# Sheaf cohomology in o-minimal structures

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**Abstract**

Here we prove the existence of sheaf cohomology theory in arbitrary o-minimal structures.

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# 1 Introduction

We work over an o-minimal structure $\mathcal{N} = (N, <, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}})$ where $(N, <)$ is a dense totally ordered set without end points, $\mathcal{C}$ is a collection of constants, $\mathcal{F}$ is a collection of functions from the cartesian products of $N$ into $N$ and $\mathcal{R}$ is a collection relations in the cartesian products of $N$. By definition, the structure $\mathcal{N} = (N, <, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}})$ is o-minimal if every definable subset of $N$ is a finite union of points and intervals with end points in $N \cup \{-\infty, +\infty\}$. The definable sets of $\mathcal{N}$ are the subsets of the cartesian products of $N$ whose elements satisfy a formula of first-order logic in the language $\{=, <, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}}\}$. The first-order formulas in this language are, roughly, the formulas that one can write down using these symbols, using symbols for variables, parameters from $N$, the logic connectives $\wedge$ (and), $\vee$ (or) and $\neg$ (not) and the quantifiers $\forall$ (for all) and $\exists$ (there exists). For example, a real closed field $(N, <, 0, 1, +, \cdot)$ is an o-minimal structure and definable sets in this real closed field are, by the Tarski-Seidenberg theorem, the semi-algebraic sets i.e., sets of the form $\{x \in N^k : f_1(x) = \cdots = f_l(x) = 0,\ g_1(x) > 0, \dots, g_m(x) > 0\}$ where $f_1, \dots, f_l, g_1, \dots, g_m \in N[X_1, \dots, X_k]$.

Given a real closed field $(N, <, 0, 1, +, \cdot)$ one often studies the geometry of real algebraic varities over $N$ and of algebraic varieties in the algebraic closure $N[\sqrt{-1}]$ of $N$. After identifying $N[\sqrt{-1}]$ with $N^2$ one can also study the geometry of these sets in the category of semi-algebraic sets with semi-algebraic maps. A semi-algebraic map is a map between semi-algebraic sets whose graph is a semi-algebraic set. More generally, given an arbitrary o-minimal structure $\mathcal{N}$ as above, one can study the geometry of definable sets with definable maps. A definable map is a map between definable sets whose graph is a definable set. For the basic theory of o-minimal structures we refer the reader to [vdd], and for basic semi-algebraic geometry we refer mainly to [BCR] but [br1], [dk1], [dk4], [dk5] and [k] are also very good references. O-minimal structures have turned out to be a wide ranging model theoretic generalization of semi-algebraic and sub-analytic geometry as we now explain.

The process used by the algebrists to go from the classical real closed field $\overline{\mathbb{R}} = (\mathbb{R}, <, 0, 1, +, \cdot)$ of real numbers to arbitrary real closed fields has a model theoretic analogue which allows us to go from classical o-minimal structures to the corresponding non-standard models of their theories. Classical o-minimal structures here include $\overline{\mathbb{R}}_{an}$ (sub-analytic geometry), $\overline{\mathbb{R}}_{\exp}$ (the reals with restricted exponential function), $\overline{\mathbb{R}}_{an,\exp}, \overline{\mathbb{R}}_{an^*}$ and $\overline{\mathbb{R}}_{an^*,\exp}$ (see resp., [dd], [w], [dm], [ds1] and [ds2]) . In fact, by [kps], the first-order logic compactness theorem and [Sh], given an o-minimal structure $\mathcal{N}$, for each cardinal $\kappa > \max\{\aleph_0, |\mathrm{Th}(\mathcal{N})|\}$, there are up to isomorphism $2^\kappa$ models $\mathcal{M}$ of the first-order theory Th($\mathcal{N}$) of $\mathcal{N}$ such that $|M| = \kappa$ and any such model is also an o-minimal structure. The advantage of working in the general o-minimal context is that we are proving theorems about all these models, as well as the classical ones.

One of the tools to study the geometry of real algebraic varieties or general semi-algebraic sets is the semi-algebraic (co)homology theory developed by Delfs and Knebusch. The goal of this paper is to extend this theory to arbitrary o-minimal structures.

Let $\mathcal{N}$ be a fixed but arbitrary o-minimal structure. By definable we mean definable with parameters in the structure $\mathcal{N}$. We are interested here in developing sheaf cohomology theory in the category DTOP whose objects are definable sets with the o-minimal site and whose morphisms are continuous definable maps.

A site is a generalisation of a topology developed in the algebro-geometric context due to the inadequacy of the Zariski topology to support a useful cohomolgy theory. It consists of a category, $\mathcal{C}$, and for each $U \in \text{Obj}\mathcal{C}$ a collection of morphisms $(U_i \to U)_{i\in I}$, called coverings, satisfying certain compatability conditions.

For $X \in$ ObjDTOP, the o-minimal site $\text{DTOP}_X$ on $X$ is the site whose underlying category is the set of all open definable subsets of $X$ (open in the strong topology) with morphisms the inclusions and admissible coverings being those open coverings which have finite subcoverings.

We replace the strong topology on definable sets, i.e. the topology induced by the topology on $N^k$ generated by the open boxes defined using the order in $N$, by the o-minimal site because, in the strong topology, definable sets are totally disconnected unless $N = \mathbb{R}$ and never locally compact except for finite sets.

We point out another reason for not working with the strong topology: the definable continuous functions from $X$ into $N$ do not form a sheaf on a definable set $X$ with the strong topology, but they define a sheaf, denoted $\mathcal{O}_X$, with respect to the o-minimal site $\text{DTOP}_X$. We call $\mathcal{O}_X$ the structure sheaf on $X$ and, by definition, for $U \in \text{ObjDTOP}_X$, the set (resp., group if $\mathcal{N}$ is an expansion of a group or ring if $\mathcal{N}$ is an expansion of a ring) $\mathcal{O}_X(U)$ is the set (resp., group or ring) of all definable continuous functions from $U$ into $N$. This observation also explains why we should identify $X \in$ ObjDTOP with $(X, \mathcal{O}_X)$ and a definable continuous map $f : X \longrightarrow Y$ in DTOP with the corresponding morphism $(f, \mu) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ where $\mu : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ is the morphism of sheaves given by $\mu_{U,V}(h) = h \circ f_{|U}$ for $h \in \mathcal{O}_Y(V)$ and $U \in \text{ObjDTOP}_X$ such that $f(U) \subseteq V$.

If $\mathcal{N}$ is a real closed field, we will denote DTOP by SA and, for $X \in$ ObjSA, the o-minimal site $\text{DTOP}_X$ coincides with the semi-algebraic site which we will denote by $\mathrm{SA}_X$.

The category of sheaves of abelian groups on $X \in \mathrm{ObjDTOP}$ with respect to the o-minimal site will be denoted by $\mathrm{Sh}_{\mathrm{dtop}}(X)$. Similarly, we define the category $\mathrm{Sh}_{\mathrm{sa}}(X)$ of sheaves of abelian groups on $X \in \mathrm{ObjSA}$.

Sheaf cohomology theory in the category SA was completely developed by Delfs in the book [D3] following previous work in [D1] and [D2]. See also [dk2]. The first difficulty encountered there is that one must work with the semi-algebraic site instead of topological spaces in the usual sense. In particular, if $X \in \mathrm{ObjSA}$ and $\mathcal{F}$ is a sheaf in $\mathrm{Sh}_{\mathrm{sa}}(X)$, then $\mathcal{F}$ is not determined by its stalks $\mathcal{F}_x$, $x \in X$ (see [D3] Example I.1.7) and so it is not immeadiate that the category $\mathrm{Sh}_{\mathrm{sa}}(X)$ is an abelian category with enough injectives. A similar problem occours when ones tries to develop sheaf cohomology in the category DTOP.

This problem is overcome by defining a functor $\mathrm{SA} \longrightarrow \mathrm{Spec}_r(\mathrm{SA})$, where $\mathrm{Spec}_r(\mathrm{SA})$ is the category of constructible subsets of the various spaces $\mathrm{Spec}_r(N[X_1, \ldots, X_m])$. Here, for a commutative ring $A$ with identity element, $\mathrm{Spec}_r(A)$ is the real spectrum of $A$ as defined by M.Coste and M.F.Roy (cf. [cr]) equipped with the its structure sheaf $\mathcal{O}_{\mathrm{Spec}_r(A)}$ as defined by G.W.Brumfiel ([br2]) and N.Schwartz ([S1], [S2]) (see also [D2]).

The spaces $\mathrm{Spec}_r(A)$ and their constructible subsets are topological spaces in the usual sense which are $T_0$, quasi-compact and spectral spaces in the sense of [h] and [cr], i.e., (i) they have a basis of quasi-compact open subsets closed under finite intersections and (ii) each irreducible closed subset is the closure of a unique point. This implies that on these spaces sheaves are determined by their stalks. Thus, for the constructible subsets of $\mathrm{Spec}_r(A)$ we can develop classical topological sheaf theory (even if they are in general not Hausdorff) and prove two of the Eilenberg-Steenrod axioms for cohomology theories just like in the classical case, namely, the exactness and the excision axiom.

The verification of the homotopy axiom is more difficult. In fact, to prove the Vietories-Biegle theorem (or the base change theorem) and consequently the homotopy invariance of topological sheaf cohomology, paracompactness (and so Hausdorffness) assumptions are required (see [b]). This difficulty is handled in [D2] in the following way. The constructible subsets of $\mathrm{Spec}_r(A)$ are spectral spaces in which the specializations of a point form a chain. This implies that such spaces are normal, in fact, arbitrary closed disjoint subsets may be separated by disjoint constructible open neighbourhoods (see [mo], [cc], [D2]). This last property in turn implies a shrinking lemma which replaces the use of paracompactness in the proof of the base change theorem and the Vietoris-Biegle theorem (see [D2]). Recall that a topological space is paracompact if it is Hausdorff and every cover by open subsets has an open locally finite refinement; every paracompact space is normal and every normal space has a shrinking lemma for locally finite open covers.

To deduce the homotopy invariance of sheaf cohomology theory for constructible subsets of the real spectra of rings from the Vietoris-Biegle theorem one encounters another difficulty. In the topological setting, the projection $\pi : X \times [0,1] \longrightarrow X$ maps closed sets to closed sets and the fiber $\pi^{-1}(x)$ is connected and acyclic for all $x \in X$ since it is homeomorphic to $[0,1]$. Thus the Vietoris-Biegle theorem can be applied to obtain the homotopy invariance. But, if $I$ is the unit interval and $\pi : X \times I \longrightarrow X$ is the projection onto $X$ in the category of constructible subsets of the real spectra of rings, then the fiber $\pi^{-1}(x)$ is the constructible subset of $\mathrm{Spec}_r(k(x)[X])$ which corresponds to the unit interval $[0,1](k(x))$ in the real closed field $k(x)$. Here, if $X$ is a constructible subset of $\mathrm{Spec}_r(A)$ and $x \in X$, then $k(x)$ is the real closure of the field of fractions $k(\mathrm{supp}(x))$ of the ordered domain $k\langle x\rangle = A/\mathrm{supp}(x)$. The field $k(x)$ can also be characterised as the residue field of the local ring $\mathcal{O}_{X,x}$. Thus we also need to know that the unit interval over an arbitrary real closed field is acyclic.

