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The rational points of a definable set

Jonathan Pila and A. J. Wilkie

Abstract

Let $X \subset \mathbb{R}^n$ be a set that is definable in an o-minimal expansion of $\mathbb{R}$. This paper shows that, in a suitable sense, there are very few rational points of X that do not lie on some connected semialgebraic subset of X of positive dimension.

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1. Introduction

Let M be an o-minimal expansion of the ordered field $\mathbb{R}$ (see [3]). By *definable* we will always mean "M-definable with parameters from M".

Let $H : \mathbb{Q} \to \mathbb{R}$ be the usual height function, $H(a/b) = \max(|a|, b)$ for $a, b \in \mathbb{Z}$ with $b > 0$ and $(a, b) = 1$. Define $H : \mathbb{Q}^n \to \mathbb{R}$ by $H(\alpha_1, \alpha_2, \ldots, \alpha_n) = \max_{1 \le j \le n}(H(\alpha_j))$. For a set $X \subset \mathbb{R}^n$ define $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$ and, for $H \ge 1$, put

$$X(\mathbb{Q}, H) = \{P \in X(\mathbb{Q}) : H(P) \le H\},$$

$$N(X, H) = \#X(\mathbb{Q}, H).$$

This paper is concerned with the counting function $N(X, H)$ for a definable set X.

To contextualize the kind of results sought, consider the following situation. An example of an o-minimal expansion of $\mathbb{R}$ is given by the class $\mathbb{R}_{\mathrm{an}}$ of *globally subanalytic* sets (see e.g. [4]), which includes the bounded subanalytic sets, and more particularly the sets $X \subset \mathbb{R}^2$ that arise as the graph of a function $f : [0, 1] \to \mathbb{R}$ real analytic on a neighbourhood of $[0, 1]$. If such f is not algebraic, and X is the graph of f on $[0, 1]$, then an estimate

$$N(X, H) \le c(X, \epsilon)H^{\epsilon},$$

for any $\epsilon > 0$, is established in [11].

Now if f is of special form (e.g. $f(x) = e^x$, or, say, a G-function) then one may have much stronger results (or at least conjectures) on the scarcity of rational (or even algebraic) points (see e.g. [1]). At the other extreme, constructions going back to Weierstrass (see e.g. [10, 16]) show that an entire transcendental f may take rational values at every rational argument. These constructions do not take much care of the height density of points.

However, given any function $\epsilon : [1, \infty) \to \mathbb{R}$, strictly decreasing with $\epsilon(t) \to 0$ as $t \to \infty$, it is possible (see [13, 7.5]) to construct a transcendental analytic function f on $[0, 1]$ and a (rather lacunary) sequence of positive integers $H_j \to \infty$ such that

$$N(X, H_j) \geq H_j^{\epsilon(H_j)}.$$

Thus the above result cannot be much improved in general. (E.g. taking $\epsilon(t) = (\log t)^{-1/2}$ shows that for certain X no bound of form $N(X, H) \leq C(\log H)^K$ holds etc.)

Consider now X of dimension ≥ 2. A new feature arises. Namely, X may contain connected subsets A of positive dimension that are semialgebraic even if X itself is not. Such sets, e.g. lines, may contain many rational points (i.e. $N(A, H) >> H^\delta$ for some $\delta > 0$). This prompts the following definition.

1.1. Definition. Let $X \subset \mathbb{R}^n$. The *algebraic part* of X, denoted X^{a}, is the union of all subsets of X that are connected semialgebraic sets of positive dimension. The *transcendental part* of X, denoted X^{t}, is the complement $X - X^{\mathrm{a}}$.

In this paper we show that, for a definable set $X \subset \mathbb{R}^n$ of any dimension, an estimate of the same quality as the one-dimensional result above holds, provided only that the rational points in the algebraic part are excluded from the count. The following result, in which X^{a} plays a role weakly analogous to the *special set* in diophantine geometry (see e.g. [8, Ch I §3; 7, §F.5]), affirms (in the particular case $M = \mathbb{R}_{\mathrm{an}}$) a conjecture made in [13] for bounded subanalytic sets.

1.2. Theorem. (First version) *Let $X \subset \mathbb{R}^n$ be a definable set, and $\epsilon > 0$. There is a constant $c(X, \epsilon)$ such that*

$$N(X^{\mathrm{t}}, H) \leq c(X, \epsilon)\, H^\epsilon.$$

The proof of the theorem begins by showing that the points in question reside on "few" (i.e. $O_{X,\epsilon}(H^\epsilon)$) hypersurfaces of suitable degree $d(\epsilon)$; it then proceeds by induction on the dimension of X. Thus it is necessary to have an estimate of the same form as above for those hypersurface intersections but in which the implied constant is uniform over all intersections of X with hypersurfaces of fixed degree: i.e., a result for a definable *family* of sets. The following convention will be adopted. In considering subsets $Z = \{(x, y)\} \subset \mathbb{R}^n \times \mathbb{R}^m$, projection on the first factor will be denoted π_1, and on the second π_2. Put $Y = Y_Z = \pi_2(Z)$ and for $y \in Y$ put $Z_y = \{z \in Z : \pi_2(z) = y\}$, and $X_y = X_{Z,y} = \pi_1(Z_y)$ its image in $\mathbb{R}^n$. A *family* $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ of sets will mean the collection of *fibres* $\{X_y : y \in Y_Z\}$. A family Z is *definable* if the set Z is. The result to be proved is then the following.

1.3. Theorem. (Second version) *Let $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ be a definable family, and $\epsilon > 0$. There is a constant $c(Z, \epsilon)$ with the following property. Let X be a fibre of Z. Then*

$$N(X^{\mathrm{t}}, H) \leq c(Z, \epsilon)\, H^{\epsilon}.$$

The example $X = \{(x, y, z) : z = x^y, x, y \in [2, 3]\}$, for which $X^{\mathrm{a}} = \{(x, y, z) \in X : y \in \mathbb{Q}\}$, shows that X^{a} is not, in general, semialgebraic (or even definable: a definable set has only finitely many connected components). Nevertheless, it might be supposed that, for any X and ϵ, there is a semialgebraic set $X_\epsilon \subset X$ and a constant $c(X, \epsilon)$ such that $N(X - X_\epsilon, H) \leq c(X, \epsilon)H^{\epsilon}$. This is not the case: Consider $X = \{(x, y) : 0 < x < 1,\ 0 < y < e^x\}$. Then $X^{\mathrm{a}} = X$ but X is not semialgebraic (otherwise its bounding graph $y = e^x, x \in [0, 1]$ would be semialgebraic). So $N(X - X_\epsilon, H) >> H^4$ for any semialgebraic $X_\epsilon \subset X$. However, it is possible to find a definable $X_\epsilon \subset X^{\mathrm{a}}$ with the desired property; indeed, for a definable family Z the sets X_ϵ may be taken to be fibres of a definable family $W(Z, \epsilon) \subset Z$, and this is the final version of the result to be proved.

1.4. Theorem. (Final version) *Let $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ be a definable family and $\epsilon > 0$. There is a definable family $W = W(Z, \epsilon) \subset Z$ and a constant $c(Z, \epsilon)$ with the following property. Let $y \in Y$. Put $X = X_{Z,y}$ and $X_\epsilon = X_{W,y}$. Then $X_\epsilon \subset X^{\mathrm{a}}$ and*

$$N(X - X_\epsilon, H) \leq c(Z, \epsilon)H^{\epsilon}.$$

Note that this version makes a nontrivial assertion in situations, like the example above, in which $X^{\mathrm{t}}(\mathbb{Q})$ is empty but X^{a} is not definable.

The diophantine part of the proof follows the strategy of [13], which goes back to [2]. The heart of the analytic part of the proof is the possibility of a certain uniform parameterization of the fibres X in a definable family. The uniformity required is in the number of $C^{(r)}$ maps $(0, 1)^{\dim(X)} \to X$ required to cover X, and at the same time in bounds on the sizes of all their partial derivatives up to some prescribed finite order r. This is achieved in §2–5, by establishing an o-minimal version of Gromov's Algebraic Reparameterization Lemma (see [5, page 232]; itself a refinement of a method of Yomdin [20, 21]) for obtaining such parameterizations of closed semialgebraic sets.

