THE THEOREM OF THE COMPLEMENT FOR SUB-PFAFFIAN SETS

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ABSTRACT. Let $\mathcal{R}$ be an o-minimal expansion of the real field, and let $\mathcal{P}(\mathcal{R})$ be its Pfaffian closure. Let $\mathcal{L}$ be the language consisting of all Rolle leaves added to $\mathcal{R}$ to obtain $\mathcal{P}(\mathcal{R})$. We prove that $\mathcal{P}(\mathcal{R})$ is model complete in the language $\mathcal{L}$, provided that $\mathcal{R}$ admits analytic cell decomposition. We do this by proving a somewhat stronger statement, the theorem of the complement for nested sub-Pfaffian sets over $\mathcal{R}$. As a corollary, we obtain that $\mathcal{P}(\mathcal{R})$ is obtained by adding to $\mathcal{R}$ all nested Rolle leaves over $\mathcal{R}$, a one-stage process.

INTRODUCTION

The basic objects we study in this paper are nested Pfaffian sets over a given o-minimal expansion of the real field. Before defining them, let us briefly recall some of the history around the notion of Pfaffian functions: roughly speaking, Pfaffian functions are maximal solutions of triangular systems of partial differential equations with polynomial coefficients, see Khovanskii [10], Gabrielov [7] and Wilkie [17]. In his thesis [10], Khovanskii proves (among other things) that any set defined by finitely many equations and inequalities between Pfaffian functions has a finite number of connected components. In the early 1980s, Van den Dries conjectured that the expansion of the real field by all Pfaffian functions was model complete, which, together with Khovanskii's theorem, would imply that this expansion is o-minimal. Wilkie [18] proved that the real field expanded by all *totally defined* Pfaffian functions is o-minimal. Based on Lion and Rolin [11], this theorem was strengthened in the following way: given an o-minimal expansion $\mathcal{R}$ of the real field, we call a function Pfaffian over $\mathcal{R}$ if it is a maximal solution of a triangular system of partial differential equations with coefficients definable in $\mathcal{R}$. Then [16] there is an o-minimal expansion

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$\mathcal{P}(\mathcal{R})$ of $\mathcal{R}$, called the Pfaffian closure of $\mathcal{R}$, such that every Pfaffian function over $\mathcal{P}(\mathcal{R})$ is definable in $\mathcal{P}(\mathcal{R})$.

However, to our knowledge none of the above o-minimality proofs establish the model completeness of the respective structures. Based on techniques used in [13], we give here a proof of the model completeness of $\mathcal{P}(\mathcal{R})$ in the case where $\mathcal{R}$ admits analytic cell decomposition. To do this, we follow the setting in [10], where Pfaffian functions are replaced by nested Rolle leaves. We need a few definitions to state the precise theorem.

Let $M \subseteq \mathbb{R}^n$ be a differentiable submanifold of dimension m, and let d be a distribution on M of $(m-1)$-dimensional subspaces of $\mathbb{R}^n$ such that $d(x) \subseteq T_xM$ for all $x \in M$. An immersed manifold $V \subseteq M$ of dimension $m-1$ is called an **integral manifold** of d if $T_xV = d(x)$ for all $x \in V$. A **leaf** of d is a maximal connected integral manifold of d. A leaf V of d is a **Rolle leaf** (see Moussu and Roche [14]) if for every differentiable $\gamma : [0,1] \longrightarrow M$ such that $\gamma(0), \gamma(1) \in V$, there exists a $t \in [0,1]$ such that $\gamma'(t) \in d(\gamma(t))$. The following criterium for the Rolle property is crucial to our paper:

Haefliger's Theorem [9, 15]**.** *Assume that M and d are analytic and that M is simply connected. Then every leaf of d is a Rolle leaf.*

A tuple $d = (d_0, \dots, d_k)$ of distributions on a manifold $M \subseteq \mathbb{R}^n$ is called **nested**, if $d_0(x) = T_xM$ for all $x \in M$ and for $j = 1, \dots, k$, the distribution d_j is an integrable $(\dim(M) - j)$-distribution on M such that $d_j(x) \subseteq d_{j-1}(x)$ for all $x \in M$. In this situation, a tuple $V = (V_0, \dots, V_k)$ of manifolds contained in M is a **nested Rolle leaf** of d, if $V_0 = M$ and for $j = 1, \dots, k$, the set V_j is a Rolle leaf of $d_j|_{V_{j-1}}$; in particular, $\dim(V_j) = \dim(M) - j$. If, in the previous situation, both M and d are definable in an o-minimal expansion $\mathcal{R}$ of the real field, we call V a **nested Rolle leaf over** $\mathcal{R}$.

Let $\mathcal{R}$ be an o-minimal expansion of the real field; Khovanskii Theory as in [14, 16] generalises in a straightforward way to the setting of nested Rolle leaves (Section 2) over $\mathcal{R}$. A set $X \subseteq \mathbb{R}^n$ is a **basic nested Pfaffian set over** $\mathcal{R}$, if there are a definable set $A \subseteq \mathbb{R}^n$, a definable manifold $M \subseteq \mathbb{R}^n$, a definable nested distribution $d = (d_0, \dots, d_k)$ on M and a nested Rolle leaf $V = (V_0, \dots, V_k)$ of d, such that $X = A \cap V_k$. A **nested Pfaffian set over** $\mathcal{R}$ is a finite union of basic nested Pfaffian sets over $\mathcal{R}$, and a **nested sub-Pfaffian set over** $\mathcal{R}$ is a projection of a nested Pfaffian set over $\mathcal{R}$. We denote by $\mathcal{N}(\mathcal{R})$ the expansion of $\mathcal{R}$ by all nested Rolle leaves over $\mathcal{R}$. (Note that every set definable in $\mathcal{R}$ is quantifier-free definable in $\mathcal{N}(\mathcal{R})$, since every manifold definable in $\mathcal{R}$ is a nested Rolle leaf over $\mathcal{R}$.)

Main Theorem. *Assume that $\mathcal{R}$ admits analytic cell decomposition. Then the complement of every nested sub-Pfaffian set over $\mathcal{R}$ is again nested sub-Pfaffian over $\mathcal{R}$, that is, $\mathcal{N}(\mathcal{R})$ is model complete.*

Every component leaf of a nested Rolle leaf over $\mathcal{R}$ is quantifier-free definable in $\mathcal{P}(\mathcal{R})$ (Corollary 2.9 below). On the other hand, the Main Theorem implies that, in the construction of $\mathcal{P}(\mathcal{R})$ in [16], every Rolle leaf added to $\mathcal{R}$ is a nested sub-Pfaffian set (Proposition 12.1). Therefore:

Corollary 1. *If $\mathcal{R}$ admits analytic cell decomposition, then $\mathcal{N}(\mathcal{R})$ is existentially interdefinable with $\mathcal{P}(\mathcal{R})$; in particular, $\mathcal{P}(\mathcal{R})$ is model complete.* $\square$

The model completeness of $\mathcal{P}(\mathcal{R})$ or $\mathcal{N}(\mathcal{R})$ remains an open problem if $\mathcal{R}$ does not admit analytic cell decomposition. Also, even in the analytic case, we do not know if the reduct of $\mathcal{P}(\mathcal{R})$ generated by all Pfaffian functions over $\mathcal{R}$ is model complete. Finally, it is unclear to us whether our proof can be used to find a minimal model complete expansion of $\mathcal{R}$ containing any specific Pfaffian function over $\mathcal{R}$; in particular, we do not know if our proof provides an alternative proof of the model completeness of the exponential real field [17].