The relation between sheaf cohomology theory for constructible subsets of $\mathrm{Spec}_r(A)$ and sheaf cohomology theory for the category SA is given by the fact that the functor $\mathrm{SA} \longrightarrow \mathrm{Spec}_r(\mathrm{SA})$ induces an isomorphism of categories $\mathrm{Sh}_{\mathrm{sa}}(X) \longrightarrow \mathrm{Sh}_{\mathrm{Spec}_r(\mathrm{SA})}(\widetilde{X})$ for every $X \in \mathrm{ObjSA}$, where $\widetilde{X}$ is the image of $X$ under this functor - see [D3]. Hence, if $X \in \mathrm{ObjSA}$ and $\mathcal{F}$ is a sheaf in $\mathrm{Sh}_{\mathrm{sa}}(X)$, then $H^l(X,\mathcal{F}) = H^l(\widetilde{X},\widetilde{\mathcal{F}})$ for all $l \geq 0$.

Now we define our analogue of the functor $\mathrm{SA} \longrightarrow \mathrm{Spec}_r(\mathrm{SA})$, which we call the tilde functor. Let the objects of $\widetilde{\mathrm{SA}}$ be the collection of all sets $\widetilde{X}$ where $X \in \mathrm{ObjSA}$ and $\widetilde{X}$ is defined to be the set of $m$-types over $N$ which imply a formula defining $X$. We call $\widetilde{X}$ the semi-algebraic spectrum of $X$. We equipe any $\widetilde{X} \in \mathrm{Obj}\widetilde{\mathrm{SA}}$ with the topology generated by the sets $\widetilde{U}$ for $U \in \mathrm{ObjSA}_X$. See [cc] and [p].

Now we come to the observation which allows us to obtain the main results of this paper, namely that the tilde functor $\mathrm{SA} \longrightarrow \widetilde{\mathrm{SA}}$ can be generalized to arbitrary o-minimal structures giving the tilde functor $\mathrm{DTOP} \longrightarrow \widetilde{\mathrm{DTOP}}$ (see [c] and [p]). For $X \in \mathrm{ObjDTOP}$, we call its image $\widetilde{X}$ under the tilde functor $\mathrm{DTOP} \longrightarrow \widetilde{\mathrm{DTOP}}$ the o-minimal spectrum of $X$ and, by definition, if $X \subseteq N^m$, then $\widetilde{X}$ is the set of $m$-types over $N$ which imply a formula defining $X$, equipped with the topology generated by the sets $\widetilde{U}$ for $U \in \mathrm{ObjDTOP}_X$.

We show that, just like in the semi-algebraic case, the tilde functor $\mathrm{DTOP} \longrightarrow \widetilde{\mathrm{DTOP}}$ induces an isomorphism of categories $\mathrm{Sh}_{\mathrm{dtop}}(X) \longrightarrow \mathrm{Sh}(\widetilde{X})$ for every $X \in \mathrm{ObjDTOP}$ (Proposition 3.1). Therefore, if $X \in \mathrm{ObjDTOP}$ and $\mathcal{F}$ is a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$, then $H^l(X,\mathcal{F}) = H^l(\widetilde{X},\widetilde{\mathcal{F}})$ for all $l \geq 0$.

The objects of $\widetilde{\mathrm{DTOP}}$ are $T_0$, quasi-compact, spectral spaces (see [p]). Hence, we can verify the exactness and the excision axioms exactly as in topological setting. We show here that the spaces in $\widetilde{\mathrm{DTOP}}$ associated to definably normal definable sets are spaces in each point is the generalisation of unique closed point (i.e. for each point, $x$ there is a unique closed point, $\rho(x)$, such that $\rho(x)$ is a specialisation of $x$). See Theorem 2.12. This implies that such spaces are normal, in fact, arbitrary closed disjoint subsets may be separated by disjoint constructible open neighbourhoods. This last property in turn implies a shrinking lemma which, as in the semi-algebraic case, replaces the use of paracompactness in the sheaf cohomology theory for $\widetilde{\mathrm{DTOP}}$. Furthermore, as in the real algebraic case in [D2] we prove the base change theorem (Theorem 4.4) and the Vietoris-Biegle theorem (Theorem 4.5).

The normality of $\widetilde{X}$ fails if $X$ is not assumed to be a definably normal definable set. Also, in general, the specilizations of a point in $\widetilde{X}$ do not form a chain. There is a result from the second author DPhil Thesis showing that if $\mathcal{N}$ is an o-minimal expansion of a real closed field and $X$ is a definable set, then the specializations of a point in $\widetilde{X}$ form a chain.

As in the real algebraic case, to deduce the homotopy invariance of o-minimal sheaf cohomology from the Vietoris-Biegle theorem we need to show that a closed interval $I$ in an arbitrary o-minimal structure is acyclic. In fact, if $\pi : X \times I \longrightarrow X$ is the projection onto $X$ in the category DTOP, then the fiber $\widetilde{\pi}^{-1}(x)$ of $\widetilde{\pi} : \widetilde{X \times I} \longrightarrow \widetilde{X}$ is the constructible subset of $\widetilde{N(x)}$ which corresponds to the closed interval $I(N(x))$ in the o-minimal structure $\mathcal{N}(x)$. Note that to make this argument work we need to assume that $\mathcal{N}$ has definable Skolem functions. The o-minimal structure $\mathcal{N}(x)$, the definable ultrapower of $\mathcal{N}$ at $x$, is isomorphic to the prime model of $\mathrm{Th}(\mathcal{N})$ over $N \cup \{a\}$, where $a$ is some realization of the type $x$. The o-minimal structure $\mathcal{N}(x)$ is an elementary extension of $\mathcal{N}$ and is unique up to isomorphisms over $N$ (see [PiS]). We should think of $\mathcal{N}(x)$ as the o-minimal analogue of $k(x)$ from the real algebraic case. There is also an o-minimal analogue $\mathcal{N}\langle x\rangle$ of $k\langle x\rangle$ (see [p] and [c]).

Suppose that $\mathcal{N}$ is an o-minimal expansion of a field. If $\mathcal{N}$ is just a real closed field, then by construction the o-minimal sheaf cohomology coincides with the semi-algebraic sheaf cohomology because of the equivalence of the categories $\mathrm{Sh}_{\mathrm{sa}}(X)$, $\mathrm{Sh}_{\mathrm{Spec}_r(\mathrm{SA})}(\widetilde{X})$ and $\mathrm{Sh}(\widetilde{X})$ for every $X \in \mathrm{ObjSA}$. On the other hand, in the category DTOP, we have, by [Wo], the o-minimal singular homology $(H_*, d_*)$ from which one easily constructs the o-minimal singular cohomology $(H^*, d^*)$ with constant coefficients (see [ew]). By the uniqueness theorem from [ew], the o-minimal sheaf cohomology constructed here coincides with the o-minimal singular cohomology $(H^*, d^*)$ when we consider constant coefficient sheaves.

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# 2 The o-minimal spectrum of definable sets

Before we start the theory of o-minimal spectra of definable sets, we present the following probably well known result that will be required later.

**Proposition 2.1** *Every definable set $A \subseteq N^n$ is a finite union of definable sets of the form $U \cap F$ where $U$ (resp., $F$) is an open (resp., a closed) definable subset of $N^n$. Recalling that sets of the form described are called constrcutible this says that every definable set is constructible.*

Indeed, by [vdd] page 51, every cell is open in its closure. Hence each cell can be written as $U \cap F$ where $U$ (resp., $F$) is an open (resp., a closed) definable subset of $N^n$. Thus cell decomposition implies the proposition.

For other basic facts about definable sets and maps we refer the reader to [vdd].

**Definition 2.2** Let $X \subseteq N^m$ be a definable set. The *o-minimal spectrum* $\widetilde{X}$ of $X$ is the set of complete $m$-types $S_m(N)$ of the first-order theory $\mathrm{Th}_N(\mathcal{N})$ which imply a formula defining $X$ equipped with the topology generated by the basic open sets of the form

$$\widetilde{U} = \{\alpha \in \widetilde{X} : U \in \alpha\}$$

where $U$ is open (in the o-minimal site) on $X$. We call this topology on $\widetilde{X}$ the *spectral topology.*

Observe that the set $\widetilde{X}$ coincides with the set of ultrafilters of the boolean algebra of definable subsets of $X$, and clearly, the sets of the form $\widetilde{U}$ with $U$ open definable subset of $X$ generate a topology on $\widetilde{X}$. In fact, we have $\widetilde{\emptyset} = \emptyset$, $\widetilde{X}$ is open and $\widetilde{U_1} \cap \cdots \cap \widetilde{U_n} = \widetilde{U}$ where $U = U_1 \cap \cdots \cap U_n$.

It is immediate that the map $X \longrightarrow \widetilde{X}$, that sends $x \in X$ into the type $\mathrm{tp}(x/N)$ is injective and induces a homeomorphism from $X$, with its strong topology, onto its image in $\widetilde{X}$. Below, we will often identify $X$ with its image in $\widetilde{X}$ under this map.

If $X$ is a definable set, we say that a subset of $\widetilde{X}$ is *constructible* if it is a finite boolean combination of basic open subsets $\widetilde{U}$. The *constructible topology* on $\widetilde{X}$ is the topology generated by the constructible subsets of $\widetilde{X}$. Since, by Proposition 2.1, every definable set is a finite boolean combination of open definable sets, it follows that every constructible subset of $\widetilde{X}$ is of the form $\widetilde{A} = \{\alpha \in \widetilde{X} : A \in \alpha\}$ where $A$ is a definable subset of $X$ (note that $A$ is not necessarily open, as in the basis for the spectral topology).

Also notice that for any definable $A \subseteq X$ we have that $X \setminus A$ is definable, and so $\widetilde{A}$ is both open and closed in the constructible topology on $\widetilde{X}$. It is a well known model theoretic fact that $\widetilde{X}$ equipped with the constructible topology is a compact, totally disconnected Hausdorff space (see [bs]). In fact, $\widetilde{X}$ with the constructible topology is the Stone space of the boolean algebra of definable subsets of $X$.

*Unless otherwise stated, we always consider $\widetilde{X}$ equipped with its spectral topology. So when we say "constructible open" we mean open in the spectral topology and constructible, as opposed to "open in the constructible topology".*

**Remark 2.3** For definable sets $A \subseteq X$, it is easy to see that the following hold:

(1) The tilde operation is an isomorphism from the boolean algebra of definable subsets of $X$ onto the boolean algebra of constructible subsets of $\widetilde{X}$

(2) $A$ is open (resp., closed) if and only if $\widetilde{A}$ is open (resp., closed). Moreover, the tilde operation commutes with the interior and closure operations.

(3) $A$ is definably connected if and only if $\widetilde{A}$ is connected.

We also have the following characterization of open (resp., closed) subsets of $\widetilde{X}$ similar to [BCR] Proposition 7.2.7.

