ON FIELDS AND COLORS

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Abstract. We exhibit a simplified version of the construction of a field of Morley rank p with a predicate of rank $p-1$, extracting the main ideas for the construction from previous papers and refining the arguments. Moreover, an explicit axiomatization is given, and ranks are computed.

1. Introduction

Zil'ber posed the question whether or not every strongly minimal set whose geometry was not locally modular arose from an algebraic curve over an algebraically closed field. The conjecture, true in the case of Zariski Geometries [8], remained open until E. Hrushovski [7] refuted it developing a procedure, taking ideas from Fraïssé, in order to construct countable structures with a richer and more complicated geometry starting from simpler ones. Moreover, he was able to merge two algebraically closed fields of different characteristics into one strongly minimal set [6]. This procedure was later adapted by Poizat [11] to obtain an algebraically closed field of any given characteristic with a predicate (whose elements were called *black*, after some considerations on the *political correctness* of such a choice of terminology) such that the field has Morley rank $\omega 2$ and the black points ω. He then used Hrushovski's collapsing method and produced "rich" fields of rank 2 with black points of rank 1, provided the rich field is ω–saturated. A proof of ω–saturation was supplied by Baldwin and Holland ([1]). Poizat and Baldwin & Holland also explained how to obtain fields of rank p with a predicate of rank 1 and $p-1$, respectively.

The main goal of this work is to give a complete self-contained proof of the above facts simplifying as much as possible the arguments. One of the novelties of this work is exhibiting an axiomatization for the resulting theory, obtained by direct translation of Hrushovski's fusion article [6] to the case of colored fields. Actually, we use a simplified version (see [12]) of the aforementioned article, following the spirit of Poizat's black points.

All throughout this work a saturated enough algebraically closed field $\mathbb{C}$ of some given characteristic q and a natural number $p \geq 2$. We will prove the following:

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Main Theorem ([11],[1])**.** $\mathbb{C}$ has a subset N such that $(\mathbb{C}, N)$ has Morley rank p and N has Morley rank $p-1$.

This paper is structured as follows: We first consider finite partial substructures of $\mathbb{C}$ with some points colored in black. A δ function is introduced, and Hrushovski's codes [6] are used to described minimal extension (with a small correction from their original definition). The number of certain such extensions is bounded with a μ function. In this case, we can proceed with the collapse, and the resulting structure is a rich field as in [11]. We show that rich fields are exactly the ω-saturated models of a given theory, whose axioms are explictly given. Finally, we compute the Morley rank in terms of δ.

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2. Codes

In this section, we work exclusively inside $\mathbb{C}$. All formulae are L–formulas, where L is the ring language

Definition 2.1. A *code* α is a tuple consisting of the following objects: Natural numbers n_α, m_α, k_α and formulae $\varphi_\alpha(\vec{x}, \vec{y})$ and $\psi_\alpha(\vec{x}_1, \ldots, \vec{x}_{m_\alpha}, \vec{y})$ such that the following holds (We will write $\theta_\alpha(\vec{y}) = \exists \vec{x}\, \varphi_\alpha(\vec{x}, \vec{y})$):

(i) $\text{length}(\vec{x}) = \text{length}(\vec{x}_i) = n_\alpha$

(ii) If $\models \theta_\alpha(\vec{b})$, then $\varphi_\alpha(\vec{x}, \vec{b})$ has Morley rank k_α and degree 1.

(iii) Let $\vec{a} \models \varphi_\alpha(\vec{x}, \vec{b})$ be generic. For $s \subset \{1, \ldots, n_\alpha\}$, write $a_s = \{a_j\}_{j \in s}$. Then, for every $i \leq n_\alpha$ and $\vec{a}' \models \varphi_\alpha(\vec{x}, \vec{b}')$, we have that:

$$\begin{aligned} a_i \in \text{acl}(a_s, \vec{b}) &\implies a'_i \in \text{acl}(a'_s, \vec{b}') \\ a_i \in a_s\vec{b} &\implies a'_i \in a'_s\vec{b}' \\ a_i \notin a_s\vec{b} &\implies a'_i \notin a'_s\vec{b}' \end{aligned}$$

(iv) If $\models \theta_\alpha(\vec{b})$, then $\;\text{MR}\big(\varphi_\alpha(\vec{x}, \vec{b}) \vartriangle \varphi_\alpha(\vec{x}, \vec{b}')\big) < k_\alpha \implies \vec{b} = \vec{b}'$.

(v) $\models \psi_\alpha(\vec{x}_1, \ldots, \vec{x}_{m_\alpha}, \vec{b})$ implies that $\vec{b} \in \text{dcl}(\vec{x}_1, \ldots, \vec{x}_{m_\alpha})$.[1]

(vi) $\psi_\alpha(\vec{x}_1, \ldots, \vec{x}_{m_\alpha}, \vec{b})$ is consistent for all $\vec{b} \models \theta_\alpha(\vec{y})$.

(vii) Given $(\vec{a}_1, \ldots, \vec{a}_{m_\alpha}, \vec{b}) \models \psi_\alpha$ and a generic $\vec{a}'$ realizing $\varphi_\alpha(\vec{x}, \vec{b})$, it follows that

$$\bigwedge_{i=1}^{m_\alpha} \psi_\alpha(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{i-1}, \vec{a}', \vec{a}_{i+1}, \ldots, \vec{a}_{m_\alpha}, \vec{b})$$

holds.

Lemma 2.2. *If $\models \theta_\alpha(\vec{b})$, we have that $\vec{b}$ is a canonical basis of the type of Morley rank k_α determined by $\varphi_\alpha(\vec{x}, \vec{b})$.*

Proof. This follows immediately from (iv). □

[1]Note that the definable closure $\text{dcl}(\vec{x})$ is the perfect hull of the field generated by $\vec{x}$.

Lemma 2.3. *For each definable set X of n-tuples of Morley rank k and degree 1 there is a code α with $n_\alpha = n$, $k_\alpha = k$ and some $\vec{b} \models \theta_\alpha(\vec{y})$ such that $\mathrm{MR}\big(\varphi_\alpha(\vec{x}, \vec{b}) \vartriangle X\big) < k_\alpha$.*

Proof. Let X be given. We begin with a formula $\varphi(\vec{x}, \vec{b}_0)$ such that $\vec{b}_0$ is a canonical base of the type determined by X and such that $\mathrm{MR}\big(\varphi_\alpha(\vec{x}, \vec{b}_0) \vartriangle X\big) < k$. Since Morley rank and degree are definable in algebraically closed fields we may assume that (ii) holds. If, in addition, $\varphi(\vec{x}, \vec{b}_0)$ witnesses all algebraic dependencies and equalities between the components of a generic solution, property (iii) holds also. Now, $\vec{b}_0$ is a canonical base if and only if $\mathrm{MR}\big(\varphi_\alpha(\vec{x}, \vec{b}_0) \vartriangle \varphi_\alpha(\vec{x}, \vec{b}')\big) < k \ \rightarrow\ \vec{b}_0 = \vec{b}'$ for every $\vec{b}' \models \mathrm{tp}(\vec{b}_0)$. Thus (iv) is true after adding some finite part of $q(\vec{y}) = \mathrm{tp}(\vec{b}_0)$ to $\varphi(\vec{x}, \vec{y})$.