The proof of the Main Theorem goes as follows: by Corollary 2.9 of [6]—with Λ there equal to the collection of all nested Pfaffian sets over $\mathcal{R}$ contained in $[-1,1]^n$, for $n \in \mathbb{N}$—it suffices to establish Axioms (I)–(IV) there. Axioms (I)–(III) follow from Khovanskii Theory for nested Pfaffian sets over $\mathcal{R}$ (Corollary 2.7); thus, the main difficulty is to show that every nested Pfaffian set over $\mathcal{R}$ has the Λ-Gabrielov property. By the Fiber Cutting Lemma for nested Pfaffian sets over $\mathcal{R}$ (Corollary 9.3), we therefore need to establish (Theorem 11.4):

Theorem. *Let $V = (V_0, \dots, V_k)$ a nested Rolle leaf over $\mathcal{R}$, and assume that $\mathcal{R}$ admits analytic cell decomposition. Then $\operatorname{fr}(V_k)$ is contained in a finite union of nested sub-Pfaffian sets over $\mathcal{R}$ of dimension strictly less than $\dim(V_k)$.*

The theorem was proved in the special case $k = 1$ by Cano et al. [3]. For the proof of the general case, we consider $\operatorname{fr}(V_k)$ as a Hausdorff limit of a certain type of leaves of a definable nested distribution on M derived from d (Section 3). We then use the method of blowing up in jet space (Section 5), similar to [13], to recover distributions along the boundary of (blowings-up of) M, such that $\operatorname{fr}(V_k)$ is almost everywhere an integral manifold of one of these distributions. The main problems solved in this paper are the following: we did not know if (a) the distributions recoverd in this way were components of definable nested distributions, and if (b) the integral manifolds in question were Rolle leaves. Here we deal with (a) and (b) seperately; indeed, we solve (a) for the case that $\mathcal{R}$ is any o-minimal expansion of the real field, but need to assume that $\mathcal{R}$ admits analytic cell decomposition for our solution of (b).

For (a), we define the degree of a definable nested distribution d on M to be the number of component distributions of d whose associated foliation of M is *not* definable in $\mathcal{R}$ (Section 3). For example, we show in Section 4 that the nested distribution derived from d used to study $\mathrm{fr}(V_k)$, as mentioned in the previous paragraph, has at most degree equal to that of d. Moreover, we also prove in Section 4 that the negligeable set, off which $\mathrm{fr}(V_k)$ is a finite union of integral manifolds of the recovered distributions, is a union of Hausdorff limits obtained from distributions of degree at most that of d. These observations and a refinement of the blow-up method in [13] yield Theorem 6.4 by induction on the degree of d and the dimension of V_k, which implies in particular the following:

Proposition 1. *There are nested integral manifolds $W^j = (W_0^j, \dots, W_l^j)$ of corresponding definable nested distributions, for finitely many j, such that $\mathrm{fr}(V_k)$ is contained in the union of the projections $\Pi_n(W_l^j)$, and such that W_l^j is definable in $\mathcal{P}(\mathcal{R})$ and $\dim(W_l^j) < \dim(V_k)$ for each j.*

For (b), it now remains to show that each of the nested integral manifolds W_l^j in the previous proposition is in turn contained in a finite union of projections of nested Rolle leaves over $\mathcal{R}$ of dimension at most $\dim(W_l^j)$. To establish the Rolle property, we want to use Haefliger's Theorem; this is one of the reasons for our assumption that $\mathcal{R}$ admits analytic cell decomposition. If the degree of the corresponding nested distributions is 1, we can easily recover the Rolle property from Haefliger's Theorem using analytic cell decomposition. If the degree is larger than 1, however, we can only apply it once we know that W_{l-1}^j is *simply connected*—alas, the latter is not definable in $\mathcal{R}$. Proceeding by induction on l, we may assume that W_{l-1}^j is a Rolle leaf; so it remains to establish (Corollary 11.2):

Proposition 2. *Assume that $\mathcal{R}$ admits analytic cell decomposition. Then V_k is a finite union of simply connected nested sub-Pfaffian sets over $\mathcal{R}$.*

To prove this, we introduce in Section 10 the notion of *proper* nested sub-Pfaffian set. These are certain projections of nested Pfaffian subsets X of $[-1,1]^n$ that are *restricted off* $\{0\}$, that is, for every $r > 0$, the set $X \setminus (-r,r)^n$ is a restricted nested Pfaffian set similar to Gabrielov [8] or [17]. Remarkably, based on the ideas in [8]—adapted to our situation in Sections 7, 8 and 9—we obtain a cell decomposition theorem for proper nested sub-Pfaffian sets over $\mathcal{R}$ (Theorem 10.3). Proposition 2 then follows from the observation that, up to an analytic inversion of the ambient space, V_k is a restricted Pfaffian set off $\{0\}$ (Proposition 11.1).

1. Preliminaries

Throughout this paper, all manifolds, functions, maps, etc. are of class C^1, and manifolds are embedded, unless otherwise specified. We start with a lemma on o-minimal structures needed in Section 6.

Lemma 1.1. *Let $\mathcal{S}$ be an o-minimal expansion of the real field and $p \geq 1$, and let $M \subseteq \mathbb{R}^n$ be a C^p manifold of dimension d definable in $\mathcal{S}$. Let also $m \leq n$, and assume that $\Pi^n_m|_M$ is an immersion. Then M is the union of finitely many definable submanifolds N for which there exist a permutation σ of the first m coordinates and a definable C^p map $f : U \longrightarrow \mathbb{R}^{n-d}$ with $U \subseteq \mathbb{R}^d$ open, such that $\sigma(N) = \mathrm{gr}(f)$.*

Proof. Given a permutation σ of the first m coordinates, the set

$$M_\sigma := \left\{y \in M :\ \Pi_d|_{T_y\sigma(M)} \text{ is an immersion}\right\}$$

is an open subset of M. Thus by replacing M with each $\sigma(M_\sigma)$, we may assume that $\Pi_d|_M$ is an immersion. Hence $U := \Pi_d(M)$ is open, and $\Pi_d : M \longrightarrow U$ is a local diffeomorphism.

Now we let $\mathcal{C}$ be a C^p-cell decomposition of $\mathbb{R}^n$ compatible with M such that $\mathcal{D} := \Pi_d(\mathcal{C})$ is a stratification, and we put $\mathcal{C}_M := \{C \in \mathcal{C} :\ C \subseteq M\}$. For $y \in M$, we let $M(y)$ be the union of all $C \in \mathcal{C}_M$ such that $y \in \mathrm{cl}(C)$. Fix an arbitrary $y \in M$; we claim that (i) $U(y) := \Pi_d(M(y))$ is open, and (ii) $M(y)$ is the graph of a C^p function $f_y : U(y) \longrightarrow \mathbb{R}^{n-d}$. This claim implies Lemma 1, since there are only finitely many different $M(y)$ as y ranges over M.

To see (i), for $x \in U$ we let $D_x \in \mathcal{D}$ be the unique cell containing x and put

$$D(x) := \bigcup \{D \in \mathcal{D} :\ x \in \mathrm{cl}(D)\}\,.$$

Since $\Pi_d|_M$ is a local diffeomorphism, we have $U(y) = D(\Pi_d(y))$; hence it suffices to show that $D(x)$ is open for every $x \in U$. Fix an arbitrary $x \in U$; since $\mathcal{D}$ is a stratification and cells are connected, we get that

$(*)$ $D_x \subseteq D(x)$ and $D(z) = D(x)$ for all $z \in D_x$.

