naturally associated to the problem. Assuming again that G is the ambiant group of finite Morley rank with a definable connected subgroup H satisfying assumption $(*)$, this geometry is defined as follows. The set P of points is H^G and the set L of lines consists of the set of conjugates of H. As we may work inside G^{eq} throughout, the set of lines is interpreted by $G/N(H)$, so that this geometry is well definable inside the structure.

We let $\mathcal{F}$ denote the flag associated to this geometry, that is

$$\mathcal{F} = \{(y, \ell) \in P \times L \mid y \in \ell\}.$$

Let also π_P and π_L denote the projections of $\mathcal{F}$ on P and L respectively. For $r \leq \mathrm{rk}\,(G/N(H))$, let

$$H_r = \{h \in H \mid \mathrm{rk}\,(\pi_P^{-1}(h)) = r\}.$$

By definability of the rank, each H_r is definable. Notice that H_r is the set of elements h of H such that the set of conjugates of H containing h has rank r. This gives a definable partition

$$H^G = (H_0)^G \sqcup \cdots \sqcup (H_{\mathrm{rk}\,(G/N(H))})^G$$

and also a definable partition

$$\mathcal{F} = \mathcal{F}_0 \sqcup \cdots \sqcup \mathcal{F}_{\mathrm{rk}\,(G/N(H))}$$

where $\mathcal{F}_r = \pi_P^{-1}(H_r^G)$. Assume now H_r nonempty. We are going to project $\mathcal{F}_r$ on L and P, using π_L and π_P respectively, and taking advantage of the additivity of the rank.

We first project on L. As $H_r \neq \emptyset$, there exists an element $h \in H$ such that the set of conjugates of H containing h has rank r. Now each conjugate of H has the same property, and thus a preimage in $\mathcal{F}_r$. Hence the restriction of π_L to $\mathcal{F}_r$ is surjective onto $G/N(H)$. On the other hand, given a line H^g, the set of elements $y \in H^g$ contained in exactly r conjugates of H has a rank equal to $\mathrm{rk}\,(H_r^g) = \mathrm{rk}\,(H_r)$. Hence:

$$\mathrm{rk}\,(\mathcal{F}_r) = \mathrm{rk}\,(G/N(H)) + \mathrm{rk}\,(H_r).$$

We now project $\mathcal{F}_r$ on P. By definition, $\pi_P(\mathcal{F}_r) = H_r^G$. Now, given $y \in H_r^G$, the set of conjugates of H containing y has rank r and $\text{rk}\,(\pi_P^{-1}(y)) = r$. Hence:

$$\text{rk}\,(\mathcal{F}_r) = \text{rk}\,(H_r^G) + r.$$

As the above analysis used only the definability of H, we get by combining these two expressions for $\text{rk}\,(\mathcal{F}_r)$ the following general rank equality.

Proposition 2.9 *Let G be a group of finite Morley rank, H a definable subgroup, and, for $r \leq \text{rk}\,(G/N(H))$, H_r the definable subset of H consisting of those elements of H having the property that the set of conjugates of H containing them has rank r. If $H_r \neq \emptyset$, then $\text{rk}\,(H_r^G) = \text{rk}\,(G) - \text{rk}\,(N(H)) + \text{rk}\,(H_r) - r$.*

Incorporating the full assumption $(*)$, that is the connectedness of H and the genericity of H^G, we can now deduce our second

Proof of Proposition 2.1.
Assume $(*)$, that is H definable, connected, and generous in G, and define H_r as above. There exists an r such that H_r is generic in H. As H is generous in G, $\text{rk}\,(H_r) = \text{rk}\,(H) = \text{rk}\,(N(H))$ by Lemma 2.2. As H is connected and generous in G, H_r is generous in G by Lemma 2.4, that is $\text{rk}\,(H_r^G) = \text{rk}\,(G)$. Incorporating these two equalities in the equality provided by Proposition 2.9 gives $r = 0$. Hence the set $Y = H_r$ is the desired one. □

We conclude by mentioning the following converse of Proposition 2.1.

Proposition 2.10 *Let G be a group of finite Morley rank, H a definable (not necessarily connected) subgroup of G. Assume that the set H_0 of elements of H contained in finitely many conjugates of H is generic in H. Assume also H of finite index in its normalizer. Then H_0 and H are generous in G.*

Proof.
Applying Proposition 2.9 with $r = 0$ gives $\text{rk}\,(H_0^G) = \text{rk}\,(G) - \text{rk}\,(N(H)) + \text{rk}\,(H_0)$. By assumption, $\text{rk}\,(H_0) = \text{rk}\,(H) = \text{rk}\,(N(H))$, and it follows that H_0 is generous in G. In particular H is also generous in G. □

3 Main theorem

The main result of the present paper is the following.