**Proposition 2.4** *Let $U$ (resp., $F$) be an open (resp., closed) definable subset of the definable set $X$. Then the following hold:*

*(1) $\widetilde{U}$ is the largest open subset of $\widetilde{X}$ whose intersection with $X$ is $U$.*

*(2) $\widetilde{F}$ is the smallest closed subset of $\widetilde{X}$ whose intersection with $X$ is $F$.*

**Proof.** (1) Let $V$ be an open subset of $\widetilde{X}$ such that $V \cap X = U$. Since the constructible open subsets form a basis of the topology of $\widetilde{X}$, it follows that $V = \cup\{\widetilde{B} : B$ is an open definable subset of $X$ such that $\widetilde{B} \subseteq V\}$. But if $\widetilde{B} \subseteq V$, then $B = \widetilde{B} \cap X \subseteq U$, and, hence, $\widetilde{B} \subseteq \widetilde{U}$. Thus $V \subseteq \widetilde{U}$.

(2) is obtained from (1) by taking complements. □

The next result is easy and is from [p], recalling that a set $X$ in a topological space is said to be *irreducible* if and only if it is not the union of any two proper closed subsets.

**Proposition 2.5** *Let $X$ be a definable set. The space $\widetilde{X}$ is $T_0$, quasi-compact and a spectral space, i.e.: (i) it has a basis of quasi-compact open subsets, closed under taking finite intersections; and (ii) each irreducible closed subset is the closure of a unique point.*

**Proof.** First we show that $\widetilde{X}$ is $T_0$, so suppose $\alpha \neq \beta \in \widetilde{X}$. Since we can consider them as distinct complete types there must be a formula defining an open subset, $U$, of $X$ which is in, without loss, $\alpha$ and not in $\beta$. If there were no such $U$ then $\alpha$ and $\beta$ contain all the same open sets, and hence all the same closed sets. But then, since by Proposition 2.1, every definable set is constructible, they contain all the same definable sets, and so are the same. The open set $\widetilde{U} = \{\gamma \in \widetilde{X} : U \in \gamma\}$ contains $\alpha$ and not $\beta$.

The basic open subsets $\widetilde{U}$ and $\widetilde{X}$ itself, are quasi-compact since the constructible topology is finer than the spectral topology.

Now let $F \subseteq \widetilde{X}$ be closed and irreducible. Let $\Phi = \{B \subseteq X : B$ is closed, definable and $B \in \beta$ for all $\beta \in F\}$. Let $\Psi = \Phi \cup \{X \setminus C : C$ is closed, definable and $C \notin \Phi\}$. By irreducibility of $F$, $\Psi$ is consistent and thus determines a type $\gamma \in \widetilde{X}$. Clearly, $F$ is the closure of $\gamma$ and only of $\gamma$. □

Note that Hochster shows in [h] that the spectral spaces are exactly the spaces homeomorphic to the prime spectrum of a (commutative) ring with identity element, equipped with the Zariski topology.

**Definition 2.6** Let $X$ be a definable set and $\alpha, \beta \in \widetilde{X}$. We say that $\beta$ is a *specialization* of $\alpha$ (or $\alpha$ is a *generalization* of $\beta$) if and only if $\beta$ is in the closure of $\{\alpha\}$.

The notion of specialization is valid for any spectral space and defines a partial order on the set of points. The following property holds in any spectral space (compare with [BCR] Proposition 7.1.21).

**Proposition 2.7** *Let $X$ be a definable set and $C$ a constructible subset of $\widetilde{X}$. Then $C$ is closed (resp., open) in $D$ if and only if it is stable under specialization (resp., generalization) in $D$.*

We now investigate if as in the real algebraic case in [BCR] Chapter 7 the specializations of a point in the o-minimal spectrum of a definable set form a chain. The proof in [BCR] is based on real algebra and does not work in the o-minimal context. However we have the following remark.

**Remark 2.8** O-minimality implies that for any complete 1-type, $\alpha$, over $N$ we only have the following possibilities:

- $\alpha$ is the type of a point $a \in N$, in which case we often abuse notation and write $a$ for it's type;
- the type of $+\infty$ or $-\infty$, when we similarly abuse notation;
- the type of an element transcendental over $N$, but which defines a cut in it's ordering (transcendental in the model theoretic sense, that is the type of an element which is not definable over $N$);
- $\alpha$ is the type of an element infinitessimally above (or below) $a \in N$ i.e. the type containing all the open intervals $(a, b)$ (resp. $(b, a)$) for all $b > a$ (resp. $b < a$), when we say $\alpha = a^+$ (resp. $a^-$);

We get the last two possibilities from the case that $\alpha$ contains some bounded interval, say $(a, b)$. For every such open interval in $\alpha$ and every $c \in (a, b)$, either $(a, c)$ or $(c, b)$ is in $\alpha$, since it is complete. If all the subintervals of $(a, b)$ in $\alpha$ are of the form $(a, c)$ (resp. $(c, b)$) then we have that $\alpha = a^+$ (resp. $a^-$). If there is no open interval $(a, b)$ in $\alpha$ such that either $(a, c) \in \alpha$ for all $c > a$ or $(c, b) \in \alpha$ for all $c < b$, then $\alpha$ defines a cut at a transcendental element by the pair of sets: $\{x : \text{there is } (a, b) \in \alpha \text{ with } a > x\}$ and $\{x : \text{there is } (a, b) \in \alpha \text{ with } b < x\}$. Clearly the cut has to be at a transcendental element or we could define $\alpha$ to be a point definable over $N$.

Only types of the final kind are not closed points of $\widetilde{X}$. It can easily be checked that $a \in N$ is a specialisation of $a^+$ and $a^-$ but not vice-versa.

This also gives that though the sets $\overline{\{a^+\}} = \{a, a^+\}$ and $\overline{\{a^-\}} = \{a, a^+\}$ are irreducible, so is the set $\{a\}$, and the set $\{a^-, a, a^+\}$ is not.

The following example shows that without some assumptions on the structure $\mathcal{N}$, we do not have that the specializations of $\alpha \in \widetilde{X}$ form a chain if $X \subseteq N^m$ and $m > 1$.

**Example 2.9** Let $\mathcal{N} = (\mathbb{Q}, <)$ and take $X = \mathbb{Q}^2$. For any $a < b \in \mathbb{Q}$ let $\alpha$ be the type given by the ordered pair $\langle a^-, b^- \rangle$, that is the type of an infinitessimal box below and to the left of the point $\langle a, b \rangle$. Then let $\beta$ be the type given by the ordered pair $\langle a^-, b \rangle$ and $\gamma$ be the type given by the ordered pair $\langle a, b^- \rangle$. Then $\beta$ and $\gamma$ are specializations of $\alpha$ since basic the open sets in $\beta$ (respectively $\gamma$) are all of the form $(c,d)\times(e,f)$ for $c<a\le d$ and $e<b<f$ (respectively $(c,d)\times(e,f)$ for $c<a<d$ and $e<b\le f$) and all of these sets are also in $\alpha$. But neither $\beta$ nor $\gamma$ is a specialization of the other, since $(c,a)\times(e,f)$ is in $\beta$ but not $\gamma$ and $(c,d)\times(e,b)$ is in $\gamma$ but not $\beta$.

We now present an example to show that the closed specialisation of $\alpha \in \widetilde{X}$ is not necessarily unique without further assumptions on $X$. Thanks to Alf Onshuus for bringing it to our attention.

**Example 2.10** We take $\mathcal{N}$ as in Example 2.9 and $X = \{\langle x, y\rangle \in \mathbb{Q}^2 : \langle x, y\rangle \neq \langle a, b\rangle\}$ for some fixed $a$ and $b$ in $\mathbb{Q}$ with $a < b$. Again letting $\alpha = \langle a^-, b^-\rangle \in \widetilde{X}$ we get that the closure of $\{\alpha\}$ in $\widetilde{\mathbb{Q}^2}$ is the set $\{\alpha, \langle a, b^-\rangle, \langle a^-, b\rangle, \langle a, b\rangle\}$, with $\langle a, b\rangle$ being the only point closed in $\widetilde{\mathbb{Q}^2}$. But in $\widetilde{X}$ we do not have this point, and so both $\langle a^-, b\rangle$ and $\langle a, b^-\rangle$ are closed points of $\widetilde{X}$ which are specialisations of $\alpha$.

There is a result from the second author DPhil Thesis showing that if $\mathcal{N}$ is an o-minimal expansion of a real closed field and $X$ is a definable set, then the specializations of every $\alpha \in \widetilde{X}$ form a chain and $\alpha$ is the generalization of a unique closed point. Our next goal is to find weaker conditions on a definable set $X$ in an arbitrary o-minimal structure $\mathcal{N}$ such that each point of $\widetilde{X}$ is the generalisation of unique closed point.

**Lemma 2.11** *Given a definable set $X \subseteq N^m$ and $\alpha \in \widetilde{X}$ there is some closed type (i.e. a closed point in the spectral topology), $\beta$, which is a specialisation of $\alpha$, and for any $A \in \alpha \setminus \beta$ we have* $\mathrm{fr}(A) \in \beta$ *where* $\mathrm{fr}(A) = \overline{A} \setminus A$ *is the frontier of $A$.*

**Proof.** We go by induction on $\dim(\alpha) := \min\{\dim(A) : A \in \alpha\}$. If $\dim(\alpha) = 1$ then $\alpha$ contains a one dimensional set $A$, which, by cell-decomposition and the completeness of $\alpha$, we can assume is a cell. By o-minimality we have that $A$ can be definably totally ordered, and thus that any type containing $A$ is one of those described in Remark 2.8. Thus $\alpha$ is either closed (in which case it is clearly it's own unique closed specialisation) or $\alpha = a^+$ or $a^-$, for $a \in A \subseteq X$ in which case the unique close specialisation of $\alpha$ is $a$.

For $\dim(\alpha) > 1$, first note that if $\alpha$ is closed then by the same reason as above we are done, so assume $\alpha$ not closed. Thus we can find a specialisation $\beta$ of $\alpha$ distinct from $\alpha$. Then we take any $A' \in \alpha \setminus \beta$ and any $A'' \in \alpha$ realising $\dim(\alpha)$ and let $A = A' \cap A''$.

If $\mathrm{fr}(A)$ is not in $\beta$ then $N^k \setminus \mathrm{fr}(A) = (N^k \setminus \overline{A}) \cup A \in \beta$ and since we already have $N^k \setminus A \in \beta$ we have that their intersection, $N^k \setminus \overline{A} \in \beta$. But this is open, and so, as $\beta$ is a specialisation of $\alpha$, we must also have $N^k \setminus \overline{A} \in \alpha$. But since $A \in \alpha$ this contradicts the consistency of $\alpha$.