Choose generic realizations $\vec{a}_1, \ldots, \vec{a}_m$ of $\varphi(\vec{x}, \vec{b}_0)$, independent over $\vec{b}_0$. If m is large enough, we have $\vec{b}_0 = f(\vec{a}_1, \ldots, \vec{a}_m)$ for some 0–definable function f. We strengthen φ, so that $\vec{b} = f(\vec{x}_1, \ldots, \vec{x}_m)$ for every sequence $\vec{x}_1, \ldots, \vec{x}_m$ of independent generic realizations of $\varphi(\vec{x}, \vec{b})$. Finally let $\psi(\vec{x}_1, \ldots, \vec{x}_m, \vec{b})$ express

> "For every sequence $\vec{x}_{m+1}, \ldots, \vec{x}_{2m}$ of generic, independent realizations of $\varphi(\vec{x}, \vec{b})$ and every choice of distinct indices $i_1, \ldots, i_m \in \{1, \ldots, 2m\}$, we have $\vec{b} = f(\vec{x}_{i_1}, \ldots, \vec{x}_{i_m})$". □

Let α be a code and σ a permutation of $\{1, \ldots, n_\alpha\}$. We denote by α^σ the code obtained from α by permuting each of the tuples $\vec{x}$, $\vec{x}_1, \ldots, \vec{x}_{m_\alpha}$ in φ_α and ψ_α according to σ.

Definition 2.4. Two codes α and α' are equivalent if $n_\alpha = n_{\alpha'}$ and $m_\alpha = m_{\alpha'}$ and

- for all realizations $\vec{b}$ of θ_α there is a tuple $\vec{b}'$ such that (in $\mathbb{C}$) $\varphi_\alpha(\vec{x}, \vec{b}) \equiv \varphi_{\alpha'}(\vec{x}, \vec{b}')$ and $\psi_\alpha(\vec{x}_1, \ldots, \vec{x}_{m_\alpha}, \vec{b}) \equiv \psi_{\alpha'}(\vec{x}_1, \ldots, \vec{x}_{m_\alpha}, \vec{b}')$.
- the same replacing the roles of α and α'.

The following lemma is a slightly weaker as the statement of Lemma 2 in [6].

Lemma 2.5. *There is a set $\mathcal{C}$ of codes such that*

(viii) *For each (non–empty) definable set X of Morley degree 1 there is a code $\alpha \in \mathcal{C}$ and some $\vec{b}$ such that $\mathrm{MR}\big(\varphi_\alpha(\vec{x}, \vec{b}) \vartriangle X\big) < k_\alpha$.*

(ix) *If $\alpha, \alpha' \in \mathcal{C}$, $\models \theta_\alpha(\vec{b})$ and $\mathrm{MR}\big(\varphi_\alpha(\vec{x}, \vec{b}) \vartriangle \varphi_{\alpha'}(\vec{x}, \vec{b}')\big) < k_\alpha$, then $\alpha' = \alpha$.*[2]

(x) *If α belongs to $\mathcal{C}$, then each permutation of α is equivalent to a code in $\mathcal{C}$.*

Proof. We refer to the claim of (viii) as "X can be coded by α". List all non–empty definable sets of degree 1 up to conjugation by automorphisms of $\mathbb{C}$ by $X_1, X_2, \ldots$ This is possible since ACF_q is small, i.e. it has only countably many n-types for each n. It is enough to show that each X_i can be coded by some elements of $\mathcal{C}$. We will obtain $\mathcal{C}$ as the union of a sequence $\emptyset = C_0 \subset C_1 \subset \cdots$ of finite sets of codes, constructed as follows. Assume that C_{i-1} has been constructed and it is closed under permutations in the weak sense of (x). If X_i can be coded by an element of C_{i-1}, we set $C_i = C_{i-1}$. Otherwise, choose a code α and $\vec{b}_0$ such that

[2] We identify two codes if there defining formulas are equivalent in $\mathbb{C}$.

$\mathrm{MR}\big(\varphi_\alpha(\vec{x},\vec{b}_0) \vartriangle X\big) < k_\alpha$. We replace φ_α by

$$\varphi_\alpha(\vec{x},\vec{y}) \wedge \text{“}\{\vec{x} \mid \varphi_\alpha(\vec{x},\vec{y})\} \text{ cannot be coded by an element of } C_{i-1}\text{”}.$$

and obtain a new code, which still codes X_i. We may assume that no permutation of α can code a set which can also be coded by a code in C_{i-1}. Let G be the group of all $\sigma \in \mathrm{Sym}(n_\alpha)$ such that

$$\mathrm{MR}\big(\varphi_\alpha(\vec{x},\vec{b}_0) \vartriangle \varphi_{\alpha^\sigma}(\vec{x},\sigma\vec{b}_0)\big) < k_\alpha$$

for some element denoted as $\sigma\vec{b}_0$ which has the same type as $\vec{b}_0$. After adding a finite part of the type of $\vec{b}_0$ to $\varphi_\alpha(\vec{x},\vec{y})$ we may assume that for all realizations $\vec{b}$ of θ_α and all σ, there exists $\sigma\vec{b}$ with $\mathrm{MR}\big(\varphi_\alpha(\vec{x},\vec{b}) \vartriangle \varphi_{\alpha^\sigma}(\vec{x},\sigma\vec{b})\big) < k_\alpha$ iff $\sigma \in G$. Note that $\sigma\vec{b}$ is a $\emptyset$-definable function of $\vec{b}$. If we let permutations act on the right on codes, this defines a left action of G on $\theta_\alpha(\mathbb{C})$.

It is easy to check that

$$\varphi_\beta(\vec{x},\vec{y}) = \bigwedge_{\sigma\in G} \varphi_{\alpha^\sigma}(\vec{x},\sigma\vec{y})$$

and $\psi_\beta(\vec{x}_1,\ldots,\vec{y}) = \bigwedge_{\sigma\in G}\psi_{\alpha^\sigma}(\vec{x}_1,\ldots,\sigma\vec{y})$ defines a code, which again codes X. Also, for $\sigma \in G$, we have $\varphi_\beta(\vec{x},\vec{y}) \equiv \varphi_{\beta^\sigma}(\vec{x},\sigma\vec{y})$ and $\psi_\beta(\vec{x}_1,\ldots,\vec{y}) \equiv \psi_{\beta^\sigma}(\vec{x}_1,\ldots,\sigma\vec{y})$, which shows that β is equivalent to β^σ. Now choose representatives $\rho_1,\ldots,\rho_r$ for the right cosets of G in $\mathrm{Sym}(n_\alpha)$ and set $C_i = C_{i-1} \cup \{\beta^{\rho_1},\ldots,\beta^{\rho_r}\}$. □

Remark 2.6. Note that the proof holds in a more general setting of a countable strongly minimal theory with the DMP (*definable multiplicity property*) where imaginary parameters $\vec{b}$ are allowed. It is not possible to find $\mathcal{C}$ closed under permutations (as stated in [6]).

3. δ-nonsense

Let X be a set. A function $\delta : \mathcal{P}_{\mathrm{fin}}(X) \to \mathbb{Z}$ is a *δ-function* if it satisfies the following:

(1) $\delta(\emptyset) = 0$
(2) $\delta(A \cup B) + \delta(A \cap B) \leq \delta(A) + \delta(B)$

Moreover, if for all A we have that $\delta(A) \geq 0$, then we say that δ is *nonnegative*.

For finite subsets A and B, we define the *relative* δ-value of A over B by:

$$\delta(A/B) = \delta(A \cup B) - \delta(B)$$

Now, (2) is equivalent to $\delta(A/B) \leq \delta(A/A \cap B)$. It is easy to see that for any $A \cap B \subset C \subset B$, we have that $\delta(A/B) \leq \delta(A/C)$.