In [13] a conjecture is made about integer points on the dilation of a compact subanalytic set. That conjecture is essentially (though not strictly) weaker than the corresponding statement about rational points, and is also affirmed here, in §8. Integer points on definable curves are studied in [19].

While, as indicated, the estimate $N(X^{\mathrm{t}}, H) = O_{X,\epsilon}(H^{\epsilon})$ cannot be improved for globally subanalytic sets, a much better estimate might be anticipated for other o-minimal expansions of the real field where we have more control over the definable sets. For example:

1.5. Conjecture. Let $M = \mathbb{R}_{\exp}$ (i.e. the expansion of the real field by the exponential function – see [18]). If X is definable, there are $c_1(X), c_2(X) \in \mathbb{N}$ such that

$$N(X^{\mathrm{t}}, H) \leq c_1(X)(\log H)^{c_2(X)}.$$

In this paper, $A \subset B$ means that A is a subset of (possibly equal to) B. The cardinality of a set A is denoted $\#A$, and $\mathbb{N}$ denotes the set of nonnegative integers. The letters $i, j, k, \ell, m, n, r, d$ are reserved exclusively to range over $\mathbb{N}$.

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2. Reparameterization (after Yomdin-Gromov)

For §2–5 we take M to be an o-minimal expansion of an arbitrary real closed field. Although we are ultimately only interested in $\mathbb{R}$, the greater generality actually simplifies the arguments here because it guarantees a certain "uniformity in parameters" that would be absent if we restricted our attention to expansions of $\mathbb{R}$.

Recall that an element $a \in M$ is called *finite* if $|a| \leq c$ for some $c \in \mathbb{N}$ (we assume that $\mathbb{Q}$ is identified with the prime subfield of M). A finite element of M will also be called *strongly bounded*. An n-tuple of elements of M is *strongly bounded* if all its coordinates are, and a definable subset of M^n is *strongly bounded* if there is a fixed finite bound for all the coordinates of all its elements. Further, a definable function is *strongly bounded* if its graph is (equivalently, if its domain and range are).

2.1. Definition. Let $X \subset M^n$ be definable. A definable function $\phi : (0,1)^\ell \to X$, where $\ell = \dim X$, is called a *partial parameterization* of X. A finite set S of partial parametrizations of X is called a *parameterization* of X if $\cup_{\phi \in S} \mathrm{range}(\phi) = X$. (Of course standard notation like "$(0,1)$" refers to its natural interpretation in M.)

We shall be interested in various extra conditions on the functions in such an S. In particular, it is not hard to show, using the $C^{(r)}$-cell decomposition theorem ([3]), that every bounded set has a $C^{(r)}$-parameterization. We shall be interested in bounding the derivatives.

2.2. Definition. A parameterization S (of some definable set X) is called an *r-parameterization* if every $\phi \in S$ is of class $C^{(r)}$ and has the property that $\phi^{(\alpha)}$ is strongly bounded for each $\alpha \in \mathbb{N}^{\dim X}$ with $|\alpha| \leq r$, where $|\alpha|$ is the sum of the coordinates of α.

2.3. Theorem. (Reparametrization Theorem [after Gromov]) *For any $r \in \mathbb{N}$ and any strongly bounded, definable set X, there exists an r-parameterization of X.*

There is also a version for functions.

2.4. Definition. Suppose that S is an r-parameterization of the definable set $X \subset M^m$ and that $F : X \to M^n$ is a definable function. Then we say that S is an *r-reparameterization* of F if, for each $\phi \in S$, $F \circ \phi$ is of class $C^{(r)}$ and $(F \circ \phi)^{(\alpha)}$ is strongly bounded for all $\alpha \in \mathbb{N}^{\dim X}$ with $|\alpha| \leq r$.

2.5. Theorem. *For any $r \in \mathbb{N}$ and any strongly bounded, definable function F, there exists an r-reparameterization of F.*

The next 3 sections are devoted to the proof of theorems 2.3 and 2.5.

3. The unary function case

There is a very simple, but crucial, analytic trick at the heart of the proof of 2.5 which we now state and prove. Indeed, the rest of the argument is just a case of organizing the induction carefully.

3.1. Lemma. *Let $r \geq 2$ and suppose that $f : (0,1) \to M$ is a definable function of class $C^{(r)}$ with $f^{(j)}$ strongly bounded for $0 \leq j \leq r-1$. Suppose further that $|f^{(r)}|$ is (weakly) decreasing. Define $g : (0,1) \to M$ by*

$$g(x) = f(x^2).$$

Then $g^{(j)}$ is strongly bounded for $0 \leq j \leq r$.

Proof. By the chain rule (applied in M), $g^{(i)}(x) = \sum_{j=0}^{i} \rho_{i,j}(x).f^{(j)}(x^2)$, for each $i = 0, 1, \ldots, r$ and $x \in (0,1)$, where each $\rho_{i,j}$ is a polynomial with integer coefficients (of degree j, in fact).

Now, by our hypothesis on f, all summands are strongly bounded except, possibly, the one with $i = j = r$. One easily checks that this summand is $2^r x^r f^{(r)}(x^2)$. Let c be a positive integer strongly bounding the function $f^{(r-1)}$ and suppose, for a contradiction, that there is a some $x_0 \in (0,1)$ with $|f^{(r)}(x_0)| > 4c/x_0$. By the Mean Value Theorem (applied in M), there is some $\xi \in [x_0/2, x_0]$ such that $f^{(r-1)}(x_0) - f^{(r-1)}(x_0/2) = f^{(r)}(\xi).(x_0 - x_0/2)$. But by the lemma hypothesis on $f^{(r)}$ we have $|f^{(r)}(\xi)| \geq |f^{(r)}(x_0)| > 4c/x_0$. Hence

$$2c \geq |f^{(r-1)}(x_0) - f^{(r-1)}(x_0/2)| > \frac{4c}{x_0}(x_0 - x_0/2) = 2c.$$

This contradiction shows that

$$|2^r x^r f^{(r)}(x^2)| \leq 2^r\, x^r\, \frac{4c}{x^2}$$

for all $x \in (0,1)$, and the right-hand side here is bounded by $2^{r+2}c$ since $r \geq 2$. Thus $g^{(i)}$ is strongly bounded for $i = 0, 1, \ldots, r$, and the lemma is proved. □

3.2. Lemma. *Let $F : (0,1) \to M$ be a definable, strongly bounded function. Then F has a 1-reparameterization, S say, with the additional property that, for each $\phi \in S$, either ϕ or $F \circ \phi$ is a polynomial (restricted to $(0,1)$) with strongly bounded coefficients.*

Proof. By o-minimality, choose elements $a_0 = 0 < a_1 < \ldots < a_p < a_{p+1} = 1$ of M so that, for each $i = 0, 1, \ldots, p$, F is of class $C^{(1)}$ and satisfies either $|F'| \leq 1$ throughout (a_i, a_{i+1}) or $|F'| \geq 1$ throughout (a_i, a_{i+1}).

In the first case, define $\phi_i : (0,1) \to M$ by $x \mapsto (a_{i+1} - a_i)x + a_i$.

In the second case (when F is certainly strictly monotone and continuous on (a_i, a_{i+1})) we set $b_i = \lim_{x \to a_i^+} F(x), b_{i+1} = \lim_{x \to a_{i+1}^-} F(x)$ and define $\phi_i : (0,1) \to M$ by $x \mapsto F^{-1}((b_{i+1} - b_i)x + b_i)$.