In particular, $D(x)$ contains an open neighborhood of z for every $z \in D_x$. From $(*)$ and the definition of $D(x)$, it follows that $D(z) \subseteq D(x)$ for all $z \in D(x)$. On the other hand, $\dim(D_z) > \dim(D_x)$ for every $z \in D(x) \setminus D_x$. Thus (i) follows by reverse induction on $\dim(D_x)$.

For (ii), we proceed by reverse induction on $\dim(D_{\Pi_d(y)})$, using the fact that $\Pi_d|_M$ is a local diffeomorphism. $\square$

Next, we recall some basic notions from [12] and adapt them to the present situation. Let $M \subseteq \mathbb{R}^n$ be a bounded manifold, and for $i \in \mathbb{N}$, let $V_i \subseteq M$ be an closed submanifold of dimension p. Let $\eta > 0$, and assume that each V_i is **η-bounded**, that is, for all $x \in V_i$ there is a matrix $L = (l_{ij}) \in \mathrm{M}_{n-p,p}(\mathbb{R})$ such that $\|L\| := \max_{i,j} |l_{i,j}| \leq \eta$ and

$$T_x V_i = \{(u, Lu) :\ u \in \mathbb{R}^p\}\,.$$

We also assume there is an $N \in \mathbb{N}$ such that for every i and every open box $U \subseteq \mathbb{R}^n$, the set $V_i \cap U$ has at most N connected components, and that both

$\lim V_i$ and $\lim \mathrm{fr}(V_i)$ exist. (Here and throughout this paper, we write $\lim V_i$ instead of $\lim \mathrm{cl}(V_i)$.)

Under these assumptions, the proof of Lemma 5 in [12] goes through:

Lemma 1.2. *For every $x \in \lim V_i \setminus \lim \mathrm{fr}(V_i)$, there are a box $U \subseteq \mathbb{R}^n$ containing x and $p\eta$-Lipschitz functions $f_1, \dots, f_N : \Pi_p(U) \longrightarrow \mathbb{R}^{n-p}$ such that*

$$\lim V_i \cap U = (\mathrm{gr}\, f_1 \cap U) \cup \dots \cup (\mathrm{gr}\, f_N \cap U).$$

Proof. For simplicity of notation, we assume throughout the proof that $\eta = 1$; the proof for general η is similar. Let $x \in \lim V_i \setminus \lim \mathrm{fr}(V_i)$, and choose $\epsilon > 0$ such that $B(x, 3\epsilon) \cap \mathrm{fr}(V_i) = \emptyset$ for all i (after passing to a subsequence if necessary). We let $U := B(x, \epsilon)$ and $U' := W \times W'$, where $W := \Pi_p(U)$ and

$$W' := \left\{ w \in \mathbb{R}^{n-p} : \ |w_k - x_{p+k}| < 3p\epsilon \text{ for } k = 1, \dots, n-p \right\}.$$

We now fix an i. By our assumptions, for any $z \in W$

($*$) there is a $\delta > 0$ such that $V_i \cap (B(z, \delta) \times W')$ is the union of at most N disjoint graphs of p-Lipschitz functions from $B(z, \delta)$ to W'.

Let $x \in V_i \cap U$; we claim that the component C of $V_i \cap U'$ that contains x is the graph of a p-Lipschitz function $g : W \longrightarrow W'$.

To see this, we choose δ as in ($*$) for $z := \Pi_p(x)$ and let $g : B(z, \delta) \longrightarrow W'$ be the corresponding p-Lipschitz function such that $g(z) = (x_{p+1}, \dots, x_n)$. We extend g to all of W as follows: for each $v \in \mathrm{bd}\, W$, we let $v' \in [z, v]$ be the point closest to v such that g extends to a p-Lipschitz function g_v along the line segment $[z, v']$ satisfying $\mathrm{gr}(g_v) \subseteq V_i \cap U'$. Then ($*$) implies that $v' = v$ for each $v \in \mathrm{bd}\, W$.

Moreover, the extension $g : W \longrightarrow W'$ defined in this way is continuous (and hence p-Lipschitz): let $v \in W$ be such that g is continuous at v' for every $v' \in [z, v)$. Let δ' be obtained for this v in place of z as in ($*$), and let $h_1, \dots, h_q : B(v, \delta') \longrightarrow W'$ be the corresponding distinct p-Lipschitz functions. We assume that $g(v) = h_1(v)$. Shrinking δ' if necessary, we may assume that there is $\mu > 0$ such that for any $s, t \in B(v, \delta')$ and any $1 \le k < l \le q$ we have $|h_k(s) - h_l(t)| > \mu$. Let $v' \in [z, v) \cap B(v, \delta')$ be close enough to v so that $|g(v') - g(v)| < \mu/4$; then $g(v') = h_1(v')$ as well. Since g is continuous at v', it follows that $g(s) = h_1(s)$ for all s sufficiently close to v'. But then the continuity of g along the radial segments $[z, t]$, $t \in \mathrm{bd}\, W$, and our choice of δ' imply that $g = h_1$ in a neighbourhood of v. This proves the claim.

By the claim, for all i there are definable p-Lipschitz functions $f_{1,i}, \dots, f_{N,i} : W \longrightarrow \mathbb{R}^{n-p}$ such that every connected component of $V_i \cap U'$ intersecting U is the graph of some $f_{l,i}$, either $f_{l,i} = f_{l',i}$ or $\mathrm{gr}\, f_{l,i} \cap \mathrm{gr}\, f_{l',i} = \emptyset$ for all $l, l' \in \{1, \dots, N\}$, and

$$V_i \cap U = (\mathrm{gr}\, f_{1,i} \cap U) \cup \dots \cup (\mathrm{gr}\, f_{N,i} \cap U).$$

Passing to a subsequence if necessary, we may therefore assume that each sequence $(f_{l,i})_i$ converges to a p-Lipschitz function $f_l : W \longrightarrow \mathbb{R}^{n-p}$. Clearly $\mathrm{gr}\, f_l \subseteq \lim V_i$. On the other hand, if $x' \in \lim V_i \cap U$, then $x' \in \lim(V_i \cap U)$, so by the above $x' \in \lim(\mathrm{gr}\, f_{l,i} \cap U)$ for some l, that is, $x' \in \mathrm{gr}\, f_l$. $\square$

We denote by G_n^l the **Grassmannian** of all l-subspaces of $\mathbb{R}^n$, identified with its representation as an algebraic subvariety of $\mathbb{R}^{n^2}$ as in Bochnak et al. [1, Section 3.4.2]. We put $G_n := \bigcup_l G_n^l$, and below we write $\Pi : \mathbb{R}^n \times G_n \longrightarrow \mathbb{R}^n$ for the projection on the first n coordinates. For a manifold $W \subseteq M$ of dimension k, we let $g_W : W \longrightarrow G_n^k$ be the **Gauss map** defined by $g_W(x) := T_x W$, and we put $T^1 W := \mathrm{gr}(g_W)$.

Next, we let $\eta > 0$ and $d : M \longrightarrow G_n^p$ be a p-distribution.

Definition 1.3. The distribution d is called **η-bounded at** $x \in M$ if there is a matrix $L = (l_{ij}) \in \mathrm{M}_{n-p,p}(\mathbb{R})$ such that $\|L\| := \max_{i,j} |l_{i,j}| \leq \eta$ and $d(x) = \{(u, Lu) : \ u \in \mathbb{R}^p\}$. The distribution d is **η-bounded** if d is η-bounded at every $x \in M$.

Remark. If d is η-bounded, then every integral manifold of d is η-bounded.