Hence, we can extend the definition of the relative δ to subsets Y (possibly not finite) as follows:

$$\delta(A/Y) = \inf_{A\cap Y\subset C\subset Y} \delta(A/C)$$

Note that $\delta(A/Y)$ is in $\{-\infty\} \cup \mathbb{Z}$. Using notation from [7], we say that Y is *self-sufficient* in X (denoted as $Y \leq X$) if for all finite $A \subset X$, we have that $\delta(A/Y) \geq 0$. We have that δ is nonnegative iff $\emptyset \leq X$.

Y is self-sufficient iff $\delta(A) \geq \delta(A \cap Y)$ for all A. If $Y \leq X$, it follows that $Y \cap Z \leq Z$ for all Z. Hence, self-sufficiency is transitive. Moreover, the intersection of self-sufficient sets is again self-sufficient and each set S is contained in a smallest self-sufficient subset, its *self-sufficient closure* $\mathrm{cl}_X(S)$. If δ is nonnegative, finite sets have finite closures.

A proper extension $Y \leq Z$ is *minimal* if no $Y \subsetneq Y' \subsetneq Z$ is self-sufficient in Z. The extension $Z \setminus Y$ must be finite, which allows us to express minimality by

$$\delta(Z/Y') < 0 \text{ for all } Y \subsetneq Y' \subsetneq Z$$

.

4. Black points

We extend the ring language L to $L^* = L \cup \{N\}$, where N is a unary predicate. **All considered L^*–structures are *colored* subsets of $\mathbb{C}$**, i.e. subsets A of $\mathbb{C}$ endowed with an interpretation $N(A)$ for N (*les points noirs*). The notation $A \subset B$ implies $N(A) = A \cap N(B)$.

We want to amalgamate à la Fraïssé-Hrushovski finite L^*–structures A according to a function δ defined as follows:

$$\delta(A) = p \cdot \operatorname{trdeg} A - |N(A)|$$

Note that δ satisfies conditions (1) and (2) from Section 3. With this particular definition, we have that $\delta(\{a\}) \leq p$. We are in a setting as in the previous section.

Although the general amalgam was studied in careful detail in [11], we will concentrate on the collapse closer to the spirit of [6]. Hence, we will consider just sets, and not the L^*-substructures that they generate. Nonetheless, in an abuse of notation, we will call them L^*-structures (and not *partial* L^*-structures).

All the lemmas in the rest of the section are true for arbitrary, finite or infinite, L^*–structures.

Lemma 4.1. *Let $B \leq A$ be a minimal extension. We have one of the following cases:*

1. *If A contains a white point a not in B, then $A = B \cup \{a\}$. Moreover, $\delta(A/B) = 0$ or p, depending whether a is algebraic or transcendental over B.*
2. *Otherwise, $A = B \cup \{a_1, \ldots, a_n\}$ with $a_1, \ldots, a_n$ distinct black and $0 \leq \delta(A/B) \leq p - 1$. Moreover, for any $\emptyset \neq S \subsetneq \{a_1, \ldots, a_n\}$, we have that*

   $$p \cdot \operatorname{trdeg}(A/B \cup S) < n - |S|.$$

   If $\delta(A/B) = p - 1$, then $A = B \cup \{a\}$ with a transcendental over B and black.

Proof. Recall that $B \leq A$ is minimal if it is proper and for any $B \subsetneq A' \subsetneq A$, we have that $\delta(A/BA') < 0$. Equivalently, $\delta(A/B)$ is the minimum among all values of $\delta(A'/B)$, where $B \subsetneq A' \subset A$, and it is attained only at A.

If a in $A \setminus B$ is white, case (1) follows, since we have that $\delta(A \setminus \{a\}/B) \leq \delta(A/B)$, hence $A = B \cup \{a\}$. The two possibilities for $\delta(A/B)$ are now clear.

Let us assume that $A \setminus B$ contains no white point. Take some $a \in A \setminus B$. Since $B \leq A$, it follows that a is transcendental over B and $\delta(a/B) = p - 1$. By minimality, $\delta(A/B) \leq p - 1$.

If $\delta(A/B) = p - 1$, then clearly $A = B \cup \{a\}$.

□

Definition 4.2. A minimal extension $B \leq A$ of type (2) is *good* if $\mathrm{tp}(A/B)$ is stationary and $\delta(A/B) = 0$. A code α is *good* if it is "the code" of a good minimal extension. That is,

- $n_\alpha = pk_\alpha$.
- $\varphi_\alpha(\vec{x}, \vec{y})$ implies that all x_i's are different and different from the components of $\vec{y}$.
- If $\models \varphi_\alpha(\vec{a}, \vec{b})$, for each $\emptyset \neq s \subsetneq \{1, \ldots, n_\alpha\}$, we have

$$p \cdot \mathrm{trdeg}(\vec{a}/\vec{a}_s\vec{b}) < (n_\alpha - |s|).$$

Note that, by (iii), the last two conditions are true, if they hold for just one realization $\vec{b}$ of θ_α and one generic realization $\vec{a}$ of $\varphi_\alpha(\vec{x}, \vec{b})$.

Let $\mathcal{C}_g$ be the subset of good codes in $\mathcal{C}$.

The next lemma is clear from the definitions.

Lemma 4.3. *Let α be a good code, $\vec{b} \in \mathrm{dcl}(B)$ realize θ_α, and $\vec{a}$ be a B–generic black realization of $\varphi_\alpha(\vec{x}, \vec{b})$. Then, $B \cup \{a_1, \ldots, a_{n_\alpha}\}$ is a good extension of B.*

Lemma 4.4. *Let $B \leq A = B \cup \{a_1, \ldots, a_n\}$ be a good extension. Then there is a good code α and $\vec{b} \in \mathrm{dcl}(B)$ such that $\models \varphi_\alpha(\vec{a}, \vec{b})$.*

Proof. Choose $\chi(\vec{x}) \in \mathrm{tp}(\vec{a}/B)$ of Morley rank $k = \mathrm{MR}(\vec{a}/B)$ and degree 1. There is $\alpha \in \mathcal{C}$ and $\vec{b}$ such that $\mathrm{MR}\big(\chi(\vec{x}) \mathbin{\triangle} \varphi_\alpha(\vec{x}, \vec{b})\big) < k = k_\alpha$. Since $\vec{a}$ is a B–generic realization of $\chi(\vec{x})$, it is also a B–generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$. Since $\vec{b}$ is a canonical base of $\mathrm{tp}(\vec{a}/B)$, $\vec{b}$ belongs to $\mathrm{dcl}(B)$. Since A/B is good, we have that α is a good code. □

In the previous Lemma, we chose $\vec{a}$ as a B–generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$. The following result shows that this the only possibility.

Lemma 4.5 (cf. Lemma 3A in [6])**.** *Let α be a good code, $\vec{b} \in \mathrm{acl}(B)$ realize θ_α, and $\vec{a}$ be a black realization of $\varphi_\alpha(\vec{x}, \vec{b})$ which does not completely lie in B. Then, the following holds:*

(1) $\delta(\vec{a}/B) \leq 0$
(2) If $\delta(\vec{a}/B) = 0$, then $\vec{a} \cap B = \emptyset$ and $\vec{a}$ is a B–generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$.