In either case, $\mathrm{range}(\phi_i) = (a_i, a_{i+1})$ and both ϕ_i and $F \circ \phi_i$ are of class $C^{(1)}$ throughout $(0,1)$ with derivatives bounded by 1 in absolute value. Further, at least one of these functions is linear with coefficients in $[-1,1]$. It is now clear that $S = \{\phi_0, \phi_1, \ldots, \phi_p, \hat{a_1}, \ldots, \hat{a_p}\}$ is a 1-reparameterization of F with the required additional property, where each $\hat{a_i}$ denotes the constant function on $(0,1)$ with value a_i. □

3.3. Lemma. *Let $r \geq 1$ and suppose that $F : (0,1) \to M$ is a definable, strongly bounded function. Then F has an r-reparameterization (with the additional property that, for each ϕ in it, either ϕ or $F \circ \phi$ is a polynomial (restricted to $(0,1)$) with strongly bounded coefficients).*

Proof. The proof (of the whole statement, including the parenthetical property) is by induction on r. The case $r = 1$ being Lemma 3.2, suppose that $r \geq 2$ and that S is an $(r-1)$-reparameterization of F with the additional property. Let $\phi \in S$ and write $\{\phi, F \circ \phi\} = \{g, h\}$ where g is a polynomial (restricted to $(0,1)$) with strongly bounded coefficients. Thus, in particular, $g^{(i)}$ exists and is strongly bounded for all i. However, we only know that $h^{(i)}$ exists, is continuous, and is strongly bounded for $i = 0, \ldots, r-1$. In order to apply Lemma 3.1 we use o-minimality to pick elements $0 = a_0 < a_1 < \ldots < a_{p_\phi} < a_{p_\phi+1} = 1$ in M (depending on ϕ) so that, for each $i = 0. \ldots, p_\phi$, the function h is of class $C^{(r)}$ on (a_i, a_{i+1}) and $|h^{(r)}|$ is (weakly) monotonic on (a_i, a_{i+1}).

Let $\theta_{\phi,i} : (0,1) \to (0,1)$ be defined by

$$\theta_{\phi,i}(x) = \begin{cases} (a_{i+1} - a_i)x + a_i, & \text{if } |h^{(r)}| \text{ is (weakly) decreasing,} \\ (a_i - a_{i+1})x + a_{i+1}, & \text{if } |h^{(r)}| \text{ is (weakly) increasing.} \end{cases}$$

(We choose the first option, say, if $|h^{(r)}|$ is constant.)

It is immediate from the inductive hypothesis that $h \circ \theta_{\phi,i} : (0,1) \to M$ is of class $C^{(r)}$, and that $(h \circ \theta_{\phi,i})^{(i)}$ is strongly bounded for $i = 0, \dots, r-1$. Further, $|(h \circ \theta_{\phi,i})^{(r)}|$ is (weakly) decreasing. Let $\rho : (0,1) \to (0,1)$ be the $C^{(\infty)}$ bijection sending x to x^2. Then by Lemma 3.1, the function $h \circ \theta_{\phi,i} \circ \rho : (0,1) \to M$ has strongly bounded i-th derivative for $i = 0, \dots, r$. Of course, the function $g \circ \theta_{\phi,i} \circ \rho$ is still a polynomial with strongly bounded coefficients and $\{h \circ \theta_{\phi,i} \circ \rho, g \circ \theta_{\phi,i} \circ \rho\} = \{\phi \circ \theta_{\phi,i} \circ \rho, F \circ (\phi \circ \theta_{\phi,i} \circ \rho)\}$. Notice also that as i varies from 0 to p_ϕ, $\mathrm{range}(\phi \circ \theta_{\phi,i} \circ \rho)$ covers $\mathrm{range}(\phi)$ apart from finitely many points. So we only have to add finitely many constant functions (taking values in $(0,1)$) to the set $\{\phi \circ \theta_{\phi,i} \circ \rho : \phi \in S\}$ in order for it to become an r-reparameterization of F with the required additional property. This completes the induction and the proof of the lemma. □

3.4. Corollary. *Let X be a strongly bounded subset of M and $F : X \to M$ a strongly bounded function. Then for all $r \geq 1$, F has an r-reparameterization.*

Proof. Since X is a (finite) union of strongly bounded intervals and points, it clearly has an r-parameterization, S say, by linear and constant functions. Now use Lemma 3.3 to r-reparameterize each funcion $F \circ \phi : (0,1) \to M$, for $\phi \in S$, and take the union of these reparameterizations. □

We now proceed to the case of functions taking values in M^n for $n \geq 2$. However, there is nothing special about unary functions in this process, so we do the general case now.

3.5. Lemma. *Let $m, r \geq 1$ and assume that every definable, strongly bounded function with domain a subset X of M^ℓ (for some $\ell \leq m$) and range a subset of M, has an r-reparameterization. Then for any $n \geq 1$, the same is true for such functions having range a subset of M^n (and domain X).*

Proof. It is clearly sufficient (by the obvious inductive argument) to show that if $n \geq 2$ and $F : X \to M^{n-1}, f : X \to M$ are definable, strongly bounded functions such that F has an r-reparameterization, then so does the function $\langle F, f \rangle : X \to M^n$, where we may as well suppose that X is a definable (strongly bounded) subset of M^m.

So let S be an r-reparameterization of F, and let $\phi \in S$. Say $\phi : (0,1)^\ell \to X$ where $\ell = \dim(X) \leq m$. Apply the hypothesis of the lemma to the function $f \circ \phi : (0,1)^\ell \to M$, to obtain an r-reparameterization of it, T_ϕ say. Then each $\psi \in T_\phi$ has domain $(0,1)^\ell$, and it clearly follows by repeated use of the Chain Rule that each function $(\phi \circ \psi)^{(\alpha)} : (0,1)^\ell \to M^m$, for $\alpha \in \mathbb{N}^\ell$ with $|\alpha| \leq r$, is strongly bounded. It is now easy to check that $\{\phi \circ \psi : \phi \in S, \psi \in T_\phi\}$ is an r-reparameterization of $\langle F, f \rangle$, as required. □

3.6. Corollary. *Let $n \geq 1$ and suppose that $F : X \to M^n$ is a strongly bounded function, where X is a (strongly bounded) subset of M. Then for any $r \geq 1$, F has an r-reparameterization.*

Proof. This is immediate from Corollary 3.4 and the case $m = 1$ of Lemma 3.5. □

4. Some questions of convergence

In Gromov's proof things can be arranged, it seems, so that derivatives are a priori bounded, and we need to be able to reduce to this situation. We shall achieve this by first truncating our given function and finding the reparameterization for the truncation. We then let the truncations converge to the original function. So we require an observation that allows us to conclude that the reparameterizations converge as well. In fact, we lose one level of differentiability here, but this will hardly matter. The final proof of Theorems 2.3 and 2.5 are so arranged that we only require a theory of convergence for unary functions, so we only treat that case here.

So suppose that $N \geq 1, N \in \mathbb{N}$, and that $\{F_t : (0,1) \to (0,1)^N : t \in (0,1)\}$ is a definable family of functions (meaning that the map sending $\langle t, x\rangle$ to $F_t(x)$ is a definable function on $(0,1)^2$). Suppose further that $r \geq 1$, that all the functions F_t are of class $C^{(r)}$, and that their derivatives $F_t^{(i)}$ are strongly bounded for $i = 0, \ldots, r$. Clearly this implies a uniform finite bound c say. Using o-minimality we may define a function $F_0 : (0,1) \to [0,1]^N$ by $F_0(x) = \lim_{t\to 0^+} F_t(x)$.

Now the fact that $r \geq 1$ implies that F_0 is continuous. For suppose that $x_1, x_2 \in (0,1)$ are distinct. Choose $t \in (0,1)$ so that $||F_0(x_i) - F_t(x_i)|| \leq |x_1 - x_2|$ for $i = 1, 2$. (We use the sup norm $||\langle u_1, \ldots, u_p\rangle|| = \max\{|u_1|, \ldots, |u_p|\}$ on cartesian products of M throughout.) By the Mean Value Theorem (in M), we also have that $||F_t(x_1) - F_t(x_2)|| \leq N\,c\,|x_1 - x_2|$ (as c is a bound for $F_t'(x)$ for $x \in [x_1, x_2]$). Thus

$$||F_0(x_1) - F_0(x_2)|| \leq ||F_0(x_1) - F_t(x_1)|| + ||F_t(x_1) - F_t(x_2)|| + ||F_t(x_2) - F_0(x_2)||$$

$$\leq (Nc + 2)\,|x_1 - x_2|,$$

whence the continuity of F_0.

