Maps on ultrametric spaces, Hensel's Lemma, and differential equations over valued fields[1]

Franz-Viktor Kuhlmann

8. 7. 2006

Abstract. We give a criterion for maps on ultrametric spaces to be surjective and to preserve spherical completeness. We show how Hensel's Lemma and the multi-dimensional Hensel's Lemma follow from our result. We give an easy proof that the latter holds in every henselian field. We also prove a basic infinite-dimensional Implicit Function Theorem. Further, we apply the criterion to deduce various versions of Hensel's Lemma for polynomials in several additive operators, and to give a criterion for the existence of integration and solutions of certain differential equations on spherically complete differential fields, for both D-fields in the sense of Scanlon, and differentially valued fields in the sense of Rosenlicht. We modify the approach so that it also covers logarithmic-exponential power series fields. Finally, we give a criterion for a sum of spherically complete subgroups of a valued abelian group to be spherically complete. This in turn can be used to determine elementary properties of power series fields in positive characteristic.

Contents

[1] I thank Lou van den Dries for inviting me to Urbana and for making the manuscript [D] available to me. This manuscript was the main inspiration for the results of Section 7.3. I also thank Florian Pop for the key idea in the proof of Theorem 23, and the referee for his thorough reading, many useful suggestions, and the proof of Lemma 42. This paper has undergone a major revision during my stay at the Newton Institute at Cambridge; I gratefully acknowledge its support.

1 Introduction

Hensel's Lemma (see Theorem 21) is an important tool in the theory of valued fields. In recent years, at has witnessed several generalizations. For example, such generalizations are important when the valued fields are enriched by additional structure like derivations. But attempts have also been made to formulate Hensel's Lemma in situations with less structure. For instance, forgetting about multiplication one may consider valued abelian groups or modules. Another interesting case is that of a non-commutative multiplication.

In view of these developments, it is logical to ask for the underlying principle that makes Hensel's Lemma work. This principle should be formulated using as little algebraic structure as possible so that one can derive new versions of Hensel's Lemma by adding whatever structure one is interested in.

It has turned out that the structure suitable for such an underlying principle is that of ultrametric spaces. In [P2], S. Prieß-Crampe proved an ultrametric Fixed Point Theorem. This theorem works with contracting maps, and indeed the Newton algorithm used to prove Hensel's Lemma for the field of p-adic numbers readily provides such a map. But in other situations, contracting maps are not always instantly available. For example, if one looks for zeros of polynomial maps on a valued field, it can be more convenient to directly study the ultrametric properties of these maps. The problem could then be solved by showing surjectivity of such maps when restricted to suitable subsets of the field. Our Ultrametric Main Theorem (Theorem 2) is of this nature.

In the next section, we give a quick introduction to the facts about ultrametric spaces that are necessary to understand the Ultrametric Main Theorem. In Section 1.2 we will then give a summary of the various applications that are derived in this paper.

1.1 The Ultrametric Main Theorem

Let (Y, u) be an ultrametric space. That is, u is a map from $Y \times Y$ onto a totally ordered set Γ with last element ∞, satisfying that for all $x, y, z \in Y$,

(U1) $u(y, z) = \infty$ if and only if $y = z$,

(U2) $u(y,z) \geq \min\{u(y,x), u(x,z)\}$ (ultrametric triangle law),
(U3) $u(y,z) = u(z,y)$ (symmetry).

It follows that

- $u(y,z) > \min\{u(y,x), u(x,z)\} \Rightarrow u(y,x) = u(x,z)$,
- $u(y,x) \neq u(x,z) \Rightarrow u(y,z) = \min\{u(y,x), u(x,z)\}$.

We will use these properties freely. We set $uY := \{u(y,z) \mid y,z \in Y, y \neq z\} = \Gamma \setminus \{\infty\}$ and call it the **value set of** (Y,u).

We recall some definitions. For $y \in Y$ and $\alpha \in uY \cup \{\infty\}$, we define the **closed ball** around y with radius α as follows:

$$B_\alpha(y) := \{z \in Y \mid u(y,z) \geq \alpha\} .$$

To facilitate notation, we will also use

$$B(x,y) := B_{u(x,y)}(x) .$$

It follows from the ultrametric triangle law that $B_{u(x,y)}(x) = B_{u(x,y)}(y)$ and that $B(x,y)$ is the smallest closed ball containing x and y. Similarly, it follows from the ultrametric triangle law that

$$B(x,y) \subseteq B(z,t) \quad \text{if and only if} \quad x \in B(z,t) \text{ and } u(x,y) \geq u(z,t) . \tag{1}$$

(Note: the bigger $u(x,y)$, the closer x and y; this is compatible with the Krull notation of valuations.)

A **ball** is the union of any non-empty collection of closed balls which contain a common element. If B_1 and B_2 are balls with non-empty intersection, then $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

A set of balls in (Y,u) is called a **nest of balls** if it is totally ordered by inclusion; this is the case as soon as every two balls in the set have a nonempty intersection. The **intersection** of the nest is defined to be the intersection of all of its balls. If it is non-empty, then it is again a ball.

The ultrametric space (Y,u) is called **spherically complete** if every nest of balls has a nonempty intersection. It is well known and easy to prove that this holds if and only if every nest of closed balls has a nonempty intersection. If (Y,u) is spherically complete and B is a ball in Y, then also (B,u) is spherically complete.

Let (Y,u) and (Y',u') be non-empty ultrametric spaces and $f: Y \to Y'$ a map. For $y \in Y$, we will write fy instead of $f(y)$. An element $z' \in Y'$ is called **attractor for** f if for every $y \in Y$ such that $z' \neq fy$, there is an element $z \in Y$ which satisfies:

(AT1) $u'(fz,z') > u'(fy,z')$,
(AT2) $f(B(y,z)) \subseteq B(fy,z')$.

Condition (AT1) says that the approximation fy of z' from within the image of f can be improved, and condition (AT2) says that this can be done in a somewhat continuous way.

The following are our main theorems.

Theorem 1 *Assume that $z' \in Y'$ is an attractor for $f: Y \to Y'$ and that (Y,u) is spherically complete. Then $z' \in f(Y)$.*

The map f will be called **immediate** if every $z' \in Y'$ is an attractor for f.

Theorem 2 *Assume that $f: Y \to Y'$ is immediate and that (Y,u) is spherically complete. Then f is surjective and (Y',u') is spherically complete. Moreover, for every $y \in Y$ and every ball B' in Y' containing fy, there is a ball B in Y containing y and such that $f(B) = B'$.*

This theorem is a generalization of a result proved in [KU1] for additive maps on spherically complete abelian groups (see Section 3 for the definition). Theorem 2 also works in the case where the map f is not additive (or even when there is no addition at all). It is related to ultrametric fixed point theorems as proved in [P2], [PR1]. Compared to them, it has the advantage that it can be applied to situations where a natural contracting map is not at hand. There is also a variant of our "Attractor Theorem" (Theorem 1) which works for ultrametric spaces with partially ordered value sets ([PR2]). For further information and applications of ultrametric fixed point theorems, see also [SCH] and [PR3].

If f is just the embedding of an ultrametric subspace Y in an ultrametric space Y', then (AT2) will automatically hold. Hence, we will say that Y is an **immediate subspace of** Y' if it is an ultrametric subspace of Y' and for all $z' \in Y'$ and $y \in Y$ there is $z \in Y$ such that $u'(z,z') > u'(y,z')$. Now Theorem 2 yields:

Corollary 3 *Assume that Y is an immediate ultrametric subspace of Y'. If (Y,u) is spherically complete, then $Y = Y'$.*

It should be noted that an immediate subspace is not necessarily a dense subspace.

A subspace Y of Y' is said to have the **optimal approximation property** (**in** Y') if for every $z' \in Y'$ there is $z \in Y$ such that $u'(z,z') = \max\{u'(y,z') \mid y \in Y\}$. The element z need not be uniquely determined. If the set $\{u'(y,z') \mid y \in Y\}$ has no maximum, then z' is an attractor for the embedding of Y in Y'. On the other hand, if $z' \in Y$, then the maximum is $u(z',z') = \infty$. Thus, Theorem 1 yields:

Corollary 4 *Assume that Y is an ultrametric subspace of Y'. If (Y,u) is spherically complete, then it has the optimal approximation property.*

1.2 Applications

• The Additive Main Theorem

In some applications, the map f is a homomorphism of abelian groups and the ultrametric u is induced by a group (or field) valuation (see Section 3 for definitions). With the presence of addition, balls can be shifted additively to balls that contain 0. In this way, the criteria for immediate maps become much easier to formulate and to check (see

Proposition 11). In Section 3.1 we will prove the additive version of our Ultrametric Main Theorem (Theorem 12), which works for homomorphisms.

In Section 3.3 we will introduce the notion of *pseudo-derivative* for arbitrary maps on valued abelian groups. One can think of it as a derivative at a certain point "up to terms of higher order", valuation theoretically speaking. This notion will then play an essential role when we study polynomial maps.

- **Hensel's Lemma revisited**

Let (K, v) be a valued field with valuation ring $\mathcal{O}$ and valuation ideal $\mathcal{M}$. Further, take a polynomial $f \in \mathcal{O}[X]$ and $b \in \mathcal{O}$ such that $s := f'(b) \neq 0$. In Section 4.3 we consider f as a map on K and prove that *f induces an immediate injective map from $b + s\mathcal{M}$ into $f(b) + s^2\mathcal{M}$* (Proposition 19). Here, the pseudo-derivative is simply multiplication by s. From Theorem 2 we obtain that *if (K, v) is spherically complete (i.e., its underlying ultrametric is spherically complete), then this map is onto* (Theorem 20).

This allows a new look at Hensel's Lemma: while it is always true for (K, v) spherically complete and $f'(b) \neq 0$ that the above map is onto, the condition "$vf(b) \geq 2vf'(b)$" of Hensel's Lemma guarantees that $0 \in f(b) + s^2\mathcal{M}$ and consequently, there is $a \in K$ such that $f(a) = 0$ and $v(a - b) > vf'(b)$ (see Section 4.4). We generalize this result to systems of n polynomials in n variables and use it to prove that the multidimensional Hensel's Lemma holds in every spherically complete valued field (Theorem 22). By an easy argument due to F. Pop, we conclude that the multidimensional Hensel's Lemma holds in every henselian field (see Theorem 23). Further, we prove results on the surjectivity of functions defined by power series in spherically complete valued fields (see Section 4.5).

Our above approach to Hensel's Lemma has also been used in a non-commutative setting. In [VC] it is applied to skew power series fields over skew fields.

- **Towards an infinite-dimensional Implicit Function Theorem**

The n-fold product of a spherically complete ultrametric space is again spherically complete (see Section 2.2). We use this fact for the proof of the multi-dimensional Hensel's Lemma. If one thinks of generalizing this to an infinite-dimensional version, one runs into problems when trying to define a suitable product. But if one restricts the scope to valued rings with well ordered value sets, then this is possible. Using the above mentioned notion of *pseudo-derivative*, we formulate in Section 5 a principle that can be seen as a basic infinite-dimensional Implicit Function Theorem, as needed in B. Teissier's approach to local uniformization in arbitrary characteristic (cf. [T], Theorem 5.56).

- **Weak D-fields**

A weak D-field is a valued field (K, v) with an additive map $D : K \to K$ satisfying conditions that are a relaxation of T. Scanlon's axioms for D-fields (cf. [S1,2]). Scanlon's notion comprises both differential and difference fields. Essential features of weak D-fields are that the value vDa depends on the value va in a sufficiently simple way and that D induces an additive map on the residue field of K (again denoted by D). The following result, proved in Section 7.1, shows that in this setting, the notion of *immediate map* appears in a very natural way: *If* (K,v,D) *is a weak* D*-field, then* D *is immediate if and only if* D *is surjective on* Kv (Theorem 40). Hence we obtain from Theorem 2 that *if* (K,v,D) *is a spherically complete weak* D*-field such that* D *is surjective on* Kv*, then* D *is surjective on* K (see Theorem 41).

In Section 7.1 we will also prove the following version of Scanlon's D-Hensel's Lemma (cf. [S1,2]). By D^i we denote the i-th iterate of D. The residue field Kv is said to be **linearly** D**-closed** if each operator $\sum_{i=0}^n c_i D^i$ with $c_i \in Kv$ is surjective on Kv.

Theorem 5 *Let* (K,v,D) *be a spherically complete weak* D*-field whose residue field is linearly* D*-closed. Take a polynomial* $f \in \mathcal{O}[X_0,X_1,\ldots,X_n]$ *and assume that there is some* $b \in \mathcal{O}$ *such that*

$$\gamma \;:=\; \min_{0\leq i\leq n} v\frac{\partial f}{\partial X_i}(b,Db,\ldots,D^n b) \;<\; \infty \qquad \textit{and} \qquad vf(b,Db\ldots,D^n b) > 2\gamma\;.$$

Then there is an element $a \in K$ *such that* $f(a,Da,\ldots,D^n a) = 0$ *and* $v(a-b) > \gamma$*.*

In fact, we will deduce this theorem from a much more general Hensel's Lemma for polynomials in several additive operators (Theorem 34 in Section 6.2).

- **Rosenlicht valued differential fields**

A valuation v on a differential field (K,D) is a *differential valuation* in the sense of M. Rosenlicht (cf. [R1]) if it satisfies an axiom that is derived from de l'Hôpital's Rule. In this case, there is in general no simple correspondence between the values vDa and va, and there is also no suitable map induced on the residue field. Yet again, immediate maps appear naturally. We say that (K,D) **admits integration** if D is surjective, and that (K,v,D) **admits asymptotic integration** (cf. [R2]) if for every $a' \in K\setminus\{0\}$, there is some $a \in K$ such that

$$v(a'-Da) \;>\; va'\;.$$

In Section 7.2, we will give the (easy) proof of the following fact: *If* v *is a differential valuation on* (K,D)*, then* D *is immediate if and only if* (K,v,D) *admits asymptotic integration* (see Proposition 46). Hence we obtain from Theorem 2: *Let* (K,D) *be a differential field, endowed with a spherically complete differential valuation* v*. If* (K,v,D) *admits asymptotic integration, then* (K,D) *admits integration* (Theorem 47).

In Section 7.2 we will also prove a theorem about integration on the union of an increasing chain of spherically complete Rosenlicht valued differential fields (Theorem 48). It can be used to show that the derivation on the logarithmic-exponential power series field $\mathbb{R}((t))^{LE}$ (cf. [DMM3]) is surjective.

When we try to prove a "differential Hensel's Lemma" for Rosenlicht's differential valuations, we experience technical problems because of the weak correspondence between the values vDa and va. In this case, the results are not as nice and simple as in the case of weak D-fields. The main results are Theorem 51, obtained from the more general Theorem 36 proved in Section 6.3, and Theorem 53, obtained from the more general

Theorem 39 proved in Section 6.4. As a simple application we obtain a result which was proved by Lou van den Dries in [D] (see Corollary 55).

- **Sums of spherically complete valued abelian groups**

So far, we have been interested in the surjectivity of maps. Here is an application where we use that the image of the map inherits spherical completeness. It is used in [KU2] to determine elementary properties of the power series field $\mathbb{F}_p((t))$ in connection with **additive polynomials**. A polynomial f is called additive on an infinite field K if $f(a+b) = f(a) + f(b)$ for all $a, b \in K$ (cf. [L], VIII, §11). For example, the polynomials X^p and $X^p - X$ are additive on $\mathbb{F}_p((t))$ and every other field of characteristic p. For every additive polynomial f on a field K, the image $f(K)$ is a subgroup of the additive group of K. If $f_1, \ldots, f_n$ are additive polynomials with coefficients in K, then the sum $f_1(K) + \ldots + f_n(K)$ is again a subgroup of the additive group of K.

If K is a maximally valued field (like $K = \mathbb{F}_p((t))$; cf. Section 4), then the image $f(K)$ of every polynomial is spherically complete. Hence the question arises whether the subgroup $f_1(K) + \ldots + f_n(K)$ is again spherically complete. In Section 8 we will show that the sum of spherically complete subgroups of a valued abelian group is spherically complete (and hence has the optimal approximation property) if the sum is *pseudo-direct* (cf. Theorem 57). The optimal approximation property of a definable subgroup in a valued abelian group is an elementary property in the language of groups with a predicate for the valuation. If the subgroups are definable, then also the assertion that their sum is pseudo-direct is elementary. Hence, given additive polynomials $f_1, \ldots, f_n$ with coefficients in $K = \mathbb{F}_p((t))$, the assertion

if $f_1(K) + \ldots + f_n(K)$ is pseudo-direct, then it has the optimal approximation property

is elementary in the language of valued fields (enriched by names for the coefficients of the polynomials f_i). By Theorem 57, it holds for $K = \mathbb{F}_p((t))$, and for every other spherically complete valued field (K, v). See [KU2] and [KU3] for further details.

2 Ultrametric Spaces

2.1 Proof of the Ultrametric Main Theorem

For the proof of Theorem 1, we show the following more precise statement:

Lemma 6 *Assume that $z' \in Y'$ is an attractor for $f : Y \to Y'$ and that (Y, u) is spherically complete. Then for every $y \in Y$ there is $z_0 \in Y$ such that $fz_0 = z'$ and $f(B(y, z_0)) \subseteq B(fy, z')$.*

Proof: If $z' = fy$ then we set $z_0 = y$ and there is nothing to show. So assume that $z' \neq fy$. Then by assumption on z' there is $z \in Y$ such that (AT1) and (AT2) hold. Take elements $y_i, z_i \in B(y, z)$, $i \in I$, such that the balls $B(y_i, z_i)$ form a nest inside of $B(y, z)$, maximal with the following properties, for all i:

i) $z' = fy_i = fz_i$ or $u'(z', fz_i) > u'(z', fy_i)$,
ii) $f(B(y_i, z_i)) \subseteq B(fy_i, z')$,
iii) for all $j \in I$, $u(y_i, z_i) < u(y_j, z_j)$ implies that $u'(fy_i, z') < u'(fy_j, z')$.