Let Σ_n be the collection of all permutations of $\{1, \dots, n\}$. For $\sigma \in \Sigma_n$, we write $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for the map defined by $\sigma(x_1, \dots, x_n) := (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, and we define

$$M_{\sigma,\eta} := \{x \in M : \ \sigma^* d \text{ is } \eta\text{-bounded at } x\}.$$

Each set $M_{\sigma,\eta}$ is open in M; moreover, the proof of Corollary 4 in [12] yields:

Lemma 1.4. *If $\eta > 1$, then $M = \bigcup_{\sigma \in \Sigma_n} M_{\sigma,\eta}$.* $\square$

Finally, we make the following conventions in this paper: given two distributions $d, e : M \longrightarrow G_n$, we write $d \cap e : M \longrightarrow G_n$ for the distribution defined by $(d \cap e)(x) := d(x) \cap e(x)$, and we write $d \subseteq e$ if $d(x) \subseteq e(x)$ for all $x \in M$. If $d : M \longrightarrow G_n$ is a distribution, we say that d **has dimension** if $d(M) \subseteq G_n^m$ for some $m \leq n$; in this situation, we put $\dim(d) := m$.

2. Nested distributions

We fix an o-minimal expansion $\mathcal{R}$ of the real field and denote by $\mathcal{P}(\mathcal{R})$ its Pfaffian closure [16]. Let $M \subseteq \mathbb{R}^n$ be a definable manifold of dimension m and

$$d = (d_0, \dots, d_k) : M \longrightarrow G_n^m \times \cdots \times G_n^{m-k}$$

be a definable nested distribution on M.

Definitions 2.1. A tuple $V = (V_0, \dots, V_k)$ is a **nested integral manifold of** d if each V_i is an integral manifold of d_i and $V_0 \supseteq \cdots \supseteq V_k$. Moreover, V is a **nested (Rolle) leaf of** d if $V_0 = M$, V_1 is a (Rolle) leaf of d_1 and for all $i = 2, \dots, k$, V_i is a (Rolle) leaf of $d_i|_{V_{i-1}}$.

A set $W \subseteq M$ is a **Rolle leaf of** d_k if there is a nested Rolle leaf $(W_0, \dots, W_k)$ of d with $W_k = W$; in this situation, the leaves $W_0, \dots, W_k$ are uniquely determined by W. Let $d = (d_0, \dots, d_k)$ be a nested distribution on M.

We call d **integrable** if for every $x \in M$, there is a nested integral manifold $V = (V_0, \dots, V_k)$ of d such that $x \in V_k$.

Remark. The theorem of Froebenius implies the following: if M and d are of class C^2, then d is integrable if and only if each d_i is integrable in the sense of Definition 1.3 in [13], that is, if and only if for each i, the definable set $I(d_i) \subseteq M$ is equal to M.

In view of the previous remark, we call d **nowhere integrable** if $I(d_i) = \emptyset$ for some $i \in \{1, \dots, k\}$.

Convention. From now on, we always assume that a definable nested distribution on M is integrable unless explicitely stated otherwise.

Example 2.2. Let $M \subseteq \mathbb{R}^n$ be a manifold of dimension m, and let $\Omega = (\omega_1, \dots, \omega_k)$ be a nested Pfaffian system on M. For $i = 1, \dots, k$, we put

$$d_i(x) := \ker \omega_1(x) \cap \cdots \cap \ker \omega_i(x);$$

then $d := (g_M, d_1, \dots, d_k)$ is a nested distribution on M.

Conversely, let $d = (d_0, \dots, d_k)$ be a nested distribution on M. We define unit vector fields $a_i = (a_{i1}, \dots, a_{in})$ on M, for $i = 1, \dots, n$, by induction on i as follows: let a_1 be the unit vector field orthogonal to d_1, and for $i > 1$ let a_i be the unit vector field orthogonal to the vector space spanned by $d_i \cup \{a_1, \dots, a_{i-1}\}$. Finally, put $\omega_i := a_{i1}dx_1 + \cdots + a_{in}dx_n$ for $i = 1, \dots, k$. Then $\Omega := (\omega_1, \dots, \omega_k)$ is a nested Pfaffian system on M.

In the above notation, it is clear that d is definable if and only if Ω is, and that d is integrable if and only if Ω is. Moreover, $(V_0, V_1, \dots, V_k)$ is a nested integral manifold (leaf, Rolle leaf) of d if and only if $(V_1, \dots, V_k)$ is a nested integral manifold (leaf, Rolle leaf) of Ω.

Let $N \subseteq M$ be a submanifold of M and assume that, for all $i = 0, \dots, k$, the distribution $g_N \cap d_i$ has dimension. In this situation, the **restriction** $d|_N$ **of** d **to** N is the nested distribution on N obtained by listing, in order of decreasing dimension, the set $\{g_N \cap d_0|_N, \dots, g_N \cap d_k|_N\}$. (Note that $d|_N$ can be a tuple of length less than k.)

Let $\mathcal{D}$ be a set of distributions on M. We adapt the following definition from [14] to our setting: a submanifold N of M is **compatible with** $\mathcal{D}$ if $g_N \cap g|_N$ has dimension for every $g \in \mathcal{D}$. A collection $\mathcal{C}$ of submanifolds of M is **compatible with** $\mathcal{D}$ if every $C \in \mathcal{C}$ is compatible with $\mathcal{D}$. Similarly, we say that N (resp. $\mathcal{C}$) is **compatible with** d if N (resp. $\mathcal{C}$) is compatible with the set $\{d_0, \dots, d_k\}$.

Lemma 2.3. *Let $A \subseteq \mathbb{R}^n$ be a definable set, and assume that each $g \in \mathcal{D}$ is definable. Then there is a finite partition $\mathcal{P}$ of M into definable C^2 cells such that $\mathcal{P}$ is compatible with A and each $\mathcal{D}$.*

Proof. This is a straightforward adaptation of Lemma 2.1 in [16] along the lines of Example 2.2. □

Next, we review Khovanskii Theory for nested Rolle leaves. Let $\Delta = \{d^1, \dots, d^q\}$ be a family of definable nested distributions on M; we write $d^p = (d^p_0, \dots, d^p_{k(p)})$ for $p = 1, \dots, q$.

We associate to Δ the following set of distributions on M:

$$\mathcal{D}_\Delta := \left\{ d^0_0 \cap \cdots \cap d^{p-1}_{k(p-1)} \cap d^p_j : \ p = 1, \dots, q \text{ and } j = 0, \dots, k(p) \right\},$$

where we put $d^0_0 := g_M$. If $N \subseteq M$ is a submanifold such that $g_N \cap g$ has dimension for every $g \in \mathcal{D}_\Delta$, we let $d^{\Delta,N} = \left(d^{\Delta,N}_0, \dots, d^{\Delta,N}_{k(\Delta,N)}\right)$ be the nested distribution on N obtained by listing the set $\{g|_N : \ g \in \mathcal{D}_\Delta\}$ in order of decreasing dimension. In this situation, if V_p is an integral manifold of $d^p_{k(p)}$, for $p = 1, \dots, q$, then the set $N \cap V_1 \cap \cdots \cap V_q$ is an integral manifold of $d^{\Delta,N}_{k(\Delta,N)}$.

Definition 2.4. Let $C \subseteq \mathbb{R}^n$ be a definable manifold. A definable function $\phi : C \longrightarrow (0, \infty)$ is a **carpeting function for** C if ϕ is continuous and $1/\phi$ is proper.