One can now go on to show that for each $i = 0, \ldots, r-1$, F_0 is of class $C^{(i)}$, $F_t^{(i)}$ is strongly bounded and, indeed, that $F_0^{(i)}(x) = \lim_{t\to 0^+} F_t^{(i)}(x)$ for each $x \in (0,1)$. (This result properly belongs to the theory of "definably Banach" spaces (over o-minimal structures) currently being developed by the second author and Margaret Thomas. The simplest example is the set $\Omega^{(r)}$ of all M-definable functions $F : (0,1) \to M$ with continuous and bounded derivatives up to order r, which is naturally a normed vector space for the field structure on M and with norm $||F||^{(r)} := \sup_{x\in(0,1), i=0,\ldots,r} |F^{(i)}(x)|$. If $\sigma = \{F_t : t \in (0,1)\}$ is a definable family contained in $\Omega^{(r)}$ then it is clear what we should mean by saying that σ is *Cauchy* (as $t \to 0^+$), and it is routine to check that the pointwise limit of of σ is, indeed, the $||\cdot||^{(r)}$–limit and lies in $\Omega^{(r)}$ if σ is Cauchy. More importantly for us here, however, is the fact, borrowed and modified from the classical theory, that $\Omega^{(r)}$ is "*definably compactly*" contained in $\Omega^{(r-1)}$ for all $r \geq 1$. In other words, *every* $||\cdot||^{(r)}$-bounded, definable family σ in $\Omega^{(r)}$ is Cauchy in $\Omega^{(r-1)}$, and hence the pointwise limit of σ lies in $\Omega^{(r-1)}$. The crucial point in the o-minimal setting is that one knows, a priori, that this limit function is $(r-1)$–times continuously differentiable at all but finitely many $x \in (0,1)$.)

We now consider, for each $t \in (0,1)$, the *set* S_t of co-ordinate functions of F_t. Let us suppose that it parameterizes $(0,1)$, so that it is an r-parameterization of $(0,1)$. We define S_0 to be the set of functions $\phi|_{\phi^{-1}[(0,1)]}$ for $\phi : (0,1) \to [0,1]$ a co-ordinate function of F_0. (Clearly ϕ cannot take values outside the closed interval $[0,1]$.) Then

(A) $\bigcup_{\psi \in S_0} \mathrm{range}(\psi) = (0,1) \setminus T$ for some finite set $T \subset (0,1)$. (For otherwise, by o-minimality, there would be a non-empty, open subinterval of $(0,1)$ missed by each $\psi \in S_0$ and hence missed by each corresponding co-ordinate function ϕ of F_0. But this easily contradicts the facts that each S_t parameterizes $(0,1)$ and $\lim_{t \to 0^+} F_t(x) = F_0(x)$ (for $x \in (0,1)$), bearing in mind the fact that as $r \geq 1$, the derivative of each co-ordinate function of F_t has a uniform finite bound.)

Notice also that

(B) each function $\psi \in S_0$ has domain an open subset of $(0,1)$ (which might have infinite complement in $(0,1)$), is of class $C^{(r-1)}$ and is such that $\psi^{(i)}$ is strongly bounded for $i = 0, \ldots, r-1$.

We now apply these remarks to set up the inductive process involved in the proofs of 2.3 and 2.5. We fix $m \geq 1$ in 4.1, 4.2 and 4.3.

4.1. Notation.

(1) For U a definable, open subset of M^{m+1}, we write $V \subset\subset U$ to mean that V is a definable, open subset of M^{m+1} with $V \subset U$ and $\dim(U \setminus V) \leq m$.

(2) For $\phi : (0,1) \to M$ a definable function, we define $I_\phi : (0,1)^{m+1} \to (0,1)^m \times M$ by $\langle x_1, \ldots, x_m, x_{m+1} \rangle \mapsto \langle x_1, \ldots, x_m, \phi(x_{m+1}) \rangle$. If $X \subset M^{m+1}$ and $f : X \to M^n$ are definable, f_ϕ denotes $f \circ I_\phi$ (having domain $I_\phi^{-1}[X]$).

4.2. Lemma. *Suppose that $n \geq 1, U \subset\subset (0,1)^{m+1}$ and that $f : U \to M^n$ is a definable, strongly bounded function. Suppose further that for each $i = 1, \ldots, m$, $\partial f / \partial x_i$ exists, is continuous and is strongly bounded (on U).*

Then for each $r \geq 2$, there exists an $(r-1)$-parameterization of a cofinite subset of $(0,1)$, S say, and a set $V \subset\subset U$ such that for each $\phi \in S$, $I_\phi[V] \subset U$, f_ϕ is of class $C^{(1)}$ on V, and **all** *its first partial derivatives $\partial f / \partial x_i$, $i = 1, \ldots, m+1$ are strongly bounded (on V).*

Proof. We treat only the case $n = 1$. The general case follows using an argument similar to that in the proof of Lemma 3.5. Our S will be constructed from a certain limit set S_0 (of a suitable family $S_t : t \in (0,1)$) as described above. (Notice that properties (A) and (B) are not quite the conditions for an $(r-1)$-parameterization of $(0,1)$, though (A) is precisely what we are asking for here, and (B) can easily be modified by composing with linear functions, as we shall see.)

Now, by o-minimality, let $W \subset\subset U$ be such that f is of class $C^{(1)}$ on W and, for each $t, y \in (0,1)$, let $W_t(y)$ denote the set of those $\overline{x} \in (0,1)^m$ such that the point $\langle \overline{x}, y\rangle$ is at a distance at least t from the set $([0,1]^m \times \{y\}) \backslash W$. It follows that the map $\overline{x} \mapsto |\partial f/\partial x_{m+1}(\overline{x}, y)|$ is defined and continuous on $W_t(y)$ and hence achieves its maximum value at some point $s_t(y) \in W_t(y)$, provided that this set is non-empty. Since M admits definable Skolem functions it follows that s may be taken to be a definable function in both t and y (taking the value $\langle 1/2, \ldots, 1/2\rangle$, say, if $W_t(y) = \emptyset$) and that (in all cases)

$$(*) \qquad \begin{array}{c} \forall t \in (0,1), \forall y \in (0,1), \forall \overline{x} \in W_t(y), \text{ we have } \langle s_t(y), y\rangle \in W \text{ and} \\ |\partial f/\partial x_{m+1}(s_t(y), y)| \geq |\partial f/\partial x_{m+1}(\overline{x}, y)|. \end{array}$$

Now consider the definable family $\{g_t : (0,1) \to (0,1)^m \times M : t \in (0,1)\}$ given by $g_t(y) := \langle s_t(y), f(s_t(y), y)\rangle$ (where we give f the value 0, say, if $\langle s_t(y), y)\rangle \notin U$), and apply Corollary 3.6 to obtain an r-reparameterization, S_t say, of g_t, for each $t \in (0,1)$. Now if we assume that M is $\aleph_0$-saturated (which is harmless–see below) then it follows easily (using the fact that M admits definable Skolem functions) that for some $N \in \mathbb{N}$, S_t may be taken as the set of co-ordinate functions of some definable function $F_t : (0,1) \to (0,1)^N$, where the family $\{F_t : t \in (0,1)\}$ is also definable. Let S_0 be the limit, as $t \to 0^+$, of this family as described at the beginning of this section. By splitting the functions in S_0, we may suppose that they are all either constant or injective and have domains an open subinterval of $(0,1)$. Now throw away the constant functions and compose each remaining function with a suitable injective linear function (with coefficients in $[-1,1]$), thereby arriving at an $(r-1)$-parameterization, S say, of a cofinite subset of $(0,1)$.

Now set $V := \left((0,1)^{m+1} \backslash \bigcup_{\phi \in S} I_\phi^{-1}[(0,1)^{m+1} \backslash W]\right) \cap U$. (See 4.1(2).)

Then the injectivity (and continuity) of the ϕ's imply that $V \subset\subset U$. Clearly $I_\phi[V] \subset W \subset U$ and so, also, the function f_ϕ is of class $C^{(1)}$ on V (for $\phi \in S$). It only remains to show that if $\phi \in S$ and $\langle \overline{x}_0, y_0\rangle \in V$, then $\partial f_\phi/\partial x_i(\overline{x}_0, y_0)$ is finite, for $i = 1, \ldots, m+1$.