So $\mathrm{fr}(A) \in \beta$, and since $\dim(\mathrm{fr}(A)) < \dim(A)$ we must have $\dim(\beta) < \dim(\alpha)$. By the induction hypothesis $\beta$ has a closed specialisation, which is thus also a closed specialisation of $\alpha$, and we are done. $\square$

**Theorem 2.12** *Given $X \subseteq N^k$ the following are equivalent:*

(1) *For any $\alpha \in \widetilde{X}$ there is a unique closed point in $\widetilde{X}$, denoted by $\rho(\alpha)$, such that $\rho(\alpha)$ is a specialization of $\alpha$. (i.e. the closed point from the Lemma 2.11 is unique, so defines a map, $\rho$).*

(2) *$\widetilde{X}$ is normal. In fact, if $F$ and $G$ are two disjoint closed subsets of $\widetilde{X}$ then there exist two disjoint constructible open (i.e. open in the spectral topology and constructible) subsets $U$ and $V$ of $\widetilde{X}$ such that $F \subseteq U$ and $G \subseteq V$.*

(3) *$X$ is definably normal (i.e. for disjoint definable closed sets $F, G$ in $X$ there are disjoint definable open sets $U$ and $V$ such that $F \subseteq U$ and $G \subseteq V$).*

**Proof.** We first show (1) $\implies$ (2), so suppose the conclusion of (2) does not hold. Then for any constructible open sets $U$ and $V$ such that $F \subseteq U$ and $G \subseteq V$ we have $U \cap V \neq \emptyset$. So for any finite collections $U_1, \dots, U_n$ and $V_1, \dots, V_m$ of constructible opens such that $F \subseteq U_i$ and $G \subseteq V_i$ we have $F \subseteq \bigcap U_i$ and $G \subseteq \bigcap V_i$, and the intersections are constructible open, so that $\bigcap U_i \cap \bigcap V_i \neq \emptyset$. Since any constructible set is closed in the constructible topology, which is also compact, the intersection of all the constructible open sets $U$ and $V$ such that $F \subseteq U$ and $G \subseteq V$ is non-empty. Take $\theta$ in this intersection.

Now $\overline{\{\theta\}} \cap F \neq \emptyset$ as if it were then $\widetilde{X} \setminus \overline{\{\theta\}}$ would be an open constructible set containing $F$ but excluding $\theta$. So we can take $\beta \in \overline{\{\theta\}} \cap F$ and similarly $\gamma \in \overline{\{\theta\}} \cap G$. Then $\beta, \gamma \in \overline{\{\theta\}}$ and so $\rho(\theta) = \rho(\beta) = \rho(\gamma)$ (by the uniqueness in (1)).

As $F$ and $G$ are closed we have $\overline{\{\beta\}} \subseteq F$ and $\overline{\{\gamma\}} \subseteq G$, so $\rho(\beta) \in F$ and $\rho(\gamma) \in G$. But $\rho(\beta) = \rho(\gamma) \in F \cap G$, so the hypothesis of (2) does not hold, and we have (1) $\implies$ (2).

That (2) implies (3) follows from the fact that any sets $F$ and $G$ as in the statement of (3) give rise to disjoint $\widetilde{F}$ and $\widetilde{G}$ closed in the spectral topology.

By (2) these can be seperated by constructible open sets $\widetilde{U}$ and $\widetilde{V}$ which, by their constructibility, come from open sets $U$ and $V$ which seperate $F$ and $G$ in $X$.

Now we prove that (1) is implied by (3) by induction on $\dim(X)$. Let $(1_\mathrm{m}), (2_\mathrm{m}), (3_\mathrm{m})$ be the statements restricted to $X$ of dimension $m$. The proof of (1) $\implies$ (2) and (2) $\implies$ (3) above shows that $(1_\mathrm{k}) \implies (2_\mathrm{k})$ and $(2_\mathrm{k}) \implies (3_\mathrm{k})$ for all $k$. We show now that if $(1_\mathrm{k})$, $(2_\mathrm{k})$, $(3_\mathrm{k})$ are equivalent for all $k \leq m$ and $(3_\mathrm{m+1})$ holds, then $(1_\mathrm{m+1})$ holds, which will complete the proof.

We first show that $(1_1)$ and so $(2_1)$ and $(3_1)$ hold in any case. If $\dim(X) = 1$ then by cell-decomposition $X$ is a finite union of disjoint 1 and 0 dimensional sets, $U_i$, each of which is definably totally ordered. For any $\alpha \in \widetilde{X}$ is in some unique $\widetilde{U_i}$. By o-minimality, types in any of the $\widetilde{U_i}$ are given in the same way as the types in some $\widetilde{Y}$ for $Y \subseteq N$, with respect to this new order. By Remark 2.8 any such type is either closed or equals $a^+$ or $a^-$ for some $a \in Y$, and thus has unique closed specialisation $a$. In either case the type has a unique closed specialisation. This proves $(1_1)$.

We now show that $(2_\mathrm{k}) \implies (2'_\mathrm{k})$ for all $k$, where $(2'_\mathrm{k})$ is the statement:

$(2'_\mathrm{k})$ *Given* $\beta \in O \subseteq \widetilde{X}$, *where* $\dim(X) = k$, $\beta$ *is a closed point and* $O$ *is a constructible open set, there is a constructible open* $W_\beta$ *set such that* $\beta \in W_\beta \subseteq \overline{W_\beta} \subseteq O$.

We use $(2_\mathrm{k})$ with $F = \{\beta\}$ and $G = \widetilde{X} \setminus O$ to get constructible open $U$ and $V$ such that $\beta \in U$ and $\widetilde{X} \setminus O \subseteq V$ and $U \cap V = \emptyset$. Putting $W_\beta = U$ we get the result, since $\overline{U} \subseteq \widetilde{X} \setminus V$, as the set on the right is closed and contains $U$, and also $\widetilde{X} \setminus V \subseteq O$, since $\widetilde{X} \setminus O \subseteq V$.

Assume that $(1_\mathrm{k})$, $(2_\mathrm{k})$, $(3_\mathrm{k})$ are equivalent for all $k \leq m$ and $X$ is a definably normal definable set with $\dim(X) = m+1$. Let $\alpha \in \widetilde{X}$ and take two distinct closed types, $\beta$ and $\gamma$, which are both specialisations of $\alpha$. Take any $A' \in \alpha$ such that $\dim(A') = \dim(\alpha)$, any $B \in \alpha \setminus \beta$ and any $C \in \alpha \setminus \gamma$. Then let $A$ be the unique cell in $A' \cap B \cap C$ which is in $\alpha$ (there is such a thing, by cell-decomposition, because $\alpha$ is consistent and complete). Then $\dim(A) = \dim(\alpha)$, $A \in \alpha \setminus \beta$, $A \in \alpha \setminus \gamma$ and $\mathrm{fr}(A)$ is closed (as $A$ is a cell and cells are open in their closures) of dimension $k$ strictly less than $A$, and hence less than or equal to $m$. Also by Lemma 2.11 we have $\mathrm{fr}(A) \in \beta$ and $\mathrm{fr}(A) \in \gamma$, so $\beta, \gamma \in \widetilde{\mathrm{fr}(A)}$.

Since $\mathrm{fr}(A)$ is a closed definable subset of $X$ and $X$ is definably normal, $\mathrm{fr}(A)$ is definably normal, i.e., $(3_\mathrm{k})$ holds. Thus we can use $(2_\mathrm{k})$ to get disjoint $U_\beta \ni \beta$ and $U_\gamma \ni \gamma$ constructible open in $\widetilde{\mathrm{fr}(A)}$. Then by $(2'_\mathrm{k})$ we get $W_\beta, W_\gamma$

constructible open in $\widetilde{\mathrm{fr}(A)}$ such that $\beta \in W_\beta \subseteq \overline{W_\beta} \subseteq U_\beta$, and similarly for $\gamma$. As $W_\beta, W_\gamma$ are constructible, by Remark 2.3 (2), we have that there are definable $C_\beta$ and $C_\gamma$ in $X$ such that $\overline{W_\beta} = \widetilde{C_\beta}$ and $\overline{W_\gamma} = \widetilde{C_\gamma}$, with $C_\beta$ and $C_\gamma$ closed in $\mathrm{fr}(A)$, and hence closed in $X$. So by definable normality of $X$, $(3_{\mathrm{m+1}})$, there are disjoint definable sets $V_\beta$ and $V_\gamma$ open in $X$ such that $C_\beta \subseteq V_\beta$ and $C_\gamma \subseteq V_\gamma$. Then $\widetilde{V_\beta}$ and $\widetilde{V_\gamma}$ are disjoint and $\beta \in \widetilde{C_\beta} \subseteq \widetilde{V_\beta}$. But as $\beta$ is a specialisation of $\alpha$ this gives $\alpha \in \widetilde{V_\beta}$, and arguing similarly with $\gamma$ in place of $\beta$ gives $\alpha \in \widetilde{V_\gamma}$, a contradiction. □

The proof of (1) $\Longrightarrow$ (2) in Theorem 2.12, is exactly the same as [BCR] Proposition 7.1.24. We included the details only for completeness. In fact, [cc] Proposition 2 shows that an arbitrary spectral space is normal if and only if every point is the generalisation of a unique closed point.

**Example 2.13** We note here that $X$ from Example 2.10 is not definably normal, explaining the lack of uniqueness of closed specialisations. Simply notice that the closed line segments $[\langle a-\epsilon, b\rangle, \langle a+\epsilon, b\rangle]$ and $[\langle a, b-\epsilon\rangle, \langle a, b+\epsilon\rangle]$ are disjoint closed definable subsets of $X$ but not seperable by open definable sets.

If $\mathcal{N}$ is an o-minimal expansion of an ordered group, then by [vdd] Chapter VI, (3.5), every definable set is definably normal.

We obtain from Theorem 2.12 the following corollary.

**Proposition 2.14** *Let $X$ be a definably normal definable set. The subspace $X^c$ of closed points of $\widetilde{X}$ is a Hausdorff compact topological space and the mapping $\rho : \widetilde{X} \longrightarrow X^c$ is a continuous and closed retraction which sends every constructible subset of $\widetilde{X}$ into a closed subset of $X^c$.*

Indeed, as shown in [cc] Proposition 3, the statment holds in any normal spectral space.

An important corollary of Theorem 2.12, is the following result which will play the role of paracompactness in o-minimal sheaf cohomology. The proof is similar to the one for constructible subsets of real spectra. See [br2], [D2], [dk3].

**Proposition 2.15 (The Shrinking Lemma)** *Let $X$ be a definably normal definable set. If $\{U_i : i = 1, \ldots, n\}$ is a covering of $\widetilde{X}$ by open subsets of $\widetilde{X}$, then there are constructible open subsets $V_i$ and constructible closed subsets $K_i$ of $\widetilde{X}$ $(1 \le i \le n)$ with $V_i \subseteq K_i \subseteq U_i$ and $\widetilde{X} = \cup\{V_i : i = 1, \ldots, n\}$.*

**Proof.** We define open subsets $V_i$ and closed subsets $K_i$ of $\widetilde{X}$ $(1 \leq i \leq n)$ by induction. Assume that the sets $V_i$ and $K_i$ are already constructed for $i = 1, \ldots, m$ with $0 \leq m \leq n-1$ and have the following propertes: (i) $V_i \subseteq K_i \subseteq U_i$ $(1 \leq i \leq m)$; (ii) $V_i$ and $\widetilde{X} \setminus K_i$ are constructible $(1 \leq i \leq m)$ and (iii) $(\cup\{V_i : i = 1, \ldots, m\}) \cup (\cup\{U_i : i = m+1, \ldots, n\}) = \widetilde{X}$.