Adapting Lemma 2.5 in [16] to our setting, we obtain:

Lemma 2.5. *Let N be a definable C^2 cell contained in U and compatible with $\mathcal{D}_\Delta$, and suppose that* $\dim\left(d^{\Delta,N}_{k(\Delta,N)}\right) > 0$. *Then there is a C^1 carpeting function $\phi : N \longrightarrow (0, \infty)$ for N such that the definable set*

$$\begin{aligned} B &:= \left\{ a \in N : \ d^{\Delta,N}_{k(\Delta,N)}(a) \subseteq \ker d\phi(a) \right\} \\ &= \left\{ a \in N : \ \nabla_N \phi(a) \textit{ is orthogonal to } d^{\Delta,N}_{k(\Delta,N)}(a) \textit{ in } T_a N \right\} \end{aligned}$$

has dimension less than $\dim(N)$. □

Theorem 2.6. *Let $A \subseteq \mathbb{R}^n$ be a definable set. Then there exists a $K \in \mathbb{N}$ such that, whenever L_p is a Rolle leaf of $d^p_{k(p)}$ for each $p = 1, \dots, q$, then $A \cap L_1 \cap \cdots \cap L_q$ is a union of at most K connected manifolds.*

Proof. We proceed by induction on $\dim(A)$ and $k := k(1) + \cdots + k(q)$. The cases $d = 0$ or $k = 0$ being trivial, we assume that $d > 0$ and $k > 0$ and that the result holds for lower values of d and k. After shrinking p, we may also assume that $k(p) > 0$ for each $p = 1, \dots, q$. By a straightforward adaptation of Lemmas 1.4 and 1.6 in [16] to the present setting and by Lemma 2.3, it suffices to consider the case where $A = N$ is a C^2 cell contained in U and compatible with $\mathcal{D}_\Delta$. For each $p = 1, \dots, q$, we let L_p be a Rolle leaf of $d^p_{k(p)}$, and we put $L := L_1 \cap \cdots \cap L_q$.

Case $\dim\left(d^{\Delta,N}_{k(\Delta,N)}\right) = 0$**.** Let $\Delta' := \{d^1, \dots, d^{p-1}, (d^p_0, \dots, d^p_{k(p)-1})\}$, let L'_p be the Rolle leaf of $d^p_{k(p)-1}$ containing L_p and put $L' := L_1 \cap \cdots \cap L_{p-1} \cap L'_p$. Then $\dim(N \cap L') \le 1$; if $\dim(N \cap L') = 0$, we are done by Lemma 1.6 in [16] and the inductive hypothesis, so we assume that $\dim(N \cap L') = 1$. Since N is compatible with $\mathcal{D}_\Delta$ and $N \cap L'$ is an integral manifold of $d^{\Delta',N}_{k(\Delta',N)}$, it follows that $\dim\left(d^{\Delta',N}_{k(\Delta',N)}\right) = 1$ as well.

By the inductive hypothesis, there is a $K \in \mathbb{N}$ (depending only on N and Δ', but not on the particular Rolle leaves) such that the manifold $N \cap L'$ has at most K components. Let C be a component of $N \cap L'$. If $|C \cap L_q| > 1$, then by the Rolle property of L_q in L'_q (and the fact that C is a connected C^1 manifold of dimension 1), C is tangent at some point $x \in C$ to $d^p_{k(p)}|_{L'_q}$, which contradicts the assumption that $\dim\left(d^{\Delta',N}_{k(\Delta',N)}\right) = 1$. So $|C \cap L_q| \le 1$ for each component C of $N \cap L'$. Hence $|N \cap L| \le K$.

Case $\dim\left(d^{\Delta,N}_{k(\Delta,N)}\right) > 0$**.** Let ϕ and B be obtained from Lemma 2.5. Then $\dim(B) < \dim(A)$; so by the inductive hypothesis, there is a $K \in \mathbb{N}$, independent of the particular Rolle leaves chosen, such that $B \cap L$ has at most K components. Since $N \cap L$ is a closed, embedded submanifold of N, ϕ attains a maximum on every component of $N \cap L$, and any point in $N \cap L$ where ϕ attains a local maximum belongs to B. Hence $N \cap L$ has at most K components. ☐

Corollary 2.7. (1) *Let $\mathcal{C}$ be a partition of M into definable C^2 cells compatible with $\mathcal{D}_\Delta$. Then there is a $K \in \mathbb{N}$ such that, for every $C \in \mathcal{C}$ and every Rolle leaf L_p of $d^p_{k(p)}$ with $p = 1, \dots, q$, the set $C \cap L_1 \cap \cdots \cap L_q$ is a union of at most K Rolle leaves of $d^{\Delta,C}_{k(\Delta,C)}$.*

(2) *Let $\mathcal{A}$ be a definable family of sets. Then there is a $K \in \mathbb{N}$ such that whenever $A \in \mathcal{A}$ and L_p is a Rolle leaf of $d^p_{k(p)}$ for each p, the set $A \cap L_1 \cap \cdots \cap L_q$ is a union of at most K connected manifolds.*

Proof. For (2), proceed as in the proof of Corollary 2.7 of [16]. ☐

It follows from part (1) above that the collection of all nested Pfaffian sets over $\mathcal{R}$ is closed with respect to taking finite intersections and Cartesian products. Since definable C^p cells are definable diffeomorphic to some appropriate $\mathbb{R}^l$, we also get the following corollary from part (1) above:

Corollary 2.8. *Let $A \subseteq M$ be a definable set and $p \in \mathbb{N} \cup \{\infty, \omega\}$ with $p \ge 1$, and assume that $\mathcal{R}$ admits C^p cell decomposition. Then there are $N \in \mathbb{N}$ and* *a finite collection* $(C_j, \psi_j, e_j)_{1 \le j \le s}$ *such that the collection* $(C_j)_j$ *is a* C^p *cell decomposition of* A *and for each* $j = 1, \dots, s$,

(i) $\psi_j : \mathbb{R}^{d_j} \longrightarrow C_j$ *is a definable* C^p *diffeomorphism;*
(ii) $e_j = (e_{j,0}, \dots, e_{j,k_j})$ *is a definable nested distribution on* $\mathbb{R}^{d_j}$ *of class* C^{p-1};
(iii) *if* V *is a Rolle leaf of* d_k, *there are Rolle leaves* $V_{j,r}$ *of* e_{j,k_j} *for* $j = 1, \dots, s$ *and* $r = 1, \dots, N$ *such that* $A \cap V = \bigcup_{j,r} \psi_j(V_{j,r})$.

□

Finally, we let Ω be associated to d as in Example 2.2, and we let $V = (V_0, V_1, \dots, V_k)$ be a nested Rolle leaf of d (and hence of Ω). Then in the notation of Definition 4.3 in [16], we have $V_i \in \mathcal{L}(\mathcal{R}_{i-1})$ for $i = 1, \dots, k$. Therefore:

Corollary 2.9. *Every Rolle leaf of* d_k *in* M *is quantifier-free definable in* $\mathcal{P}(\mathcal{R})$. □

3. Admissible nested Pfaffian limits

Let $M \subseteq \mathbb{R}^n$ be a manifold of dimension m definable in $\mathcal{R}$, let $g : M \longrightarrow G_n^p$ be a p-distribution on M tangent to M, and suppose that g is integrable.

There is an equivalence relation $\sim_g$ on M associated to g, given by

$$x \sim_g y \quad \text{iff} \quad x \text{ and } y \text{ belong to the same leaf of } g.$$

Clearly, if $\sim_g$ is definable, then so is g; however, the converse is not true in general.