Now since $\langle \overline{x}_0, \phi(y_0)\rangle \in W \subset U$, this is clear (by the lemma hypothesis) for $i = 1, \ldots, m$. For the remaining case we note that there is some linear function λ (with finite coefficients) and some function ψ in S_0 (or, rather, a subfunction of a function in S_0) such that $\phi(y) = \psi(\lambda(y))$ (for all $y \in (0,1)$), and so it is clearly sufficient to show that if $y_1 \in \mathrm{dom}(\psi)$, then

$\psi'(y_1).\partial f/\partial x_{m+1}(\overline{x}_0, \psi(y_1))$ is finite, where we also know that $\langle \overline{x}_0, \psi(y_1)\rangle \in W$. Since W is open it follows that

(i) $\overline{x}_0 \in W_t(\psi(y_1))$ for all sufficiently small $t \in (0,1)$.

Now by definition of S_0, there is, for each $t \in (0,1)$, (uniformly) a function $\phi_t \in S_t$ such that $\lim_{t\to 0^+} \phi_t(y_1) = \psi(y_1)$ and (as $r \geq 2$), $\lim_{t\to 0^+} \phi_t'(y_1) = \psi'(y_1)$. Hence

(ii) $|\partial f/\partial x_{m+1}(\overline{x}_0, \psi(y_1)) - \partial f/\partial x_{m+1}(\overline{x}_0, \phi_t(y_1))| \leq 1$ (and $\langle \overline{x}_0, \phi_t(y_1)\rangle \in W$) for sufficiently small $t \in (0,1)$ (by the continuity of $\partial f/\partial x_{m+1}$ on W), and

(iii) $|\phi_t'(y_1) - \psi'(y_1)| \leq |\partial f/\partial x_{m+1}(\overline{x}_0, \psi(y_1))|^{-1}$ for sufficiently small $t \in (0,1)$, and

(iv) $\overline{x}_0 \in W_t(\phi_t(y_1))$ for all sufficiently small $t \in (0,1)$ (since if (i) holds for some $t_0 \in (0,1)$, then (iv) holds for any $t < t_0/2$ satisfying $|\psi(y_1) - \phi_t(y_1)| < t_0/2$).

Thus, if we select some $t \in (0,1)$ such that (ii)–(iv) all hold simultaneously, we see that

$$|\psi'(y_1).\frac{\partial f}{\partial x_{m+1}}(\overline{x}_0, \psi(y_1))| \leq |\phi_t'(y_1)|.|\frac{\partial f}{\partial x_{m+1}}(\overline{x}_0, \psi(y_1))| + 1, \text{ (by (iii))},$$

$$\leq |\phi_t'(y_1)|.|\frac{\partial f}{\partial x_{m+1}}(\overline{x}_0, \phi_t(y_1))| + |\phi_t'(y_1)| + 1, \text{ (by (ii))},$$

$$\leq |\phi_t'(y_1)|.|\frac{\partial f}{\partial x_{m+1}}(s_t(\phi_t(y_1)), \phi_t(y_1))| + |\phi_t'(y_1)| + 1, \text{ (by (iv) and (*))}.$$

However, $|\phi_t'(y_1)|$ is certainly finite (since $\phi_t \in S_t$), so it suffices to show that

$$\phi_t'(y_1).\partial f/\partial x_{m+1}(s_t(\phi_t(y_1)), \phi_t(y_1))$$

is finite. But since S_t is an r-reparameterization of g_t it follows that

(v) $(s_t \circ \phi_t)'(y_1)$ is finite, and

(vi) $(d/dy)|_{y=y_1} f(s_t \circ \phi_t(y), \phi_t(y))$ is finite.

Now by (vi), the quantity

$$(s \circ \phi)'(y_1) \cdot \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}\rangle(s_t(\phi_t(y_1)), \phi_t(y_1)) + \phi_t'(y_1) \cdot \frac{\partial f}{\partial x_{m+1}}(s_t(\phi_t(y_1)), \phi_t(y_1))$$

is finite. Also the scalar product term here is finite by (v) and the strong boundedness of the functions $\partial f/\partial x_i$ (for $i = 1, \ldots, m$) as given by the Lemma hypothesis. (Note that $\langle s_t(\phi_t(y_1)), \phi_t(y_1)\rangle \in W \subset U$ by (iv) and (*).) Hence the second term is finite, which is what we had to show. □

Remark. The assumption of $\aleph_0$-saturation here is harmless because the hypothesis and conclusion are definable properties of U, f, V and S and, further, dimension is uniformly definable in parameters (which is needed to express "$V \subset\subset U$"). Hence it is sufficient to establish the result in any elementary extension of M.

4.3. Corollary. *Suppose that $r, n \geq 1$, $U \subset\subset (0,1)^{m+1}$ and that $f : U \to M^n$ is a definable, strongly bounded function. Suppose further that for each $\alpha = \langle \alpha_1, \ldots, \alpha_{m+1} \rangle \in \mathbb{N}^{m+1}$ with $|\alpha| \leq r$ and $\alpha_{m+1} = 0$, $f^{(\alpha)}$ exists, is continuous and is strongly bounded (on U).*

Then for each $k \geq 0$ there exists a set $V_k \subset\subset U$ and an r-parameterization of a cofinite subset of $(0,1)$, S_k say, such that for each $\phi \in S_k$, $I_k[V_k] \subset U$, f_ϕ is of class $C^{(r)}$ on V_k and all its derivatives $f_\phi^{(\alpha)}$ (for $\alpha = \langle \alpha_1, \ldots, \alpha_{m+1} \rangle \in \mathbb{N}^{m+1}, |\alpha| \leq r, \alpha_{m+1} \leq k$) are strongly bounded (on V_k).

Proof. We may take $V_0 \subset\subset U$ such that f is a function of class $C^{(r)}$ on V_0 (by o-minimality), and $S_0 = \{\mathrm{id}|_{(0,1)}\}$. So suppose, inductively, that V_k and S_k have been constructed with the required properties.

Let $\Delta := \{\alpha = \langle \alpha_1, \ldots, \alpha_{m+1} \rangle \in \mathbb{N}^{m+1} : |\alpha| \leq r-1, \alpha_{m+1} \leq k\}$, set $\tilde{n} := \#\Delta \cdot \#S_k$, and let $F = \langle F_1, \ldots, F_{\tilde{n}} \rangle : V_k \to M$ be an enumeration of all the functions $f_\phi^{(\alpha)} : V_k \to M$ for $\phi \in S_k$ and $\alpha \in \Delta$. Then the hypotheses of Lemma 4.2 obtain (with F for f, V_k for U, $\tilde{n} \cdot n$ for n and $r+1$ in place of r) — note the "$r-1$" in the definition of Δ — so we may choose an r-parameterization, S say, of a cofinite subset of $(0,1)$ and a set $V_{k+1} \subset\subset V_k$ such that for each $\psi \in S$, $I_\psi[V_{k+1}] \subset V_k$ (so that, in particular $f_{\phi\circ\psi} = (f_\phi)_\psi = f_\phi \circ I_\psi$ is of class $C^{(r)}$ on V_{k+1}, being the composition of $C^{(r)}$ functions) and so that each function $(f_\phi^{(\alpha)})_\psi$ is of class $C^{(1)}$ with

(†) $\dfrac{\partial}{\partial x_i}((f_\phi^{(\alpha)})_\psi)$ strongly bounded on V_{k+1} for $i = 1, \ldots, m+1, \alpha \in \Delta,$ and $\phi \in S_k$.

Thus, we define

$$S_{k+1} := \{\phi \circ \psi : \phi \in S_k, \psi \in S\},$$

and it remains to show that if $\alpha = \langle \alpha_1, \ldots, \alpha_{m+1} \rangle \in \mathbb{N}^{m+1}$, with $|\alpha| \leq r$ and $\alpha_{m+1} \leq k+1$, and if $\phi \circ \psi \in S_{k+1}$ then $(f_{\phi\circ\psi})^{(\alpha)}$ is strongly bounded on V_{k+1}.