The sets $A = \widetilde{X} \setminus U_{m+1}$ and $B = \widetilde{X} \setminus [(\cup\{V_i : i = 1, \ldots, m\}) \cup (\cup\{U_i : i = m+2, \ldots, n\})]$ are closed and disjoint subsets of $\widetilde{X}$. Hence, by Theorem 2.12, there exist an open constructible neighbourhoods $W$ of $A$ and $V_{m+1}$ of $B$ with $W \cap V_{m+1} = \emptyset$. Define $K_{m+1} = \widetilde{X} \setminus W$. Then properties (i), (ii) and (iii) are fulfilled with $m$ replaced by $m+1$. Since $V_i$ and $\widetilde{X} \setminus K_i$ are constructible open subsets of $\widetilde{X}$, the sets $V_i$ and $K_i$ are constructible. □

Observe that since $X$ is definably normal there is a shrinking lemma for finite covers of $X$ by open definable subsets (compare with [vdd] Chapter VI,(3.6)), which gives a shrinking lemma for finite covers of $\widetilde{X}$ by open *constructible* subsets. Similarly, since $\widetilde{X}$ is a normal space, there is a topological shrinking lemma for $\widetilde{X}$ which does not give that the $V_i$'s are necessarily constructible.

We end the section with the o-minimal spectrum of definable maps. The following is the o-minimal analogue of [BCR] Proposition 7.2.8.

**Definition 2.16** Let $f : X \longrightarrow Y$ be a definable map. Then there exists a unique mapping $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$, called the *o-minimal spectrum of f*, such that for $\alpha \in \widetilde{X}$ and for every definable subset $B$ of $Y$ we have $B \in \widetilde{f}(\alpha)$ if and only if $f^{-1}(B) \in \alpha$.

**Remark 2.17** Let $f : X \longrightarrow Y$ be a definable map. Then the following hold:

- $\widetilde{f}^{-1}(\widetilde{B}) = \widetilde{f^{-1}(B)}$ for every definable subset $B$ of $Y$.
- $\widetilde{f}(\widetilde{A}) = \widetilde{f(A)}$ for every definable subset $A$ of $X$.
- By Remark 2.3 (2), if the definable map $f : X \longrightarrow Y$ is continuous, then the mapping $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ is continuous.

The main property of the o-minimal spectrum of definable maps that we will require later, to get the base change theorem, is the following proposition.

**Proposition 2.18** *If $f : X \longrightarrow Y$ is a continuous definable map then for any $\alpha \in \widetilde{Y}$ we have that $\widetilde{f}^{-1}(\alpha)$ is quasi-compact.*

**Proof.** Let $\mathcal{N}^\star$ be a sufficiently saturated elementary extension of $\mathcal{N}$ where the type $\alpha$ is realised. We denote by $\alpha$ the realisation of the type $\alpha$ in $\mathcal{N}^\star$, and by $X^\star$, $Y^\star$ and $f^\star : X^\star \longrightarrow Y^\star$ we will denote the interpretations of $X$, $Y$ and $f : X \longrightarrow Y$ in $\mathcal{N}^\star$. Consider the natural "forgetful" restriction map $\epsilon_Y : \widetilde{Y^\star} \longrightarrow \widetilde{Y}$, which sends a type $\beta^\star \in \widetilde{Y^\star}$ to the type $\beta \in \widetilde{Y}$ obtained from $\beta^\star$ by forgetting any formulas with parameters from $N^\star \setminus N$. We denote by $\epsilon_X$ the same restriction map from $\widetilde{X^\star}$ to $\widetilde{X}$. These maps are clearly surjective as every type $\beta \in \widetilde{Y}$ has some extension over $N^\star$. They are continuous since for any open $U \subseteq \widetilde{Y}$ we have $U = \bigcup_{i\in I} \widetilde{U}_i$ for $N$-definable $U_i$, and then $\epsilon_Y^{-1}(\widetilde{U}_i) = \widetilde{U_i^\star}$ is open in $\widetilde{Y^\star}$ so that $\epsilon_Y^{-1}(U) = \bigcup_{i\in I} \widetilde{U_i^\star}$ is also open. These arguments clearly work equally well for $X$ in place of $Y$.

We note that $\alpha$ is a closed point in $\widetilde{Y^\star}$, and thus $\widetilde{f^\star}^{-1}(\alpha)$ is a closed set in $\widetilde{X^\star}$, which is a quasi-compact space, and so $\widetilde{f^\star}^{-1}(\alpha)$ is quasi-compact.

Now let $\{\widetilde{U}_i\}_{i\in I}$ be an open cover of $\widetilde{f}^{-1}(\alpha)$ in $\widetilde{X}$ (there is clearly no harm in assuming that the elements of the cover are basic open). We claim that $\{\epsilon_X^{-1}(\widetilde{U}_i)\}_{i\in I} = \{\widetilde{U_i^\star}\}_{i\in I}$ is an open cover of $\widetilde{f^\star}^{-1}(\alpha)$. They are clearly open as $\epsilon_X$ is continuous and to get that they cover we show that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X^\star} & \xrightarrow{\widetilde{f^\star}} & \widetilde{Y^\star} \\ \downarrow^{\epsilon_X} & & \downarrow^{\epsilon_Y} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y}. \end{array}$$

Take $\beta \in \widetilde{X^\star}$ and note that $\epsilon_X(\beta) = \{B \in \beta : B \text{ defined over } N\}$, and that $\widetilde{f}(\epsilon_X(\beta)) = \{B \subseteq Y : f^{-1}(B) \in \epsilon_X(\beta)\}$. Also note that, by definition, $\widetilde{f^\star}(\beta) = \{B^\star \subseteq Y^\star : (f^\star)^{-1}(B^\star) \in \beta\}$ and $\epsilon_Y(\widetilde{f^\star}(\beta)) = \{B \subseteq Y : B^\star \in \widetilde{f^\star}(\beta)$ and $B$ defined over $N\}$. Thus $B \in \epsilon_Y(\widetilde{f^\star}(\beta))$ if and only if $B$ is defined over $N$ and $(f^\star)^{-1}(B^\star) \in \beta$, and this is the case if and only if $f^{-1}(B) \in \beta$ if and only if $f^{-1}(B) \in \epsilon_X(\beta)$ if and only if $B \in \widetilde{f}(\epsilon_X(\beta))$. Thus the diagram commutes.

Now take any $\gamma \in \widetilde{f^\star}^{-1}(\alpha)$, so that $\epsilon_Y(\widetilde{f^\star}(\gamma)) = \alpha \in \widetilde{Y}$. As the diagram commutes we then have $\widetilde{f}(\epsilon_X(\gamma)) = \alpha$. So $\epsilon_X(\gamma) \in \widetilde{f}^{-1}(\alpha)$ and so we have $\epsilon_X(\gamma) \in \widetilde{U}_i$ for some $i$. Thus $U_i \in \epsilon_X(\gamma)$ and so $\epsilon_X^{-1}(U_i) = U_i^\star \in \gamma$, i.e., $\gamma \in \widetilde{U_i^\star}$, and we have that $\{\widetilde{U_i^\star}\}_{i\in I}$ is an open cover for $\widetilde{f^\star}^{-1}(\alpha)$. Now as this set is quasi-compact we can find finitely many amongst the $\widetilde{U_i^\star}$ which cover it, whose images under $\epsilon_X$ form a finite subcover of $\widetilde{f}^{-1}(\alpha)$. □

If we knew that, as in the real algebraic case or as in the case of o-minimal expansions of fields (by the second author DPhil thesis), the closure $\overline{\{\alpha\}}$ of every $\alpha \in \widetilde{Y}$ is always finite, then the proof of Proposition 2.18 could be simplified. Indeed, either $\alpha$ is closed in which case the result follows or, by Lemma 2.11, there exists a definable subset $A$ of $Y$ such that for every proper specialization $\beta$ of $\alpha$ we have $A \in \alpha \setminus \beta$ and $\mathrm{fr}(A) \in \beta$. Thus, replacing $Y$ by $Z = Y \setminus (\mathrm{fr}(A))$, $\alpha$ would became a closed point in $\widetilde{Z}$ and $\widetilde{f}^{-1}(\alpha)$ would be closed and hence quasi-compact in $\widetilde{f}^{-1}(\widetilde{Z})$. So $\widetilde{f}^{-1}(\alpha)$ would be quasi-compact in $\widetilde{X}$.

# 3 sheaves on definable sets

Definitions 2.2, 2.16 and Remark 2.17 give us the *o-minimal tilde functor* $\mathrm{DTOP} \longrightarrow \widetilde{\mathrm{DTOP}}$ where $\widetilde{\mathrm{DTOP}}$ is the category whose objects are the o-minimal spectra of definable sets and the morphisms are the o-minimal spectra of continuous definable maps between definable sets.

Let $X$ be a definable set. We denote by $\mathrm{Sh}_{\mathrm{dtop}}(X)$ the category of sheaves of abelian groups on $X$ with respect to the o-minimal site on $X$. For the o-minimal spectrum $\widetilde{X}$ of $X$, since it is a topological space, we use the classical notation $\mathrm{Sh}(\widetilde{X})$ to denote the category of sheaves of abelian groups on $\widetilde{X}$.

Since the topology on the o-minimal spectrum $\widetilde{X}$ of $X$ is generated by the constructible open subsets, i.e., sets of the form $\widetilde{U}$ with $U$ an open definable subset of $X$, a sheaf on $\widetilde{X}$ is determined by its values on the sets $\widetilde{U}$ with $U \in \mathrm{ObjDTOP}_X$. Thus, for a definable set $X$, we define the functors of the categories of sheaves of abelian groups

$$\mathrm{Sh}_{\mathrm{dtop}}(X) \longrightarrow \mathrm{Sh}(\widetilde{X})$$

which sends $\mathcal{F} \in \mathrm{Sh}_{\mathrm{dtop}}(X)$ into $\widetilde{\mathcal{F}}$ where, for $U \in \mathrm{ObjDTOP}_X$, we define $\widetilde{\mathcal{F}}(\widetilde{U}) = \{\widetilde{s} : s \in \mathcal{F}(U)\} \simeq \mathcal{F}(U)$, and

$$\mathrm{Sh}(\widetilde{X}) \longrightarrow \mathrm{Sh}_{\mathrm{dtop}}(X)$$

which sends $\widetilde{\mathcal{F}}$ into $\mathcal{F}$ where, for $U \in \mathrm{ObjDTOP}_X$, we define $\mathcal{F}(U) = \{s : \widetilde{s} \in \widetilde{\mathcal{F}}(\widetilde{U})\} \simeq \widetilde{\mathcal{F}}(\widetilde{U})$.

**Proposition 3.1** *Let $X$ be a definable set. The functor* $\mathrm{Sh}_{\mathrm{dtop}}(X) \longrightarrow \mathrm{Sh}(\widetilde{X})$ *is a well defined isomorphism of categories with inverse given by* $\mathrm{Sh}(\widetilde{X}) \longrightarrow \mathrm{Sh}_{\mathrm{dtop}}(X)$*, hence* $\mathrm{Sh}_{\mathrm{dtop}}(X)$ *is an abelian category with enough injectives.*

**Proof.** Let $\mathcal{F}$ be a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$, $U \in \mathrm{Obj}\mathrm{DTOP}_X$ and suppose that $\{\widetilde{U}_i : i \in I\}$ is an (admissible) open cover of $\widetilde{U}$ in $\widetilde{X}$ and $\widetilde{s}_i \in \widetilde{\mathcal{F}}(\widetilde{U}_i)$ are sections such that $\widetilde{s}_{i|\widetilde{U}_i \cap \widetilde{U}_j} = \widetilde{s}_{j|\widetilde{U}_i \cap \widetilde{U}_j}$. Since $\widetilde{U}$ is quasi-compact, we may assume that $I$ is finite. Hence, $\{U_i : i \in I\}$ is an admissible open cover of $U$ in $\mathrm{DTOP}_X$. But then, the sections $s_i \in \mathcal{F}(U_i)$ can be glued together to give a section $s \in \mathcal{F}(U)$. Consequently, the sections $\widetilde{s}_i \in \widetilde{\mathcal{F}}(\widetilde{U}_i)$ can be glued together to give a section $\widetilde{s} \in \widetilde{\mathcal{F}}(\widetilde{U})$.