Non-empty nests with these properties exist. Indeed, the singleton $\{B(y, z)\}$ is such a nest. Maximal nests with these properties exist by Zorn's Lemma. Take one such maximal nest. As soon as we find $z_0 \in B(y, z)$ such that $z' = fz_0$ we are done because $f(B(y, z_0)) \subseteq f(B(y, z)) \subseteq B(fy, z')$.

Assume first that this nest has a minimal ball, say, $B(y_0, z_0)$. If $z' = fz_0$ then we are done. So assume that $z' \neq fz_0$, and set $\tilde{y} := z_0$. Then by assumption on z', we can find $\tilde{z} \in Y$ such that

$$u'(f\tilde{z}, z') > u'(f\tilde{y}, z') \quad \text{and} \quad f(B(\tilde{y}, \tilde{z})) \subseteq B(f\tilde{y}, z') .$$

We have that

$$u'(f\tilde{y}, z') = u'(fz_0, z') > u'(fy_0, z') = u'(f\tilde{y}, fy_0) , \tag{2}$$

where the last equality follows from the ultrametric triangle law. So we know that $fy_0 \notin B(f\tilde{y}, z')$ and thus, $y_0 \notin B(\tilde{y}, \tilde{z})$. This shows that $u(\tilde{y}, \tilde{z}) > u(\tilde{y}, y_0) = u(z_0, y_0)$, and since $\tilde{y} = z_0 \in B(z_0, y_0)$, it follows that $B(\tilde{z}, \tilde{y}) \subsetneqq B(z_0, y_0)$. So we can enlarge our nest of balls by adding $B(\tilde{z}, \tilde{y})$, and conditions i) and ii) hold for the new nest. From iii) we see that $u'(fy_0, z')$ is maximal among the $u'(fy_i, z')$, $i \in I$; so (2) shows that also iii) holds for the new nest. But this contradicts the maximality of the chosen nest.

Now assume that the nest contains no smallest ball. Since (Y, u) is spherically complete by assumption, there is some $z_0 \in \bigcap_{i \in I} B(y_i, z_i)$. Suppose that $fz_0 \neq z$. Then we set $\tilde{y} := z_0$. For all i, we have $\tilde{y} \in B(y_i, z_i)$ and $f\tilde{y} \in f(B(y_i, z_i)) \subseteq B(fy_i, z')$, showing that $u'(f\tilde{y}, z') \geq u'(fy_i, z')$. We choose $\tilde{z}$ as before. We have $f(B(\tilde{y}, \tilde{z})) \subseteq B(f\tilde{y}, z') \subseteq B(fy_i, z')$ for all i. On the other hand, since the nest contains no smallest ball, the set $\{u(y_i, z_i) \mid i \in I\}$ has no maximal element. So iii) implies that also the set $\{u'(fy_i, z') \mid i \in I\}$ has no maximal element. Consequently, for all $i \in I$ there is $j \in I$ such that $u'(f\tilde{y}, z') \geq u'(fy_j, z') > u'(fy_i, z')$. Consequently, $fy_i \notin B(f\tilde{y}, z')$, which yields that $y_i \notin B(\tilde{y}, \tilde{z})$. Therefore, $B(\tilde{y}, \tilde{z}) \subsetneqq B(y_i, z_i)$ and $u(\tilde{y}, \tilde{z}) > u(y_i, z_i)$ for all i. So we can enlarge our nest of balls by adding $B(\tilde{y}, \tilde{z})$, and conditions i), ii) and iii) hold for the new nest. This again contradicts the maximality of the chosen nest. Hence, $fz_0 = z'$ and we are done. □

Corollary 7 *Assume that $f : Y \to Y'$ is immediate and that (Y, u) is spherically complete. Then the following holds:*

(BB) *for every $y \in Y$ and every ball B' in Y' around fy, there is a ball B in Y around y such that $f(B) = B'$.*

Proof: Assume that $y \in Y$ and that B' is any ball in Y' which contains fy. Then we can write

$$B' = \bigcup_{z' \in B'} B(z', fy) .$$

According to the foregoing lemma, for every z' there is $z_0 \in Y$ such that $z' \in f(B(y,z_0)) \subseteq B(fy,z') \subseteq B'$. Take B to be the union over all such balls $B(y,z_0)$ when z' runs through all elements of B'. Then B is a ball around y satisfying $f(B) = B'$. □

The next lemma proves Theorem 2:

Lemma 8 *Assume that $f: Y \to Y'$ is a map which satisfies (BB), and that (Y,u) is spherically complete. Then f is surjective, and (Y',u') is spherically complete.*

Proof: Taking $B' = Y'$, we obtain the surjectivity of f.

Now we take any nest of balls $\{B'_j \mid j \in J\}$ in Y'. We have to show that this nest has a nonempty intersection. We claim that in Y there exists a nest of balls B_i, $i \in I$, maximal with the property that

$$I \subseteq J, \text{ and for all } i \in I,\ f(B_i) = B'_i\,. \tag{3}$$

To show this, we first take any $j \in J$ and choose some $y_j \in Y$ such that $fy_j \in B'_j$, making use of the surjectivity of f. As f satisfies (BB), we can choose a ball B_j in Y around y_j and such that $f(B_j) = B'_j$. So the nest $\{B_j\}$ has property (3). Hence, a maximal nest $\{B_i \mid i \in I\}$ with property (3) exists by Zorn's Lemma.

We wish to show that the balls B'_i, $i \in I$, are coinitial in the nest B'_j, $j \in J$, that is, for every ball B'_j there is some $i \in I$ such that $B'_i \subseteq B'_j$. Once we have shown this we are done: as Y is spherically complete, there is some $y \in \bigcap_{i\in I} B_i$, and

$$fy \in \bigcap_{i\in I} f(B_i) = \bigcap_{i\in I} B'_i = \bigcap_{j\in J} B'_j$$

shows that $\bigcap_{j\in J} B'_j$ is non-empty.

Suppose the balls B'_i, $i \in I$, are not coinitial in the nest B'_j, $j \in J$. Then there is some $j \in J$ such that $B'_j \subsetneqq B'_i$ for all $i \in I$. Since Y is spherically complete, there is some $y \in \bigcap_{i\in I} B_i$. We have that $fy \in \bigcap_{i\in I} B'_i =: B'$, and also that $B'_j \subseteq B'$. By assumption, there is a ball B around y such that $f(B) = B'$. If B' happens to be the smallest ball among the B'_i, say, $B' = B'_{i_0}$ with $i_0 \in I$, then we just take $B = B_{i_0}$. If $B' \subsetneqq B'_i$, then it follows that $B \subsetneqq B_i$. Hence in all cases, $B \subseteq B_i$ for all i. Since $B'_j \subseteq B'$, we can choose $\tilde{y} \in B$ such that $f\tilde{y} \in B'_j$. By assumption, there is a ball B_j around $\tilde{y}$ such that $f(B_j) = B'_j$. Since $\tilde{y} \in B_i$ for all $i \in I$, we know that B_i, $i \in I \cup \{j\}$ is a nest of balls. By construction, it has property (3). Since $j \notin I$, this contradicts our maximality assumption on I. This proves that the balls B'_i, $i \in I$, must be coinitial in the nest B'_j, $j \in J$. □

2.2 Products

Let (Y_i,u_i), $i \in I$, be ultrametric spaces whose value sets u_iY_i are all contained in a common ordered set, and assume that I is finite or that $\bigcup_{i\in I} u_iY_i$ is wellordered. Then their **direct product** will be the cartesian product $\prod_{i\in I} Y_i$ equipped with the ultrametric

$$u:\ \prod_{i\in I} Y_i \times \prod_{i\in I} Y_i \ \to\ \bigcup_{i\in I} u_i Y_i \cup \{\infty\}$$

defined by

$$u\left((y_i)_{i\in I}\,,\,(z_i)_{i\in I}\right) := \min_{i\in I} u_i(y_i,z_i)\ .$$

We leave it to the reader to verify that this map satisfies (U1), (U2) and (U3). Note that indeed every element of $\bigcup_{i\in I} u_i Y_i$ appears as the distance of two suitably chosen elements of $\prod_{i\in I} Y_i$.

Lemma 9 *Take $k \in I$ and let $\pi : \prod_{i\in I} Y_i \to Y_k$ denote the projection onto the k-th component. If B is a ball in $(\prod_{i\in I} Y_i, u)$, then for every $k \in I$, $\pi_k B$ is a ball in (Y_i, u_i), and*

$$B \ =\ \prod_{i\in I} \pi_i B\ . \tag{4}$$

Proof: Since $B \neq \emptyset$, we have that $\pi_k B \neq \emptyset$ and we can pick an element $y_k \in \pi_k B$ which is the projection of some $y = (y_i)_{i\in I} \in B$. We claim that

$$\pi_k B \ =\ \bigcup_{z\,\in\, B} B(y_k, \pi_k z)\ , \tag{5}$$

where $B(y_k, \pi_k z)$ is understood to designate a ball in (Y_k, u_k). Since $\pi_k z \in B(y_k, \pi_k z)$, the inclusion "$\subseteq$" is trivial. Now take $z = (z_i)_{i\in I} \in B$ and some $x_k \in B(y_k, \pi_k z)$. Set $x = (x_i)_{i\in I}$ with $x_i := y_i$ for $k \neq i \in I$. Then $u(y,x) = u_k(y_k, x_k) \geq u_k(y_k, \pi_k z) \geq u(y,z)$ and therefore, $x \in B$ and $x_k \in \pi_k B$. This proves that "$\supseteq$", and hence equality holds in (5). As a union of balls with common element y_k, $\pi_k B$ is itself a ball.

The inclusion "$\subseteq$" in (4) is trivial. For the converse, pick an element $x = (x_i)_{i\in I} \in \prod_{i\in I} \pi_i B$. Then there are elements $z^i \in B$ such that $x_i = \pi_i z^i$ for all $i \in I$. Pick an arbitrary element $y \in B$. Then for some $j \in I$, $u(y,x) = \min u_i(y_i, x_i) = \min u_i(y_i, \pi_i z^i) = u_j(y_j, \pi_j z^j) \geq u(y, z^j)$. Since $y, z^j \in B$, it follows that $x \in B$. This proves the inclusion "$\supseteq$" and hence equality in (4). □

Proposition 10 *If the ultrametric spaces (Y_i, u_i), $i \in I$, are spherically complete, then the same holds for their direct product $(\prod_{i\in I} Y_i\,,\, u)$.*

Proof: Let $\mathbf{B} = \{B_j \mid j \in J\}$ be a nest of balls in the direct product. We have to show that the intersection of $\mathbf{B}$ is nonempty. For every $i \in I$ we consider the projections $\pi_i B_j$ which by the foregoing lemma are balls in (Y_i, u_i). Since $\mathbf{B}$ is a nest, all intersections $B_j \cap B_k$ are non-empty and therefore, all intersections $\pi_i B_j \cap \pi_i B_k$ are non-empty. This proves that for each $i \in I$, $\{\pi_i B_j \mid j \in J\}$ is a nest of balls in (Y_i, u_i). By our assumption that the ultrametric spaces (Y_i, u_i) are spherically complete, there exist elements $x_i \in \bigcap_{j\in J} \pi_i B_j$ for each i. By equation (4) of the foregoing lemma, $(x_i)_{i\in I} \in B_j$ for every $j \in J$, hence $(x_i)_{i\in I} \in \bigcap_{j\in J} B_j$. □

2.3 Embeddings and isomorphisms

Take ultrametric spaces (Y, u) and (Y', u') and a map $f : Y \to Y'$. A map $\varphi : uY \to u'Y'$ will be called a **value map for** f if it preserves $\le$ and satisfies $u'(fy, fz) = \varphi u(y, z)$ for all $y, z \in Y$, $y \neq z$. From the latter it follows that f is injective since $u'(fy, fz) = \varphi u(y, z) \in u'Y'$ means that $u'(fy, fz) \neq \infty$, i.e., $fy \neq fz$. We call f an **embedding of ultrametric spaces (with value map φ)** if in addition, φ preserves $<$ and hence is itself injective. An embedding f is called an **isomorphism of ultrametric spaces** if it is onto. In this case, also φ is onto. We set $\varphi\infty = \infty$.

3 Immediate maps on valued abelian groups

A **valued abelian group** (G, v) is an abelian group G endowed with a **valuation** v. That is, $a \mapsto va$ is a map from G onto $vG \cup \{\infty\}$, where vG is a totally ordered set and ∞ is an element bigger than all elements of vG, and the following laws hold:

(V1) $va = \infty \Leftrightarrow a = 0\,,$

(V2) $v(a - b) \ge \min\{va, vb\}$ (ultrametric triangle law).

The **value set** of (G, v) is vG. For every valued abelian group (G, v), the set G endowed with the map

$$u : G \times G \to vG \cup \{\infty\}\,, \quad u(a, b) := v(a - b)$$

is an ultrametric space. We note the following translations of properties of the ultrametric:

- $v(a - b) > \min\{va, vb\} \;\Rightarrow\; va = vb$,
- $va \neq vb \;\Rightarrow\; v(a - b) = \min\{va, vb\}$,
- $va = v(-a)$.

A valued abelian group (G, v) is called **spherically complete** if the underlying ultrametric space (G, u) is spherically complete. Standard examples for spherically complete abelian groups are the Hahn products (see, e.g., [KU4]).

Observe that in a valued abelian group, any ball around 0 is a subgroup. Since balls are unions of closed balls, this has only to be proved for closed balls. Note that

$$B_\alpha(0) \;=\; \{z \in G \mid u(0, z) \ge \alpha\} \;=\; \{z \in G \mid vz \ge \alpha\}$$

since $u(0, z) = v(0 - z) = v(-z) = vz$. Take $a, b \in B_\alpha(0)$. Then $va \ge \alpha$ and $vb \ge \alpha$, whence $v(a - b) \ge \alpha$ by (V2), that is, $a - b \in B_\alpha(0)$. This proves that every $B_\alpha(0)$ and every other ball B containing 0 is a subgroup of G. Let us note that since every ball B containing 0 is a union of closed balls $B_\alpha(0)$, it follows that

$$y \in B \text{ and } vz \ge vy \;\Rightarrow\; z \in B\,.$$

Every ball $\tilde{B}$ in (G, v) can be written in the form $b + B$ where $b \in \tilde{B}$ and $B = \{a - b \mid a \in \tilde{B}\}$ is a ball around 0. Hence the balls in (G, v) are precisely the cosets with respect to the subgroups that are balls.

3.1 Immediate homomorphisms

In this section we will give a handy criterion for group homomorphisms to be immediate.

Proposition 11 *Let (G,v) and (G',v') be valued abelian groups and $f : G \to G'$ a group homomorphism. Then f is immediate if and only if for every $a' \in G \setminus \{0\}$ there is some $a \in G$ such that*

(IH1) $v'(a' - fa) > v'a'$,

(IH2) *for all $b \in G$, $va \leq vb$ implies $v'fa \leq v'fb$.*

Proof: Suppose first that f is immediate, and take any $a' \in G'$, $a' \neq 0$. Set $z' := a'$ and $y := 0$. Take $z \in G$ such that conditions (AT1) and (AT2) hold, and set $a := z$. Then $v'(a' - fa) = u'(z', fz) > u'(z', fy) = v'(a' - f0) = v'a'$. Hence, (IH1) holds. Also, we obtain from the ultrametric triangle law that $v'a' = v'fa$. Further, condition (AT2) shows that

$$\begin{aligned} f(\{b \mid vb \geq va\}) &= f(B(0,a)) = f(B(y,z)) \\ &\subseteq B(fy,z') = B(0,a') = \{b' \mid v'b' \geq v'a' = v'fa\} . \end{aligned}$$

That is, $va \leq vb \Rightarrow v'fa \leq v'fb$, i.e., (IH2) holds.

For the converse, take any $y \in G$ and $z' \in G' \setminus \{fy\}$. Set $a' := z' - fy \neq 0$. Choose $a \in G$ such that conditions (IH1) and (IH2) hold, and set $z := y + a$. Then $u'(z', fz) = v'(z' - fz) = v'(z' - fy - fa) = v'(a' - fa) > v'a' = v'(z' - fy) = u'(z', fy)$. So (AT1) holds. Also, we obtain from the ultrametric triangle law that $v'fa = v'(z' - fy)$. To show that (AT2) holds, take any $x \in B(y,z)$. Then $v(x - y) \geq v(z - y) = va$. Hence by (IH2), $v'(fx - fy) = v'f(x - y) \geq v'fa = v'(z' - fy)$, so $fx \in B(fy, z')$. □

By Theorem 2, we obtain:

Theorem 12 *Let (G,v) and (G',v') be valued abelian groups and $f : G \to G'$ a group homomorphism which satisfies (IH1) and (IH2). Assume further that (G,v) is spherically complete. Then f is surjective and (G',v') is spherically complete.*

Take valued abelian groups (G,v) and (G',v'). For an arbitrary map $f : G \to G'$ we will say that $a \in G$ is ***f*-regular** if it is non-zero and satisfies condition (IH2). We will denote the set of all f-regular elements by $\mathrm{Reg}\,(f)$. Then the following holds:

Proposition 13 *If f is an immediate group homomorphism, then*

$$va \;\mapsto\; v'fa$$

for $a \in \mathrm{Reg}\,(f)$ induces a well defined and $\leq$-preserving map from $\{va \mid a \in \mathrm{Reg}\,(f)\}$ onto $v'G'$.

Proof: If $a, b \in \mathrm{Reg}\,(f)$ such that $va = vb$, then by (IH2), $v'fa \leq v'fb$ and $v'fa \geq v'fb$, whence $v'fa = v'fb$. This shows that the map is well defined. Again because of (IH2), it preserves $\leq$. Now take any $a' \in v'G'$, $a' \neq 0$. Then by (IH1), there is $a \in G$ such that $v'(a' - fa) > v'a'$, whence $v'a' = v'fa$ by the ultrametric triangle law. This proves that the map is onto. □

3.2 Basic criteria

Even if the map f that we consider on a valued abelian group is not a homomorphism, the presence of addition helps us to give handy and natural criteria for the map to be immediate. We just have to work a little harder. In this section, we present basic criteria that will cover all our applications in the non-additive case.