Now if $\alpha_{m+1} = 0$, then this is clear because $(f_{\phi\circ\psi})^{(\alpha)} = (f_\phi^{(\alpha)})_\psi$ and $f_\phi^{(\alpha)}$ is strongly bounded. If $\alpha_{m+1} > 0$, then $(f_{\phi\circ\psi})^{(\alpha)} = \partial/\partial x_{m+1}(f_{\phi\circ\psi}^{(\beta)})$ for some $\beta \in \Delta$. Further, for $\overline{a} := \langle a_1, \ldots, a_{m+1} \rangle \in V_{k+1}$,

$$(f_{\phi\circ\psi})^{(\beta)}(\overline{a}) = \psi^{(\alpha_{m+1}-1)}(a_{m+1}) \cdot (f_\phi^{(\beta)})_\psi(\overline{a}).$$

Thus

$$(f_{\phi\circ\psi})^{(\alpha)}(\overline{a}) = \psi^{(\alpha_{m+1})}(a_{m+1}) \cdot (f_\phi^{(\beta)})_\psi(\overline{a}) + \psi^{(\alpha_{m+1}-1)}(a_{m+1}) \cdot \frac{\partial}{\partial x_{m+1}}(f_\phi^{(\beta)})_\psi(\overline{a})$$

which is finite since we have $\alpha_{m+1} \leq |\alpha| \leq r$ and $\beta \in \Delta$ (see (†)), and $\psi \in S$, so $\psi^{(\alpha_{m+1}-1)}$ and $\psi^{(\alpha_{m+1})}$ are strongly bounded. □

5. The proofs of 2.3 and 2.5

For each $m \geq 1$ consider the following two statements.

$(\mathrm{I})_m$ For all $r, n \geq 1$ and all definable, strongly bounded functions $F : (0,1)^m \to M^n$, there exists an r-reparameterization of F.

$(\mathrm{II})_m$ For all $r \geq 1$, every definable, strongly bounded subset $X \subset M^{m+1}$, there exists an r-parameterization of X.

Note that $(\mathrm{I})_1$ holds by Corollary 3.6. Also, $(\mathrm{II})_m$ makes sense for $m = 0$ and clearly holds in this case (via linear functions). We proceed by induction to show that the statements hold for all $m \geq 1$. So suppose that $m \geq 1$ and that $(\mathrm{I})_\ell$ holds for all $\ell \leq m$ and that $(\mathrm{II})_\ell$ holds for all $\ell < m$. We shall show that $(\mathrm{II})_m$ holds and then that $(\mathrm{I})_{m+1}$ holds.

For $(\mathrm{II})_m$, let $r \geq 1$ and $X \subset M^{m+1}$ be definable and strongly bounded. We may clearly assume that X is a cell in M^{m+1}, and we do the more difficult of the two cases, namely $X = (f, g)_Y$ where Y is a (strongly bounded) cell in M^m, and leave the other case, $X = \mathrm{graph}(f|Y)$, to the reader.

So let S be an r-parameterization of Y (using $(\mathrm{II})_{m-1}$) and for each $\phi \in S$ let T_ϕ be an r-reparameterization of the function $\langle f \circ \phi, g \circ \phi \rangle : (0,1)^\ell \to M^2$, where $\ell = \dim(Y)$ (using $(\mathrm{II})_\ell$). Then for each $\psi \in T_\phi$, define $\theta_{\phi,\psi} : (0,1)^{\ell+1} \to X$ by

$$\theta_{\phi,\psi}(\overline{x}) := \langle \phi \circ \psi(x_1, \ldots, x_\ell), (1 - x_{\ell+1}) f \circ \phi \circ \psi(x_1, \ldots, x_\ell) + x_{\ell+1} g \circ \phi \circ \psi(x_1, \ldots, x_\ell) \rangle$$

where $\overline{x} = \langle x_1, \ldots, x_{\ell+1} \rangle$. Then the set $\{\theta_{\phi,\psi} : \phi \in S, \psi \in T_\phi\}$ is readily seen to be an r-parameterization of X.

For $(\mathrm{I})_{m+1}$ we need only do the case $n = 1$ (by Lemma 3.5), so let $r \geq 1$ and $F : (0,1)^{m+1} \to M$ be a definable, strongly bounded function. By $(\mathrm{I})_m$ there exists, for each $u \in (0,1)$, an r-reparameterization, S_u say, of the function $F_u : (0,1)^m \to M : \overline{x} \mapsto F(\overline{x}, u)$, where $\overline{x} = \langle x_1, \ldots, x_m \rangle$ and by using a saturation and Skolem function argument (just as in the proof of Lemma 4.2) we may suppose that there exist definable families of functions $\{{}^{(1)}\phi_u : u \in (0,1)\}, \ldots, \{{}^{(N)}\phi_u : u \in (0,1)\}$ such that $S_u = \{{}^{(1)}\phi_u, \ldots, {}^{(N)}\phi_u\}$.

Now, for $j = 1, \ldots, N$ define the function ${}^{(j)}F : (0,1)^{m+1} \to M$ by ${}^{(j)}F(\overline{x}, u) := F({}^{(j)}\phi(\overline{x}, u), u)$. Let

$${}^{*}F := \langle {}^{(1)}\phi, \ldots, {}^{(N)}\phi, {}^{(1)}F, \ldots, {}^{(N)}F \rangle : (0,1)^{m+1} \to M^{mN+N}$$

and notice that the hypotheses of Corollary 4.3 hold with ${}^{*}F$ for f, $(0,1)^{m+1}$ for U, $mN + N$ for n. (This is just a restatement of the fact that S_u is an r-reparameterization of F_u, uniformly in u.) So we apply 4.3 with $k = r$, to obtain $V_r \subset\subset (0,1)^{m+1}$ and S_r with the properties stated. Now if $V_r = (0,1)^{m+1}$ and S_r were an r-parameterization of all of $(0,1)$, then we could simply take our required r-reparameterization of F to consist of the functions ${}^{(j)}\phi_\psi$ for $j = 1, \ldots, N$ and $\psi \in S_r$. As it is, we at least know that the union of the ranges of these functions (on $(0,1)^{m+1}$) covers $(0,1)^{m+1}$ apart from finitely many planes $\{x_{m+1} = a\}$, and it follows that if we restrict them to the (open) set V_r (where they are all of class $C^{(r)}$ and satisfy the bounding condition for r–reparameterizability) then they still cover a subset of $(0,1)^{m+1}$ of codimension ℓ, for some $\ell \leq m$.

Using the (now proven) $(\mathrm{II})_m$, let T_1 be an r-parameterization of V_r and T_2 an r-parameterization of the ℓ-dimensional set $(0,1)^{m+1} - \bigcup_{1\leq j\leq N} {}^{(j)}\phi_\psi[V_r]$.

For each $\theta \in T_2$ we may apply $(\mathrm{I})_\ell$ to obtain an r-reparameterization, U_θ say, of the function $F \circ \theta : (0,1)^\ell \to M$. The required r-reparameterization of F is now given by

$$\{{}^{(j)}\phi_\psi \circ \chi : j = 1\ldots, N, \psi \in S_r, \chi \in T_1\} \cup \{\hat{\theta} \circ \hat{\lambda} : \theta \in T_2, \lambda \in U_\theta\}$$

where the $\hat{}$ denotes extension of the domain of a function from $(0,1)^\ell$ to $(0,1)^m$ (but leaving its values independent of the last $m - \ell$ variables).

This completes the proof of $(\mathrm{I})_{m+1}$, and the induction is complete. In particular, Theorem 2.3 is now proven. Theorem 2.5 requires one more step and we leave this to the reader. □

5.1. Corollary. *Let $m, r \geq 1$ and suppose that $X \subset (0,1)^m$ is a definable set. Then there exists a finite set S of functions, each mapping $(0,1)^{\dim(X)}$ to X and of class $C^{(r)}$ such that*

(1) $\bigcup_{\phi\in S} \mathrm{range}(\phi) = X$ and

(2) $|\phi^{(\alpha)}(\overline{x})| \leq 1$ for each $\phi \in S, \alpha \in \mathbb{N}^{\dim(X)}$ with $|\alpha| \leq r$ and all $\overline{x} \in (0,1)^{\dim(X)}$.