Theorem 3.1 *In any group of finite Morley rank, generous Carter subgroups are conjugate and generically disjoint.*

Our proof of Theorem 3.1 uses a crucial ingredient which will fully exploit the finiteness results obtained in Section 2.

3.1 The fondamental lemma

The next lemma is the crucial ingredient which encapsulates the main idea of the present paper. It is indeed a mere elaboration on the following well known basic tool.

Fact 3.2 [BN94, Lemma 5.9] *Let G be a connected group of finite Morley rank acting definably on a finite set S. Then G fixes S pointwise.*

The fondamental lemma in question is the following.

Lemma 3.3 *Let G be a group of finite Morley rank, H a definable subgroup of G, and Y the definable subset of H consisting of those elements of H which are in only finitely many conjugates of H. Then, for any definable subset U of H meeting Y in a nonempty subset, $N^\circ(U) \leq N^\circ(H)$.*

Proof.
We may work inside G^{eq} throughout, and identify the set of conjugates of H with $G/N(H)$. Let $U_1 = U \cap Y$. As U_1 is exactly the subset of U consisting of those elements of U which are in finitely many conjugates of H, $N^\circ(U)$ normalizes U_1. In particular, its action on G by conjugation induces an action on the set of conjugates of H containing U_1. But there are finitely many such conjugates of H containing U_1, so $N^\circ(U)$ fixes all of them by Fact 3.2. In other words, $N^\circ(U)$ normalizes each of these conjugates. As H is one of them, our claim follows. $\square$

3.2 Proof of Theorem 3.1

The last ingredient for the proof of Theorem 3.1 will be the definition of Carter subgroups, the fact that they are of finite index in their normalizers on the one hand, and their nilpotence on the other hand. Concerning the nilpotence, we will more specifically use the following version of the normalizer condition in nilpotent groups of finite Morley rank.

Fact 3.4 [BN94, Lemma 6.3] *Let G be a connected nilpotent group of finite Morley rank, and H a proper definable subgroup of G. Then $[N(H) : H]$ is infinite.*

Gluing together the fondamental Lemma 3.3 with the two main properties of Carter subgroups gives the following conclusion.

Lemma 3.5 *Let G be a group of finite Morley rank, H a definable connected subgroup of G of finite index in its normalizer, and C a definable connected nilpotent subgroup of G. Let Y be the definable subset of H consisting of those elements of H which are in finitely many conjugates of H. If $Y \cap C \neq \emptyset$, then $C \leq H$. In this case, C is also a Carter subgroup of H whenever it is a Carter subgroup of G, and $C = H$ whenever C and H are two Carter subgroups of G.*

Proof.
Let $U = H \cap C$. As $U \cap Y$ is nonempty, $N^\circ(U) \leq N^\circ(H)$ by Lemma 3.3. As $N^\circ(H) = H$, $N^\circ(U) \leq H$. In particular $N_C^\circ(U) \leq U^\circ$, and hence U is of finite index in its normalizer in C. By Fact 3.4, $U = C$ and our first claim follows. Our second claim is immediate. Our third claim is now another application of Fact 3.4 in H. □

Corollary 3.6 *Let G be a group of finite Morley rank, C a Carter subgroup of G, and y an element in C which is contained in only finitely many conjugates of C. Then C is the unique maximal definable connected nilpotent subgroup of G containing y.*

As Carter subgroups are maximal definable connected nilpotent subgroups by Fact 3.4, it is noticeable that, under the circumstances of Corollary 3.6, C is in particular the unique Carter subgroup of G containing y.

Now it suffices to shake the preceding preparations together with the genericity ingredient to show our main result.

Proof of Theorem 3.1.
Let C_1 and C_2 be two generous Carter subgroups of a group G of finite Morley rank. For $i = 1$ and 2, let Y_i denote the definable subset of C_i consisting of those elements of C_i which are in finitely many conjugates of C_i. By Proposition 2.1, Y_i is generic in C_i for each i. By Lemma 3.5 or Corollary 3.6,

$$(\dagger) \qquad Y_1 \cap Y_2 = \emptyset \text{ or } C_1 = C_2.$$

For each i, Y_i is generic in C_i, and thus Y_i is generous in G by Lemma 2.4. Of course, $Y_i^G \subseteq G^\circ$, and it follows that Y_i^G is generic in G°. By connectedness of G°, Y_1^G meets Y_2^G nontrivially. Hence, after conjugacy in G, we have $Y_1 \cap Y_2$ nonempty, and hence $C_1 = C_2$ by (†). Our conjugacy result is proved.