Clearly, $\mathrm{Sh}(\widetilde{X}) \longrightarrow \mathrm{Sh}_{\mathrm{dtop}}(X)$ is the inverse to $\mathrm{Sh}_{\mathrm{dtop}}(X) \longrightarrow \mathrm{Sh}(\widetilde{X})$.

Since the category of sheaves of abelian groups on a topological space is an abelian category with enough injectives (see [ks] Proposition 2.2.4 and 2.4.3 or [b] Chapter II, Theorem 3.2) and $\widetilde{X}$ is a topological space, it follows from the isomorphism $\mathrm{Sh}_{\mathrm{dtop}}(X) \longrightarrow \mathrm{Sh}(\widetilde{X})$, that the same holds for $\mathrm{Sh}_{\mathrm{dtop}}(X)$. $\Box$

Given a continuous definable map $f : X \longrightarrow Y$, we can define the *direct image*

$$f_* : \mathrm{Sh}_{\mathrm{dtop}}(X) \longrightarrow \mathrm{Sh}_{\mathrm{dtop}}(Y)$$

and the *inverse image*

$$f^* : \mathrm{Sh}_{\mathrm{dtop}}(Y) \longrightarrow \mathrm{Sh}_{\mathrm{dtop}}(X)$$

morphisms via the isomorphism of Proposition 3.1 from the direct image and inverse image morphisms in the category of sheaves of abelian groups in topological spaces treated in [b] Chapter I, Section 3 and 4:

$$\widetilde{f_*\mathcal{F}} = \widetilde{f}_*\widetilde{\mathcal{F}} \text{ and } \widetilde{f^*\mathcal{G}} = \widetilde{f}^*\widetilde{\mathcal{G}}$$

for $\mathcal{F} \in \mathrm{Sh}_{\mathrm{dtop}}(X)$ and $\mathcal{G} \in \mathrm{Sh}_{\mathrm{dtop}}(Y)$.

The direct image and the inverse image are adjoint to each other

$$\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}) \simeq \mathrm{Hom}(f^*\mathcal{G}, \mathcal{F})$$

and we have functoriality $\mathrm{id}_* = \mathrm{id}$, $(f \circ g)_* = f_* \circ g_*$, $\mathrm{id}^* = \mathrm{id}$ and $(f \circ g)^* = g^* \circ f^*$. Furthermore, from the fact that the inverse image and the direct image are adjoint, it follows that there are natural morphisms of functors $f^* \circ f_* \to \mathrm{id}$ and $\mathrm{id} \to f_* \circ f^*$ called the *adjunction morphisms.*

If $Z \subseteq X$ are definable sets, $j : Z \longrightarrow X$ is the inclusion and $\mathcal{F} \in \mathrm{Sh}_{\mathrm{dtop}}(X)$ the *restriction* is $\mathcal{F}_{|Z} = j^*\mathcal{F}$ and if $Z$ is closed, the *extension by zero* $\mathcal{F}_Z = j_*j^*\mathcal{F}$ is the sheaf such that the sequence $0 \longrightarrow \mathcal{F}_Z(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}(U \setminus (Z \cap Z)) \longrightarrow 0$ is exact for every $U \in \mathrm{Obj}\mathrm{DTOP}_X$. We also use the notation $\Gamma(U; \mathcal{F})$ and $\Gamma_Z(U; \mathcal{F})$ for $\mathcal{F}(U)$ and $\mathcal{F}_Z(U)$ respectively.

For details on all of the above see [ks] Chapter II, Section 2.3 or [b] Chapter I, Section 3 and 4.

The results we present below are in the category $\widetilde{\mathrm{DTOP}}$ and by the isomorphism of Proposition 3.1 they have a suitable, but more restrictive, analogue in the category DTOP.

**Lemma 3.2** *Assume that $X$ is a subspace of a normal space in $\widetilde{\mathrm{DTOP}}$, $\mathcal{F}$ is a sheaf on $X$ and $Y$ is a quasi-compact subset of $X$. Then for every $s \in \Gamma(Y, \mathcal{F}_{|Y})$ there exists an open neighbourhood $W$ of $Y$ in $X$ and $t \in \Gamma(W, \mathcal{F}_{|W})$ such that $t_{|Y} = s$.*

The proof of this result is an immediate consequence of the shrinking lemma (Proposition 2.15) and holds in any normal spectral space. For details see its analogue in [D2] Lemma 2.2. As pointed out in [D2], this fact is well known if $X$ and $Y$ are Hausdorff topological spaces and $Y$ has a fundamental system of paracompact neighbourhoods in $X$ ([g] Chapter II, 3.3.1) and the proof is the same.

A sheaf $\mathcal{F}$ on a topological space $X$ is *soft* (resp., *flabby*) if and only if for every closed (resp., open) subset $Y$ of $X$ the restriction $\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(Y, \mathcal{F}_{|Y})$ is surjective. Lemma 3.2 above implies that any flabby sheaf on a subspace $X$ of normal space in $\widetilde{\mathrm{DTOP}}$ is soft.

**Proposition 3.3** *Assume that $X$ is a subspace of a normal space in $\widetilde{\mathrm{DTOP}}$. Then the full additive subcategory of $\mathrm{Sh}(X)$ of injective or of flabby, or of soft sheaves is $\Gamma(X; -)$-injective, i.e.:*

(1) *For every $\mathcal{F} \in \mathrm{Sh}(X)$ there exists an injective (resp., flabby and soft) $\mathcal{F}' \in \mathrm{Sh}(X)$ and an exact sequence $0 \to \mathcal{F} \to \mathcal{F}'$.*

(2) *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence in $\mathrm{Sh}(X)$ with $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{F}''$ injective (resp., flabby and soft), then we have an exact sequence $0 \longrightarrow \Gamma(X; \mathcal{F}') \longrightarrow \Gamma(X; \mathcal{F}) \longrightarrow \Gamma(X; \mathcal{F}'') \longrightarrow 0$.*

(3) *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence in $\mathrm{Sh}(X)$ with $\mathcal{F}'$ and $\mathcal{F}$ injective (resp., flabby and soft), then $\mathcal{F}''$ is injective (resp., flabby and soft).*

**Proof.** The result for the injective and flabby case is classical for topological spaces. Indeed, since there are enough injectives, (1) holds for the injective case; since every injective sheaf is flabby ([b] Chapter II, Proposition 5.3), (1) also holds for the flabby case; (2) and (3) for the flabby (and hence for the injective case) are proved in [b] Chapter II, Theorem 5.4.

By Lemma 3.2 any flabby sheaf on $X$ is soft. Thus (1) holds for the soft case. On the other hand, (2) for the soft case is another consequence of the shrinking lemma (Proposition 2.15) and we refer the reader to [D2] Lemma 2.3 for details. Finally, (3) follows at once from (2) (see [D2] Lemma 2.4). $\square$

**Corollary 3.4** *In the situation of Proposition 3.3, let $Y$ be a closed subset of $X$. Then the full additive subcategory of* $\mathrm{Sh}(X)$ *of injective (resp., flabby and soft) sheaves is* $\Gamma_Y(X;-)$*-injective.*

This is purely algebraic and follows immeadiately from the exact sequence in Proposition 3.3 (2) together with the exact sequence defining the extension by zero. For details compare with [ks] Corollary 2.4.8.

Note that there is an analogue of Proposition 3.3 for paracompact (Hausdorff) topological spaces and the proof is similar once one replaces paracompactness by the shrinking lemma (see [D2] and [g], 3.5). In fact, paracompact spaces are normal and in normal spaces there is a shrinking lemma for locally finite open covers.

# 4 O-minimal sheaf cohomology

In this section we prove the existence of o-minimal sheaf cohomology satisfying the Eilenberg-Steenrod axioms adapted to the o-minimal site.

## 4.1 O-minimal sheaf cohomology

Let $X$ be a definable set and $\mathcal{F}$ a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$. We define the *o-minimal sheaf cohomology groups* by

$$H^n(X;\mathcal{F}) = R^n\Gamma(X;\mathcal{F}) \text{ for all } n \in \mathbb{N}$$

where $R^n\Gamma(X;-)$ is the $n$-th right derived functor of the global sections functor $\Gamma(X;-)$. Since $\mathrm{Sh}_{\mathrm{dtop}}(X)$ is an abelian category with enough injectives (Proposition 3.1), to compute $H^n(X;\mathcal{F})$, one takes an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

of $\mathcal{F}$ and the group $H^n(X;\mathcal{F})$ is the $n$-th cohomology group of the chain complex

$$0 \xrightarrow{\iota} \Gamma(X;\mathcal{I}^0) \to \Gamma(X;\mathcal{I}^1) \to \Gamma(X;\mathcal{I}^2) \to \cdots$$

where $0 \stackrel{\iota}{\to} \Gamma(X;\mathcal{I}^0)$ is the composition $0 \to \Gamma(X;\mathcal{F}) \to \Gamma(X;\mathcal{I}^0)$. By the isomorphism of Proposition 3.1, we have $R^n\Gamma(X;\mathcal{F}) = R^n\Gamma(\widetilde{X};\widetilde{\mathcal{F}})$ for all $n \in \mathbb{N}$. So the group $H^n(X;\mathcal{F})$ can also be computed by taking a flabby resolution of $\mathcal{F}$ since ([b] Chapter II, Theorem 5.5) flabby sheaves are acyclic (see [b] Chapter II, Theorem 4.1). If $X$ is definably normal, as in [D2] Remark 2.6, Proposition 3.3, implies that soft sheaves are acyclic and hence, by [b] Chapter II, Theorem 4.1, one can take a soft resolution of $\mathcal{F}$ to compute $H^n(X;\mathcal{F})$.

Let $f : X \longrightarrow Y$ be a continuous definable map. By the isomorphism of Proposition 3.1 and the fact that the inverse image functor in the topological case is exact ([b] Chapter I, Section 3), the inverse image functor $f^* : \mathrm{Sh}_{\mathrm{dtop}}(Y) \longrightarrow \mathrm{Sh}_{\mathrm{dtop}}(X)$ is exact. Thus, if $\mathcal{F}$ a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(Y)$, we get as in [b] Chapter II, 6.3 the induced homomorphism

$$f^* : H^*(Y;\mathcal{F}) \longrightarrow H^*(X; f^*\mathcal{F})$$

in cohomology which when we consider cohomology as a functor of sheaves in $\mathrm{Sh}_{\mathrm{dtop}}(Y)$ is a natural transformation of functors compatible with connecting homomorphisms. Since $\widetilde{R^l f_*\mathcal{F}} = R^l\widetilde{f}_*\widetilde{\mathcal{F}}$ for all $l \geq 0$ where $R^l f_*$ is the $l$-th right derived functor of $f_*$, the induced homomorphism in cohomology is given by the composition $\epsilon \circ \nu$ where

$$\epsilon : H^p(Y; f_*f^*\mathcal{F}) \longrightarrow H^p(X; f^*\mathcal{F})$$

is the canonical edge homomorphism $E_2^{p,0} \to E^p$ in the Leray spectral sequence

$$H^p(Y; R^q f_*(f^*\mathcal{F})) \implies H^{p+q}(X; f^*\mathcal{F})$$

of $f^*\mathcal{F}$ with respect to $f$ and

$$\nu : H^p(Y;\mathcal{F}) \longrightarrow H^p(Y; f_*f^*\mathcal{F})$$

is induced by the adjunction homomorphism $\mathcal{F} \to f_*f^*\mathcal{F}$. (See [D2], [b] Chapter IV, Section 6 or [g] Chapter II, 4.17). By construction we have that $f^* : H^*(Y;\mathcal{F}) \longrightarrow H^*(X; f^*\mathcal{F})$ is the same as $\widetilde{f}^* : H^*(\widetilde{Y};\widetilde{\mathcal{F}}) \longrightarrow H^*(\widetilde{X}; \widetilde{f}^*\widetilde{\mathcal{F}})$.