Proof. Let S^* be an r-parameterization of X (as given by Theorem 2.3). Then (1) holds for S^* and (2) holds with c in place of 1, for some $c \in \mathbb{N}$. Cover $(0,1)^{\dim(X)}$ with $(2c)^{\dim(X)}$ cubes of side $1/c$ and for each such cube K let $\lambda_K : (0,1)^{\dim(X)} \to K$ be the obvious linear bijection. Then the set of all $\phi \circ \lambda_K$'s, as ϕ varies over S^* and K over the cover, is the required S. The details are left to the reader. □

As usual, the existence of definable Skolem functions and a saturation argument imply a uniform version.

5.2. Corollary. *Let $n, m, r \geq 1$ and suppose that $X \subset (0,1)^n \times M^m$ is a definable family. Then there exists $N \in \mathbb{N}$ and, for each $\overline{y} \in M^m$, a set $S_{\overline{y}}$ of N functions, each mapping $(0,1)^{\dim(X_{\overline{y}})}$ to $X_{\overline{y}}$ and each of class $C^{(r)}$, such that*

(1) $\bigcup_{\phi\in S_{\overline{y}}} \mathrm{range}(\phi) = X_{\overline{y}}$ and

(2) $|\phi^{(\alpha)}(\overline{x})| \le 1$ *for each* $\phi \in S_{\overline{y}}, \alpha \in \mathbb{N}^{\dim(X_{\overline{y}})}$ *with* $|\alpha| \le r$ *and all* $\overline{x} \in (0,1)^{\dim(X_{\overline{y}})}$.

Further, the functions comprising $S_{\overline{y}}$ *depend definably on* $\overline{y}$. □

6. The "Main Lemma"

By a *hypersurface of degree* d *(in* $\mathbb{R}^n$*)* we mean a set of the form $\{x \in \mathbb{R}^n : f(x) = 0\}$ where f is a nonzero polynomial over $\mathbb{R}$ of degree d in n variables. If $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ is a family (cf §1), the *fibre dimension* of Z means the maximum dimension of a fibre of Z (in situations where this makes sense).

The main device in the diophantine part of the argument here, as in [2, 11, 13, 14], is that the rational points of height $\le H$ in the image of a (sufficiently smooth) map $\phi : [-1,1]^k \to \mathbb{R}^n$, where $k < n$, reside on "few" hypersurfaces of prescribed degree d relative to norms of ϕ and its derivatives up to some suitable order (depending on d). A similar result is achieved by p-adic means in the algebraic setting in [6].

Already in [2], where $k = 1$, the dependence of the estimate on these norms was eliminated by the observation that, for an algebraic or compact analytic curve, the controlled oscillation implies that intervals on which derivatives are "large" have to be "short" and "few". (Another manifestation of "tameness" in [2] is the compactness argument in the proof Theorem 1.) This device has also been used to obtain bounds for the rational points of a pfaff curve in [15].

Here we use the r-parameterization results of §2–5.

6.1. Proposition. *Let* $k, n \in \mathbb{N}$ *with* $k < n$. *Then there is for each* $d \in \mathbb{N}, d \ge 1$ *a nonnegative integer* $r = r(k,n,d)$ *and positive constants* $\epsilon(k,n,d), C(k,n,d)$ *with the following property.*

Suppose $\phi : (0,1)^k \to \mathbb{R}^n$ *is a function of class* $C^{(r)}$ *with* $|\phi^{(\alpha)}(x)| \le 1$ *for all* $x \in (0,1)^k$ *and all* $\alpha \in \mathbb{N}^k$ *with* $|\alpha| \le r$. *Let* $X = \phi((0,1)^k)$ *and* $H \ge 1$. *Then* $X(\mathbb{Q}, H)$ *is contained in the union of at most*

$$C(k,n,d)\, H^{\epsilon(k,n,d)}$$

hypersurfaces of degree $\le d$. *Further,* $\epsilon(k,n,d) \to 0$ *as* $d \to \infty$.

Proof. This follows from [13, 4.2], with $r(k,n,d)$ taken to be one more than the $b(k,n,d)$ therein. The constant c_{16} in that result corresponding to $C(k,n,d)$ here depends, in addition to k, n, d, on the domain of ϕ, and the size of the derivatives up to order r. So the conditions of the hypothesis on those derivatives and fixed domain mean that here it may be taken to depend only on k, n, d. That $\epsilon(k,n,d) \to 0$ as $d \to \infty$ is observed just before the proof of [13, 4.2]. □

6.2. Proposition. ("Main Lemma"). *Let* $Z \subset (0,1)^n \times M^m$ *be a definable family of fibre dimension* $k < n$. *Let* $\epsilon > 0$. *There is a* $d = d(\epsilon, k, n) \in \mathbb{N}$ *and a constant* $K(Z, \epsilon)$ *with the* *following property. For any $y \in Y$ and $H \geq 1$, the set $X(\mathbb{Q}, H)$, where $X = X_y$, is contained in the union of at most*

$$K(Z, \epsilon)H^{\epsilon}$$

hypersurfaces of degree $\leq d$.

Proof. Take d such that $\epsilon(k, n, d) \leq \epsilon$ and set $r = r(k, n, d)$ as in 6.1. By Corollary 5.2, there is an $N \in \mathbb{N}$ such that, for every $y \in Y$, there is an $r(k, n, d)$-parameterization, S_y say, of X_y consisting of at most N maps $\phi : (0, 1)^k \to \mathbb{R}^n$ having all derivatives up to order $r(k, n, d)$ of absolute value bounded by 1. To each map $\phi \in S_y$, by 6.1, we have that $\phi[(0, 1)^k](\mathbb{Q}, H)$ is contained in the union of at most $C(k, n, d)H^{\epsilon}$ hypersurfaces of degree $\leq d$. This establishes the result with $K(Z, \epsilon) = N \cdot C(k, n, d)$. □

7. Proof of Theorem 1.4

If $X \subset \mathbb{R}^n$ is definable and $k \leq n$, we denote by $\mathrm{reg_k(X)}$ the subset of C^1-smooth points of X of dimension k ([4, 1.8]).

7.1. Proof of 1.4.

Since the rational points of height $\leq H$ are stable under the maps $x \mapsto \pm x^{\pm 1}$, as are the algebraic parts of a set, we may suppose that $Z \subset [0, 1]^n \times \mathbb{R}^m$, and so, by a suitable induction on n, that $Z \subset (0, 1)^n \times \mathbb{R}^m$.

Consider first the situation in which $A, B, C \subset (0, 1)^n \times \mathbb{R}^m$ are definable sets with $A \cup B = C$. For any $y \in Y_C$, it is immediate that $X^{\mathrm{a}}_{A,y} \cup X^{\mathrm{a}}_{B,y} \subset X^{\mathrm{a}}_{C,y}$. Therefore if the theorem holds for A and B and ϵ, with sets $W(A, \epsilon), W(B, \epsilon)$ and constants $c(A, \epsilon), c(B, \epsilon)$ then it holds for C with

$$W(C, \epsilon) = W(A, \epsilon) \cup W(B, \epsilon), \quad c(C, \epsilon) = c(A, \epsilon) + c(B, \epsilon).$$

The proof will be by induction on the fibre dimension of Z. If the fibre dimension of Z is zero then there is a uniform bound c for the number of points in any fibre, and the theorem holds with $c(Z, \epsilon) = c$. Suppose then that $k > 0$ and the theorem holds for all families of fibre dimension $\leq k - 1$ and let $Z \subset (0, 1)^n \times \mathbb{R}^m$ be a definable family with fibre dimension k, and $\epsilon > 0$.