Proof. If $\vec{a}$ is not disjoint from B we have

$$\delta(\vec{a}/B) \leq \delta(\vec{a}/\vec{a}'\vec{b}) < 0.$$

for $\vec{a} \cap B = \vec{a}'$. Hence, $\delta(\vec{a}/B) = 0$ yields that $\vec{a} \cap B = \emptyset$. In this case, we have $\delta(\vec{a}/B) \leq p \cdot k_\alpha - n_\alpha = 0$. Therefore, $\mathrm{trdeg}(\vec{a}/B) = k_\alpha$ and $\vec{a}$ is a B–generic solution of $\varphi_\alpha(\vec{x}, \vec{b})$. □

5. The (in)famous μ function

We now fix a function $\mu^* : \mathcal{C}_g \to \mathbb{N}$ which is finite-to-one on the set of all α with $n_\alpha = n$ for each n in $\mathbb{N}$. Moreover, $\mu^*(\beta) = \mu^*(\alpha)$ must hold if β is equivalent to a permutation of α, and

$$\mu^*(\alpha) \geq m_\alpha - 1.$$

The function μ is then defined by

$$\mu(\alpha) = ((p-1)(n_\alpha - 1) + 1)m_\alpha + \mu^*(\alpha).$$

We note that $\mu(\alpha) \geq m_\alpha$.

Note 5.1. One can replace in the following μ by $\mu'(\alpha) = F(\alpha) + \mu^*(\alpha)$ for any function F which satisfies $F(\alpha^\sigma) = F(\alpha)$ and $F(\alpha) \geq ((p-1)(n_\alpha - 1) + 1)m_\alpha$. The class of functions μ is not increased by this, only the complete theories T^μ (see Section 7) get weaker, but equivalent, axiomatizations.

We recover the definition introduced in [11] for approximations to a Morley sequence of a given good minimal extension.

Definition 5.2. Let α be a good code and $\vec{b} \models \theta_\alpha$. A *pseudo-morley sequence* for α over $\vec{b}$ is a (finite) sequence $\vec{a}_1, \ldots, \vec{a}_r$ of disjoint realizations of $\varphi_\alpha(\vec{x}, \vec{b})$ painted in black such that any distinct m_α elements among $\{\vec{a}_1, \ldots, \vec{a}_r\}$ realize $\psi_\alpha(\vec{x}_1, \ldots, \vec{x}_{m_\alpha}, \vec{b})$.

It follows that $\vec{b}$ is in the definable closure of the pseudo-morley sequence if $r \geq m_\alpha$ from part (v) of 2.1.

We now consider the class of L^*-structures on which δ is non-negative and for any good code in $\mathcal{C}$, we cannot find a pseudo-morley sequence that is longer than the value of μ at this code.

Definition 5.3. The class $\mathcal{K}^\mu$ is the class of all L^*-structures M (i.e colored subsets of $\mathbb{C}$) such that:

- $\emptyset \leq M$.
- No α in $\mathcal{C}_g$ has a pseudo-morley sequence in M of length longer than $\mu(\alpha)$.

We denote by $\mathcal{K}^\mu_{\text{fin}}$ the class of all finite L^*-structures in $\mathcal{K}^\mu$.

Recall that the first condition means that for any finite set $A \subset M$, we have $\delta(A) \geq 0$. Clearly, $\mathcal{K}^\mu_{\text{fin}}$ is not empty ($\emptyset$ is an element of this class). In fact all finite subsets of $\mathbb{C}$ with no black points are in the class.

Since ACF_q is small, $\mathcal{K}^\mu_{\text{fin}}$ contains at most countably many structures up to isomorphism.

The following result resumes the ingredients used in [6] stating them in a form closer to the original idea of Fraïssé's amalgamation procedure to construct a countable ultrahomogeneous model whose age is exactly $\mathcal{K}^\mu_{\text{fin}}$. Moreover, it yields explicit conditions for an L^*-structure to be a member of $\mathcal{K}^\mu$, which will be useful for exhibiting an axiomatization of this class.

Lemma 5.4. *Let M be in $\mathcal{K}^\mu$ and $M \leq M'$ a minimal extension.*

If M' contains a new white point, then M' is in $\mathcal{K}^\mu$.

Otherwise, M' is in $\mathcal{K}^\mu$ if and only if none of the following two conditions holds:

- *a) There is a code $\alpha \in \mathcal{C}_g$ and a realization $\vec{b} \in \text{dcl}(M)$ of θ_α, such that:*
  - *i) $M' \setminus M$ contains a realization $\vec{a}$ of $\varphi_\alpha(\vec{x}, \vec{b})$.*
  - *ii) M contains a pseudo-morley sequence for α over $\vec{b}$ of length $\mu(\alpha)$.*
- *b) There is some code $\alpha \in \mathcal{C}_g$ and a pseudo-morley sequence for α in M' of length $\mu(\alpha) + 1$, such that there are more than $\mu^*(\alpha)$ many elements of the sequence contained in $M' \setminus M$.*

If a) holds, $\vec{a}$ is an enumeration of $M' \setminus M$ and an M–generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$.

Since $n_\alpha \geq p$ for good codes, the lemma implies that $M' \in \mathcal{K}^\mu$ if $\delta(M'/M) = p-1$.

Proof. If M' contains a new white point a, by 4.1 (1), we get that $M' = M \cup \{a\}$. If M' is not in $\mathcal{K}^\mu$, it contains a pseudo-morley sequence of length $\mu(\alpha)+1$ for some code $\alpha \in \mathcal{C}_g$. Since $\{a\}$ adds no black points, the sequence is contained in M, which is a contradiction.

Suppose now that $M' \setminus M$ has no new white points. If b) holds, M' is not in $\mathcal{K}^\mu$ by definition. If we have case a), $\vec{a}$ is generic over M by Lemma 4.5 (2) and we can extend the sequence of ii) by $\vec{a}$, thanks to condition (vii). This shows that M' is not in $\mathcal{K}^\mu$. Also, since M'/M is minimal and $\delta(\vec{a}/M) = 0$, we have that $M' = M \cup \{a_1, \ldots, a_{n_\alpha}\}$.

For the other direction, if M' as above is not in $\mathcal{K}^\mu$, there exists a code $\alpha \in \mathcal{C}_g$ and a pseudo-morley sequence $\vec{e}_0, \ldots, \vec{e}_{\mu(\alpha)}$ for α in M' over some $\vec{b} \in \operatorname{dcl}(M')$. We may rearrange the sequence as follows:

- $\vec{e}_0, \ldots, \vec{e}_{r_0-1}$ are contained in M.
- $\vec{e}_{r_0}, \ldots, \vec{e}_{r_1-1}$ are not in M, but have at least one coordinate in M.
- $\vec{e}_{r_1}, \ldots, \vec{e}_{\mu(\alpha)}$ are in $\{a_1, \ldots, a_{n_\alpha}\}$.

Since M is in $\mathcal{K}^\mu$, we have that $r_0 \leq \mu(\alpha)$. There are two possibilities:

Case 1. $m_\alpha \leq r_0$. In this case, $\vec{b} \in \operatorname{dcl}(\vec{e}_0, \ldots, \vec{e}_{m_\alpha - 1})$ is in $\operatorname{dcl}(M)$. By 4.5 (1), we have that $\delta(\vec{e}_{r_0}/M) \leq 0$. Since $M \leq M'$, we have that $\delta(\vec{e}_{r_0}/M) = 0$. Hence, for each i, we conclude from 4.5 (2) that either $\vec{e}_i$ is disjoint from or contained in M. That is, $r_0 = r_1$. As above, we conclude $M' = M \cup \vec{e}_{r_0}$. Hence, $r_0 = \mu(\alpha)$ by disjointness of the pseudo-morley sequence. Therefore, a) holds.