Lemma 3.1. *Assume that* $\sim_g$ *is definable. Then there is a finite partition* $\mathcal{N}$ *of* M *into definable manifolds such that, for each* $N \in \mathcal{N}$,

(1) $\dim(N) \ge p$ *and* g *is tangent to* N, *and*
(2) *there is a definable nested distribution* $e = (e_0, \dots, e_k)$ *on* N *such that each* $\sim_{e_j}$ *is definable and* $e_k = g|_N$.

Proof. By o-minimality, there is a definable set $E \subseteq M$ such that $x \nsim_g y$ for all $x, y \in E$ and for all $x \in M$, there is a $y \in E$ with $x \sim_g y$. Let $\mathcal{C}$ be a finite decomposition of E into definable C^1 cells. Refining $\mathcal{C}$ if necessary, we may assume that for each $C \in \mathcal{C}$, the distribution $g_C \cap g$ has dimension 0. For $C \in \mathcal{C}$, we now set $F_C := \{x \in M : \ x \sim_g y \text{ for some } y \in C\}$ and put

$$\mathcal{N} := \{F_C : \ C \in \mathcal{C}\}.$$

We claim that this $\mathcal{N}$ works. To see this, first note that $\mathcal{N}$ is a partition of M. Next, it follows from foliation theory (see for instance Chapter III in Camacho and Lins Neto [2]) that each $N \in \mathcal{N}$ is a manifold, and clearly g is tangent to each $N \in \mathcal{N}$. Since each $N \in \mathcal{N}$ is definable, we assume from now on, after replacing M by N, m by $\dim(N)$ and g by $g|_N$ for each $N \in \mathcal{N}$, that E is a definable C^1 cell in M that is transverse to g, and we prove the lemma for this case with $\mathcal{N} = \{M\}$; in particular, $\dim(E) = m - p$. We let $p_E : M \longrightarrow E$ be the definable map defined by

$$p_E(x) := \text{ the unique } y \in E \text{ such that } x \sim_g y.$$

Since E is a cell, there is a definable diffeomorphism $\phi_E : E \longrightarrow \mathbb{R}^p$ for some $p < n$. This implies that there is a definable nested distribution $e' = (e'_0, \dots, e'_{m-p})$ on E such that each $\sim_{e'_j}$ is definable. For each $j = 1, \dots, m-p$, we now define an equivalence relation $\sim_j$ on M by

$$x \sim_j y \quad \text{iff} \quad p_E(x) \sim_{e'_j} p_E(y).$$

Then each $\sim_j$ is definable, and by foliation theory again, $\sim_j = \sim_{e_j}$ for some definable distribution $e_j : M \longrightarrow G_n^{m-j}$ such that g is tangent to e_j. The lemma now follows with $e := (g_M, e_1, \dots, e_{m-p})$. $\square$

For the rest of this section, we fix a definable nested distribution $d = (d_0, \dots, d_k)$ on M. We put $\dim(d) := m - k$, and we define the **degree of d** as

$$\deg(d) := |\, \{i \in \{0, \dots, k\} : \sim_{d_i} \text{ is not definable}\}\,|.$$

We call d **separated** if there is an $l \in \{0, \dots, k\}$ such that $\sim_{d_i}$ is definable if and only if $i \in \{0, \dots, l\}$.

Remark. If d is separated, then so is $d' := (d_0, \dots, d_{k-1})$, and $\deg(d') < \deg(d)$.

For a p-distribution g on M tangent to M, we put

$$\deg(g) := \inf\{\deg(e) : \ e = (e_0, \dots, e_{m-p}) \text{ is a definable nested distribution on } M \text{ and } g = e_{m-p}\} \in \mathbb{R} \cup \{\infty\}.$$

Lemma 3.1 now implies:

Corollary 3.2. *Let g be a p-distribution on M tangent to M such that $\deg(g) < \infty$, and let $e = (e_1, \dots, e_{m-p})$ be a definable nested distribution on M such that $g = e_{m-p}$ and $\deg(e) = \deg(g)$. Then e is separated.* $\square$

Remark 3.3. Let $C \subseteq \mathbb{R}^n$ be a definable cell of dimension $n - k$, where $0 \le k \le n$. Then there are a definable nested distribution $e = (e_0, \dots, e_k)$ of degree 0 on an open set $U \subseteq \mathbb{R}^n$ and a nested leaf $V = (V_0, \dots V_k)$ of e such that $C = V_k$. (The proof is elementary and left to the reader.)

Let $e = (e_0, \dots, e_l)$ be a definable nested distribution on M with $l \le k$. We call e a **core distribution of** d if

(i) $\sim_{d_i}$ is definable for $i = 1, \dots, k - l$, and
(ii) $d_i = d_{k-l} \cap e_{i-k+l}$ for $i = k - l + 1, \dots, k$.

Remarks. (1) Let e be a core distribution of d. Then $\deg(d) \le \deg(e)$, and if e is separated, then so is d. Moreover, if f is a core distribution of e, then f is also a core distribution of d.

(2) Let $N \subseteq M$ be a definable submanifold of M such that $g_N \cap d_i$ has dimension for each i. If d has core distribution $e = (e_0, \dots, e_l)$ and each $g_N \cap e_j$ has dimension, then $d|_N$ has core distribution $e|_N$.

Example 3.4. Let $\phi : M \longrightarrow \mathbb{R}$ be a definable function, and define $d_\phi : M \longrightarrow G_n$ by $d_\phi(x) := \ker d\phi(x) \subseteq T_x M$. Let $\mathcal{C}$ be a cell decomposition of M compatible with d and such that for every $C \in \mathcal{C}$ and $j = 0, \dots, k$, the distribution $g_C \cap d_j \cap d_\phi : C \longrightarrow G_n$ has dimension.

Let $\mathcal{C}'$ be the set of all $C \in \mathcal{C}$ such that $g_C \cap d_k \not\subseteq d_\phi$, and fix an arbitrary $C \in \mathcal{C}'$. We associate to C a nested distribution d^C on C derived from d as follows: writing $d|_C = (e_0^C, \dots, e_l^C)$ with $l = l(C) \le k$, we put $d^C := (d_0^C, \dots, d_{l+1}^C)$, where $d_0^C := g_C$, $d_1^C := g_C \cap d_\phi$ and $d_{j+1}^C := e_j^C \cap d_\phi$ for $j = 1, \dots, l$. Then d^C has core distribution $d|_C$ and $\dim(d^C) < \dim(d)$; in particular, $\deg(d^C) \le \deg(d)$.

Finally, after refining $\mathcal{C}$ we may assume that $\mathcal{C}$ is a Whitney stratification (Theorem 4.8 in Van den Dries and Miller [5]). In this situation, the union of all cells in $\mathcal{C}'$ is an open subset M' of M, and $d' = (d'_0, \dots, d'_{k+1})$ is a definable nested distribution on M' with core distribution $d|_{M'}$, where $d'_0 := g_{M'}$ and $d'_{j+1} := (d_j \cap d_\phi)|_{M'}$ for $j = 0, \dots, k$.

Remarks. In the situation of the previous example, if d is separated, then $d|_{M'}$ and d' are separated, and if d has a separated core distribution, then so do $d|_{M'}$ and d'.

An integral manifold V of d_k is **admissible** if either

(i) V is a definable leaf of d_k, or
(ii) V is a Rolle leaf of d_k, or
(iii) $k > 1$ and d has a core distribution $e = (e_0, \dots, e_l)$ with $l < k$, and there is a definable leaf B of d_{k-l} and an admissible integral manifold W of e_l such that $V = W \cap B$.

In case (iii) above, we call W an **core** of V (**corresponding to** e).