Proposition 14 *Take valued abelian groups (G, v) and (G', v'), an element $b \in G$, a ball B around 0 in G, a ball B' around 0 in G', and a map $f : b+B \rightarrow fb+B'$. Assume that $\phi : B \rightarrow B'$ is a map such that for all $a' \in B' \setminus \{0\}$ there is $a \in \mathrm{Reg}\,(\phi)$ with the following properties:*

$$v'(a' - \phi a) \; > v'a' \; = \; v'\phi a \; , \tag{6}$$

and

$$v'(fy - fz \; - \; \phi(y - z)) \; > \; v'\phi a \;\; \text{for all} \;\; y, z \in b + B \;\; \text{such that} \;\; v(y - z) \geq va \; . \tag{7}$$

Then f is immediate.

If $\phi 0 = 0$ then (7) only needs to be checked for $y \neq z$.

Proof: Take $z' \in fb + B'$ and $y \in b + B$ such that $z' \neq fy$. Applying our assumption to $a' := fy - z'$ we find that there is some $a \in \mathrm{Reg}\,(\phi)$ such that by (6),

$$v'(fy - z' \; - \; \phi a) \; > \; v'(fy - z') \; = \; v'\phi a \; , \tag{8}$$

and such that (7) holds. Set $z := y - a \in y - B = y + B = b + B$. Then $y - z = a$ and hence by (7) and (8),

$$v'(fy - fz - \phi(y - z)) \; > \; v'\phi a \; = \; v'(fy - z') \; .$$

Consequently,

$$\begin{aligned} v'(z' - fz) \;\; &\geq \;\; \min\{v'(z' - fy \; + \; \phi a) \, , \, v'(fy - fz - \phi a)\} \\ &= \;\; \min\{v'(fy - z' \; - \; \phi a) \, , \, v'(fy - fz - \phi(y - z))\} \\ &> \;\; v'(fy - z') \; = \; v'(z' - fy) \; . \end{aligned}$$

Hence (AT1) holds. Now take $x \in B(y,z) \subseteq b+B$, i.e., $v(y-x) \geq v(y-z) = va$. Then $v'\phi(y-x) \geq v'\phi a$ because $a \in \mathrm{Reg}\,(\phi)$, and $v'(fy-fx\,-\,\phi(y-x)) > v'\phi a$ by (7). Therefore,

$$v'(fy-fx) \;\geq\; \max\{v'(fy-fx\,-\,\phi(y-x))\,,\, v'\phi(y-x)\} \;\geq\; v'\phi a \;=\; v'(fy-z')\,,$$

whence $fx \in B(fy,z')$. Hence (AT2) holds.

Assume that $\phi 0 = 0$. Observe that $\phi a \neq 0$ since $a' \neq 0$ and $v'a' = v'\phi a$. Hence if $y = z$ then $v'(fy-fz\,-\,\phi(y-z)) = v'0 = \infty > v'\phi a$, which shows that (7) need only be checked for $y \neq z$. □

Note that by the ultrametric triangle law, the equality in (6) is a consequence of the inequality. Further, observe that this proposition proves the direction "⇐" of Proposition 11: if we take $B = G$, $B' = G'$ and $\phi = f$, then (IH1) implies (6) and (IH2) implies that $a \in \mathrm{Reg}\,(\phi)$, while (7) is trivially satisfied. Hence if for every $a' \in G' \setminus \{0\}$ there is $a \in G$ such that (IH1) and (IH2) hold, then the above proposition shows that f is immediate.

Proposition 15 *Take valued abelian groups (G,v) and (G',v'), an element $b \in G$, a ball B around 0 in G and a map $f : b+B \to G'$. Assume that there are*

- *a ball B' around 0 in G',*
- *a map $\varphi : vB \to v'B'$ which preserves $\leq$,*
- *a map $\phi : B \to B'$ such that $\phi 0 = 0$ and*

(BC1) $v'\phi a \;\geq\; \varphi va$ *for* $0 \neq a \in B$,

(BC2) *for all $a' \in B' \setminus \{0\}$ there is $a \in B \setminus \{0\}$ such that*

$$v'(a'-\phi a) > v'a' \quad \text{and} \quad \varphi va \;=\; v'\phi a\,,$$

(BC3) *for all distinct $y,z \in b+B$,*

$$v'(fy-fz\,-\,\phi(y-z)) \;>\; \varphi v(y-z)\,.$$

Then $f(b+B) \subseteq fb+B'$, and $f : b+B \to fb+B'$ is immediate.

If in addition equality always holds in (BC1), *then f is injective and φ is a value map for f. If in this situation, φ preserves $<$, then f is an immediate embedding of ultrametric spaces with value map φ.*

Proof: By (BC3) and (BC1),

$$v'(fy-fz) \;\geq\; \min\{v'(fy-fz\,-\,\phi(y-z))\,,\, v'\phi(y-z)\} \;\geq\; \varphi v(y-z) \tag{9}$$

for all distinct $y,z \in b+B$. Take $y \in b+B$, $y \neq b$, and set $z = b$. Then $v'(fy-fb) \geq \varphi v(y-b) \in v'B'$. Since B' is a ball around 0, this implies that also $fy-fb \in B'$, that is, $fy \in fb+B'$.

Observe that if $a \in B$ such that $\varphi va = v'\phi a$, then $a \in \mathrm{Reg}\,(\phi)$. Indeed, if $\varphi va = v'\phi a$ and $va \leq v\tilde{a}$, then by (BC1) and since φ preserves $\leq$, we have $v'\phi a = \varphi va \leq \varphi v\tilde{a} \leq v'\phi\tilde{a}$, hence $a \in \mathrm{Reg}\,(\phi)$.

Pick some $a' \in B' \setminus \{0\}$ and choose a according to (BC2). Then (6) is satisfied and by what we have shown, $a \in \mathrm{Reg}\,(\phi)$. Take distinct $y, z \in b + B$ such that $v(y - z) \geq va$. Then (BC3) yields $v'(fy - fz - \phi(y - z)) > \varphi v(y - z) \geq \varphi va = v'\phi a$, hence (7) is satisfied. Since $\phi 0 = 0$, we have to check (7) only for $y \neq z$. Now it follows from Proposition 14 that $f : b + B \to fb + B'$ is immediate.

Assume in addition that equality always holds in (BC1). Then $v'\phi(y - z) = \varphi v(y - z)$, and we obtain the following stronger version of (9):

$$v'(fy - fz) \;=\; \min\{v'(fy - fz \;-\; \phi(y - z))\,,\, v'\phi(y - z)\} \;=\; \varphi v(y - z)\;.$$

If $y \neq z$ then $v(y - z) \in vB$ and $v'(fy - fz) = \varphi v(y - z) \in v'B'$ which implies $fy \neq fz$. This proves that f is injective. If also φ preserves $<$, then it follows that f is an immediate embedding of ultrametric spaces with value map φ. □

3.3 Pseudo-derivatives

In this section, we will present a special case of the basic criterion, with nicer properties. Take valued abelian groups (G, v) and (G', v'), an element $b \in G$, a ball B in G around 0, a ball B' in G' around 0, and a map $f : b + B \to G'$. We will say that a map ϕ is a **pseudo-derivative of f at b on $b + B$** if it satisfies:

(PD1) $\phi : B \to B'$ is an isomorphism of ultrametric spaces,

(PD2) $v'(fy - fz - \phi(y - z)) \;>\; v'(fy - fz) \;=\; v'\phi(y - z)$ for all distinct $y, z \in b + B$.

(Recall that by the ultrametric triangle law, the equality in (PD2) is a consequence of the inequality).

Proposition 16 *Take f, b and B as above and assume that $\phi : B \to G'$ is a pseudo-derivative of f at b on $b + B$ with value map φ. Then $f(b + B) \subseteq fb + B'$, and $f : b + B \to fb + B'$ is an immediate embedding of ultrametric spaces with value map φ.*

If in addition (G, v) is spherically complete, then f is an isomorphism of ultrametric spaces from $b + B$ onto $fb + B'$.

Proof: Since φ is the value map of the isomorphism $\phi : B \to B'$ of ultrametric spaces, we have that $\varphi va = \varphi v(a - 0) = \varphi u(a, 0) = u'(a, 0) = v'a$ for all $a \in B$. Hence, $\varphi : vB \to v'B'$ is a bijection which preserves $<$. In particular, condition (BC1) of Proposition 15 holds with equality. Since $v'\phi(y - z) = \varphi v(y - z)$ for all distinct $y, z \in b + B$, (PD2) implies (BC3). Since ϕ is surjective, we can also satisfy (BC2) by simply taking a such that $\phi a = a'$. Now the first assertion of our proposition follows from Proposition 15.

In the case of (G, v) spherically complete, the surjectivity follows from Theorem 2. □

4 Immediate maps on valued fields and their finite-dimensional vector spaces

Let (K, v) be a valued field. That is, v is a valuation of its additive group, vK is a totally ordered abelian group, and the following additional law holds:

(V3) $v(ab) = va + vb$.

The **value group** of (K, v) is $vK := v(K^\times)$. Throughout this paper, its **valuation ring** $\{y \in K \mid vy \geq 0\}$ will be denoted by $\mathcal{O}$, and its **valuation ideal** $\{y \in K \mid vy > 0\}$ by $\mathcal{M}$. The field $\mathcal{O}/\mathcal{M}$ is called the **residue field** and is denoted by Kv. Note that $c\mathcal{O} = \{y \in K \mid vy \geq vc\} = B_{vc}(0)$ and $c\mathcal{O} = \{y \in K \mid vy > vc\}$.

A valued field (K, v) is called **spherically complete** if the underlying valued additive group is spherically complete (i.e., if the underlying ultrametric space is spherically complete).

Main examples for spherically complete fields are the **power series fields** $k((G))$ with their **canonical valuation**. Here, k can be any field and G any ordered abelian group, and $k((G))$ consists of all formal sums $a = \sum_{g \in G} c_g t^g$ with $c_g \in k$ and well-ordered **support** $\mathrm{supp}(a) = \{g \in G \mid c_g \neq 0\}$. The canonical valuation on $k((G))$ is given by $va := \min \mathrm{supp}(a) \in G$ and $v0 := \infty$. Its value group is G, and its residue field is k.

An extension $(L, w) \supset (K, v)$ of valued fields is called **immediate** if the canonical embedding of vK in wL and the canonical embedding of Kv in Lw are onto. It is well known that this holds if and only if as ultrametric spaces, (K, v) is an immediate subspace of (L, v) (cf. [KU4]). A valued field is called **maximally valued** if it admits no proper immediate extensions. It was shown by Krull ([KR]; see also [G]) that for every valued field (K, v) there is a maximal immediate extension field; this is maximally valued by definition.

A valued field is maximally valued if and only if it is spherically complete (cf. [P1], [P2], [KU4]). This was essentially proved by Kaplansky in [KA], using the notion of "pseudo Cauchy sequence" instead of "nest of balls". Every power series field is spherically complete (cf. [P2], [KU4]). Hence it is maximally valued.

4.1 The minimum valuation

For every $n \in \mathbb{N}$, the valuation v of K induces a valuation of the n-dimensional K-vector space K^n, called the **minimum valuation**:

$$v(a_1, \ldots, a_n) := \min_{1 \leq i \leq n} va_i \tag{10}$$

for all $(a_1, \ldots, a_n) \in K^n$. This valuation satisfies (V1) and (V2) for all $a, b \in K^n$, so (K^n, v) is a valued abelian group. Instead of (V3), it satisfies

(V3′) $v(ca) = vc + va$ for all $c \in K$, $a \in K^n$.

Again, $u(a, b) := v(a - b)$ makes K^n into an ultrametric space with value set vK. If $0 \neq c \in K$, then we write $(c\mathcal{O})^n$ for the n-fold product $c\mathcal{O} \times \ldots \times c\mathcal{O}$ which is the subgroup of vectors in K^n whose entries all have value $\geq vc$; $(c\mathcal{M})^n$ is defined similarly. Note that $(c\mathcal{O})^n = \{ca \mid a \in \mathcal{O}^n\} = c\mathcal{O}^n$ and $(c\mathcal{M})^n = c\mathcal{M}^n$. For $b \in K^n$, $c \in K$,

$$b + c\mathcal{O}^n = \{a \in K^n \mid v(a-b) \geq vc\} = B_{vc}(b) \text{ and } b + c\mathcal{M}^n = \{a \in K^n \mid v(a-b) > vc\} .$$

We will say that (K^n, v) is **spherically complete** if its underlying ultrametric space (K^n, u) is. Proposition 10 of Section 2.2 implies:

Lemma 17 *If (K, v) is spherically complete, then so is (K^n, v).*

4.2 Pseudo-linear maps

Take $Y \subseteq K^n$, $0 \neq s \in K$ and f a map from Y into K^n. We will say that f is **pseudo-linear with pseudo-slope** s if for all $y, z \in Y$ such that $y \neq z$,

$$v(fy - fz - s(y - z)) > v(fy - fz) = vs(y - z) . \tag{11}$$

If B is any ball in (K^n, v) around 0, then sB is again a ball in (K^n, v) around 0 and the map $B \ni x \mapsto sx \in sB$ is an isomorphism of ultrametric spaces with value map $\varphi : \alpha \mapsto \alpha + vs$. Hence pseudo-linear maps are maps with a particularly simple pseudo-derivative given by multiplication with a suitable scalar. From Proposition 16 we obtain:

Proposition 18 *Take $b \in K^n$ and B a ball in (K^n, v) around 0. Assume that $f : b + B \to K^n$ is pseudo-linear with pseudo-slope s. Then $f(b + B) \subseteq fb + sB$, and*

$$f : b + B \to fb + sB$$

is an immediate embedding of ultrametric spaces with value map $\varphi : \alpha \mapsto \alpha + vs$.

If in addition, (K, v) is spherically complete, then f is an isomorphism of ultrametric spaces from $b + B$ onto $fb + sB$.

4.3 Polynomial maps

Take any $n \in \mathbb{N}$. For any system $f = (f_1, \ldots, f_n)$ of n polynomials in n variables with coefficients in K, we denote by $J_f(b)$ its Jacobian matrix at $b \in K^n$. We will denote by $J_f^*(b)$ the adjoint matrix of $J_f(b)$.

Proposition 19 *a) Take a polynomial $f \in \mathcal{O}[X]$ and $b \in \mathcal{O}$ such that*

$$s := f'(b) \neq 0 .$$

Then f induces a pseudo-linear map with pseudo-slope s from $b + s\mathcal{M}$ into $f(b) + s^2\mathcal{M}$, and

$$f_{\langle b \rangle}(y) := \frac{1}{s^2} f(b + sy) .$$

induces a pseudo-linear map with pseudo-slope 1 from $\mathcal{M}$ into $s^{-2}f(b) + \mathcal{M}$.

b) Take n polynomials in n variables $f_1,\dots,f_n\in\mathcal{O}[X_1,\dots,X_n]$ and $b\in\mathcal{O}^n$ such that

$$s \;:=\; \det J_f(b)\;\neq\;0$$

for $f=(f_1,\dots,f_n)$. If $vs=0$, then $J_f(b)$ is a pseudo-derivative of f on $b+\mathcal{M}$ and f induces an embedding from $b+\mathcal{M}$ into $f(b)+\mathcal{M}$ with value map $\varphi=\mathrm{id}$.

In the general case, $J_f^*(b)\,f$ induces a pseudo-linear map with pseudo-slope s from $b+s\mathcal{M}^n$ into $J_f^*(b)f(b)+s^2\mathcal{M}^n$, and

$$F(y)\;:=\;f\left(b+J_f^*(b)(y-b)\right) \tag{12}$$

induces a pseudo-linear map with pseudo-slope s from $b+s\mathcal{M}^n$ into $f(b)+s^2\mathcal{M}^n$. Further,

$$f_{\langle b\rangle}(y)\;:=\;\frac{1}{s^2}\,J_f^*(b)\,f(b+sy)\;.$$

induces a pseudo-linear map with pseudo-slope 1 from $\mathcal{M}^n$ into $s^{-2}J_f^*(b)f(b)+\mathcal{M}^n$.

Proof: Note that whenever we prove pseudo-linearity, the assertions about the range of the functions will follow from Proposition 18.

a): For a polynomial f in one variable over a field of arbitrary characteristic, we denote by $f^{[i]}$ its i-th formal derivative (cf. [KA], [KU4]). These polynomials are defined such that the following Taylor expansion holds in arbitrary characteristic:

$$f(b+\varepsilon)\;=\;f(b)+\sum_{i=1}^{\deg f}\varepsilon^i f^{[i]}(b)\;. \tag{13}$$

Note that $f'=f^{[1]}$. Since $f\in\mathcal{O}[X]$, we have that $f^{[i]}\in\mathcal{O}[X]$. Since $b\in\mathcal{O}$, we also have that $f^{[i]}(b)\in\mathcal{O}$. Now take $y,z\in b+s\mathcal{M}$. Write $y=b+\varepsilon_y$ and $z=b+\varepsilon_z$ with $\varepsilon_y,\varepsilon_z\in s\mathcal{M}$. Then by (13),

$$f(y)-f(z)\;=\;(\varepsilon_y-\varepsilon_z)f'(b)+\sum_{i=2}^{\deg f}(\varepsilon_y^i-\varepsilon_z^i)f^{[i]}(b)\;=\;s(y-z)+S(b,\varepsilon_y,\varepsilon_z)\;. \tag{14}$$

Since

$$\varepsilon_y^i-\varepsilon_z^i\;=\;(\varepsilon_y-\varepsilon_z)(\varepsilon_y^{i-1}+(i-1)\varepsilon_y^{i-2}\varepsilon_z+\ldots+(i-1)\varepsilon_y^{i-2}\varepsilon_z^{i-2}+\varepsilon_y^{i-1})\in(\varepsilon_y-\varepsilon_z)s\mathcal{M}$$

for every $i\geq 2$, and since $f^{[i]}(b)\in\mathcal{O}$, we find that

$$S(b,\varepsilon_y,\varepsilon_z)\in(\varepsilon_y-\varepsilon_z)s\mathcal{M}\;=\;s(y-z)\mathcal{M}\;.$$

This proves that

$$v(f(y)-f(z)-s(y-z))\;=\;vS(b,\varepsilon_y,\varepsilon_z)\;>\;vs(y-z) \tag{15}$$

which implies that (11) holds. This proves the first assertion of a).