Case 2. $r_0 \leq m_\alpha$. Define $\delta(i) = \delta(\vec{e}_i/M\vec{e}_0, \ldots, \vec{e}_{i-1})$. Then, since $M \leq M'$, we have that:

$$0 \leq \delta(\vec{e}_0, \ldots, \vec{e}_{r_1-1}/M) = \sum_{i<r_1} \delta(i) = \sum_{i<m_\alpha} \delta(i) + \sum_{m_\alpha \leq i < r_1} \delta(i)$$

For $i < m_\alpha$, we have that $\delta(i) \leq (p-1)(n_\alpha - 1)$ (Note that if $\vec{d}$ is a tuple of black points, we always have that $\delta(\vec{d}/B) \leq (p-1) \cdot \operatorname{trdeg}(\vec{d}/B)$ for any set B).

For $m_\alpha \leq i < r_1$, it follows that $\vec{b}$ belongs to $\operatorname{dcl}(M\vec{e}_0 \ldots \vec{e}_{i-1})$. But there is some coordinate of $\vec{e}_i$ in M, and hence, again from Lemma 4.5 (2), we conclude that $\delta(i) < 0$.

From the inequalities above, we get:

$$0 \leq (p-1)(n_\alpha - 1)m_\alpha - (r_1 - m_\alpha)$$

That is, $r_1 \leq ((p-1)(n_\alpha - 1) + 1)m_\alpha$. Now,

$$\mu(\alpha) - r_1 + 1 \geq \mu(\alpha) - ((p-1)(n_\alpha - 1) + 1)m_\alpha + 1 \geq \mu^*(\alpha) + 1$$

This yields b). □

Corollary 5.5. *Let M be in $\mathcal{K}^\mu$, $\alpha \in \mathcal{C}_g$, $\vec{b} \in \operatorname{dcl} M$ a realization of θ_α and $\vec{a}$ a black M–generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$. Then $M' = M \cup \{a_1, \ldots, a_{n_\alpha}\}$ is in $\mathcal{K}^\mu$ if and only if none of the following two conditions holds:*

a) M contains a pseudo-morley sequence for α over $\vec{b}$ of length $\mu(\alpha)$.

b) *There is some code $\beta \in \mathcal{C}_g$ and a pseudo-morley sequence for β in M' of length $\mu(\beta)+1$, such that there are more than $\mu^*(\beta)$ many elements of the sequence contained in $M' \setminus M$.*

Proof. M' is a minimal extension of M by 4.3, so we can apply the last lemma. We need only show the following: If α' is a code in $\mathcal{C}_g$, $\vec{b}' \in \mathrm{dcl}(M)$ such that $\vec{a}$ is a permuted M-generic realization of $\varphi_{\alpha'}(\vec{x}, \vec{b}')$ and if α' has a pseudo-morley sequence of length $\mu(\alpha')$ in M over $\vec{b}'$, then α has a pseudo-morley sequence of length $\mu(\alpha)$ in M over $\vec{b}$.

Let σ be a permutation of α such that $\vec{a}$ realizes φ_{α^σ}. By (x) there is a code $\alpha'' \in \mathcal{C}_g$ which is equivalent to α^σ. So there is $\vec{b}''$ such that $\varphi_{\alpha^\sigma}(\vec{x}, \vec{b}') \equiv \varphi_{\alpha''}(\vec{x}, \vec{b}'')$ and $\psi_{\alpha^\sigma}(\vec{x}_1, \ldots, \vec{b}') \equiv \psi_{\alpha''}(\vec{x}_1, \ldots, \vec{b}'')$. The permuted pseudo-morley sequence of α' is a pseudo-morley sequence of α'' over $\vec{b}'' \in \mathrm{dcl}(M)$, and $\vec{a}$ is an M-generic realization of $\varphi_{\alpha''}(\vec{x}, \vec{b}'')$. The properties (ix) and (iv) of $\mathcal{C}$ imply $\alpha'' = \alpha$ and $\vec{b}'' = \vec{b}$. Finally, we have $\mu(\alpha) = \mu(\alpha'') = \mu(\alpha^\sigma)$. □

6. Fraïssé limits for $\mathcal{K}^\mu$

In this section, we show that the class $\mathcal{K}^\mu$ (and hence, $\mathcal{K}^\mu_{\mathrm{fin}}$) has the Amalgamation Property, and hence, we can obtain *rich* fields as introduced by Poizat in [11] (We apologize for translating notation into other languages).

An *isomorphism* between two colored subsets A and B of $\mathbb{C}$ is a bijection which maps $N(A)$ onto $N(B)$ and is elementary as a partial map defined on $\mathbb{C}$. A *self-sufficient embedding* from A to B is an isomorphism between A and a self-sufficient subset of B.

Theorem 6.1. *The class $\mathcal{K}^\mu$ has the amalgamation property with respect to self-sufficient embeddings.*

Proof. Let $B \leq M$ and $B \leq A$ be structures in $\mathcal{K}^\mu$. We need to show that there is an extension M' of M in $\mathcal{K}^\mu$, with $M \leq M'$ and some $B \leq A' \leq M'$ such that A and A' are isomorphic over B. By splitting the extension $B \leq A$ into minimal ones, we may assume it is minimal.

Case 1. $B \leq A$ has a new white point a. Let p be the type of a over B. We distinguish two (non-exclusive) cases.

Subcase 1.1. p is algebraic and realized in M, say by a'. Self-sufficiency of B in M yields that a' is white. So $A' = B \cup \{a'\}$ is isomorphic to A. Since $\delta(a'/B) = 0$, it implies that $A' \leq M$.

Subcase 1.2. p can be realized in an extension of M by a new element a'. We paint a' white and set $M' = B \cup \{a'\}$.

Case 2. $B \leq A$ has no new white points. Since B is self-sufficient in A, no element of $A \setminus B$ is algebraic over B. So we can take for M' be the *free amalgam* (as in [11]) of M and A over B, that is, we assume M and A to be algebraically independent over B and let M' be their union. It is easy to see that M and A are self-sufficient in M' and that M'/M is minimal. We are done if M' belongs to $\mathcal{K}^\mu$. Otherwise, by Lemma 5.4, there are two cases:

Subcase 2.1) There is a code $\alpha \in \mathcal{C}_g$, a realization $\vec{b}$ of θ_α in $\mathrm{dcl}(M)$, a pseudo-morley sequence for α in M over $\vec{b}$ of length $\mu(\alpha)$ and $M' \setminus M = \vec{a}$ is a M–generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$. Since $\vec{a}$ is independent from M over B and $\vec{b}$ is the canonical parameter of $\mathrm{tp}(\vec{a}/M)$), we have that $\vec{b} \in \mathrm{acl}(B)$. The sequence cannot be contained in B, since A is in $\mathcal{K}^\mu$. Hence, there is some *black* realization $\vec{a}'$ of $\varphi_\alpha(\vec{x}, \vec{b})$ in M not completely contained in B. Now, $\delta(\vec{a}'/B) = 0$ since $B \le M$, therefore $\vec{a}'$ is generic over B by Lemma 4.5. So, $B \cup \{a'_1, \ldots, a'_{n_\alpha}\}$ is self-sufficient in M and it is isomorphic to A over B.