If $X$ be a definable set, $A$ is a closed definable subset of $X$ and $\mathcal{F}$ a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$, we define as above the *relative o-minimal sheaf cohomology groups*

$$H^n(X, A; \mathcal{F}) \text{ for all } n \in \mathbb{N}$$

by replacing $\Gamma(X; -)$ by $\Gamma_A(X; -)$. Similarly, if $f : (X, A) \longrightarrow (Y, B)$ is a continuous definable map of closed pairs of definable sets (i.e., $A \subseteq X$

and $B \subseteq Y$ are closed definable subsets and $f : X \longrightarrow Y$ is a continuous definable map such that $f(A) \subseteq B$) and $\mathcal{F}$ a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(Y)$, then the induced homomorphisms

$$f^* : H^*(Y, B; \mathcal{F}) \longrightarrow H^*(X, A; f^*\mathcal{F})$$

in cohomology are defined as above by replacing $\mathcal{F}$ by $\mathcal{F}_B$.

We have the following useful characterisation of the o-minimal cohomology groups.

**Proposition 4.1** *Let $X$ be a definably normal definable set and $\mathcal{F}$ a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$. Then for all $n \in \mathbb{N}$, the cohomology group $H^n(X; \mathcal{F})$ is isomorphic to the Čech cohomology group $\check{H}^n(X; \mathcal{F})$ relative to the o-minimal site on $X$, i.e., calculated using finite covers by open definable subsets of $X$.*

This is the same as its semi-algebraic analogue in [cc] Proposition 5 and the proof is the same since one only uses that fact that $\widetilde{X}$ is a normal spectral space (Theorem 2.12). Similarly we have the following vanishing theorem which is the o-minimal version of [cc] Corollary 3. Here we use the fact that for a definable set $X$, we have $\dim X = \dim_{\mathrm{Krull}} \widetilde{X}$, where $\dim_{\mathrm{Krull}} \widetilde{X}$, the *Krull dimension* of $\widetilde{X}$, is the maximal lenght of proper specialisations of points in $\widetilde{X}$. To see this use Lemma 2.11 and the fact that a cell of dimension $k$ is definably homeomorphic to an open cell in $N^k$ ([vdd] Chapter III, (2.7)).

**Proposition 4.2 (Vanishing theorem)** *Let $X$ be a definably normal definable set and $\mathcal{F}$ a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$. Then*

$$H^n(X; \mathcal{F}) = 0 \quad \text{for all } n > \dim X.$$

## 4.2 Base change and Vietoris-Biegle theorem

We start with the following o-minimal version of [D2] Theorem 3.1. The proof is the same and one uses Lemma 3.2 in place of its real algebraic version in [D2] Lemma 2.2.

**Proposition 4.3** *Assume that $X$ is a subspace of a normal space in $\widetilde{\mathrm{DTOP}}$, $\mathcal{F}$ is a sheaf in $\mathrm{Sh}(X)$ and $Y$ is a quasi-compact subset of $X$. Then the canonical homomorphism*

$$\lim_{Y \subseteq U, U \text{ open in } X} H^q(U; \mathcal{F}) \longrightarrow H^q(Y; \mathcal{F}_{|Y})$$

*is an isomorphism for every $q \geq 0$.*

We now state the base change theorem. The proof is similar to [D2] Theorem 3.5 but we include it to illustrate the use of Proposition 2.18.

**Theorem 4.4 (Base Change theorem)** *Let $f : X \longrightarrow Y$ be a morphism in $\widetilde{\mathrm{DTOP}}$. Assume that $f$ maps constructible closed subsets of $X$ onto closed subsets of $Y$. Let $\mathcal{F}$ be a sheaf in* $\mathrm{Sh}(X)$ *and suppose that $Y$ is a subspace of a normal space in* $\widetilde{\mathrm{DTOP}}$. *Then, for every $\beta \in Y$, the canonical homomorphism*

$$(R^q f_* \mathcal{F})_\beta \longrightarrow H^q(f^{-1}(\beta); \mathcal{F}_{|f^{-1}(\beta)})$$

*is an isomorphism, where $(R^q f_* \mathcal{F})_\beta$ denotes the stalk of the higher direct image $R^q f_* \mathcal{F}$ in $\beta$.*

**Proof.** By Proposition 2.18 the fiber $f^{-1}(\beta)$ is quasi-compact and so has a fundamental system of constructible open neighbourhoods in $X$. If $U$ is a constructible open neighbourhood of $f^{-1}(\beta)$, then $X \setminus U$, by assumption, is mapped onto a closed subset of $Y$ not containing $\beta$. Thus the collection of all $f^{-1}(U)$ with $U$ a constructible open neighbourhood of $\beta$ in $Y$ is a fundamental system of open neighbourhoods of $f^{-1}(\beta)$ in $X$. The result follows from Proposition 4.3, since as a pre-sheaf $R^q f_* \mathcal{F}(U) = H^q(f^{-1}(U); \mathcal{F})$ for all $U \in \mathrm{ObjDTOP}_Y$. □

The Vietoris-Biegle theorem follows from the base change theorem using classical arguments:

**Theorem 4.5 (Vietoris-Biegle theorem)** *Let $f : X \longrightarrow Y$ be a morphism in* $\widetilde{\mathrm{DTOP}}$ *that maps constructible closed subsets of $X$ onto closed subsets of $Y$. Let $\mathcal{F}$ be a sheaf in* $\mathrm{Sh}(Y)$ *and suppose that $Y$ is a subspace of a normal space in* $\widetilde{\mathrm{DTOP}}$. *Assume that $f^{-1}(\beta)$ is connected and $H^q(f^{-1}(\beta); f^* \mathcal{F}_{|f^{-1}(\beta)}) = 0$ for $q > 0$ and all $\beta \in Y$. Then the following hold:*

(1) *$R^q f_*(f^* \mathcal{F}) = 0$ for all $q > 0$.*

(2) *The adjunction homomorphism $\mathcal{F} \to f_* f^* \mathcal{F}$ is an isomorphism.*

(3) *$f^* : H^*(Y; \mathcal{F}) \longrightarrow H^*(X; f^* \mathcal{F})$ is an isomorphism.*

**Proof.** Since on topological spaces sheaves are determined by their stalks ([ks] Proposition 2.2.2), for (1) and (2) it is enough to show that for all $\beta \in Y$ and $q > 0$,

$$(R^q f_*(f^* \mathcal{F}))_\beta = 0 \text{ and } \mathcal{F}_\beta \to (f_* f^* \mathcal{F})_\beta \text{ is an isomorphism.}$$

But by the base change theorem (Theorem 4.4) and the assumptions, we have

$$(R^q f_*(f^*\mathcal{F}))_\beta = H^q(f^{-1}(\beta); f^*\mathcal{F}_{|f^{-1}(\beta)}) = 0 \text{ for all } q > 0,$$

and since $f^*\mathcal{F}_{|f^{-1}(\beta)}$ is the constant sheaf $\mathcal{F}_\beta$ on $f^{-1}(\beta)$ (because $f \circ k = l \circ f$ where $k : f^{-1}(\beta) \longrightarrow X$ and $l : \{\beta\} \longrightarrow Y$ are the inclusions) we have

$$(f_* f^*\mathcal{F})_\beta = (R^0 f_*(f^*\mathcal{F}))_\beta = H^0(f^{-1}(\beta); f^*\mathcal{F}_{|f^{-1}(\beta)}) = \mathcal{F}_\beta.$$

By (1) the the Leray spectral sequence splits and the edge homomorphism $\epsilon : H^p(Y; f_*f^*\mathcal{F}) \longrightarrow H^p(X; f^*\mathcal{F})$ is an isomorphism. By (2) the homomorphism $\nu : H^p(Y;\mathcal{F}) \longrightarrow H^p(Y; f_*f^*\mathcal{F})$ induced by the adjunction isomorphism $\mathcal{F} \to f_*f^*\mathcal{F}$ is an isomorphism. Hence,

$$f^* = \epsilon \circ \nu : H^*(Y;\mathcal{F}) \longrightarrow H^*(X; f^*\mathcal{F})$$

is an isomorphism. □

## 4.3 The Eilenberg-Steenrod axioms

Finally we are ready to prove the main result of the paper, namely that the o-minimal cohomology functor $H^*$ constructed above satisfies the o-minimal Eilenberg-Steenrod axioms:

**Theorem 4.6** *If $X$ is a definable set and $\mathcal{F}$ is a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$, then the following hold:*

**Exactness Axiom**. *Let $A \subseteq X$ be a closed definable subset. If $i : (A,\emptyset) \longrightarrow (X,\emptyset)$ and $j : (X,\emptyset) \longrightarrow (X,A)$ are the inclusions, then we have a natural exact sequence*

$$\cdots \longrightarrow H^n(X,A;\mathcal{F}) \stackrel{j^*}{\to} H^n(X;\mathcal{F}) \stackrel{i^*}{\to} H^n(A;\mathcal{F}) \stackrel{d^n}{\to} H^{n+1}(X,A;\mathcal{F}) \longrightarrow \cdots .$$

**Excision Axiom**. *For every closed definable subset $A \subseteq X$ and definable open subset $U \subseteq X$ such that $U \subseteq A$, the inclusion $(X-U, A-U) \longrightarrow (X,A)$ induces isomorphisms*

$$H^n(X,A;\mathcal{F}) \longrightarrow H^n(X-U, A-U;\mathcal{F})$$

*for all $n \in \mathbb{N}$.*

**Homotopy Axiom**. *Let $[a,b] \subseteq N$ be a closed interval and $A \subseteq X$ a closed definable subset. Assume that $\mathcal{N}$ has definable Skolem functions, $X$ is* *definably normal and the projection $X \times [a,b] \longrightarrow X$ maps closed definable subsets of $X \times [a,b]$ onto closed definable subsets of $X$. If for $c \in [a,b]$,*

$$i_c : (X, A) \longrightarrow (X \times [a,b], A \times [a,b])$$

*is the continuous definable map given by $i_c(x) = (x, c)$ for all $x \in X$, then*

$$i_a^* = i_b^* : H^n(X \times [a,b], A \times [a,b]; \pi^*\mathcal{F}) \longrightarrow H^n(X, A; \mathcal{F})$$

*for all $n \in \mathbb{N}$.*

**Dimension Axiom**. *If $X$ is a one point set, then $H^n(X;\mathcal{F}) = 0$ for all $n > 0$ and $H^0(X;\mathcal{F}) = \mathcal{F}$.*

**Proof.** Once we pass to $\widetilde{\text{DTOP}}$ the proofs of the exactness and excision axioms are purely algebraic. See [b] Chapter II, Section 12, (22) and 12.8 respectively. The dimension axiom is also immeadiate.