If $y, z \in \mathcal{M}$, then by what we just have proved,

$$\begin{aligned} v(f_{\langle b\rangle}(y) - f_{\langle b\rangle}(z) - (y-z)) &= v(s^{-2}f(b+sy) - s^{-2}f(b+sz) - (y-z)) \\ &= v(f(b+sy) - f(b+sz) - s(sy-sz)) \;-\; vs^2 \\ &> vs(sy-sz) - vs^2 \;=\; v(y-z)\;. \end{aligned}$$

This proves the second assertion of a).

b): We write $J = J_f(b)$ and $J^* = J_f^*(b)$. Then $JJ^* = (\det J)E = sE$ where E is the $n\times n$ identity matrix. Note that $J, J^* \in \mathcal{O}^{n\times n}$ by our assumptions on f and b. If $y \in K^n$ then we can write $y = cz$ with $c \in K$, $vc = vy$, $z \in \mathcal{O}^n$ and $vz = 0$. Then $Jy = cJz \in c\mathcal{O}^n$, hence $vJy = vc + vJz \geq vc = vy$. Similarly, $vJ^*y \geq vy$ for all $y \in K^n$.

Take $\varepsilon_1, \varepsilon_2 \in s\mathcal{M}^n$. The multidimensional Taylor expansion gives the following analogue of (14):

$$f(b+\varepsilon_1) - f(b+\varepsilon_2) \;=\; J(\varepsilon_1 - \varepsilon_2) + S(b,\varepsilon_1,\varepsilon_2) \tag{16}$$

with

$$vS(b,\varepsilon_1,\varepsilon_2) \;>\; vs(\varepsilon_1 - \varepsilon_2)\;. \tag{17}$$

Assume first that $vs = 0$. Then also $J^{-1} = \frac{1}{s}J^* \in \mathcal{O}^{n\times n}$, so for all $y \in K^n$, $vJ^{-1}y \geq vy$. But then, $vy = vEy = vJ^{-1}Jy \geq vJy \geq vy$, so equality must hold. We find that for all $y \in K^n$, $vJy = vy$ and similarly, $vJ^*y = vy$. In particular, this yields that J induces a value-preserving automorphism of the valued abelian group $(\mathcal{M}^n, +)$, and an isomorphism of ultrametric spaces from $\mathcal{M}^n$ onto $\mathcal{M}^n$ with value map $\varphi = \mathrm{id}$, with inverse maps induced by J^{-1}. From (16) and (17) we obtain that for $y = b + \varepsilon_1$ and $z = b + \varepsilon_2$ in $b + \mathcal{M}$,

$$v(f(y) - f(z) \;-\; J(y-z)) \;>\; vs(y-z) \;=\; v(y-z) \;=\; vJ(y-z)\;.$$

This proves that J is a pseudo-derivative of f on $b + \mathcal{M}$. From Proposition 16 we infer that f induces an embedding from $b + \mathcal{M}$ into $f(b) + J\mathcal{M} = f(b) + \mathcal{M}$ with value map $\varphi = \mathrm{id}$.

Now we turn to the general case. We compute:

$$\begin{aligned} J^*f(y) - J^*f(z) &= J^*(f(b+y-b) - f(b+z-b)) \\ &= J^*J(y-z) \;+\; J^*S(b, y-b, z-b) \\ &= s(y-z) \;+\; J^*S(b, y-b, z-b)\;. \end{aligned}$$

By (17),

$$vJ^*S(b, y-b, z-b) \;\geq\; vS(b, y-b, z-b) \;>\; vs(y-z)\;.$$

Hence,

$$v\left(J^*f(y) - J^*f(z) - s(y-z)\right) \;=\; vJ^*S(b, y-b, z-b) \;>\; vs(y-z)\;.$$

This proves our assertion for the map $J_f^*(b)\,f$. Further,

$$\begin{aligned} F(y)-F(z) &= f(b+J^*(y-b))-f(b+J^*(z-b)) \\ &= J(J^*(y-b)-J^*(z-b)) \,+\, S(b,J^*(y-b),J^*(z-b)) \\ &= s(y-z) \,+\, S(b,J^*(y-b),J^*(z-b))\;. \end{aligned}$$

As before, it follows that also $J^*(y-b)\,,\ J^*(z-b)\in s\mathcal{M}^n$, and that $vJ^*(y-z)\geq v(y-z)$. Hence,

$$\begin{aligned} v\left(F(y)-F(z)-s(y-z)\right) &= vS(b,J^*(y-b),J^*(z-b)) \\ &> vs(J^*(y-b)-J^*(z-b)) \;=\; vsJ^*(y-z) \;\geq\; vs(y-z)\;, \end{aligned}$$

which implies that (11) holds for F in the place of f. This proves our assertion for the map F.

To prove the last assertion, we take $y,z\in\mathcal{M}^n$. We apply (16) with $\varepsilon_1=sy$ and $\varepsilon_2=sz$. Using that $J^*J=(\det J)E=sE$, we obtain:

$$\begin{aligned} f_{\langle b\rangle}(y)-f_{\langle b\rangle}(z) &= s^{-2}J^*(f(b+sy)-f(b+sz)) \\ &= s^{-2}J^*J(sy-sz)+s^{-2}J^*S(b,sy,sz) \\ &= s^{-2}s(sy-sz)+s^{-2}J^*S(b,sy,sz) \;=\; y-z+s^{-2}J^*S(b,sy,sz)\;, \end{aligned}$$

so that

$$\begin{aligned} v\left(f_{\langle b\rangle}(y)-f_{\langle b\rangle}(z)-(y-z)\right) &= vJ^*S(b,sy,sz)-vs^2 \;\geq\; vS(b,sy,sz)-vs^2 \\ &> vs(sy-sz)-vs^2 \;=\; v(y-z)\;. \end{aligned}$$

This proves our assertion for the map $f_{\langle b\rangle}$. □

Note that in the one-dimensional case ($n=1$), we may write $\det J_f(b)=f'(b)$ and $J_f^*(b)=1$; in this way, the definition of $f_{\langle b\rangle}$ in the one-dimensional case becomes a special case of the definition for the multi-dimensional case.

If $vs>0$ in the multi-dimensional case, then in general $J_f(b)$ will not be a pseudo-derivative of f. It is necessary to transform f in order to obtain suitable pseudo-derivatives. We have shown above that this can be done so that one even obtains pseudo-linear functions.

From Proposition 19 together with Propositions 18 and 16, we obtain:

Theorem 20 *Assume that (K,v) is spherically complete.*
a) Take a polynomial $f\in\mathcal{O}[X]$ and $b\in\mathcal{O}$ such that $s:=f'(b)\neq 0$. Then f induces an isomorphism of ultrametric spaces from $b+s\mathcal{M}$ onto $f(b)+s^2\mathcal{M}$, and $f_{\langle b\rangle}$ induces an isomorphism of ultrametric spaces from $\mathcal{M}$ onto $s^{-2}f(b)+\mathcal{M}$.
b) Take n polynomials in n variables $f_1,\ldots,f_n\in\mathcal{O}[X_1,\ldots,X_n]$ and $b\in\mathcal{O}^n$ such that $s:=\det J_f(b)\neq 0$ for $f=(f_1,\ldots,f_n)$. If $vs=0$, then f induces an embedding of ultrametric spaces from $b+\mathcal{M}$ onto $f(b)+\mathcal{M}$.

In the general case, $J_f^*(b)\, f$ induces an isomorphism of ultrametric spaces from $b+s\mathcal{M}^n$ onto $J_f^*(b)\, f(b)+s^2\mathcal{M}^n$, the map F defined in (12) induces an isomorphism of ultrametric spaces from $b+s\mathcal{M}^n$ onto $f(b)+s^2\mathcal{M}^n$, and $f_{\langle b\rangle}$ induces an isomorphism of ultrametric spaces from $\mathcal{M}^n$ onto $J_f^*(b)s^{-2}f(b)+\mathcal{M}^n$.

4.4 Hensel's Lemma revisited

Let us apply Theorem 20 to prove that Hensel's Lemma holds for every spherically complete valued field (K,v). We prove the following version of Hensel's Lemma, which is often called "Newton's Lemma":

Theorem 21 *Let (K,v) be a spherically complete valued field. Then (K,v) satisfies the one-dimensional Newton's Lemma:*
Take $f\in\mathcal{O}[X]$ and assume that $b\in\mathcal{O}$ is such that $vf(b)>2vf'(b)$. Then there exists a unique root a of f such that $v(a-b)>vf'(b)$.

Proof: The inequality $vf(b)>2vf'(b)$ implies that $s:=f'(b)\neq 0$. Hence by Theorem 20, f induces an isomorphism of ultrametric spaces from $b+s\mathcal{M}$ onto $f(b)+s^2\mathcal{M}$. Since $vf(b)>2vf'(b)=vs^2$, we have that $f(b)\in s^2\mathcal{M}$, that is, $f(b)+s^2\mathcal{M}=s^2\mathcal{M}$. Therefore, $0\in f(b)+s^2\mathcal{M}$. Since f induces a bijection from $b+s\mathcal{M}$ onto $f(b)+s^2\mathcal{M}$, there is a unique $a\in b+s\mathcal{M}$ such that $f(a)=0$. As "$a\in b+s\mathcal{M}$" is equivalent to "$v(a-b)>vf'(b)$", this proves our assertion. □

Here is the multi-dimensional version:

Theorem 22 *Let (K,v) be a spherically complete valued field. Then (K,v) satisfies the multi-dimensional Newton's Lemma:*
Let $f=(f_1,\dots,f_n)$ be a system of n polynomials in n variables with coefficients in $\mathcal{O}$. Assume that $b\in\mathcal{O}^n$ is such that $vf(b)>2v\det J_f(b)$. Then there exists a unique $a\in\mathcal{O}^n$ such that $f(a)=0$ and $v(a-b)>v\det J_f(b)$.

Proof: The inequality $vf(b)>2v\det J_f(b)$ implies that $s:=\det J_f(b)\neq 0$. Hence by Theorem 20, J^*f induces an isomorphism of ultrametric spaces from $b+s\mathcal{M}^n$ into $J^*f(b)+s^2\mathcal{M}^n$, where $J^*=J_f^*(b)$. Since $vf(b)>vs^2$, we have that $f(b)\in s^2\mathcal{M}^n$ and hence also $J^*f(b)\in s^2\mathcal{M}^n$ (since $J^*\in\mathcal{O}^{n\times n}$). That is, $J^*f(b)+s^2\mathcal{M}^n=s^2\mathcal{M}^n$. Therefore, $0\in J^*f(b)+s^2\mathcal{M}^n$. Since J^*f induces a bijection from $b+s\mathcal{M}^n$ onto $J^*s^{-2}f(b)+\mathcal{M}^n$, there is a unique $a\in b+s\mathcal{M}^n$ such that $J^*f(a)=0$. Since J^* is invertible, we have that $f(a)=0\Leftrightarrow J^*f(a)=0$. Hence, a is the unique element in $b+s\mathcal{M}^n$ such that $f(a)=0$. As "$a\in b+s\mathcal{M}^n$" is equivalent to "$v(a-b)>v\det J_f(b)$", this proves our assertion. □

Note that like in the one-dimensional case, also in the multi-dimensional case the proof of Newton's Lemma can be reduced by transformation to a simpler case where we would in fact obtain the identity as a pseudo-derivative. But as we have already shown that even in the general case we can derive suitable pseudo-linear maps from f, it is much easier to employ them directly in the proof of the multidimensional Newton's Lemma.

A valued field (K,v) is called **henselian** if the extension of v to the algebraic closure $\tilde{K}$ of K is unique. It is well known that this holds if and only if (K,v) satisfies the one-dimensional Newton's Lemma (see, e.g., [KU4]). We are now going to show that the multi-dimensional Newton's Lemma holds in every henselian field.

Theorem 23 *A valued field (K,v) is henselian if and only if it satisfies the multidimensional Newton's Lemma.*

Proof: $\Rightarrow$: Let (K,v) be henselian. Take (L,v) to be a maximal immediate extension of (K,v). Then (L,v) is spherically complete. By the foregoing theorem, (L,v) satisfies the multidimensional Newton's Lemma. Denote by $\mathcal{O}$ the valuation ring of K, and by $\mathcal{O}_L$ that of L. Now assume that the hypothesis of the multidimensional Newton's Lemma is satisfied by a system f of polynomials with coefficients in $\mathcal{O}$ and by $b\in\mathcal{O}^n$. It follows that there is a unique $a\in\mathcal{O}_L^n$ such that $f(a)=0$ and $v(a-b)>v\det J_f(b)$. From the latter, it follows that $v\det J_f(a)=v\det J_f(b)$ and in particular, $\det J_f(a)\neq 0$. Now [L], Chapter X, §7, Proposition 8, shows that the elements $a_1,\ldots,a_n$ are separable algebraic over K. On the other hand, for every $\sigma\in\mathrm{Aut}\,(\tilde{K}|K)$, the element $\sigma a=(\sigma a_1,\ldots,\sigma a_n)$ satisfies $f(\sigma a)=\sigma f(a)=0$ and $v(\sigma a-b)=\min_i v(\sigma a_i-b_i)=\min_i v\sigma(a_i-b_i)=\min_i v(a_i-b_i)=v(a-b)>v\det J_f(b)$ (note that $v\sigma=v$ because (K,v) is henselian). By the uniqueness of a, it follows that $\sigma a=a$ for every $\sigma\in\mathrm{Aut}\,(\tilde{K}|K)$, that is, $a\in K^n$, as required.

$\Leftarrow$: If $n=1$, then $\det J_f(b)=f_1'(b_1)$, and the assertion is precisely the assertion of the one-dimensional Newton's Lemma. Hence the multidimensional Newton's Lemma implies that (K,v) is henselian. □

4.5 Power series maps on valuation ideals

Take any field k and any ordered abelian group G. We endow $k((G))$ with the canonical valuation v and denote the valuation ideal by $\mathcal{M}$. Every power series

$$f(X)=\sum_{i\in\mathbb{N}}c_iX^i\in k[[X]] \tag{18}$$

defines in a canonical way a map $f:\mathcal{M}\to\mathcal{M}$. This can be shown by use of Neumann's Lemma, cf. [DMM1]. We note that for every integer $r>1$ and every $y,z\in\mathcal{M}$,

$$v(y^r-z^r)\;>\;v(y-z)\;. \tag{19}$$

Therefore, if $c_1\neq 0$, we have that

$$v(f(y)-f(z)-c_1(y-z))\;=\;v\sum_{i\geq 2}c_i(y^i-z^i)\;>\;v(y-z)\;=\;vc_1(y-z) \tag{20}$$

because $vc_i=0$ for all i. So we see that f is pseudo-linear with slope c_1 if $c_1\neq 0$. By Proposition 18, we obtain:

Theorem 24 *If $f : \mathcal{M} \to \mathcal{M}$ is defined by the power series (18), then f is an isomorphism of ultrametric spaces.*

A similar result holds for power series with generalized exponents (which for instance are discussed in [DS]). Take any subgroup G of $\mathbb{R}$ and a generalized power series of the form

$$f(X) = \sum_{i\in\mathbb{N}} c_i X^{r_i} \in k[[X^G]] \tag{21}$$

where r_i, $i \in \mathbb{N}$, is an increasing sequence of positive real numbers in G. Suppose that the power functions $y \mapsto y^{r_i}$ are defined on $\mathcal{M}$ for all i. Then again, the generalized power series (21) defines a map $f : \mathcal{M} \to \mathcal{M}$. We note that (19) also holds for every real number $r > 1$ for which $y \mapsto y^r$ is defined on $\mathcal{M}$. Hence if $c_1 \neq 0$ and $r_1 = 1$, then (20) holds, with the exponent i replaced by r_i. This shows again that f is pseudo-linear with pseudo-slope c_1. If, however, $r_1 \neq 1$, we may think of writing $f(y) = \tilde{f}(y^{r_1})$ with

$$\tilde{f}(X) \;=\; \sum_{i\in\mathbb{N}} c_i X^{r_i/r_1} \;.$$

If the power functions $y \mapsto y^{r_i/r_1}$ are defined on $\mathcal{M}$ for all i, then $\tilde{f}$ defines a pseudo-linear map from $\mathcal{M}$ to $\mathcal{M}$ with pseudo-slope c_1. So we obtain:

Theorem 25 *Suppose that the power functions $y \mapsto y^{r_i}$ and $y \mapsto y^{r_i/r_1}$ are defined on $\mathcal{M}$ for all i, and that $y \mapsto y^{r_1}$ is surjective. If $f : \mathcal{M} \to \mathcal{M}$ is defined by the power series (21) with $c_1 \neq 0$, then f is surjective.*

5 Towards an infinite-dimensional Implicit Function Theorem

From our result in Section 2.2 it follows that an infinite power Y^I of an ultrametric space Y can be equipped with an ultrametric u^I (analogous to the minimum valuation) if the value set uY is well ordered. In this case, if (Y, u) is spherically complete, then so is (Y^I, u^I). So we obtain the following corollary to our Main Theorem 2 and to Proposition 16:

Corollary 26 *a) Take two ultrametric spaces (Y, u) and (Y', u'), and an arbitrary index set I. Assume that uY is well ordered, $f : Y^I \to Y'$ is immediate and that (Y, u) is spherically complete. Then f is surjective and (Y', u') is spherically complete.*
b) Take two valued abelian groups (G, v) and (G', v'), and an arbitrary index set I. Assume that vG is well ordered, $b \in G^I$, B is a ball around 0 in G^I, $f : G^I \to G'$ has a pseudo-derivative at b on $b + B$, and that (G, v) is spherically complete. Then f is surjective and (G', v') is spherically complete.