Remark. Let $e = (e_0, \dots, e_l)$ be a core distribution of d, and let W be an admissible integral manifold of e_l. By Corollary 2.7(2), there is an $N \in \mathbb{N}$ such that every admissible integral manifold of d_k with core W is the union of at most N (embedded) leaves of d_k.

A sequence (V_i) of integral manifolds of d_k is **admissible** if $k > 1$ and there are a core distribution $e = (e_0, \dots, e_l)$ of d with $l < k$, a sequence of definable leaves (B_i) of d_{k-l} and an admissible integral manifold W of e_l such that $V_i = W \cap B_i$ for every $i \in \mathbb{N}$. In this situation, we call W an **core** of the admissible sequence (V_i) (**corresponding to** e).

Remark. By the previous remark, for every admissible sequence (V_i) of integral manifolds of d_k, there is an $N \in \mathbb{N}$ such that each V_i is the union of at most N (embedded) leaves of d_k.

Let (V_i) be an admissible sequence of integral manifolds of d_k. If the sequence $(\operatorname{cl} V_i)$ converges in the Hausdorff metric to a compact set K, we call K an **admissible nested Pfaffian limit over** $\mathcal{R}$, or **admissible limit over** $\mathcal{R}$ for short, and write $K = \lim V_i$. (In this paper, if $\mathcal{R}$ is clear from context, we will not explicitely mention it.) In this situation, we say that K **is obtained from** d, and we put

$$\deg(K) := \min \{\deg(f) : \ K \text{ is obtained from } f\}.$$

Also in the above situation, if W is a core of the sequence (V_i), we say that K **has core** W.

Remarks.
(1) We leave it to the reader to verify that if K is an admissible limit of degree p, then there is a definable nested distribution $d = (d_1, \ldots, d_k)$ on a definable manifold M with core distribution e such that e is separated and $\deg(e) = p$, and there is an admissible sequence (V_i) of integral manifolds of d_k with core corresponding to e, such that $K = \lim V_i$.
(2) We think of the core W above as representing the "non-definable content" of the admissible sequence (V_i) or the admissible limit K, and it is crucial to our arguments in Section 4 that only the "definable content", represented by the sequence (B_i) above, is allowed to vary with i.
(3) Following Example 2.2, it follows from Corollary 2.7(2) that every Pfaffian limit over $\mathcal{R}$, as defined in [13], is a finite union of admissible limits over $\mathcal{R}$.

The following situation is central to our use of admissible limits.

Example 3.5. In the situation of Example 3.4, we let $e = (e_0, \ldots, e_l)$ be a core distribution of d and W an admissible leaf of e_l; note that $e|_{M'}$ is a core distribution of $d|_{M'}$ and hence of d'. Then $W \cap M'$ is a finite union of admissible integral manifolds $W_1', \ldots, W_q'$ of $e|_{M'}$, and by Corollary 2.7(2), there is a $\nu \geq 0$ such that for every admissible integral manifold V of d_k with core W, the set $V \cap M'$ is the union of ν admissible integral manifolds $V_{1,V}', \ldots, V_{\nu,V}'$ (not necessarily distinct) of d_k' with cores among $W_1', \ldots, W_q'$ corresponding to $e|_{M'}$. By o-minimality, there is a $\mu \geq 0$ such that for every $r > 0$, the set $\phi^{-1}(r) \cap M'$ has μ components $\Phi_{1,r}', \ldots, \Phi_{\mu,r}'$ (not necessarily distinct). Then each $\Phi_{l,r}' \cap V_{j,V}'$ is an admissible integral manifold of d_{k+1}' with core $V_{j,V}'$ corresponding to $d|_{M'}$ (and hence with core among $W_1', \ldots, W_q'$ corresponding to $e|_{M'}$).

Assume now that $\phi(x) > 0$ for all $x \in M$. Let V be an admissible integral manifold of d_k with core W, and let (r_ι) be a sequence of positive real numbers such that $r_\iota \to 0$ and $K := \lim_\iota (\phi^{-1}(r_\iota) \cap V)$ exists. (This situation arises for instance when expressing $\mathrm{fr}(V)$ as an admissible limit; see the beginning of Section 4.) Then for each $j = 1, \dots, \nu$ and $l = 1, \dots, \mu$, the sequence $\left(\Phi'_{l,r_\iota} \cap V'_{j,V}\right)$ is an admissible sequence of integral manifolds of d'_{k+1} with core $V'_{j,V}$ corresponding to $d|_{M'}$ (and hence with core among $W'_1, \dots, W'_q$ corresponding to $e|_{M'}$). Passing to a subsequence, we may assume that $K_{j,l}(V) := \lim \left(\Phi'_{l,r_\iota} \cap V'_{j,V}\right)$ exists for each j and l.

Lemma 3.6. *In the above situation, we have $K = \bigcup_{j=1}^{\nu} \bigcup_{l=1}^{\mu} K_{j,l}(V)$.*

Proof. To see this, let $N \in \mathbb{N}$ be such that for every $C \in \mathcal{C}$, the set $V \cap C$ is the union of N admissible integral manifolds $V_1^C, \dots, V_N^C$ (not necessarily distinct) of e_l^C, where $l := l(C)$, and for every $r > 0$ the set $\phi^{-1}(r) \cap C$ has N components $\Phi_{1,r}^C, \dots, \Phi_{N,r}^C$ (not necessarily distinct). Moreover, if $C \in \mathcal{C}'$ then each $\Phi_{j,r}^C$ is a leaf of d_1^C; in particular, for each j and l the sequence $\left(V_j^C \cap \Phi_{l,r_\iota}^C\right)_\iota$ is an admissible sequence of integral manifolds of d_{l+1}^C. Since M is bounded and after passing to a subsequence if necessary, we may assume that $K_{j,l}^C := \lim_\iota \left(V_j^C \cap \Phi_{l,r_\iota}^C\right)$ exists for every $C \in \mathcal{C}'$ and $j, l = 1, \dots, N$. Clearly

$$\bigcup_{j=1}^{\nu} \bigcup_{l=1}^{\mu} K_{j,l}(V) = \bigcup_{C \in \mathcal{C}'} \bigcup_{j,l=1}^{N} K_{j,l}^C.$$

Thus, to finish our argument, we let $z \in K$ and show that $z \in K_{j,l}^C$ for some $C \in \mathcal{C}'$ and $j, l \in \{1, \dots, N\}$. Let $x_\iota \in V \cap \phi^{-1}(r_\iota)$ be such that $\lim x_\iota = z$. Let $C \in \mathcal{C}$ be such that infinitely many x_ι belong to C; passing to a subsequence, we may assume that $x_\iota \in C$ for all ι. Then $C \in \mathcal{C}'$: otherwise, we have $g_C \cap d_k \subseteq d_\phi$, which implies that $V \cap \phi^{-1}(r) \cap C = \emptyset$ for all but finitely many r. (It is crucial here that V, representing the "non-definable content" of K, is fixed, while only the "definable content", represented by ϕ, is allowed to vary.) Thus, passing again to a subsequence, we may assume that there are $j, l \in \{1, \dots, N\}$ such that $x_\iota \in V_j^C \cap \Phi_{l,r_\iota}^C$ for all i. Hence $z \in K_{j,l}^C$, as required. $\square$

4. Some basic facts about admissible limits

Let $M \subseteq \mathbb{R}^n$ be a bounded, definable manifold and $d = (d_0, \dots, d_k)$ be a definable nested distribution on M with core distribution $e = (e_0, \dots, e_l)$; we assume that M, d and e are of class C^p with $p \geq 1$. The goal of this section is to prove that the various exceptional sets obtained in Sections 5 and 6 have "small" dimension and degree. These are technical results; the reader may wish to go directly to Sections 5 and 6 and refer back here as needed.