The homotopy axiom will follow once we show that the projection map $\pi : (X \times [a,b], A \times [a,b]) \longrightarrow (X, A)$ induces an isomorphism

$$\pi^* : H^n(X, A; \mathcal{F}) \longrightarrow H^n(X \times [a,b], A \times [a,b]; \pi^*\mathcal{F})$$

since by functoriality we obtain

$$i_a^* = i_b^* = (\pi^*)^{-1} : H^n(X \times [a,b], A \times [a,b]; \pi^*\mathcal{F}) \longrightarrow H^n(X, A; \mathcal{F})$$

for all $n \in \mathbb{N}$. By the exactness axiom it suffices to show that we have an isomorphism $\pi^* : H^n(X;\mathcal{F}) \longrightarrow H^n(X \times [a,b]; \pi^*\mathcal{F})$. Equivalently we need to show that $\widetilde{\pi}^* : H^n(\widetilde{X}; \widetilde{\mathcal{F}}) \longrightarrow H^n(\widetilde{X \times [a,b]}; \widetilde{\pi}^*\widetilde{\mathcal{F}})$ is an isomorphism. For this we verify the hypothesis of the Vietoris-Biegle theorem (Theorem 4.5).

By Theorem 2.12, $\widetilde{X}$ is normal. By the assumption, $\pi : X \times [a,b] \longrightarrow X$ maps closed definable subsets of $X \times [a,b]$ onto closed definable subsets of $X$. Therefore, $\widetilde{\pi} : \widetilde{X \times [a,b]} \longrightarrow \widetilde{X}$ maps constructible closed subset of $\widetilde{X \times [a,b]}$ onto (constructible) closed subsets of $\widetilde{X}$.

Since for each $\alpha \in \widetilde{X}$, $\widetilde{\pi}^*\widetilde{\mathcal{F}}_{|\widetilde{\pi}^{-1}(\alpha)}$ is the constant sheaf $\widetilde{\mathcal{F}}_\alpha$, it remains to show that $\widetilde{\pi}^{-1}(\alpha)$ is connected and acyclic i.e., $H^q(\widetilde{\pi}^{-1}(\alpha); F) = 0$ for every $q > 0$ and every abelian group $F$.

**Claim 4.7** *Let $\alpha \in \widetilde{X}$ and let $\mathcal{N}^\star$ be the prime model of the first-order theory of $\mathcal{N}$ over $N \cup \{e\}$, where $e$ is an element realising the type $\alpha$. Then there exists a homeomorphism $t : \widetilde{\pi}^{-1}(\alpha) \longrightarrow \widetilde{[a,b]^\star}$, and so $\widetilde{\pi}^{-1}(\alpha)$ is quasi-compact, connected and acyclic.*

Define the map $t : \widetilde{\pi}^{-1}(\alpha) \longrightarrow \widetilde{[a,b]^\star}$ by sending $\gamma \in \widetilde{\pi}^{-1}(\alpha)$ to the type $t(\gamma) = \mathrm{tp}(c/N^\star)$ such that $(e,c)$ realises $\gamma$ in some saturated elementary extension of $\mathcal{N}^\star$. Since $\mathcal{N}$ has definable Skolem functions, every element in $N^\star$ is defined over $N \cup \{e\}$. Hence the map $t$ is a well defined. Indeed, suppose that $(e,c)$ and $(e,d)$ realise $\gamma$ in some saturated elementary extension of $\mathcal{N}^\star$ but $\mathrm{tp}(c/N^\star)$ is different from $\mathrm{tp}(d/N^\star)$. Then there is a first-order formula $\phi(u,m)$ with parameter $m \in N^\star$ such that $\phi(c,m)$ holds but $\phi(d,m)$ doesn't hold. As $m$ is defined over $N \cup \{e\}$, there exists a first-order formula $\psi(v,w)$ with parameters in $N$ such that $\psi(v,e)$ defines $m$. So $\exists v \psi(v,w) \wedge \phi(u,v)$ is realised by $(e,c)$ but not by $(e,d)$ which is a contradiction.

A similar argument shows that $t$ is injective. Let us show that $t$ is surjective. Let $\beta$ be a type in $\widetilde{[a,b]^\star}$. As above, every formula $\phi(u,m)$ in $\beta$ is equivalent to a formula of the form $\tau(e,u)$ where $\tau(w,u)$ is a formula over $N$. Clearly the type $\alpha$ is consistent with the collection $\Sigma(w,u)$ of all such formulas obtained from the formulas in $\beta$. Furhermore, $\alpha \cup \Sigma(w,u)$ determines a type $\gamma$ over $N$ such that $t(\gamma) = \beta$. In fact, let $\theta(w,u)$ be a first-order formula over $N$ and let $c$ be a realisation of $\beta$. Then either $\theta(e,c)$ holds in which case $\theta(e,u) \in \beta$ and $\theta(w,u) \in \Sigma(w,u)$ or $\theta(e,c)$ doesn't hold in which case $\neg\theta(e,u) \in \beta$ and $\neg\theta(w,u) \in \Sigma(w,u)$.

Noting that if $U$ is an open definable subset of $X \times [a,b]$, then $t(\widetilde{U} \cap \widetilde{\pi}^{-1}(\alpha)) = \widetilde{r^\star(U^\star)}$ where $r : X \times [a,b] \longrightarrow [a,b]$ is the projection, it follows that $t$ is a open map. To show that $t$ is a homeomorphism, it remains to show that $t$ is continuous. Let $a < c_1 < c_2 < b$ be elements of $[a,b]^\star$ over $N^\star$. Since $c_1$ and $c_2$ are defined over $N \cup \{e\}$ (because $\mathcal{N}$ has definable Skolem functions), there are definable functions $f_1, f_2 : A \subseteq X \longrightarrow [a,b]$ over $N$ such that $f_i(e) = c_i$ for $i = 1,2$. As pointed out in the proof of [p] Proposition 2.2, the proof of [p] Proposition 2.1 shows that there exists an open definable subset $U$ of $X$ over $N$ containing $A$ and continuous definable functions $g_1, g_2 : U \subseteq X \longrightarrow [a,b]$ over $N$ such that $g_{i|A} = f_i$ for $i = 1,2$. But then $t(\widetilde{W} \cap \widetilde{\pi}^{-1}(\alpha)) = \widetilde{(c_1,c_2)}$ where $W = (g_1,g_2)_U$ is an open definable subset of $X \times [a,b]$ over $N$. Similarly, there are open definable subset $W_i$ $(i = 1,2)$ of $X \times [a,b]$ such that $t(\widetilde{W_1} \cap \widetilde{\pi}^{-1}(\alpha)) = \widetilde{[a,c_1)}$ and $t(\widetilde{W_2} \cap \widetilde{\pi}^{-1}(\alpha)) = \widetilde{(c_2,b]}$.

Since $t$ is a homeomorphism and $\widetilde{[a,b]^\star}$ is quasi-compact and connected (by Remark 2.3 (3)), $\widetilde{\pi}^{-1}(\alpha)$ is quasi-compact and connected. Furthermore, we have $H^*(\widetilde{\pi}^{-1}(\alpha); F) \simeq H^*(\widetilde{[a,b]^*}; F) \simeq H^*([a,b]^*; F)$. By Proposition 4.1, $H^*([a,b]^*; F) \simeq \check{H}^*([a,b]^*; F)$. Arguing as in [D2] page 124, we conclude that $\check{H}^q([a,b]^*; F) = 0$ for all $q > 0$ as required. □

Let $f, g : (X, A) \longrightarrow (Y, B)$ be continuous definable maps with $A \subseteq X$ and $B \subseteq Y$ closed definable subsets and suppose that $s : (X \times [a, b], A \times [a, b]) \longrightarrow (Y, B)$ is a definable homotopy between $f$ and $g$, meaning that $s$ is a continuous definable map such that $s \circ i_a = f$ and $s \circ i_b = g$. If $X$ satisfies the assumptions of the homotopy axiom and $F$ is a constant sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$, then we get by functoriality

$$f^* = g^* : H^n(Y, B; F) \longrightarrow H^n(X, A; F)$$

for all $n \in \mathbb{N}$.

Observe that if $\mathcal{N}$ is an o-minimal expansion of an ordered group, then the two assumptions on $X$ in the homotopy axiom hold for every definable set.

We end the section with the exactness for triples and the Mayer-Vietoris theorem.

**Proposition 4.8 (Exactness for triples)** *Let $X$ be a definable set, $B \subseteq A \subseteq X$ closed definable subsets and $\mathcal{F}$ a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$. Then there is an exact sequence for all $n \in \mathbb{N}$*

$$\to H^n(X, A; \mathcal{F}) \to H^n(X, B; \mathcal{F}) \to H^n(A, B; \mathcal{F}) \to H^{n+1}(X, A; \mathcal{F}) \to .$$

In $\widetilde{\mathrm{DTOP}}$ the proof of Proposition 4.8 is as in [b] Chapter II, Section 12, (24). If $X$ is a definable set and $B \subseteq A \subseteq X$ are closed definable subsets, then by the excision axiom, $(X; A, B)$ is an *excisive triad* meaning that the inclusion $(A, A \cap B) \longrightarrow (A \cup B, B)$ map induces isomorphisms

$$H^*(A \cup B, B; \mathcal{F}) \simeq H^*(A, A \cap B; \mathcal{F})$$

for every sheaf $\mathcal{F}$ in $\mathrm{Sh}_{\mathrm{dtop}}(X)$. Thus, by going to $\widetilde{\mathrm{DTOP}}$, the following holds (see [b] Chapter II, Section 13, (32)).

**Proposition 4.9 (Mayer-Vietoris)** *Let $X$ and $Z$ be definable sets and let $X_2 \subseteq X_1$ and $Z_2 \subseteq Z_1$ be closed definable subsets such that $X = X_1 \cup X_2$ and $Z = Z_1 \cup Z_2$. Let $\mathcal{F}$ be a sheaf in $\mathrm{Sh}_{\mathrm{dtop}}(X)$. Assume that we have the following commutative diagram of inclusions*

$$\begin{array}{ccc} (X_1 \cap X_2, Z_1 \cap Z_2) & \longrightarrow & (X_1, Z_1) \\ \downarrow & & \downarrow \\ (X_2, Z_2) & \longrightarrow & (X, Z). \end{array}$$

*Then there is an exact sequence for all $n \in \mathbb{N}$*

$$\cdots \to H^n(X, Z; \mathcal{F}) \to H^n(X_1, Z_1; \mathcal{F}) \oplus H^n(X_2, Z_2; \mathcal{F}) \to$$
$$\to H^n(X_1 \cap X_2, Z_1 \cap Z_2; \mathcal{F}) \to H^{n+1}(X, Z; \mathcal{F}) \to \cdots .$$

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