Suppose $k = n$. If $x \in \mathrm{reg_k(X)}$ of any fibre X, then X contains an open ball in $\mathbb{R}^n$ containing x. Therefore $x \in X^{\mathrm{a}}$. Moreover, for any $k \in \mathbb{N}$, the family

$$R_k(Z) = \{(x, y) : x \in \mathrm{reg_k(X_y)}\}$$

is definable ([4]). Thus the fibres of $A = R_n(Z)$ are subsets of the algebraic parts of the fibres of Z. So the conclusion for A holds with $W(A, \epsilon) = A$. The fibre dimension of $B = Z - A$ is $\leq k - 1$ and so the theorem holds for B by induction. So it may be assumed that $k < n$.

Let $\{x_1, x_2, \ldots, x_n\}$ be the coordinate system in $\mathbb{R}^n$. For a subset $\sigma \subset \{1, 2, \ldots, n\}$ let Π_σ denote the linear coordinate subspace of $\mathbb{R}^n$ with coordinates $\{x_i : i \in \sigma\}$, and let π_σ be the projection of $\mathbb{R}^n$ onto Π_σ. Let $S = S_{k,n} = \{\sigma \subset \{1, 2, \ldots, n\} : \#\sigma = k + 1\}$, and put $q = \#S$.

By 6.2, there is $d \in \mathbb{N}$ and a constant $\alpha(Z, \epsilon)$ such that, for any fibre X of Z, any subset $\sigma \in S$ and any $H \geq 1$, $(\pi_\sigma X)(\mathbb{Q}, H)$ is contained in the union of at most

$$\alpha(Z, \epsilon) H^{\epsilon/2q}$$

intersections of $\pi_\sigma X$ with hypersurfaces of degree $\leq d$. (So $X(\mathbb{Q}, H)$ is contained in at most $\alpha(Z, \epsilon)^q H^{\epsilon/2}$ intersections of X with cylinders on hypersurfaces of degree d in each such subspace.)

Let $T \subset \mathbb{R}^p$ parameterize real hypersurfaces of degree d in $\mathbb{R}^{k+1}$. (Note that $T = \mathbb{P}^{\nu(k,d)}(\mathbb{R})$, for suitable ν, is compact, so we can take $T \subset [-1, 1]^p \subset \mathbb{R}^p$.) Then

$$t = (t_\sigma : \sigma \in S) \in \prod T_\sigma \subset (\mathbb{R}^p)^q$$

corresponds to a choice of a hypersurface $L = L(t_\sigma)$ of degree d in each $(k + 1)$-dimensional linear coordinate subspace Π_σ of $\mathbb{R}^n$. We have the definable family

$$\Sigma = \{(x, (y, t)) : \pi_\sigma(x) \in L(t_\sigma), \text{ all } \sigma \in S\} \subset \mathbb{R}^n \times \big(\mathbb{R}^m \times (\mathbb{R}^p)^q\big).$$

Consider a fibre X of Σ. Since any choice of $k + 1$ coordinates is algebraically dependent, X is a closed algebraic set in $\mathbb{R}^n$ of dimension $\leq k$.

Replace Z by

$$\{(x, (y, t)) \in \mathbb{R}^n \times \big(\mathbb{R}^m \times (\mathbb{R}^p)^q\big) : (x, y) \in Z, t \in \prod T_\sigma\}$$

which has the same fibres (and so $Z \subset (-1, 1)^n \times \mathbb{R}^{m+pq}$).

The fibre dimension of $Z \cap \Sigma$ is $\leq k$. If

$$A_1 = \{(x, (y, t)) \in Z \cap \Sigma : x \notin \mathrm{reg_k(X_{Z \cap \Sigma, (y,t)})}\}$$

then the fibre dimension of A_1 is $\leq k - 1$ and, by induction, an estimate

$$c(A_1, \epsilon/2) H^{\epsilon/2}$$

holds for the number of rational points of height $\leq H$ on a fibre of A_1 outside (the fibre of) some suitable family $W(A_1, \epsilon/2)$. This includes in particular the case of an intersection of a fibre X of Z with cylinders on hypersurfaces of degree d when the intersection has dimension $\leq k-1$.

Similarly, the fibre dimension of

$$A_2 = \{(x, (y, t)) \in Z \cap \Sigma : x \notin \mathrm{reg_k(X_{\Sigma,(y,t)})}\}$$

is $\leq k-1$, and an estimate of the above form holds. Likewise for

$$A_3 = \{(x, (y, t)) \in Z \cap \Sigma : x \notin \mathrm{reg_k(X_{Z,(y,t)})}\}.$$

Let then B be the subset of $Z \cap \Sigma$ of points that are regular (of dimension k) in their fibres in $Z, \Sigma, Z \cap \Sigma$. Consider a point $P = (x, (y, t)) = (x, u) \in B$. In some small neighbourhood Δ of x in $\mathbb{R}^n$, each of the fibres

$$X_{Z\cap\Sigma,u}, \quad X_{Z,u}, \quad X_{\Sigma,u}$$

is a C^1 submanifold of $\mathbb{R}^n$ of dimension k. Since $X_{Z\cap\Sigma,u} \subset X_{Z,u}, X_{\Sigma,u}$ the sets locally coincide. But the intersection $X_{\Sigma,u} \cap \Delta$ is a semialgebraic set of dimension $k \geq 1$ if Δ is taken to be a small ball. Therefore $P \in X^{\mathrm{a}}_{B,u} \subset X^{\mathrm{a}}_{Z,u}$. The theorem holds for B with $W(B, \epsilon) = B$.

Let now $y \in Y, X = X_{Z,y}, H \geq 1$. Let $P \in X(\mathbb{Q}, H)$. So $\pi_\sigma(P) \in (\pi_\sigma X)(\mathbb{Q}, H)$ for any $\sigma \in S$, and therefore lies on one of the hypersurfaces t_σ. So P lies in one of

$$(\alpha(Z, \epsilon))^q \, H^{\epsilon/2}$$

fibres of $Z \cap \Sigma$. Further, either P lies in a family A_1, A_2, A_3 for which as estimate

$$c(A_i, \epsilon) H^{\epsilon/2}$$

holds for the number of rational points of height $\leq H$ outside the corresponding fibre of $W(A_i, \epsilon/2)$, or P lies in B. This completes the proof. □

7.2. Remark. In the one-dimensional case, application of the method to the function $y = e^x$, for which all intersection multiplicities can be precisely controlled, led to natural proofs of (the real versions of) classical transcendence statements [12]. (A similar method was found a little earlier by Laurent [9] (see also [17])). It would be interesting to make the present argument fully quantitative for e.g. the threefold $\log x \log y = \log w \log z, x, y, z, w > 0$ associated with the "four exponentials" conjecture, with a view to showing there can be only "few" solutions in some more general sense than the "six exponentials" theorem ([17]).

8. Dilation-integer points

The *homothetic dilation* tX of a set $X \subset \mathbb{R}^n$ by t is defined by

$$tX = \{(tx_1, \ldots, tx_n) : (x_1, \ldots, x_n) \in X\}.$$

It will always be assumed that $t \geq 1$. A *dilation-integer* point of X of *height* t is a point $x \in X$ such that $tx \in (tX)(\mathbb{Z})$. (Here $X(\mathbb{Z}) = X \cap \mathbb{Z}^n$.) Note that t need not be rational. The following result affirms a conjecture made in [13].

8.1. Theorem. *Let $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ be a definable family and $\epsilon > 0$. There is a definable family $W = W(Z, \epsilon) \subset Z$ and a constant $c(Z, \epsilon)$ with the following property. Let $X = X_{Z,y}$ and put $X_\epsilon = X_{W,y}$. Then $X_\epsilon \subset X^{\mathrm{a}}$ and*

$$\#(tX - tX_\epsilon)(\mathbb{Z}) \leq c(Z, \epsilon) t^\epsilon.$$

Proof. This follows by the method of proof of 1.4, using a result for dilation-integer points in the image of a map $\phi : (0,1)^k \to \mathbb{R}^n$ with suitably bounded derivatives adapted from [13, 4.1] in the same way that 6.1 above adapts [13, 4.2] for rational points. □

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School of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
UK

j.pila@bristol.ac.uk

Mathematical Institute
University of Oxford
24-29 St Giles'
Oxford OX1 3LB
UK

wilkie@maths.ox.ac.uk