The generic disjointness follows also, as the definable generic subset Y_1 of C_1 has the property that its elements are in no other generous Carter subgroups than C_1. □

Of course, we would prefer a conjugacy theorem inside connected components.

Corollary 3.7 *In any group G of finite Morley rank, generous Carter subgroups in G are generous in G° and, thus, conjugate in G°.*

Proof.
If C denotes a generous Carter subgroup in G, then, as C is connected, C^G is a generic subset of G°. As G is the union of finitely many right cosets of G°, our first claim follows. Our second claim corresponds now to Theorem 3.1 applied in G°. □

We finish this subsection with necessary and sufficient conditions for a Carter subgroup to be generous.

Corollary 3.8 *Let G be a group of finite Morley rank and C a Carter subgroup of G. Then the following are equivalent:*

a. C is generous in G.

b. There exists a definable generic subset Y of C such that, for each $y \in Y$, C is the unique maximal definable connected nilpotent subgroup containing y.

c. C is generically disjoint from its conjugates.

d. There exists a definable generic subset Y of C such that each $y \in Y$ is contained in only finitely many conjugates of C.

Proof.
Clause (a) implies clause (b) by Proposition 2.1 and Corollary 3.6. Clearly, clause (b) implies clause (c) and clause (c) implies clause (d). Finally, clause (d) implies clause (a) by Proposition 2.10. □

3.3 Applications

The presence of a generous Carter subgroup in a group of finite Morley rank has a strong impact on its overall structure. This subsection is devoted to miscellaneous such corollaries which can be derived from Theorem 3.1. We first take afresh by rephrasing the early general lemmas of Subsection 2.1.

Lemma 3.9 *Let $L \leq H \leq G$ be groups of finite Morley rank, with L and H definable. Then:*

a. If H is connected, L generous in H, and H generous in G, then L is generous in G.

b. Conversely, if L is generous in G, then L is generous in H and H is generous in G.

In particular, in any of these two cases, L and H are both of finite index in their normalizers in G.

Proof.
The first item, concerning transitivity, is a mere rephrasing of Lemma 2.4. The converse item is a mere rephrasing of Lemma 2.3, or can also be seen as a corollary of Lemma 2.2 and Propositions 2.1 and 2.10. Our final claim is just Lemma 2.2. □

We now return to generous Carter subgroups. Let start with centralizers of their generic elements.

Lemma 3.10 *Let G be a group of finite Morley rank with a generous Carter subgroup C. Then $C(x) \leq N(C)$ holds for x generic in C*

Proof.
By Theorem 3.1, C is the only conjugate of C containing x. □

Theorem 3.1 gives an extra conjugacy result.

Lemma 3.11 *Let $C \leq H \leq G$ be groups of finite Morley rank, with C a generous Carter subgroup of G, and H a definable subgroup of G. Then any G-conjugate of C in H is H°-conjugate to C.*

Proof.
Let $C^g \leq H$ with $g \in G$. Then C and C^g are both generous in H° by Lemma 3.9 (b). It follows that they are H°-conjugate by Theorem 3.1. □

By analogy with algebraic groups, it is natural to call $W = N(C)/C$ the *Weyl* group associated to a given Carter subgroup C. This group is finite by definition of Carter subgroups. By Theorem 3.1, the Weyl group associated to a generous Carter subgroup is well defined.

Lemma 3.12 *Let $C \leq H_1$, $H_2 \leq G$ be groups of finite Morley rank, with C a generous Carter subgroup of G, and H_1 and H_2 two definable connected subgroups containing C. If H_1 and H_2 are G-conjugate, then they are $N(C)$-conjugate (and, thus, in particular $N(C)/C$-conjugate).*

Proof.
We assume $H_2 = H_1^g$ for some g in G, and $C \leq H_1 \cap H_1^g$. Then $C^{g^{-1}} = C^\gamma$ for some $\gamma \in H_1$ by Lemma 3.11, $\gamma g \in N(C)$ and $H_1^g = H_1^{\gamma g}$. □

As another application of Lemma 3.11, a Frattini Argument precises the last item of Lemma 3.9 (b) in the presence of a generous Carter subgroup.