Subcase 2.2) There is a code $\alpha \in \mathcal{C}_g$, a canonical basis $\vec{b}$ in $\mathrm{dcl}(M')$ for α such that there is a pseudo-morley sequence $\vec{e}_0, \ldots, \vec{e}_{\mu(\alpha)}$ for α over $\vec{b}$ in M' with more than $\mu^*(\alpha)$ many elements coming from $M' \setminus M$. Again, since $\mu^*(\alpha) + 1 \ge m_\alpha$, we have that $\vec{b}$ is in $\mathrm{dcl}(A)$. There must be at least one member $\vec{e}_i$ not contained in A (because A is in $\mathcal{K}^\mu$). Since $A \le M'$, it follows from 4.5 (2) that $\vec{e}_i$ is an A–generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$ in M. But $\vec{e}_i$ and A are independent over B, therefore the canonical basis $\vec{b}$ of α is in $\mathrm{acl}(B)$.
Pick some $\vec{e}_j$ in $M' \setminus M = A \setminus B$. Again it follows that $\vec{e}_j$ is a B–generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$. Since M'/M is minimal, $\vec{e}_j$ enumerates $M' \setminus M$ and we are in subcase 2.1. Note that all e_k, $k \neq j$, are in M, which implies $\vec{b} \in \mathrm{dcl}(M)$. □

We call M in $\mathcal{K}^\mu$ *rich* if for any $B \le M$ finite and any finite extension $B \le A$ of members of $\mathcal{K}^\mu$, there is a self-sufficient substructure $A' \le M$ with $B \le A'$ and B-isomorphic to A.

Corollary 6.2. *There is a unique* (*up to isomorphism*) *countable rich structure M in $\mathcal{K}^\mu$.*

We will see in Theorem 7.2 that rich structures are colored algebraically closed fields. We will call them *rich fields*.

Remark 6.3. Let M be a rich field[3], α be a code in $\mathcal{C}_g$ and $\vec{b}$ be a realization of θ_α in M. Let $\dim_\alpha(M/\vec{b})$ be the maximal length of a pseudo-morley sequence of α over $\vec{b}$ in M and $B = \mathrm{cl}_M(\vec{b})$ the (finite) self-sufficient closure of $\vec{b}$ in M. Then there are two cases, either

$$\dim_\alpha(M/\vec{b}) = \dim_\alpha(B/\vec{b})$$

or

$$\dim_\alpha(M/\vec{b}) = \mu(\alpha).$$

Proof. Choose a black B–generic realization $\vec{a}$ of $\varphi_\alpha(\vec{x}, \vec{b})$ outside M. If $A = B \cup \vec{a}$ is not in $\mathcal{K}^\mu$, all realizations of $\varphi_\alpha(\vec{x}, \vec{b})$ in M are contained in B (by Lemma 4.5 (2)). Therefore $\dim_\alpha(M/\vec{b}) = \dim_\alpha(B/\vec{b})$.

If A belongs to $\mathcal{K}^\mu$, let $C \le M$ be a finite extension of B with $\dim_\alpha(M/\vec{b}) = \dim_\alpha(C/\vec{b})$ and let C' be the free amalgam of C and A over B. Since M is rich, C' does not belong to $\mathcal{K}^\mu$. The proof of 6.1 (applied to C instead of M) and of 5.5 shows that $\dim_\alpha(C/\vec{b}) = \mu(\alpha)$. □

[3]The remark is true for all models of T^μ, as defined in Section 7.

7. A theory for $\mathcal{K}^{\mu}$

In this section, we will show that the class $\mathcal{K}^{\mu}$ is axiomatizable and we will give explicit axioms that describe some completion. Rich fields will then be ω-saturated models of this theory. First, a foreword about the choice of axioms:

We will see in Section 8 that extensions with $\delta = 0$ will become algebraic. We know (by reducing it to the case of good minimal extensions) that at most there are μ many realizations. If we are given a minimal extension $B \leq A$, where $B \leq M$, we could amalgamate A and M freely over B and the amalgam could be potentially an element of $\mathcal{K}^{\mu}$. By richness, this cannot happen, since there is one realization too many in the amalgam not in M. Hence, we need to prohibit the amalgam to be an element of $\mathcal{K}^{\mu}$. We know exactly by 5.4 and 5.5 when this happens. Therefore, our axioms should state that such an amalgam cannot happen.

The theory T^{μ} in the extended language $L^* = L \cup \{N\}$ has the following axioms (more precisely, *axiom schemes*):

Universal Axioms:

(1) Any model is an integral domain of characteristic q.
(2) $\emptyset$ is self-sufficient in any model of T^{μ}.
(3) Given a code $\alpha \in \mathcal{C}_g$, any pseudo-morley sequence for α has length at most $\mu(\alpha)$.

$\forall\exists$ Axioms:

(4) Any model of T is an algebraically closed field of characteristic q.
(5) Given a code $\alpha \in \mathcal{C}_g$ and $\vec{b}$ realizing $\theta_\alpha(\vec{y})$, one of the following holds:
  a) α has a pseudo-morley sequence of length $\mu(\alpha)$ over $\vec{b}$.
  b) Given a realization $\vec{a}$ of $\varphi_\alpha(\vec{x}, \vec{b})$ generic over the model that we are considering, if we paint $\vec{a}$ in black, there is a code $\beta \in \mathcal{C}_g$ and a pseudo-morley sequence for β of length $\mu(\beta) + 1$ in the L^*-structure consisting of the model and $\vec{a}$ such that there are more than $\mu^*(\beta)$ many elements of the sequence contained $\{a_1, \ldots, a_{n_\alpha}\}$.

Note 7.1. We discuss here why the above axioms are first-order and their meaning. Since our final theory will have finite Morley rank, it follows from [10] that Axiom (4) needs to be included. Axiom (3) will yield that the types of $\delta = 0$ will become algebraic, and hence of Morley rank 0.

Why is Axiom (5) axiomatizable? In order to encode β, we need to determine *a priori* how many variables we will use. Equivalently, how many β's need to be considered. We cannot use more than n_α variables. On the other hand, we have n_β many variables to consider for each element of the pseudo-morley sequence, and there are at least $\mu^*(\beta) + 1$ many such members. That is,

$$(\mu^*(\beta) + 1) n_\beta \leq n_\alpha$$

By the finite-to-one condition on μ^*, there are only finitely many β's that satisfy the above inequality, and we are done.

Moreover, it follows from 6.3 that in order to get a complete theory, we do not need to determine how many realizations of a code there must be in a model, since we implicitly do so.

Theorem 7.2. *An L^*-structure is rich if and only if it is an ω-saturated model of T^{μ}.*

Proof. Let $M \models T^\mu$ be ω-saturated. Let $B \leq M$ and $B \leq A$ be finite sets. We need to find a self-sufficient B-copy of A in M. Splitting $B \leq A$ into minimal extensions, we are reduced to the minimal case. We can distinguish four different cases:
If $B \leq A$ is algebraic, we are done (by Axiom (4)).
If $B \leq A = B \cup \{\vec{a}\}$ is of type (2) (see Lemma 4.1) with $\delta(A/B) = 0$, consider the free amalgam M' of M and A over B. Since M is algebraically closed, M' is a good extension of M. By (the proof of) Lemma 4.4 and 4.5 there is a code $\alpha \in \mathcal{K}_g$ and $\vec{b} \in M$ such that $\vec{a}$ is an M-generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$. By Axiom (5) M' does not belong to $\mathcal{K}^\mu$. Theorem 6.1 implies that A has a strong embedding over B into M.
For $1 \leq \delta(A/B) \leq p - 1$, we need to *approximate* the extension by extensions of $\delta < \delta(A/B)$ and apply induction. We know by 4.1 that A contains no new white points. Choose some element $a \in A \setminus B$. Since a is transcendental over B, a^n is not in A for large n. We can paint a^n in black and consider $A_n = A \cup \{a^n\}$. It is easy to check that $B \leq A_n$ is minimal and $\delta(A_n/B) < \delta(A/B)$. The sequence A_n/B converges (in the space of L-types) to the extension A_∞/B, where $A_\infty = A \cup \{c\}$, with c transcendental over A and black. Clearly $A \leq A_\infty \in \mathcal{K}^\mu$, by 5.4. Since there is only a finite number of codes $\alpha \in \mathcal{C}_g$ for which there could be a pseudo-morley sequence of length longer than $\mu(\alpha)$ in any A_n (bounded only in terms of $|A|$), we have that A_n is in $\mathcal{K}^\mu$ for large n. Hence, by induction, we can find self-sufficient B-copies of A_n in M for large n. By saturation of M, A_∞ is also self-sufficiently embedable over B. Since $A \leq A_\infty$, we conclude that there is a self-sufficient B-copy of A in M.
For the last case, let $A = B \cup \{a\}$ with a white transcendental over B. Consider for each n the extension

$$B \leq B \cup \{c\} \leq B \cup \{c, c^n\},$$

where c is black transcendental over B and c^n is white. $B \cup \{c, c^n\}$ belongs to $\mathcal{K}^\mu$ by 5.4. By the above, that we can realize $B \leq B \cup \{c, c^n\}$ self-sufficiently in M. Since these extensions *converge* to $B \leq B \cup \{c, a\} = A'$ where c and a are algebraically independent over B, we can realize $B \leq A'$ self-sufficiently in M. Since $A \leq A'$, we are done.