In the case of a valued field (K, v) we cannot do the same since if the valuation is non-trivial, the value group will not be well ordered. If the valuation is not discrete (i.e., its value group is not isomorphic to $\mathbb{Z}$), then not even the value set $v\mathcal{O} := v(\mathcal{O}\setminus\{0\})$ of the valuation ring is well ordered. But we may be interested in infinite systems of polynomials with coefficients in a subring R of $\mathcal{O}$ with well ordered value set $vR := v(R \setminus \{0\})$. Note that in this case (R,v) is not necessarily spherically complete if (K,v) is. So we will assume that (R,v) is spherically complete.

We generalize the definitions of **minimum valuation** and of **pseudo linear map** in the obvious way. If $a = (a_i)_{i\in I} \in R^I$, then $va := \min_{i\in I} va_i$. If $Y \subseteq R^I$, $0 \neq s \in R$ and f a map from Y into R^I, then f is pseudo-linear with pseudo-slope s if (11) holds for all $y,z \in Y$ such that $y \neq z$. We then have the following application of Proposition 16 together with Proposition 10:

Proposition 27 *Take $b \in R^I$ and B a ball in (R^I, v) around 0. Assume that $f : b+B \to R^I$ is pseudo-linear with pseudo-slope $s \in R$ and that (R,v) is spherically complete. Then f is an isomorphism of ultrametric spaces from $b+B$ onto $fb+sB$.*

If the map is given by an infinite system of polynomials $f = (f_i)_{i\in I}$ in infinitely many variables X_i , $i \in I$, and with coefficients in R, then we may consider the infinite matrix $J_f(b) \in R^{I\times I}$. Note that this matrix has only finitely many non-zero entries in every row. Nevertheless, we may not be able to use determinants here. Still, we can use our approach if $J_f(b)$ is invertible in $R^{I\times I}$. Actually, we do not need that R is a subring of a valued field. It suffices to assume that it is a valued abelian group with multiplication satisfying (V3), and that its value set is a well ordered subset of an ordered abelian group. It then follows that the value set does not contain negative elements. From the invertibility of $J_f(b)$ in $R^{I\times I}$ it follows that the value set must contain 0. We set $\mathcal{M}_R := \{a \in R \mid va > 0\}$.

Proposition 28 *Take any index set I and a system of polynomials $f = (f_i)_{i\in I}$ in variables X_i , $i \in I$, and with coefficients in R. Assume that (R,v) is spherically complete. If $J_f(b)$ admits an inverse in $R^{I\times I}$, then $J_f(b)$ is a pseudo-derivative of f on $b+\mathcal{M}_R^I$ and f induces an isomorphism from $b+\mathcal{M}_R^I$ onto $f(b)+\mathcal{M}_R^I$. The system f has a zero on $b+\mathcal{M}_R^I$ (which then is unique) if and only if $vf(b) > 0$.*

We leave the proof to the reader. It just needs an adaptation of the proof of the first assertion of part b) of Proposition 19 to the infinite case, followed by an application of part b) of Corollary 26.

6 Polynomials in additive operators

In this section, we will consider polynomials $f \in \mathcal{O}[X_0, X_1, \ldots, X_n]$ over valued fields (K,v) and additive operators $\sigma_i : K \to K$ and try to solve equations of the form

$$f(\sigma_0 X, \sigma_1 X, \ldots, \sigma_n X) \;=\; 0\;.$$

6.1 Basic results

For any polynomial f in $n+1$ variables over a field of arbitrary characteristic, we denote by $f^{[\underline{i}]}$ its $\underline{i}$-th formal derivative, where $\underline{i} = (i_0, \ldots, i_n)$ is a multi-index. These polynomials are defined such that the following analogue of (13) holds in arbitrary characteristic:

$$f(b+\varepsilon) \;=\; f(b) + \sum_{i\in I} \varepsilon^{\underline{i}} f^{[\underline{i}]}(b) \quad \text{for all } b,\varepsilon \in K^{n+1}\;, \tag{22}$$

where $I = \{0,1,\ldots,\deg f\}^{n+1} \setminus \{(0,\ldots,0)\}$ and $\varepsilon^{\underline{i}} = \varepsilon_0^{i_0} \cdot \ldots \cdot \varepsilon_n^{i_n}$. Note that if $\underline{i} = (0,\ldots,0,1,0,\ldots,0)$ with the 1 in the j-th place, then $f^{[\underline{i}]} = \frac{\partial f}{\partial X_j}(X_0,\ldots,X_n)$.

Lemma 29 *Take $f \in \mathcal{O}[X_0,\ldots,X_n]$ and $\underline{b} \in \mathcal{O}^{n+1}$, $s \in \mathcal{O}$ such that*

$$vs \;=\; \min_{0\le i\le n} v\frac{\partial f}{\partial X_i}(\underline{b}) \;<\; \infty\,.$$

Then for all distinct $\underline{y} = (y_0,\ldots,y_n)$ and $\underline{z} = (z_0,\ldots,z_n)$ in $\underline{b} + s\mathcal{M}^{n+1}$,

$$v\left(f(\underline{y}) - f(\underline{z}) \;-\; \sum_{i=0}^{n} (y_i - z_i)\frac{\partial f}{\partial X_i}(\underline{b})\right) \;>\; vs + \min_{0\le i\le n} v(y_i - z_i) \tag{23}$$

and

$$v(f(\underline{y}) - f(\underline{z})) \;\ge\; vs + \min_{0\le i\le n} v(y_i - z_i)\,. \tag{24}$$

Proof: Since $f \in \mathcal{O}[X_0,\ldots,X_n]$, we have that $f^{[\underline{i}]} \in \mathcal{O}[X_0,\ldots,X_n]$. Since $\underline{b} \in \mathcal{O}^{n+1}$, we also have that $f^{[\underline{i}]}(\underline{b}) \in \mathcal{O}$. Write $\underline{y} = \underline{b} + \underline{\delta}$ and $\underline{z} = \underline{b} + \underline{\varepsilon}$ with $\underline{\delta} = (\delta_0,\ldots,\delta_n), \underline{\varepsilon} = (\varepsilon_0,\ldots,\varepsilon_n) \in s\mathcal{M}^{n+1}$. Then by (22),

$$f(\underline{y}) - f(\underline{z}) \;=\; \sum_{i=0}^{n} (\delta_i - \varepsilon_i)\frac{\partial f}{\partial X_i}(\underline{b}) \;+\; \sum_{i\in I'} (\underline{\delta}^{\underline{i}} - \underline{\varepsilon}^{\underline{i}}) f^{[\underline{i}]}(\underline{b})$$

where $I' = \{\underline{i} \in I \mid |\underline{i}| \ge 2\}$ with $|\underline{i}| := i_0 + \ldots + i_n$.

Choose $c \in \mathcal{M}$ such that $vc = \min_i v(\delta_i - \varepsilon_i) = \min_i v(y_i - z_i)$. Pick $j \in \{0,\ldots,n\}$ and take $\underline{i} \in I'$ such that $i_j \ne 0$. Let $\underline{i}' \in I$ be the multi-index obtained from $\underline{i}$ by subtracting 1 in the j-th place. Then

$$\underline{\delta}^{\underline{i}} - \underline{\varepsilon}^{\underline{i}} \;=\; \delta_j \underline{\delta}^{\underline{i}'} - \varepsilon_j \underline{\varepsilon}^{\underline{i}'} \;=\; (\delta_j - \varepsilon_j)\underline{\delta}^{\underline{i}'} + \varepsilon_j(\underline{\delta}^{\underline{i}'} - \underline{\varepsilon}^{\underline{i}'})$$

Suppose we have already shown by induction on $|\underline{i}|$ that $\underline{\delta}^{\underline{i}'} - \underline{\varepsilon}^{\underline{i}'} \in c\mathcal{O}$. Since $\delta_j - \varepsilon_j \in c\mathcal{O}$ and $\underline{\delta}^{\underline{i}'}, \varepsilon_j \in s\mathcal{M}$, we then find that

$$\underline{\delta}^{\underline{i}} - \underline{\varepsilon}^{\underline{i}} \;\in\; sc\mathcal{M}$$

for every multi-index $\underline{i}$ with $|\underline{i}| \ge 2$. Since also $f^{[\underline{i}]}(\underline{b}) \in \mathcal{O}$, we obtain that

$$f(\underline{y}) - f(\underline{z}) \;-\; \sum_{i=0}^{n} (\delta_i - \varepsilon_i)\frac{\partial f}{\partial X_i}(\underline{b}) \;=\; \sum_{i\in I'} (\underline{\delta}^{\underline{i}} - \underline{\varepsilon}^{\underline{i}}) f^{[\underline{i}]}(\underline{b}) \;\in\; sc\mathcal{M}\,.$$

This proves (23). To prove (24), we observe that

$$v\sum_{i=0}^{n}(y_i - z_i)\frac{\partial f}{\partial X_i}(\underline{b}) \;\geq\; \min_{0\leq i\leq n} v(y_i - z_i)\frac{\partial f}{\partial X_i}(\underline{b}) \;\geq\; vs + \min_{0\leq i\leq n} v(y_i - z_i)$$

and therefore,

$$\begin{aligned} & v(f(\underline{y}) - f(\underline{z})) \;\geq \\ \geq\; & \min\left\{ v\left(f(\underline{y}) - f(\underline{z}) \;-\; \sum_{i=0}^{n}(y_i - z_i)\frac{\partial f}{\partial X_i}(\underline{b})\right) \,,\, v\sum_{i=0}^{n}(y_i - z_i)\frac{\partial f}{\partial X_i}(\underline{b}) \right\} \\ \geq\; & vs + \min_{0\leq i\leq n} v(y_i - z_i)\;. \end{aligned}$$

□

Proposition 30 *Take*

- *additive operators* $\sigma_i : \mathcal{O} \to \mathcal{O}\,,\;\; 0 \leq i \leq n$,
- $f \in \mathcal{O}[X_0, \ldots, X_n]$,
- $b \in \mathcal{O}$ *such that at least one of the following derivatives is not zero:*

$$d_i \;:=\; \frac{\partial f}{\partial X_i}(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) \qquad (0 \leq i \leq n), \tag{25}$$

- $s \in \mathcal{O}$ *such that*

$$vs \;=\; \min_{0\leq i\leq n} vd_i\;. \tag{26}$$

Suppose that
(V≥) $v\sigma_i a \;\geq\; va$ *for all* $a \in \mathcal{O}$ $\qquad (0 \leq i \leq n)\,,$
holds and that the additive operator

$$\phi \;:=\; \sum_{i=0}^{n} d_i\sigma_i :\; s\mathcal{M} \longrightarrow s^2\mathcal{M}$$

has the property that for all $a' \in s^2\mathcal{M}$ *there is some* $a \in s\mathcal{M}$ *such that* $v(a' - \phi a) > va'$ *and* $va = va' - vs$. *Then the map*

$$b + s\mathcal{M} \ni x \mapsto f(\sigma_0 x, \sigma_1 x, \ldots, \sigma_n x) \in f(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) + s^2\mathcal{M}$$

is immediate.

Proof: We define $\varphi : vs\mathcal{M} \to vs^2\mathcal{M}$ by $\varphi\alpha := vs + \alpha$. Then φ is an bijection which preserves $<$. For all $a \in s\mathcal{M}$, the definition of s together with (V≥) yields

$$v\phi a \;=\; v\sum_{i=0}^{n} d_i\sigma_i a \;\geq\; \min_{0\leq i\leq n} vd_i\sigma_i a \;\geq\; \min_{0\leq i\leq n} vd_i \;+\; va \;=\; vs + va \;=\; \varphi va\;.$$

Hence $\phi(s\mathcal{M}) \subseteq s^2\mathcal{M}$. We also see that ϕ satisfies (BC1) of Proposition 15. By assumption, it also satisfies (BC2). We wish to apply Proposition 15 to the map g defined by

$$gx \;:=\; f(\sigma_0 x, \sigma_1 x, \ldots, \sigma_n x)\;.$$

Take distinct elements $y, z \in b + s\mathcal{M}$. From (V≥) it follows that $b_i := \sigma_i b \in \mathcal{O}$, $y_i := \sigma_i y \in \mathcal{O}$, $z_i := \sigma_i z \in \mathcal{O}$ with $y_i - b_i = \sigma_i(y-b) \in s\mathcal{M}$ and $z_i - b_i = \sigma_i(z-b) \in s\mathcal{M}$, so $(y_0, \ldots, y_n)\,,\,(z_0, \ldots, z_n) \in (b_0, \ldots, b_n) + s\mathcal{M}^{n+1}$. Thus we can apply Lemma 29 to obtain

$$\begin{aligned}
& v(gy - gz \,-\, \phi(y-z)) \;= \\
&= \;\; v\left(f(\sigma_0 y, \ldots, \sigma_n y) - f(\sigma_0 z, \ldots, \sigma_n z) - \sum_{i=0}^{n} d_i \sigma_i (y-z)\right) \\
&= \;\; v\left(f(\sigma_0 y, \ldots, \sigma_n y) - f(\sigma_0 z, \ldots, \sigma_n z) - \sum_{i=0}^{n} (\sigma_i y - \sigma_i z)\frac{\partial f}{\partial X_i}(\sigma_0 b, \ldots, \sigma_n b)\right) \\
&> \;\; vs + \min_i v(\sigma_i y - \sigma_i z) \;=\; vs + \min_i v\sigma_i(y-z) \\
&\geq \;\; vs + v(y-z) \;=\; \varphi v(y-z)\;.
\end{aligned}$$

Hence, also (BC3) is satisfied, with g in the place of f. Our assertion now follows from Proposition 15. □

In the next section, we give a criterion which guarantees that the hypothesis of Proposition 30 on the operator ϕ is satisfied.

6.2 The case of operators compatible with a weak coefficient map

Let us start with the following useful observation.

Lemma 31 *Let (K, v) be any valued field. For all $\alpha \in vK$, $\alpha \neq 0$, choose elements*

$$m_\alpha \in K \;\;\textit{such that}\;\; vm_\alpha \;=\; \alpha \;\;\textit{and}\;\; m_0 = 1\;. \tag{27}$$

Define $\mathrm{co}\,0 := 0$ *and*

$$\mathrm{co}\,a \;:=\; (m_{-va}\,a)\,v \;\;\textit{for all}\;\; a \in K \setminus \{0\}\;.$$

Then co *has the following properties:*

(WCM0) $\mathrm{co}\,a = 0$ *if and only if* $a = 0$,
(WCM1) *if* $va = 0$, *then* $\mathrm{co}\,a = av$,
(WCM2) *if* $va_1 = va_2 = \ldots = va_k$ *and* $\sum_{i=1}^k \mathrm{co}\,a_i \neq 0$, *then* $\mathrm{co}\,(\sum_{i=1}^k a_i) = \sum_{i=1}^k \mathrm{co}\,a_i$,
(WCM3) *if* $\mathrm{co}\,a = \mathrm{co}\,b$ *and* $va = vb$, *then* $v(a-b) > va$,
(WCM4) *if* $\gamma \in vK$ *and* $0 \neq \overline{a} \in Kv$, *then* $\exists a \in K : \mathrm{co}\,a = \overline{a}$ *and* $va = \gamma$.

Proof: Since $(m_{-va}\,a)v \neq 0$ for $a \neq 0$, (WCM0) holds. Since $m_0 = 1$, also (WCM1) holds.

If $va_1 = va_2 = \ldots = va_k$ and $\sum_{i=1}^k \mathrm{co}\, a_i \neq 0$, then $m_{-va_1} = m_{-va_2} = \ldots = m_{-va_k}$ and

$$0 \neq \sum_{i=1}^k \mathrm{co}\, a_i \;=\; \sum_{i=1}^k (m_{-va_i}\, a_i)\, v \;=\; \sum_{i=1}^k (m_{-va_1}\, a_i)\, v \;=\; \left(m_{-va_1} \sum_{i=1}^k a_i \right) v\;,$$

whence $vm_{-va_1} \sum_{i=1}^k a_i = 0$ and therefore, $v \sum_{i=1}^k a_i = va_1$. Hence,

$$\sum_{i=1}^k \mathrm{co}\, a_i \;=\; \left(m_{-va_1} \sum_{i=1}^k a_i \right) v \;=\; \mathrm{co}\, \left(\sum_{i=1}^k a_i \right)\;.$$

This shows that (WCM2) holds.

If $\mathrm{co}\, a = \mathrm{co}\, b$ and $va = vb$, then

$$(m_{-va}\, a)\, v \;=\; \mathrm{co}\, a \;=\; \mathrm{co}\, b \;=\; (m_{-vb}\, b)\, v \;=\; (m_{-va}\, b)\, v\;,$$

so $0 < v(m_{-va}a - m_{-va}b) = vm_{-va} + v(a - b) = -va + v(a - b)$, that is, $v(a - b) > va$. This shows that (WCM3) holds.