Corollary 3.13 *Let $C \leq H \leq G$ be groups of finite Morley rank, with C a generous Carter subgroup and H definable. Then $N(H) \subseteq N(C)H^\circ$.*

Proof.
If $g \in N(H)$, then $C^g = C^\gamma$, with $\gamma \in H^\circ$, and $g = g\gamma^{-1}\gamma \in N(C)H^\circ$. □

If G is a group of finite Morley rank with a generous Carter subgroup C, then it is natural to call *standard* any Borel subgroup containing a conjugate of C. Similarly, a *standard parabolic* subgroup would be a definable connected subgroup containing a standard Borel subgroup, a *standard root* group a minimal definable connected subgroup normalized by a conjugate of C, and so on. Probably, most of the theory of algebraic groups *à la Tits* could be recovered in this context. One would like first a conjugacy theorem of standard Borel subgroups and a finiteness theorem on the number of standard parabolic subgroups, up to conjugacy. Unfortunately, nothing like this has been proved yet.

4 Around genericity

We finish this paper with a few prospects concerning the relations between the genericity conjectures and Carter subgroups.

4.1 Genericity conjectures

As it has been said in the introduction already, there are several natural genericity conjectures around Carter subgroups. The most natural one is the one formulated in the introduction, and corresponds to item (b) in the list of successively weaker genericity conjectures below.

There are many issues concerning nonconnected subgroups. Recently some pathological configurations involving Sylow 2-subgroups in groups of finite Morley rank have been disposed of in [BBC05], allowing to kill some pathological configurations involving torsion in general. These results probably have to deal with issues relating genericity conjectures and nonconnected subgroups. Here we prefer to concentrate on connected groups throughout.

Genericity Conjecture 4.1 *One of the following statement is true for any connected group G of finite Morley rank:*

a. Any definable connected subgroup of G of finite index in its normalizer is generous in G.

b. Any Carter subgroup of G is generous in G.

c. There is a generous Carter subgroup in G.

d. If G is nonnilpotent, there is a proper definable connected generous subgroup in G.

e. There is a definable generic subset of G all of whose elements are contained in a Carter subgroup.

f. There is a definable generic subset of G all of whose elements are contained in a definable connected nilpotent subgroup of G.

The fact that the above conjectures are successively weaker follows readily, once one has noticed that Conjectures 4.1 (c) and (d) are equivalent. Indeed, assuming Conjecture 4.1 (d) verified, than a *minimal* definable connected subgroup has to be nilpotent and selfnormalizing, thanks to Lemma 2.2, and hence a Carter subgroup of the ambiant group.

Conjecture 4.1 (a) is really strong. For example, it has the following implication.

Lemma 4.2 *Let G be a group of finite Morley rank, all of those definable connected subgroups verify Conjecture 4.1 (a). If $C \leq H \leq G$, with H a definable connected subgroup of finite index in its normalizer, and C a Carter subgroup of H, then C is a generous Carter subgroup of G.*

Proof.
By Lemma 3.9 (a), C is generous in G. Then C is of finite index in its normalizer by Lemma 2.2, and C is also a Carter subgroup of G. □

For example, this would kill the problematic configurations left in [CJ04]. Such configurations would a priori not be killed by a verification of the weaker Conjecture 4.1 (b), but this latter conjecture would kill the configurations left in [Jal01].

A verification of any of Conjectures 4.1 (a)–(d) would imply a generic covering of groups by a single conjugacy class of nilpotent subgroups. This is a priori lost with the weaker Conjecture 4.1 (e), where it becomes unclear how to prove a conjugacy theorem like Theorem 3.1.

Conjecture 4.1 (f) a priori loses all connection with Carter subgroups. This latter conjecture has been verified in [BBC05] in the case of minimal simple groups.

4.2 Mini minimal counterexample theory

We conclude this paper with a primitive study of a minimal counterexample to the Genericity Conjecture. We could of course study minimal couterexamples to each of the Genericity Conjectures 4.1 (a)-(f), but we focus only on Conjecture 4.1 (b). Hence we work here with the following assumption:

G is a connected minimal counterexample to the Genericity Conjecture 4.1 (b), with a nongenerous Carter subgroup C. (1)

The thesis here is that the problem looks very much like the analogous well known problem in the specific case of minimal simple groups. We start by a mere application of the inductive assumption.