Suppose now that M is a rich field. We first show that M is algebraically closed. Let $a \in \mathrm{acl}(M)$. Choose a finite set B in M such that a is in $\mathrm{acl}(B)$. Taking the closure of B in M, we can assume that $B \leq M$. Paint a in white. It is clear that $B \cup \{a\}$ is in $\mathcal{K}^\mu$ (since B is) and $B \leq B \cup \{a\}$. By richness, we find a copy of a in M over B. This yields (4).
For Axiom 5, let α and $\vec{b}$ be as in the statement such that neither a) nor b) hold. Choose some generic *black* realization of $\varphi_\alpha(\vec{x}, \vec{b})$ over M. By 5.5, we have that $M \cup \vec{a}$ is in $\mathcal{K}^\mu$. Choose some finite set $B \leq M$ containing $\vec{b}$. Again, $B \leq B \cup \vec{a}$, and by richness, we get a B-copy of $\vec{a}$ in M, say $\vec{a}'$. Take now some finite $C \leq M$ containing $B \cup \vec{a}'$. We have that $C \leq C \cup \vec{a}$. We can iterate and obtain a pseudo-morley sequence in M for α of arbitrarily large length. This contradicts that M is in $\mathcal{K}^\mu$.
Now, M is elementarily equivalent to an ω–saturated structure M', which is by the above a model of T^μ and therefore rich. So M is ∞-equivalent to M' and therefore ω–saturated itself. □

Corollary 7.3. *Let M be an L^*-structure in $\mathcal{K}^\mu$. Then $M \models T^\mu$ iff every existential $L^*(M)$-formula ϕ true in some $M \leq N$ with $N \models$ "Universal Axioms of T^μ" holds also in M.*

Proof. Let M be a model of T^μ, N and ϕ as above. We can assume M and N are saturated. Let $B \leq M$ contain all parameters in ϕ. Choose some $B \subset A \leq N$ containing a realization of ϕ. Since $B \leq A$ and M is rich, we can embed A in M over B. Hence, we have a solution for ϕ in M.

If M is existentially closed among self-sufficient extensions, it satisfies Axiom (4), since $\operatorname{acl}(M)$ (new elements painted in white) is a self-sufficient extension. If M does not satisfy Axiom (5), there is an $\alpha \in \mathcal{C}_g$, $\vec{b} \in M$ and a black M-generic solution $\vec{a}$ of $\varphi_\alpha(\vec{x}, \vec{b})$ such that $M \leq M \cup \vec{a}$ is in $\mathcal{K}^\mu$. Considering finite sets $C \leq M$ containing $\vec{b}$ and using the existential closedness of M, we can find infinitely many disjoint black realizations of $\varphi_\alpha(\vec{x}, \vec{b})$. (We may assume that φ_α is quantifier free.) This contradicts that $M \in \mathcal{K}^\mu$. □

Corollary 7.4. *The theory T^μ is complete.*

Two tuples $\vec{a}$ and $\vec{b}$ in two models M and N of T^μ have the same L^*-type iff there is an isomorphism $f : \operatorname{cl}(\vec{a}) \to \operatorname{cl}(\vec{b})$ which maps $\vec{a}$ to $\vec{b}$.

An extension $M \subset N$ of models of T^μ is elementary iff M is self-sufficient in N[4]

Proof. T^μ is complete, since any two countable saturated models are elementarily equivalent, by richness.

Consider two models M, N of T^μ. If M is subset of N, but no self-sufficient, there is a finite $A \leq M$ and a tuple $\vec{b} \in N$ sucht that $\delta(\vec{b}/A) < 0$. Responsible is a finite part of the L^*-type of $\vec{b}$ over A. So $M \prec N$ would imply the existence of an $\vec{a} \in M$ with $\delta(\vec{a}/A) < 0$, which is not possible.

If $\vec{a}$ and $\vec{b}$ have the same L^*–type, it is easy to see that the map $\vec{a} \mapsto \vec{b}$ extends to an isomorphism $f : \operatorname{cl}(\vec{a}) \to \operatorname{cl}(\vec{b})$. Conversely, let f be given. Choose rich extensions $M \prec M'$ and $N \prec N'$. We know that M and N are self-sufficient in these extension and therefore also $cl(\vec{a})$ and $\operatorname{cl}(\vec{b})$. Since isomorphisms between finite self–sufficent subsets of M' and N' have the back-and-forth property, f is elementary map.

Finally assume that $M \leq N$. Since cl_M is the restriction of cl_N to M, all finite tuples $\vec{a} \in M$ have the same L^*-type in M as in N, i.e. $M \prec N$. □

8. Computing ranks

In this section, we compute the Morley rank of types in T^μ. In order to avoid confusion, we will denote it by MR^*, since we work with L^*-types $\mathrm{tp}^*(\vec{a}/B)$. We work inside a sufficiently saturated model M of T^μ.

Lemma 8.1. *T^μ has finite Morley rank.*

Proof. It is clear that $\operatorname{cl}(A)$ is contained in $\operatorname{acl}^*(A)$. This implies

$$\mathrm{MR}^*(\vec{a}/C) = \mathrm{MR}^*(\operatorname{cl}(C\vec{a})/\operatorname{cl}(C)).$$

[4]By an observation of M. Hils T^μ is model complete, i.e. *all* extensions of models of T^μ are self-sufficient. See Remark 8.4 for a proof.

So it is enough to compute Morley ranks $\mathrm{MR}^*(A/B)$, where $B \leq A \leq M$ and $A \setminus B$ is finite. We will show that the rank is bounded by a function of $\delta(A/B)$.