If $\gamma \in vK$ and $0 \neq \overline{a} \in Kv$, we choose $a_0 \in \mathcal{O}^\times$ such that $a_0 v = \overline{a}$. Then we set $a = m_{-\gamma}^{-1} a_0$. This gives $va = -vm_{-\gamma} = \gamma$ and $\mathrm{co}\, a = (m_{-\gamma}(m_{-\gamma}^{-1} a_0))v = a_0 v = \overline{a}$. Hence, (WCM4) holds. □

A map co with properties (WCM0) – (WCM4) will be called a **weak coefficient map**. We will assume that the operators σ_i satisfy (V$\geq$); hence they induce additive operators $\overline{\sigma}_i$ on Kv:

$$\text{for all } a \in \mathcal{O}, \quad \overline{\sigma}_i(av) \;=\; (\sigma_i a)v \qquad (0 \leq i \leq n)\;. \tag{28}$$

We will need some stronger compatibility of the σ_i with the weak coefficient map:

Lemma 32 *Assume that the operators σ_i satisfy* (V$\geq$) *and that the elements m_α in (27) can be chosen such that*

$$\textit{for all } a \in \mathcal{O}, \quad v(\sigma_i m_{-va} a \;-\; m_{-va}\sigma_i a) \;>\; 0 \qquad (0 \leq i \leq n)\;. \tag{29}$$

Then

$$\textit{for all } a \in \mathcal{O} \textit{ and all } d \in \mathcal{O}^\times, \quad (\mathrm{co}\, d)\, \overline{\sigma}_i\, \mathrm{co}\, a \;=\; \begin{cases} \mathrm{co}\,(d\sigma_i a) & \textit{if } v\sigma_i a = va \\ 0 & \textit{if } v\sigma_i a > va \end{cases} \tag{30}$$

Proof: Take any $d \in \mathcal{O}^\times$; then $vd = 0$ and hence, $\mathrm{co}\, d = dv$. We have that

$$\begin{aligned} (\mathrm{co}\, d)\, \overline{\sigma}_i \mathrm{co}\, a \;&=\; (dv)\, \overline{\sigma}_i((m_{-va}a)v) \;=\; (dv)\, (\sigma_i m_{-va} a)v \\ &=\; (dv)\, (m_{-va}\sigma_i a)v \;=\; (m_{-va} d\sigma_i a)v\;. \end{aligned}$$

Here, the second equality holds by equation (28), and the third equality holds by (29). Now we distinguish two cases. Suppose first that $v\sigma_i a = va$. Then

$$(m_{-va} d\sigma_i a)v \;=\; (m_{-v\sigma_i a} d\sigma_i a)v \;=\; (m_{-vd\sigma_i a} d\sigma_i a)v \;=\; \mathrm{co}\,(d\sigma_i a)\;.$$

Now suppose that $v\sigma_i a > va$. Then $vm_{-va} d\sigma_i a > 0$ and hence, $(m_{-va} d\sigma_i a)v = 0$. This proves that (30) holds. □

Property (30) can be expressed by saying that unit multiples of the additive operators commute with the coefficient map.

Proposition 33 *Let the assumptions on f, b, d_i and s be as in Proposition 30. Assume that the additive operators σ_i satisfy* (V≥)*, that* co *is a weak coefficient map and that (30) holds. Suppose further that the additive operator*

$$\sum_{i=0}^{n} c_i \overline{\sigma}_i \quad \textit{with} \quad c_i = \begin{cases} \mathrm{co}\, s^{-1} d_i & \textit{if} \;\; vd_i = vs \\ 0 & \textit{if} \;\; vd_i > vs \end{cases}$$

on the residue field Kv is surjective. Then the map

$$b + s\mathcal{M} \ni x \mapsto f(\sigma_0 x, \sigma_1 x, \ldots, \sigma_n x) \in f(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) + s^2 \mathcal{M}$$

is immediate.

Proof: We define ϕ as in Proposition 30. Now we just have to show that ϕ satisfies the assumptions of that proposition. So take any $a' \in s^2\mathcal{M}$, $a' \neq 0$. Since $\sum_{i=0}^{n} c_i \overline{\sigma}_i$ is surjective on Kv by assumption, there is some $\overline{a} \in Kv$ such that $\sum_{i=0}^{n} c_i \overline{\sigma}_i\, \overline{a} = \mathrm{co}\, s^{-1} a'$. Property (WCM4) of the coefficient map allows us to choose $a \in K$ such that $\mathrm{co}\, a = \overline{a}$ and $va = va' - vs$. Thus, $0 \neq a \in s\mathcal{M}$. Set $I = \{i \mid 0 \leq i \leq n \text{ with } vd_i = vs \text{ and } \overline{\sigma}_i\, \mathrm{co}\, a \neq 0\}$. Then by the definition of the c_i ,

$$\begin{aligned} \mathrm{co}\, s^{-1} a' \;&=\; \sum_{i=1}^{n} c_i \overline{\sigma}_i\, \overline{a} \;=\; \sum_{i\in I} \mathrm{co}\,(s^{-1} d_i)\, \overline{\sigma}_i\, \mathrm{co}\, a \\ &=\; \sum_{i\in I} \mathrm{co}\,(s^{-1} d_i \sigma_i a) \;=\; \mathrm{co}\,(\sum_{i\in I} s^{-1} d_i \sigma_i a)\;, \end{aligned}$$

where the third equality holds by (30). The last equality follows from (WCM2) since the left hand side is non-zero, being equal to $\mathrm{co}\, s^{-1} a'$, and because for each $i \in I$, $\overline{\sigma}_i\, \mathrm{co}\, a \neq 0$ implies $v\sigma_i a = va$ by (30), and $vd_i = vs$ then yields $vs^{-1} d_i \sigma_i a = va$ so that all values are equal. By (WCM3), it follows that

$$v\left(s^{-1} a' - \sum_{i\in I} s^{-1} d_i \sigma_i a\right) \;>\; vs^{-1} a'\;.$$

Consequently,

$$v\left(a' - \sum_{i\in I} d_i\sigma_i a\right) \;=\; v\left(s^{-1}a' - \sum_{i\in I} s^{-1}d_i\sigma_i a\right) + vs \;>\; vs^{-1}a' + vs \;=\; va' \;.$$

On the other hand, take $i \in I' := \{0,\ldots,n\}\setminus I$. In the case of $vd_i > vs$, since $v\sigma_i a \geq va = va' - vs$, we find that $vd_i\sigma_i a \geq vd_i + va' - vs > va'$. Observe that $a \neq 0$ implies $d\sigma_i a \neq 0$, and this implies $\operatorname{co} d\sigma_i a \neq 0$. Hence in the case of $\overline{\sigma}_i \operatorname{co} a = 0$, (30) shows that $v\sigma_i a > va$ and we obtain that $vd_i\sigma_i a > vd_i + va = vd_i + va' - vs \geq va'$. Therefore,

$$v\sum_{i\in I'} d_i\sigma_i a \;\geq\; \min_{i\in I'} vd_i\sigma_i a \;>\; va' \;.$$

This gives us

$$v(a' - \phi a) \;=\; v\left(a' - \sum_{i=0}^{n} d_i\sigma_i a\right) \;\geq\; \min\left\{v\left(a' - \sum_{i\in I} d_i\sigma_i a\right), v\sum_{i\in I'} d_i\sigma_i a\right\} \;>\; va' \;.$$

So the conditions of Proposition 30 are satisfied and we are done. □

In the same way as for the original Hensel's Lemma (except for the uniqueness assertion), Proposition 33 yields the following generalized Hensel's Lemma in the present setting:

Theorem 34 *In addition to the assumptions of Proposition 33, suppose that (K,v) is spherically complete and that*

$$vf(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) > 2vs \;.$$

Then there is an element $a \in K$ such that $f(\sigma_0 a, \sigma_1 a, \ldots, \sigma_n a) = 0$ and $v(a-b) > vs$.

6.3 The case of a dominant operator

In this section, we consider the case where one of the additive operators, say σ_n (without loss of generality), is dominant on some ball B around 0, that is,

$$\forall\, a \in B: \quad v\sigma_n a < \min_{0\leq j\leq n-1} v\sigma_j a \quad \text{or} \quad \sigma_0 a = \sigma_1 a = \ldots = \sigma_n a = 0 \;. \tag{31}$$

We will not assume that (V≥) holds, so we cnnot apply Proposition 30. Instead, we prove:

Proposition 35 *Let $\sigma_i : \mathcal{O} \to \mathcal{O}$, $0 \leq i \leq n$, be additive operators satisfying condition (31). With f, b and d_i as in Proposition 30, assume that*

$$vd_n \;=\; \min_{0\leq i\leq n} vd_i \;. \tag{32}$$

Suppose further that for some balls $B, B' \subseteq d_n\mathcal{M}$ around 0, the map $\sigma_n : B \to B'$ is immediate. Then the map

$$b + B \ni x \mapsto f(\sigma_0 x, \sigma_1 x, \ldots, \sigma_n x) \in f(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) + d_n B' \tag{33}$$

is immediate. If σ_n is injective on B, then (33) is injective, too.

Proof: We set $b' := f(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) \in \mathcal{O}$ and $s = d_n$. Take distinct elements $y, z \in b + B \subseteq b + s\mathcal{M}$ and set $b_i := \sigma_i b \in \mathcal{O}$, $y_i := \sigma_i y \in \mathcal{O}$, $z_i := \sigma_i z \in \mathcal{O}$. It follows from (31) that $v(y_i - b_i) = v\sigma_i(y - b) \geq v\sigma_n(y - b)$, and our assumption on σ_n yields that $y_i - b_i \in B'$ for $0 \leq i \leq n$. We obtain $y_i \in b_i + B' \subseteq b_i + s\mathcal{M}$ and similarly, $z_i \in b_i + s\mathcal{M}$. Thus we can apply Lemma 29 to g in the place of f, and (24) shows that

$$gx \;:=\; f(\sigma_0 x, \sigma_1 x, \ldots, \sigma_n x) \;\in\; b' + d_n B' \quad \text{for every } x \in b + B \;.$$

We shall apply Proposition 14 in order to show that $g : b + B \to b' + d_n B'$ is immediate. We set $\phi := d_n \sigma_n$. Pick any $a' \in d_n B'$, $a' \neq 0$. Since $\sigma_n : B \to B'$ is immediate, Proposition 11 shows that there is some $a \in B$ such that $a \neq 0$ and

$$v\left(\frac{a'}{d_n} - \sigma_n a\right) \;>\; v\,\frac{a'}{d_n} \tag{34}$$

and

$$va \leq vb \implies v\sigma_n a \leq v\sigma_n b \;. \tag{35}$$

From (34) we obtain that $v(a' - \phi a) > va'$, which shows that (6) of Proposition 14 is satisfied for g in the place of f. Now take distinct $y, z \in b + B$. As in the proof of Proposition 30, we can apply Proposition 29 to obtain that

$$v\left(gy - gz - \sum_{i=0}^{n} d_i \sigma_i (y - z)\right) \;>\; vs + \min_i v(\sigma_i y - \sigma_i z) \;=\; vs + \min_i v\sigma_i (y - z) \;.$$

By (31),

$$vs + \min_i v\sigma_i(y - z) \;=\; vs + v\sigma_n(y - z) \;=\; vd_n\sigma_n(y - z) \;.$$

Again by (31),

$$v\sum_{i=0}^{n-1} d_i \sigma_i (y - z) \;>\; vd_n \sigma_n (y - z) \;,$$

and we conclude that

$$v(gy - gz - d_n\sigma_n(y - z)) \geq \min\{v(gy - gz - \sum_{i=0}^{n} d_i\sigma_i(y - z)) \,,\, \sum_{i=0}^{n-1} d_i\sigma_i(y - z)\} > vd_n\sigma_n(y - z) \;. \tag{36}$$

If $v(y - z) \geq va$, then by (35), $vd_n\sigma_n(y - z) \;\geq\; vd_n\sigma_n a$ and thus, (36) yields

$$v(gy - gz \;-\; \phi(y - z)) \;=\; v(gy - gz \;-\; d_n\sigma_n(y - z)) \;>\; vd_n\sigma_n a \;=\; v\phi a \;.$$

Since $\phi 0 = 0$ as ϕ is additive, this shows that (7) is satisfied for g in the place of f. Now Proposition 14 proves that g is immediate.

If σ_n is injective on B, then $y \neq z$ implies $vd_n\sigma_n(y - z) < \infty$, whence $gy \neq gz$ by (36). Hence in this case, (33) is injective. □

Proposition 35 yields the following Hensel's Lemma for the case of a dominant operator:

Theorem 36 *In addition to the assumptions of Proposition 35, suppose that* (K, v) *is spherically complete and that for some* $e \in B$,

$$vf(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) \geq vd_n + v\sigma_n e \,. \tag{37}$$

Then there is an element $a \in b + B$ *such that* $f(\sigma_0 a, \sigma_1 a, \ldots, \sigma_n a) = 0$ *and* $v\sigma_n(a - b) \geq v\sigma_n e$. *If* σ_n *is injective on* B, *then* a *is unique.*

Proof: It just remains to show that $v\sigma_n(a - b) \geq v\sigma_n e$. By (37),

$$\begin{aligned} vd_n + v\sigma_n e \;&\leq\; vf(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) \;=\; v(f(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) \,-\, f(\sigma_0 a, \sigma_1 a, \ldots, \sigma_n a)) \\ &=\; vd_n + v\sigma_n(b - a) \,, \end{aligned}$$

where the last equality follows from (36) by the ultrametric triangle law. Hence, $v\sigma_n(a - b) = v\sigma_n(b - a) \geq v\sigma_n e$. □

In Section 7.3 we will deduce from this theorem a Hensel's Lemma for Rosenlicht valued differential fields. But this Hensel's Lemma is not strong enough. To improve it, we consider also the values of the higher derivatives of f. So we need to modify our approach, which we will do in the next section.

6.4 Rosenlicht systems of operators

We will call $\sigma_0, \sigma_1, \ldots, \sigma_n$ a **Rosenlicht system of operators** if each $\sigma_i : \mathcal{O} \to \mathcal{O}$ is additive and there exist elements $e_i \in \mathcal{O}$ such that

$$e_n \;=\; 1 \quad \text{and} \quad ve_0 \;\geq\; ve_1 \;\geq \ldots \geq\; ve_n \;=\; 0 \,, \tag{38}$$

and

$$ve_i + v\sigma_i a \;>\; v\sigma_n a \quad \text{for all } a \in \mathcal{M},\ a \neq 0 \,. \tag{39}$$

The latter implicitly includes the condition that σ_n is injective on $\mathcal{M}$.

The following is an adaptation of Lemma 29.

Lemma 37 *Take* $f \in \mathcal{O}[X_0, X_1, \ldots, X_n]$ *and* $\underline{b} \in \mathcal{O}^{n+1}$ *such that*

$$d_n \;=\; \frac{\partial f}{\partial X_n}(\underline{b}) \;\neq\; 0$$

and for all $\underline{i} \in I = \{0, \ldots, \deg f\}^{n+1} \setminus \{(0, \ldots, 0)\}$,

$$vf^{[\underline{i}]}(\underline{b}) \;\geq\; vd_n + ve_k \quad \text{if } \; k = \min\{j \mid i_j \neq 0\} \tag{40}$$

where the elements $e_i \in K$ *satisfy (38). Take* $\underline{y} = (y_0, \ldots, y_n)$ *and* $\underline{z} = (z_0, \ldots, z_n)$ *in* $\underline{b} + \mathcal{M}^{n+1}$ *such that*

$$ve_i + v(y_i - z_i) \;>\; v(y_n - z_n) \quad \text{for } \; 0 \leq i < n \,. \tag{41}$$

Then the following holds:

$$v(f(\underline{y}) - f(\underline{z}) \,-\, d_n(y_n - z_n)) \;>\; vd_n(y_n - z_n) \;=\; v(f(\underline{y}) - f(\underline{z})) \,. \tag{42}$$

Proof: Write $\underline{y} = \underline{b} + \underline{\delta} \in \underline{b} + \mathcal{M}^{n+1}$ and $\underline{z} = \underline{b} + \underline{\varepsilon} \in \underline{b} + \mathcal{M}^{n+1}$, where $\underline{\delta} = (\delta_0, \ldots, \delta_n)$ and $\underline{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_n)$ satisfy

$$ve_i + v(\delta_i - \varepsilon_i) \;=\; ve_i + v(y_i - z_i) \;>\; v(y_n - z_n) \quad \text{for } \; 0 \leq i < n\;. \tag{43}$$

We note that $ve_n + v(\delta_n - \varepsilon_n) = v(y_n - z_n)$; so we have

$$ve_i + v(\delta_i - \varepsilon_i) \;\geq\; v(y_n - z_n) \quad \text{for } \; 0 \leq i \leq n\;. \tag{44}$$

Take $\underline{i} \in I$, $|\underline{i}| \geq 2$, and let $\underline{i}'$ be the multi-index obtained from $\underline{i}$ by subtracting 1 in the k-th place, where $k = \min\{j \mid i_j \neq 0\}$. Then

$$\underline{\delta}^{\underline{i}} - \underline{\varepsilon}^{\underline{i}} \;=\; (\delta_k - \varepsilon_k)\underline{\delta}^{\underline{i}'} + \varepsilon_k(\underline{\delta}^{\underline{i}'} - \underline{\varepsilon}^{\underline{i}'})\;.$$

Suppose that we have already shown by induction on $|\underline{i}'|$ that

$$ve_\ell + v(\underline{\delta}^{\underline{i}'} - \underline{\varepsilon}^{\underline{i}'}) \;\geq\; v(y_n - z_n) \quad \text{for } \ell = \min\{j \mid i'_j \neq 0\}\;,$$

with the induction start for $|\underline{i}'| = 1$ being covered by (44). We have that $\ell \geq k$, hence $ve_k \geq ve_\ell$ by (38); therefore, also $ve_k + v(\underline{\delta}^{\underline{i}'} - \underline{\varepsilon}^{\underline{i}'}) \geq v(y_n - z_n)$. Since $ve_k + v(\delta_k - \varepsilon_k) \geq v(y_n - z_n)$ by (44), and since $\underline{\delta}^{\underline{i}'}, \varepsilon_k \in \mathcal{M}$, we then find

$$\begin{aligned} ve_k + v(\underline{\delta}^{\underline{i}} - \underline{\varepsilon}^{\underline{i}}) &\;\geq\; \min\{ve_k + v(\delta_k - \varepsilon_k) + v\underline{\delta}^{\underline{i}'}\,,\; ve_k + v\varepsilon_k + v(\underline{\delta}^{\underline{i}'} - \underline{\varepsilon}^{\underline{i}'})\} \\ &\;>\; v(y_n - z_n)\;. \end{aligned} \tag{45}$$

Take $\underline{i} \in I' := I \setminus \{(0, \ldots, 0, 1)\}$. Then because of (43), inequality (45) also holds in the case of $|\underline{i}| = 1$. Hence by hypothesis (40),

$$v(\underline{\delta}^{\underline{i}} - \underline{\varepsilon}^{\underline{i}})f^{[\underline{i}]}(\underline{b}) \;\geq\; vd_n + ve_k + v(\underline{\delta}^{\underline{i}} - \underline{\varepsilon}^{\underline{i}}) \;>\; vd_n + v(y_n - z_n)\;.$$

Since

$$f(\underline{y}) - f(\underline{z}) \;=\; d_n(\delta_n - \varepsilon_n) \;+\; \sum_{i \in I'} (\underline{\delta}^{\underline{i}} - \underline{\varepsilon}^{\underline{i}}) f^{[\underline{i}]}(\underline{b})$$

by (22), this yields

$$\begin{aligned} v(f(\underline{y}) - f(\underline{z}) \;-\; d_n(y_n - z_n)) &\;=\; v(f(\underline{y}) - f(\underline{z}) \;-\; d_n(\delta_n - \varepsilon_n)) \\ &\;=\; v\sum_{i \in I'} (\underline{\delta}^{\underline{i}} - \underline{\varepsilon}^{\underline{i}}) f^{[\underline{i}]}(\underline{b}) \;>\; vd_n + v(y_n - z_n)\;, \end{aligned}$$

which gives the inequality in(42). The equality in (42) follows from the inequality by the ultrametric triangle law. □

Proposition 38 *Let $\sigma_0, \ldots, \sigma_n$ be a Rosenlicht system of operators satisfying (38) and (39). Take f, b and d_n as in Proposition 37 such that (40) holds. Suppose further that for some balls $B, B' \subseteq \mathcal{M}$ around 0, the map $\sigma_n : B \to B'$ is immediate. Then*

$$b + B \ni x \mapsto f(\sigma_0 x, \sigma_1 x, \ldots, \sigma_n x) \in f(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) + d_n B' \tag{46}$$

is immediate and injective.