Lemma 4.3 *In any proper definable connected subgroup of G, Carter subgroups are generous, generically disjoint, and conjugate. (And Sylow subgroups of the minimal unipotence degree are also conjugate in such proper definable connected subgroups).*

Proof.
Immediate from Theorem 3.1. For Sylow subgroups and unipotence degree, the reader may consult [Bur05] and [FJ05]. The point concerning Sylow subgroups of the minimal unipotence degree is that they are always in a Carter subgroup by the construction of Carter subgroups in [FJ05]. □

It is worth mentioning that nongenerocity propagates to supergroups, as well as some consequence of the Frattini Argument.

Lemma 4.4 *Let H be a proper definable subgroup of G containing C. Then H° is not generous in G, and $N(H) \subseteq N(C)H^\circ$.*

Proof.

As $H^\circ < G$, C is generous in H° and Lemma 3.9 (a) gives our first claim. If $g \in N(H)$, then C^g, $C \leq H^\circ$, $C^g = C^\gamma$ for some $\gamma \in H^\circ$, and $g = g\gamma^{-1}\gamma \in N(C)H^\circ$. □

Of course, the group G has tendancy to be simple.

Lemma 4.5 *One can assume G simple.*

Proof.

We first claim that $R^\circ(G) = 1$, where $R(G)$ denotes the solvable radical of G (cf. [BN94, Section 7.2]). Notice first that $CR^\circ(G) < G$, as otherwise G is solvable and the theory of solvable groups applies (cf. [CJ04, Lemma 3.5]). By that theory again, or merely by induction here, Carter subgroups of $CR^\circ(G)$ are generous in $CR^\circ(G)$. By Corollary 3.8, the $CR^\circ(G)$-conjugates of C distinct from C do not cover C generically. By Lemma 4.4, $N^\circ(CR^\circ(C)) \leq N^\circ(C)CR^\circ(G) = CR^\circ(G)$. Hence $\overline{C}$ is a Carter subgroup of $\overline{G} = G/R^\circ(G)$, generically covered by its distinct conjugates by Corollary 3.8. By Corollary 3.8 again, $\overline{C}$ is nongenerous in $\overline{G}$. By minimality, $R^\circ(G) = 1$, as claimed.

Hence $R(G)$ is finite and equal to $Z(G)$. Now $G/Z(G)$ has a finite solvable radical, which must be trivial, and we may replace G by $G/Z(G)$ to get a semisimple group satisfying our assumptions.

Let now H be a definable normal subgroup of G. We claim that $H^\circ = 1$. Assume $H^\circ \neq 1$. As H° is normal and proper in G, its Carter subgroups are not generous in G, and we may assume that C is one of them. By Lemma 4.4, $G = N(C)H^\circ$, and it follows that H° has finite index in G. This gives $H^\circ = G$, a contradiction. Hence $H^\circ = 1$ as claimed and H is finite. As H is normal in G, $H \leq Z(G) = 1$. □

One more trivial structural information, showing the nontriviality of the problem:

Lemma 4.6 $G = \langle C^g \mid g \in G \rangle$.

Proof.

$\langle C^g \mid g \in G \rangle \trianglelefteq G$, and Lemma 4.4 applies. □

We can also remark the following on Sylow subgroups.

Lemma 4.7 *Any Sylow subgroup of the minimal unipotence degree of any Carter subgroup of G is a Sylow subgroup of G.*

Proof.

Let T be the Sylow subgroup of the minimal unipotence degree of a Carter subgroup C_1 of G. As G is semisimple, $N(T) < G$. Of course $C_1 \leq N^\circ(T)$, and C_1 is a Carter subgroup of $N^\circ(T)$. As Carter subgroups of $N^\circ(T)$ are all conjugate in $N^\circ(T)$, C_1 can be obtained by the construction of Carter subgroups as in [FJ05]. In this construction, a Sylow subgroup of the minimal unipotence degree of a Carter subgroup is a Sylow subgroup of the ambiant group. Hence T is a Sylow subgroup of $N^\circ(T)$ and it follows by a normalizer condition [Bur05, Lemma 2.4] that T is a Sylow subgroup of G. □

Of course, one may wish to specialize the above theory to the specific cases C abelian and/or W trivial, where W denotes the Weyl group associated to C. But it seems that the failure of *none* of these too assumptions could be the real *core* of the problem. In the same vein, one may really wonder whether the nilpotency of C is really important, and should maybe try to focus on the more ambitious Conjecture 4.1 (a).

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