We prove first that $\delta(A/B) = 0$ implies that A is algebraic in M over B, i.e. $\mathrm{MR}^*(A/B) = 0$. For this we may assume that A/B is minimal. If A has a new white element, then A/B is algebraic (in the field sense). Otherwise, $A \setminus B$ contains only black points and we may assume – after adding algebraic elements to B – that A/B is good. By 4.4, $A \setminus B$ is enumerated by a generic solution $\vec{a}$ of a code $\alpha \in \mathcal{C}_g$ over some $b \in \mathrm{dcl}(B)$. By 4.5 any sequence A_i of different conjugates of A over B in M yields a sequence of B–generic realizations of $\varphi_\alpha(\vec{x}, \vec{b})$ which is (in $\mathbb{C}$) independent over B. So the sequence is a pseudo-morley sequence of α over $\vec{b}$ and cannot be longer that $\mu(\alpha)$. This proves that A/B is algebraic in M.[5]

Now, assume that $\delta(A/B) = d > 0$ and that $\mathrm{MR}^*(A'/B') \leq r$ for all $B' \leq A' \leq M$ with $\delta(A'/B') < d$. The above case shows that we may also assume that for all $C \leq A$ with $B \subsetneq C \subsetneq A$ we have $\delta(C/B) > 0$ and $\delta(A/C) > 0$. We distinguish three cases:

Case 1: A/B is not minimal. Then we find $B \leq C \leq A$ with $\delta(C/B) < d$ and $\delta(A/C) < d$. Enumerate $C \setminus B$ by $\vec{c}$ and $A \setminus C$ by $\vec{a}$ and choose L^*-formulas $\rho(\vec{z})$ and $\chi(\vec{z}, \vec{x})$ over B which are satisfied by $\vec{c}$ and $\vec{c}\vec{a}$ and which imply $\delta(\vec{z}/B) < d$ and $\delta(\vec{x}/B\vec{z}) < d$. By above inductive assumption, we have $\mathrm{MR}^*(\vec{c}'/B) \leq r$ and $\mathrm{MR}^*(\vec{a}'/B\vec{c}') \leq r$ for all realization $\vec{c}'\vec{a}'$ of $\rho(\vec{z}) \wedge \chi(\vec{z}, \vec{x})$. Now we can apply Erimbetov's inequalities [5] and obtain $\mathrm{MR}^*(\rho(\vec{z}) \wedge \chi(\vec{z}, \vec{x})) \leq r(r+1)$. Hence $\mathrm{MR}^*(A/B) \leq r(r+1)$.

Case 2: A/B is minimal and $1 \leq d \leq p-1$. We fix an enumeration $\vec{a}$ of $A \setminus B$. We may again assume that $p = \mathrm{tp}(\vec{a}/B)$ is stationary. Choose an L-formula $\phi(\vec{x})$ in p of the same Morley rank and of degree 1 which satifies 2.1 (iii). It follows from Lemma 4.5 that for every black realization $\vec{a}'$ of φ, only two possibilities may occur: Either we have $\delta(\vec{a}'/B) < d$, which implies $\delta(\mathrm{cl}(B\vec{a}')/B) < d$ and $\mathrm{MR}^*(\vec{a}'/B) = \mathrm{MR}^*(\mathrm{cl}(\vec{a}')/B) \leq r$, or we have $\delta(\vec{a}'/B) = d$, which implies $\mathrm{tp}(\vec{a}'/B) = p$. If $B\vec{a}'$ is not self-sufficient in M, we conlude again $\mathrm{MR}^*(\mathrm{cl}(\vec{a}')/B) \leq r$. Otherwise, by 7.4, we have that $\mathrm{tp}^*(\vec{a}'/B) = \mathrm{tp}^*(\vec{a}/B)$. This shows that the L^*–type of $\vec{a}$ over B is not an accumulation point of types over B of rank bigger than r. If B were an ω-saturated elementary substructure of M, we could conclude that $\mathrm{MR}^*(A/B) \leq r+1$. Hence, consider any ω-saturated elementary substructure N which contains B. Since A is either contained in N or intersects N in B, we have $\delta(A/N) \leq \delta(A/B)$, which implies $\mathrm{MR}^*(A/n) \leq r+1$. Since N was arbitrary, it follows $\mathrm{MR}^*(A/B) \leq r+1$.

Case 3: A/B is minimal and $d = p$. Then $A = B \cup \{a\}$ with a white and transcendental. All 1-Types different from $\mathrm{tp}^*(a/B)$ have rank $\leq r$. So, if B were an ω–saturated elementary model, we could conclude $\mathrm{MR}^*(a/B) \leq r+1$. By the same argument as above, we show that the claim holds. □

For any set of parameters C and any finite tuple $\vec{a}$ we define

$$\mathrm{d}(\vec{a}/C) = \delta(\mathrm{cl}(C\vec{a})/\mathrm{cl}(C)).$$

[5]Note that the number of conjugates of $\vec{a}$ over B is bounded by $n_\alpha! \cdot \mu(\alpha)$.

Theorem 8.2. $\mathrm{MR}^*(\vec{a}/C) = \mathrm{d}(\vec{a}/C)$.

Proof. Since M is a field of finite Morley rank, Morley rank satisfies the Lascar inequalities by a result of Lascar [9]. It follows now from the proof of 8.1 and the additive character of δ that $\mathrm{MR}^* \leq d$.

For the other inequality, let $B \leq A \leq M$ be minimal, and $\delta(A/B) = d > 0$. It follows from the proof of 7.2 (and from 7.4) there is a sequence of extensions $B \leq A_n \leq M$, such that $\delta(A_n/B) = d - 1$ and

$$\lim_{n\to\infty} \mathrm{tp}^*(A_n/B) = \mathrm{tp}^*(A/B).$$

Since $\mathrm{MR}^*(A_n/B) \geq d - 1$ by induction, we have $\mathrm{MR}^*(A/B) \geq d$. □

Proof of the Main Theorem: For any a in M, we have:

$$\mathrm{d}(a) \leq \delta(a) = \begin{cases} p & \text{if } a \text{ is white} \\ p-1 & \text{if } a \text{ is black} \end{cases}$$

This shows $\mathrm{MR}^*(T^\mu) \leq p$ and $\mathrm{MR}^*(N) \leq p - 1$. On the other hand the structures $\{a\}$, $\{b\}$ with a white, transcendental and with b black transcendental are both in $\mathcal{K}^\mu$. So we find them as self-sufficient subsets of M. Then $\mathrm{d}(a) = \delta(a) = p$ and $\mathrm{d}(b) = \delta(b) = p - 1$. This proves the result. □

Example 8.3. It was observed in Theorem 18 [2] that every generic white point is the sum of p independent black points of Morley rank 1. We want to give a simpler proof of this fact.

Let $a_1, \dots, a_p$ be generic independent elements of $\mathbb{C}$. It follows trivially from 5.4 that the black $A_i = \{a_i^r\}_{1\leq r\leq p-1}$ and $\bigcup_{i=1}^{p} A_i$ with all elements painted in black belong to $\mathcal{K}^\mu$.

Take now $a = a_1 + \cdots + a_p$ painted in white. Again by 5.4, $\bigcup_{i=1}^{p} A_i \cup \{a\}$ belongs to $\mathcal{K}^\mu$. We may assume that $\bigcup_{i=1}^{p} A_i \cup \{a\} \leq M$.

Since $\{a\} \leq M$, we have $d(a) = p$. Hence, a is an generic white element. Since $\mathrm{cl}(a_i) = A_i \leq M$, we have $d(a_i) = \delta(A_i) = 1$. So each a_i has Morley rank 1.

Remark 8.4 (added January 23, 2005)**.** Martin Hils made the following oberservation: A theory of fields of finite Morley rank is $\aleph_1$–categorical. Since T^μ is $\forall\exists$–axiomatizable, a theorem of Lindström implies that *T^μ is model complete.*

Let us give a direct proof. Assume that M is a model of T^μ and N an extension which belongs to $\mathcal{K}^\mu$. We want to show that M is self–sufficient in N. We may assume that $\delta(N/M) < 0$ and that N is minimal with this property. Then $N = N' \cup \{a\}$, where $M \leq N'$ with $\delta(N'/M) = 0$, and a is black and algebraic over N' in the field sense. Choose a rich field $N' \leq N''$. Then N'' is an elementary extension of M. On the other hand, we have $\mathrm{MR}^*(N'/M) = \mathrm{d}(N'/M) = \delta(N'/M) = 0$, so N' is, in N'', algebraic over M and therefore contained in M. This implies that a is field-algebraic over M, which is impossible, since M is an algebraically closed field.

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