Proof: We modify the proof of Proposition 35 as follows. As before, we set $b' := f(\sigma_0 b, \sigma_1 b, \ldots, \sigma_n b) \in \mathcal{O}$. In order to apply Lemma 37, we set $y_i = \sigma_i y$ and $z_i = \sigma_i z$. From (39) it follows that

$$\begin{aligned} ve_i + v(y_i - z_i) &= ve_i + v(\sigma_i y - \sigma_i z) = ve_i + v\sigma_i(y-z) \\ &> v\sigma_n(y-z) = v(\sigma_n y - \sigma_n z) = v(y_n - z_n) \end{aligned}$$

for all $y, z \in b + B$ and $0 \leq i < n$. Therefore, we can apply Lemma 37, and (42) shows that

$$v(f(\underline{y}) - f(\underline{z})) = vd_n\sigma_n(y-z) = vd_n + v\sigma_n(y-z) \tag{47}$$

for all $y, z \in b + B$. It follows that

$$gx := f(\sigma_0 x, \sigma_1 x, \ldots, \sigma_n x) \in b' + d_n B' \quad \text{for every } x \in b + B .$$

As in the proof of Proposition 35 we use Proposition 14 to show that $g : b+B \to b'+d_nB'$ is immediate. The proof that (6) and (7) hold can be taken over literally, except that instead of deducing (36) we just apply inequality (42) of Lemma 37 to obtain that

$$v(gy - gz - d_n\sigma_n(y-z)) > vd_n\sigma_n(y-z) .$$

Since σ_n is injective on $\mathcal{M}$ (as a consequence of condition (39)), it follows as in the proof of Proposition 35 that g is injective. □

Proposition 38 yields the following generalized Hensel's Lemma for the case of a Rosenlicht system of operators:

Theorem 39 *The assertion of Theorem 36 also holds under the assumptions of Proposition 38.*

7 Immediate differentiation

7.1 Weak D-fields

We will call a valued field (K, v) with an additive map $D : K \to K$ a **weak** D**-field** if the following conditions are satisfied:

(WDF1) $vDa \geq va$ for all $a \in K$,

(WDF2) $vK = \{va \mid a \in K \text{ with } vDa > va\}$,

(WDF3) there is $e \in \mathcal{O}$ such that $D(ab) = aDb + bDa + e(Da)(Db)$ for all $a, b \in K$.

Together with (WDF1), the additivity of D implies:

(WDF4) D induces an additive map on Kv, again denoted by D, such that $(Da)v = D(av)$,

Proposition 40 *Let (K, v, D) be a weak D-field. Then D is immediate if and only if D is surjective on Kv.*

Proof: "$\Rightarrow$": Take any $a' \in \mathcal{O}$; we have to show that $D(av) = a'v$ for some $a \in \mathcal{O}$. Condition (IH1) implies that there is $a \in K$ such that $v(a' - Da) > va' \geq 0$, whence $a'v = (Da)v = D(av)$.

"$\Leftarrow$": Take any $a' \in K \setminus \{0\}$. By (WDF2), we choose $c \in K$ such that $vc = va'$ with $vDc > vc$, and set $a'_0 = a'/c$. Then $va'_0 = 0$, and since D is surjective on Kv, there is some $a_0 \in \mathcal{O}$ such that $a'_0 v = D(a_0 v) = (Da_0)v$. Hence, $v(a'_0 - Da_0) > 0$. We set $a = ca_0$. We have that $va_0 Dc = va_0 + vDc \geq vDc > vc$ and $ve(Dc)(Da_0) = ve + vDc + vDa_0 \geq vDc > vc$. Hence,

$$\begin{aligned} v(a' - Da) &= v(ca'_0 - Dca_0) \;=\; v(ca'_0 - cDa_0 - a_0Dc - e(Dc)(Da_0)) \\ &\geq \min\{vc + v(a'_0 - Da_0)\,,\, va_0Dc\,,\, ve(Dc)(Da_0)\} \;>\; vc \;=\; va'\,. \end{aligned}$$

This shows that (IH1) holds. Since $D(a_0v) = a'_0 v \neq 0$, we know that $a_0 v \neq 0$, that is, $va_0 = 0$. Therefore, $vDa = va' = vc = vca_0 = va$. So we obtain from (WDF1) that $va \leq vb$ implies $vDa = va \leq vb \leq vDb$, for all $b \in K$. Hence, also (IH2) is satisfied. □

The next theorem is an immediate consequence of this proposition and Theorem 2.

Theorem 41 *Let (K, v, D) be a spherically complete weak D-field. Assume that D is surjective on Kv. Then D is surjective on K.*

As a preparation for our "D-Hensel's Lemma", we need the following facts:

Lemma 42 *In every weak D-field, $D1 = 0$.*

Proof: Suppose that $D1 \neq 0$. From (WDF3) with $b = 1$ we then obtain $eDa = -a$ for all $a \in K$. With $a = 1$ this yields $e = -(D1)^{-1}$, so $ve \leq 0$ since $vD1 \geq v1 = 0$ by (WDF1). But by (WDF2), $e \in \mathcal{O}$, so we get $ve = 0$. But then $eDa = -a$ shows that $vDa = va$ for all $a \in K$, in contradiction to (WDF2). □

Recall that by D^i we denote the i-th iterate of D, with D^0 being the identity map.

Lemma 43 *Let (K, v) be a weak D-field and $m \in K$ such that $vDm > vm$. Then*

$$v\left(D^i(ma) - mD^ia\right) \;>\; vma \tag{48}$$

for all $a \in K^\times$, and

$$vDm^{-1} \;>\; vm^{-1}\,. \tag{49}$$

Proof: By assumption, $vaDm = va + vDm > va + vm = vma$ and $ve(Dm)(Da) = ve + vDm + vDa \geq vDm + va > vm + va = vma$. Hence by (WDF3),

$$v\,(D(ma) - mDa) \geq \min\{vaDm\,,\, ve(Dm)(Da)\} \;>\; vma\,.$$

Now we proceed by induction on i. Suppose that $j > 1$ and that we have already shown (48) for all $i < j$ and all $a \in K$. Then

$$\begin{aligned} & v\left(D^j(ma) - mD^j a\right) \;=\\ &= \;v\left(DD^{j-1}(ma) - D(mD^{j-1}a) + D(mD^{j-1}a) - mDD^{j-1}a\right)\\ &\geq \;\min\{vD\left(D^{j-1}(ma) - mD^{j-1}a\right), v\left(D(mD^{j-1}a) - mDD^{j-1}a\right)\}\\ &> \;\min\{vma\,,\; vmD^{j-1}a\} \;=\; vma \end{aligned}$$

since $vD^{j-1}a \geq va$. This proves (48).

By Lemma 42 and (WDF3),

$$0 \;=\; D1 \;=\; D(mm^{-1}) \;=\; mDm^{-1} + m^{-1}Dm + e(Dm)(Dm^{-1})\;.$$

From this together with $veDm \geq vDm > vm$, we infer

$$vDm^{-1} \;=\; vm^{-1}Dm - v(m + eDm) \;=\; vm^{-1} + vDm - vm \;>\; vm^{-1}\;,$$

which proves (49). □

In every weak D-field, condition (V$\geq$) holds for the additive operators $\sigma_i = D^i$. This follows by induction on i (and we have used it already in the last proof). Again by induction on i, (WDF4) implies that

$$(D^i a)v \;=\; D^i(av) \quad \text{for every } i \geq 1\;, \tag{50}$$

that is, the map induced by D^i on Kv is the i-th iterate of the map induced by D on Kv. Indeed, having already shown that $(D^{i-1}a)v = D^{i-1}(av)$, we obtain $(D^i a)v = (D(D^i a))v = D((D^i a)v) = D(D^i(av)) = D^i(av)$.

Now we can prove the following theorem:

Theorem 44 *Let (K, v, D) be a spherically complete weak D-field. Take a polynomial $f \in \mathcal{O}[X_0, X_1, \ldots, X_n]$ and assume that*
1) there is $b \in \mathcal{O}$ and $s \in K$ with $vDs > vs$ such that

$$vs = \min_{0\leq i\leq n} v\frac{\partial f}{\partial X_i}(b, Db, \ldots, D^n b) < \infty \quad \textit{and} \quad vf(b, Db\ldots, D^n b) > 2vs\;,$$

2) the differential operator

$$\sum_{i=0}^{n} c_i D^i \quad \textit{with} \quad c_i = \left(s^{-1}\frac{\partial f}{\partial X_i}(b, Db\ldots, D^n b)\right)v \tag{51}$$

on the residue field Kv is surjective.
Then there is an element $a \in K$ such that $f(a, Da, \ldots, D^n a) = 0$ and $v(a - b) > vs$.

Proof: By (WDF2), we can choose elements m_α with $vm_\alpha = \alpha$ and $vDm_\alpha > vm_\alpha$ for $\alpha \in vK$; we set $m_0 = 1$. By Lemma 31, this gives rise to a weak coefficient map co. Inequality (48) of Lemma 43 shows that condition (29) of Lemma 32 holds for the elements m_α and the additive operators $\sigma_i = D^i$. Therefore, co satisfies (30) for these operators. Since $vDs > vs$, inequality (49) of Lemma 43 shows that $vDs^{-1} > vs^{-1}$. Thus, we can choose $m_{vs} = s$ and obtain that $\mathrm{co}\, a = (s^{-1}a)v$ whenever $va = vs$. With d_i defined as in Proposition 30, we thus obtain that the elements c_i defined above coincide with the elements c_i defined in Proposition 33 and that the operator $\sum_{i=0}^n c_i D^i$ coincides with the operator $\sum_{i=0}^n c_i \overline{\sigma}_i$ of Proposition 33. The former being surjective on Kv, our theorem now follows from Theorem 34. □

This theorem yields Theorem 5. Indeed, if the assumptions of that theorem are satisfied, then by use of (WDF2) we pick $s \in K$ with $vDs > vs$ such that $vs = \gamma$. Since Kv is assumed to be linearly D-closed, the operator (51) on Kv is surjective, and we can apply Theorem 44.

7.2 Integration on Rosenlicht valued differential fields

Let (K, D) be a differential field with field of constants $C = \{a \in K \mid Da = 0\}$. Following M. Rosenlicht [R1], a valuation v of K is called a **differential valuation** if C is a field of representatives for the residue field of (K, v) (that is, v is trivial on C and for every $y \in K$ with $vy = 0$ there is a unique $c \in C$ s.t. $v(y - c) > 0$), and v satisfies

$$\forall a, b \in K : \ va \geq 0 \wedge vb > 0 \wedge b \neq 0 \ \Rightarrow \ v\left(\frac{bDa}{Db}\right) > 0 \,. \tag{52}$$

Because of our assumption on C, this condition is equivalent to

$$\forall a, b \in K \setminus \{0\}, va \neq 0, vb \neq 0 : \ va \leq vb \ \Leftrightarrow \ vDa \leq vDb \,. \tag{53}$$

Lemma 45 *Assume that v is a differential valuation with respect to D. Then for every $\tilde{a} \in K$ there is some $a \in K$ such that $va \neq 0$ and $Da = D\tilde{a}$. Moreover,*

$$\forall a, b \in K : \ (0 \neq va \wedge va \leq vb) \ \Rightarrow \ vDa \leq vDb \,. \tag{54}$$

This shows that $\{a \in K \mid va \neq 0\} \subseteq \mathrm{Reg}\,(D)$.

Proof: If $v\tilde{a} = 0$ then by our assumption that the field of constants is a field of representatives for the residue field, there is some constant c such that $v(\tilde{a} - c) > 0$; hence for $a := \tilde{a} - c$ we have that $va \neq 0$ and $Da = D\tilde{a} - Dc = D\tilde{a}$.

To prove (54), assume that $0 \neq va$ and $va \leq vb$. If $vb = 0$, then we choose a constant c such that $v(b - c) > 0$. So we can infer from (53) that $vDa \leq vD(b - c) = vDb$. □

Proposition 46 *Let v be a differential valuation on (K,D). Then $D:\ (K,v)\to(K,v)$ is immediate if and only if (K,D,v) admits asymptotic integration.*

Proof: "⇒": Condition (IH1) implies that (K,D,v) admits asymptotic integration.
"⇐": Take any $a'\in K\setminus\{0\}$. Since (K,D,v) admits asymptotic integration, there is some $a\in K$ such that $v(a'-Da)>va'$, that is, (IH1) holds. By Lemma 45, a can be chosen such that $va\neq 0$ and (IH2) holds. □

The next theorem is an immediate consequence of this proposition and Theorem 2.

Theorem 47 *Let (K,D) be a differential field, endowed with a spherically complete differential valuation v. Assume further that (K,D) admits asymptotic integration. Then (K,D) admits integration.*

For certain applications, one has to work with a field K which is a union of an increasing sequence of power series fields K_i, $i\in\mathbb{N}$. If this sequence does not become stationary, then K itself will not be spherically complete. However, we still can prove the following:

Theorem 48 *Let (K,v) be the union of an increasing chain (K_i,v) of spherically complete valued fields, $i\in\mathbb{N}$. Let D be a derivation on K such that v is a differential valuation with respect to D. Assume further that for each i there are elements $a_{i,j}\in K_{i+1}$, $j\in I_i$, such that*
1) $Da_{i,j}\in K_i$ for all $j\in I_i$,
2) the valued K_i-subvector space $V_i:=K_i+\sum_{j\in I_i}K_ia_{i,j}$ of K_{i+1} is spherically complete,
3) for every $b\in K_i$ there is some $a\in V_i$ such that $v(b-Da)>vb$.
Then (K,D) admits integration.

Proof: It suffices to show that for each i, D is a surjective map from V_i onto K_i. Since $K=\bigcup_{i\in\mathbb{N}}K_i$ it then follows that D is surjective on K.

Because of 1), we have that $DV_i\subseteq K_i$. We set $Y=V_i$ and $Y'=K_i$. As in the proof of Proposition 46 one uses 3) to show that $D:Y\to Y'$ is immediate. From 2) together with Theorem 2, one obtains that $DV_i=K_i$. □

This theorem implies that the derivation on the logarithmic-exponential power series field $\mathbb{R}((t))^{LE}$ (cf. [DMM3]) is surjective. The argument is as follows. It can be shown that $\mathbb{R}((t))^{LE}$ is the union over an increasing sequence of differential power series fields K_i such that for every i there is just one $a_i\in K_{i+1}$ such that $Da_i\in K_i$ and condition 3) holds. In fact, $a_i=\log_i x$ for a certain element x, where $\log_i$ denotes the i-th iterate of log. Further, va_i is rationally independent over vK. It follows that $v(c+c'a_i)=\min\{vc,vc'a_i\}$ for all $c,c'\in K_i$, that is, the ultrametric space underlying V_i is just the direct product of the one underlying K_i and the one underlying K_ia_i. As the latter is isomorphic to the one underlying K_i, both are spherically complete. By Proposition 10, their direct product is spherically complete. The foregoing theorem now proves the surjectivity of D.

7.3 Differential equations on Rosenlicht valued differential fields

Now let us assume in addition that

$$D(\mathcal{M}) \subseteq \mathcal{M} . \tag{55}$$

If K contains an element x such that $vDx = 0$ and $vx < 0$ (as it is the case in $\mathbb{R}((t))^{LE}$, see below), then (55) is a consequence of (53). In fact, (55) also holds in every Hardy field. If (55) does not hold for a derivation D, then we may replace D by the derivation aD, with $0 \neq a \in K$; it follows from (53) that (55) will hold for aD in the place of D for every a of sufficiently high value va.

Assumption (55) implies that $D^i(\mathcal{M}) \subseteq \mathcal{M}$ for each $i \in \mathbb{N}$. We leave it to the reader to use this fact together with (53) to prove the following easy lemma by induction on i:

Lemma 49 *If (K, v, D) admits asymptotic integration, then for each $i \in \mathbb{N}$, the map*

$$D^i : \mathcal{M} \longrightarrow \mathcal{M}_{D^i} := \bigcup_{e \in \mathcal{M}} (D^i e)\mathcal{O} \subseteq \mathcal{M} \tag{56}$$

is an immediate embedding of ultrametric spaces with value map $va \mapsto vD^i a$.

Hence by Theorem 2, we have:

Lemma 50 *If (K, D, v) is spherically complete and admits asymptotic integration, then the map (56) is an isomorphism of ultrametric spaces.*

When we try to prove a differential Hensel's Lemma for Rosenlicht's differential valuations, we we have to deal with the problem that the connection between $vD^i a$ and $vD^j a$ for $i \neq j$ is not as nice as in the case of D-fields. The natural hypothesis on the partial derivatives as used in Theorem 5 may not suffice. We need to set up a relation between the values $vy, vDy, \ldots, vD^n y$. The key is definition (52) of a differential valuation. By induction, it implies that for arbitrary $e \in \mathcal{M}$,

$$vD^i y + (n-i)vDe \; > \; vD^n y \qquad \text{for } 0 \leq i < n . \tag{57}$$

Because of this relation, we will have to assume that the partial derivative of least value appears at the variable X_n which is associated with the highest power D^n of D. The following is a special case of Theorem 36 in Section 6.3:

Theorem 51 *Let (K, D) be a differential field, endowed with a spherically complete differential valuation v. Assume that (K, v, D) admits asymptotic integration. Take a polynomial $g \in \mathcal{O}[X_0, X_1, \ldots, X_n]$ and assume that there are $b \in \mathcal{O}$ and $e \in \mathcal{M}$ such that, with $d := De$,*

$$g(d^{-n}X_0, d^{1-n}X_1, \ldots, d^{-1}X_{n-1}, X_n) \in \mathcal{O}[X_0, X_1, \ldots, X_n]$$

and

$$v\frac{\partial g}{\partial X_n}(b, Db, \ldots, D^n b) \; = \; \min_{0 \leq i \leq n} vd^{i-n}\frac{\partial g}{\partial X_i}(b, Db, \ldots, D^n b) \; = \; 0 \tag{58}$$

and

$$vg(b, Db \ldots, D^n b) \;\geq\; vD^n e\,. \tag{59}$$

Then there is a unique element $a \in \mathcal{O}$ such that $g(a, Da, \ldots, D^n a) = 0$. It satisfies $v(a-b) \geq ve$.

Proof: Set $f(X_0, \ldots, X_n) = g(d^{-n}X_0, d^{1-n}X_1, \ldots, d^{-1}X_{n-1}, X_n) \in \mathcal{O}[X_0, X_1, \ldots, X_n]$. With d_i defined as in (25) of Proposition 30, it follows from (58) that $0 = vd_n = \min_i vd_i$, which shows that (32) of Proposition 35 is satisfied. Further, we set $\sigma_i := d^{n-i}D^i$, $B := \mathcal{M}$ and $B' := \mathcal{M}_{D^n} \subseteq \mathcal{M} = d_n\mathcal{M}$. Then by (57), $v\sigma_n a < v\sigma_i a$ for all $i < n$ and $a \in \mathcal{M}$, showing that (31) holds. Since $\sigma_n(\mathcal{M}) = D^n(\mathcal{M}) \subseteq B' \subseteq \mathcal{M}$, it follows that also $\sigma_i(\mathcal{M}) \subseteq \mathcal{M}$ for $0 \leq i \leq n$. Condition (59) tells us that condition (37) of Theorem 36 is satisfied. Finally, Lemma 49 tells us that $D^n : \mathcal{M} \to B'$ is immediate and injective. We have proved that all conditions of Theorem 36 are satisfied. Hence, there is a unique element $a \in b+\mathcal{M}$ such that $g(a, Da, \ldots, D^n a) = f(\sigma_0 a, \sigma_1 a, \ldots, \sigma_n a) = 0$, and it satisfies $v\sigma_n(a-b) \geq v\sigma_n e$. The latter means that $vD^n(a-b) \geq vD^n e$, which by (53) implies $v(a-b) \geq ve$ since $a-b\,,\, e \in \mathcal{M}$. □

This theorem can be improved if one also considers the values of the higher derivatives of f. The formal higher derivatives $f^{[\underline{i}]}$ have already been introduced and used in Section 6.1. We will work with the Rosenlicht system

$$\sigma_i \;:=\; D^i\,, \quad e_i \;:=\; (De)^{n-i}$$

for fixed $n \in \mathbb{N}$ and some $e \in \mathcal{M}$. Then condition (38) in Section 6.4 is trivially satisfied, and condition (39) is satisfied because of (57). We will apply Theorem 39 to prove:

Proposition 52 *Take $f \in \mathcal{O}[X_0, X_1, \ldots, X_n]$ and $b \in \mathcal{O}$ such that*

$$d_n \;=\; \frac{\partial f}{\partial X_n}(b, Db, \ldots, D^n b) \;\neq\; 0$$

and for all $\underline{i} \in I = \{0, \ldots, \deg f\}^{n+1} \setminus \{(0, \ldots, 0)\}$,

$$vf^{[\underline{i}]}(b, Db, \ldots, D^n b) \;\geq\; vd_n + ve_k \quad \textit{if} \;\; k = \min\{j \mid i_j \neq 0\}\,. \tag{60}$$

Suppose further that for some balls $B, B' \subseteq \mathcal{M}$ around 0, $D^n : B \to B'$ is immediate. Then

$$b + B \ni x \mapsto f(x, Dx, \ldots, D^n x) \in f(b, Db, \ldots, D^n nb) + d_n B' \tag{61}$$

is an immediate embedding of ultrametric spaces with value map $va \mapsto vd_n + vD^n a$. If (K, v) is spherically complete, it is an isomorphism of ultrametric spaces.

Proof: All this follows from Proposition 38 and Theorem 2. It just remains to prove that (61) is an embedding of ultrametric spaces with value map $va \mapsto vd_n + vD^n a$. But this follows from equation (47) of Proposition 38 and the fact that $va \mapsto vD^n a$ preserves $<$ for $a \in \mathcal{M}$. □

Theorem 53 *Let (K,D) be a differential field, endowed with a spherically complete differential valuation v. Assume that (K,v,D) admits asymptotic integration. Take a polynomial $f\in\mathcal{O}[X_0,X_1,\ldots,X_n]$ and assume that there are $b\in\mathcal{O}$ and $e\in\mathcal{M}$ such that*

$$vf^{[\underline{i}]}(b,Db,\ldots,D^nb)\;\geq\;v\frac{\partial f}{\partial X_n}(b,Db,\ldots,D^nb)+(n-k)vDe\quad \text{if}\;\; k=\min\{j\mid i_j\neq 0\} \tag{62}$$

and

$$vf(b,Db\ldots,D^nb)\;\geq\;vD^ne\;.$$

Then there is a unique element $a\in\mathcal{M}$ such that $f(a,Da,\ldots,D^na)=0$. It satisfies $v(a-b)\geq ve$.

Proof: As in the proof of Theorem 51, we set $B:=\mathcal{M}$ and $B':=\mathcal{M}_{D^n}$. But now we apply Theorem 39 instead of Theorem 36. □

If K is of characteristic 0, then the usual higher derivative

$$f^{(\underline{i})}(x)\;:=\;\frac{\partial^{i_0+\ldots+i_n}f}{\partial^{i_0}X_0\cdots\partial^{i_n}X_n}(x)$$

can be substituted for $f^{(\underline{i})}(\underline{x})$ in the above theorem. Indeed,

$$f^{(\underline{i})}(\underline{x})\;=\;i_0!\cdot\ldots\cdot i_n!\cdot f^{[\underline{i}]}(\underline{x})$$

and therefore,

$$vf^{(\underline{i})}(b,Db,\ldots,D^nb)\;=\;vf^{[\underline{i}]}(b,Db,\ldots,D^nb)\;.$$

In $\mathbb{R}((t))^{LE}$, the element $x=t^{-1}$ satisfies $vx<0$ and $Dx=1$. Suppose that $1<r\in\mathbb{R}$. Then $e=\frac{1}{1-r}x^{1-r}\in\mathbb{R}((t))^{LE}$ satisfies $ve>0$ and $De=x^{-r}$. With K_i as in the discussion at the end of Section 7.2, take $\mathcal{M}_i$ to be the valuation ideal of K_i. Then $\frac{1}{x}\notin(De)\mathcal{M}_i$ and it can be shown that for every $a'\in(De)\mathcal{M}_i$ there is some $a\in e\mathcal{M}_i$ such that $v(a'-Da)>va'$. As for the proof of Lemma 49, it can thus be deduced that for every $k\geq 1$, $D^k:e\mathcal{M}_i\to(D^ke)\mathcal{M}_i$ is an immediate embedding of ultrametric spaces. Hence on every ball of the form $e\mathcal{M}_i$ in K_i, differential equations of the above form can be solved without any modification of our approach.

The union of an ascending chain of henselian fields is again henselian. With the same idea of proof, working in K_i for all i large enough to contain all coefficients of h and then passing to the union of the K_i, one obtains, applying Theorem 53 with e as given above to the polynomial $f(X_0,\ldots,X_n)=g(X_0,\ldots,X_n)+c-X_n$ and $b=0$:

Theorem 54 *Let $\mathcal{O}$ denote the valuation ring of $\mathbb{R}((t))^{LE}$. Suppose that*

$$g(X_0,\ldots,X_n)\;\in\;\sum_{i=0}^{n-1}x^{-(n-i)r}X_i\,\mathcal{O}[X_i\,,\;\ldots\;,\;X_n]\;+\;X_n^2\mathcal{O}\;+\;X_n\mathcal{M} \tag{63}$$

and

$$c \in x^{-r-n+1}\,\mathcal{O}\ .$$

Then the differential equation

$$D^n y \;=\; g(y, Dy, \ldots, D^n y) \;+\; c \tag{64}$$

has a unique infinitesimal solution in $\mathbb{R}((t))^{LE}$*; this solution has value* $\geq vx^{1-r}$*.*

This theorem implies the following result, which was proved by Lou van den Dries in [D]:

Corollary 55 *Suppose that* p *is a polynomial in one variable with coefficients in* $\mathbb{R}((t))^{LE}$*, all of value* $\geq vt^r$ *for some* $r \in \mathbb{R}$*,* $r > 1$*. Then the differential equation*

$$Dy \;=\; p(y)$$

has a unique infinitesimal solution in $\mathbb{R}((t))^{LE}$*.*

8 Sums of spherically complete valued abelian groups

Let $(\mathcal{A}, v)$ be a valued abelian group and $A_1, \ldots, A_n$ be subgroups of $\mathcal{A}$. The restrictions of v to every A_i will again be denoted by v. We call the sum $A_1 + \ldots + A_n \subseteq \mathcal{A}$ **pseudo-direct** if for every $a' \in A_1 + \ldots + A_n$, $a' \neq 0$, there are $a_i \in A_i$ such that

$$v\sum_{i=1}^{n} a_i \;=\; \min_{1\leq i\leq n} va_i \quad\text{and}\quad v\left(a' - \sum_{i=1}^{n} a_i\right) \;>\; va'\ . \tag{65}$$

Proposition 56 *The sum* $A_1 + \ldots + A_n \subseteq \mathcal{A}$ *is pseudo-direct if and only if the group homomorphism* $f : A_1 \times \ldots \times A_n \to A_1 + \ldots + A_n$ *defined by* $f(a_1, \ldots, a_n) := a_1 + \ldots + a_n$ *is immediate.*

Proof: $\Rightarrow$: Assume that the sum $A_1 + \ldots + A_n$ is pseudo-direct. Take any $a' \in \sum_i A_i$ and choose $a_i \in A_i$ such that (65) holds. Then $a := (a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ satisfies (IH1). If $b = (b_1, \ldots, b_n) \in A_1 \times \ldots \times A_n$ such that $vb \geq va$, then

$$vfb \;=\; v\sum_i b_i \;\geq\; \min_i vb_i \;=\; vb \;\geq\; va \;=\; \min_i va_i \;=\; v\sum_i a_i \;=\; vfa\ .$$

This shows that a also satisfies (IH2).

$\Leftarrow$: Assume that f is immediate. Take any $a' \in \sum_i A_i$, $a' \neq 0$. Choose $a := (a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ such that (IH1) and (IH2) hold. Then $v\,(a' - \sum_i a_i) = v(a' - fa) > va'$. Now choose some j such that $va_j = \min_i va_i$. Then set $b_j = a_j \in A_j$ and $b_i = 0 \in A_i$ for $i \neq j$. For $b = (b_1, \ldots, b_n)$, we thus have that $va = \min_i va_i = va_j = vb_j = \min_i vb_i = vb$. Hence by (IH2), $v\sum_i a_i = vfa \leq vfb = vb_j = \min_i va_i$. We have proved that the elements a_i satisfy (65). $\square$

If the groups (A_i, v) are spherically complete, then by Proposition 10, the same is true for their direct product $A := A_1 \times \ldots \times A_n$, endowed with the minimum valuation as defined in (10). Hence, the foregoing proposition, Theorem 2 and Corollary 4 show:

Theorem 57 *Assume that the subgroups (A_i, v) of $(\mathcal{A}, v)$, $1 \leq i \leq n$, are spherically complete. If the sum $A_1 + \ldots + A_n$ is pseudo-direct, then it is also spherically complete and has the optimal approximation property.*

References

[D] van den Dries, L.: *Solutions of ODE's in $\mathbb{R}((t))^{LE}$ as fixed points*, manuscript (November 1997)

[DMM1] van den Dries, L. – Macintyre, A. – Marker, D.: *The elementary theory of restricted analytic functions with exponentiation*, Annals Math. **140** (1994), 183–205

[DMM2] van den Dries, L. – Macintyre, A. – Marker, D.: *Logarithmic-Exponential Power Series*, J. London Math. Soc. **56** (1997), 417–434

[DMM3] van den Dries, L. – Macintyre, A. – Marker, D.: *Logarithmic-Exponential Series*, Ann. Pure Appl. Logic **111** (2001), 61–113

[DS] van den Dries, L. – Speissegger, P.: *The real field with convergent generalized power series is model complete and o-minimal*, Trans. Amer. Math. Soc. **350** (1998), 4377–4421

[G] Gravett, K. A. H.: *Note on a result of Krull*, Cambridge Philos. Soc. Proc. **52** (1956), 379

[KA] Kaplansky, I.: *Maximal fields with valuations I*, Duke Math. J. **9** (1942), 303–321

[KR] Krull, W.: *Allgemeine Bewertungstheorie*, J. reine angew. Math. **167** (1931), 160–196

[KU1] Kuhlmann, F.-V.: *A theorem on spherically complete valued abelian groups*, The Fields Institute Preprint Series, Toronto (June 1997)

[KU2] Kuhlmann, F.-V.: *Elementary properties of power series fields over finite fields*, J. Symb. Logic **66** (2001), 771-791

[KU3] Kuhlmann, F.-V.: *Additive Polynomials and Their Role in the Model Theory of Valued Fields*, in: Proceedings of the Workshop and Conference on Logic, Algebra, and Arithmetic, Tehran, Iran 2003, Lecture Notes in Logic, Vol. **13**, Association for Symbolic Logic, AK Peters

[KU4] Kuhlmann, F.-V.: Book in preparation. Preliminary versions of several chapters available at:
`http://math.usask.ca/~fvk/Fvkbook.htm`

[L] Lang, S.: *Algebra*, Addison-Wesley, New York (1965)

[P1] Prieß-Crampe, S.: *Angeordnete Strukturen. Gruppen, Körper, projektive Ebenen*, Ergebnisse der Mathematik und ihrer Grenzgebiete **98**, Springer (1983)

[P2] Prieß-Crampe, S.: *Der Banachsche Fixpunktsatz für ultrametrische Räume*, Results in Mathematics **18** (1990), 178–186

[PR1] Prieß-Crampe, S. – Ribenboim, P.: *Fixed Points, Combs and Generalized Power Series*, Abh. Math. Sem. Hamburg **63** (1993), 227-244

[PR2] Prieß-Crampe, S. – Ribenboim, P.: *Fixed Point and Attractor Theorems for Ultrametric Spaces*, Formum Math. **12** (2000), 53–64

[PR3] Prieß-Crampe, S. – Ribenboim, P.: *Differential equations over valued fields (and more)*, J. Reine Angew. Math. **576** (2004), 123–147

[RB] Robba, P.: *Lemmes de Hensel pour les opérateur différentiels. Application a la réduction formelle des équations différentielles.*, Enseign. Math. (2) **26** (1980), 279–311

[R1] Rosenlicht, M.: *Differential valuations*, Pacific J. Math. **86** (1980), 301–319

[R2] Rosenlicht, M.: *On the value group of a differential valuation*, Amer. J. Math. **191** (1979), 258–266

[S1] Scanlon, T.: *Model theory of valued D-fields*, Ph.D. thesis, Harvard University (1997)

[S2] Scanlon, T.: *A model complete theory of valued D-fields*, J. Symb. Logic **65** (2000), 1758–1784

[SCH] Schörner, E.: *Ultrametric Fixed Point Theorems and Applications*, in: Valuation Theory and its Applications, Volume II, Proceedings of the Valuation Theory Conference Saskatoon 1999, eds. F.-V. Kuhlmann, S. Kuhlmann and M. Marshall, The Fields Institute Communications Series **33**, Publications of the Amer. Math. Soc. (2003), 353–359

[T] Teissier, B.: *Valuations, Deformations, and Toric Geometry*, in: Valuation Theory and its Applications, Volume II, Proceedings of the Valuation Theory Conference Saskatoon 1999, eds. F.-V. Kuhlmann, S. Kuhlmann and M. Marshall, The Fields Institute Communications Series **33**, Publications of the Amer. Math. Soc. (2003), 361–459

[VC] Veluscek, D. – Cimpric, J.: *Higher product Pythagoras numbers of skew fields*, preprint. Available at:

`http://www.fmf.uni-lj.si/srag/preprints/pitag_final1_popravki.ps`

Mathematical Sciences Group, University of Saskatchewan,
106 Wiggins Road, Saskatoon, Saskatchewan, Canada S7N 5E6
email: fvk@math.usask.ca — home page: http://math.usask.ca/˜fvk/index.html