We first look at the interplay between taking the frontier and taking an admissible limit. For the next lemma, suppose there is a definable C^p carpeting function ϕ on M such that the definable set $B := \{x \in M :\ d_k \subseteq d_\phi\}$ has dimension less than m, where $d_\phi(x) := \ker d\phi(x)$ for all $x \in M$. (By Lemma 2.5, such an ϕ exists whenever M is a definable C^p cell.) We adopt the notations introduced in Examples 3.4 and 3.5; refining $\mathcal{C}$, we may assume $\mathcal{C}$ is compatible with B. Then $\dim(M \setminus M') < m$, and we get:

Lemma 4.1. (1) *Let V be an admissible integral manifold of d_k, and let $V'_1 \dots, V'_q$ be the admissible integral manifolds of $d|_{M'}$ whose union is $V \cap M'$. Then*

$$\operatorname{fr}(V) = \lim_\iota \left(\phi^{-1}(r_\iota) \cap V\right) = \bigcup_{j=1}^{\nu} \bigcup_{l=1}^{\mu} K_{j,l}(V),$$

that is, $\operatorname{fr}(V)$ is a finite union of admissible limits obtained from d' whose cores corresponding to $d|_{M'}$ are among $V'_1, \dots, V'_q$.

(2) *Assume that $l < k$, let W be an admissible integral manifold of e_l, and let $W'_1, \dots, W'_q$ be the admissible integral manifolds of $e_l|_{M'}$ whose union is $W \cap M'$. Let (V_i) be an admissible sequence of integral manifolds of d_k with core W such that $K := \lim_i \operatorname{fr}(V_i)$ exists. Then K is a finite union of admissible limits obtained from d' whose cores are among $W'_1, \dots, W'_q$.*

Proof. The properties of ϕ imply that $\operatorname{fr}(V) = \lim_\iota \left(\phi^{-1}(r_\iota) \cap V\right)$, so part (1) follows from Lemma 3.6. For part (2), note that by part (1) we have

$$K = \lim_i \left(\lim_\iota \left(\phi^{-1}(r_\iota) \cap V_i\right)\right) = \bigcup_{j=1}^{\nu} \bigcup_{l=1}^{\mu} \lim_i K_{j,l}(V_i).$$

Hence $K = \bigcup_{j=1}^{\nu} \bigcup_{l=1}^{\mu} \lim_i \left(\Phi'_{l,r_{\iota(i)}} \cap V'_{j,V_i}\right)$ for some subsequence $(\iota(i))_i$. However, by Corollary 2.7(2), there is a $\rho \in \mathbb{N}$ such that each set $\Phi'_{l,r_{\iota(i)}} \cap V'_{j,V_i}$ is the union of admissible integral manifolds $G'_{l,j,1,i}, \dots, G'_{l,j,\rho,i}$ (not necessarily disjoint) of d'_{k+1} with cores among $W'_1, \dots, W'_q$. Passing to a subsequence again and reindexing, we may assume that for each l, j and s, the sequence $(G'_{l,j,s,i})$ is admissible with core among $W'_1, \dots, W'_q$, and that $\lim_i G'_{l,j,s,i}$ exists. Then $K = \bigcup_{j=1}^{\nu} \bigcup_{l=1}^{\mu} \bigcup_{s=1}^{\rho} \lim_i G'_{l,j,s,i}$, which finishes the proof. □

Next, we study some (very simple) intersections involving admissible limits. For the next two lemmas, we assume that $l < k$. We let (V_i) be an admissible sequence of integral manifolds of d_k with core W corresponding to e, and we assume that $K := \lim_i V_i$ exists. We define the following distributions on $M \times \mathbb{R}^n \times (0, \infty)$, where we write (x, y, ϵ) for the typical element of $M \times \mathbb{R}^n \times (0, \infty)$

with $x \in M$, $y \in \mathbb{R}^n$ and $\epsilon > 0$:

$$\begin{aligned}
\widetilde{d}_0 &:= g_{M\times\mathbb{R}^n\times(0,\infty)}, && \\
\widetilde{d}_1 &:= \ker d\epsilon \cap \widetilde{d}_0, && \\
\widetilde{d}_{1+j} &:= \ker dy_j \cap \widetilde{d}_j && \text{for } j = 1,\dots,n, \\
\widetilde{d}_{1+n+j}(x,y,\epsilon) &:= (d_j(x)\times\mathbb{R}^{n+1}) \cap \widetilde{d}_{n+j}(x,y,\epsilon) && \text{for } j = 1,\dots,k, \\
\widetilde{e}_j(x,y,\epsilon) &:= e_j(x)\times\mathbb{R}^{n+1} && \text{for } j = 0,\dots,l.
\end{aligned}$$

We also put $\widetilde{W} := W \times \mathbb{R}^n \times (0,\infty)$. Then $\widetilde{d} := (\widetilde{d}_0,\dots,\widetilde{d}_{1+n+k})$ is a definable nested distribution on $M \times \mathbb{R}^n \times (0,\infty)$ with core distribution $\widetilde{e} := (\widetilde{e}_0,\dots,\widetilde{e}_l)$. Thus, $\deg(\widetilde{d}) \le \deg(d)$ and if e is separated, then so is $\widetilde{e}$. Moreover, whenever $(y_i,\epsilon_i) \in \mathbb{R}^n \times (0,\infty)$ for $i \in \mathbb{N}$, the sequence $\big(V_i \times \{(y_i,\epsilon_i)\}\big)$ is an admissible sequence of integral manifolds of $\widetilde{d}$ with core $\widetilde{W}$ corresponding to $\widetilde{e}$.

Lemma 4.2. *Let $C \subseteq \mathbb{R}^n$ be a definable cell. Then there is a definable open subset $\widetilde{M}$ of $M \times \mathbb{R}^n \times (0,\infty)$ such that*

$$\operatorname{fr}(K\cap C) = \bigcup_{q=1}^{p} \Pi_n(K_q),$$

where $K_1,\dots,K_p \subseteq \mathbb{R}^{2n+1}$ are admissible limits obtained from $\widetilde{d}|_{\widetilde{M}}$ with cores among the admissible integral manifolds of $\widetilde{e}|_{\widetilde{M}}$ whose union is $\widetilde{W}\cap\widetilde{M}$.

Proof. We let ϕ be a C^1 carpeting on C (as obtained from Lemma 2.5, say). Put

$$\widetilde{M} := \left\{(x,y,\epsilon) \in M\times\mathbb{R}^n\times(0,\infty):\ d(x,\phi^{-1}(y_1)) < \epsilon\right\},$$

an open, definable subset of $M \times \mathbb{R}^n \times (0,\infty)$, where we put $d(x,\emptyset) := \infty$ for all $x \in M$. Since K is compact, we have $\operatorname{fr}(K\cap C) = \lim_{r\to 0}(\phi^{-1}(r)\cap K)$. Moreover, for every $r > 0$ we have $\phi^{-1}(r)\cap K = \lim_{\epsilon\to 0}\lim_i(V_i \cap \widetilde{M}_{\widetilde{r},\epsilon})$, where $\widetilde{r} := (r,\dots,r)$. Hence there are $r_i \to 0$ and $\epsilon_i \to 0$ such that

$$\operatorname{fr}(K\cap C) = \lim\left(V_i \cap \widetilde{M}_{\widetilde{r}_i,\epsilon_i}\right) = \lim\left(\left(V_i\times\{(\widetilde{r}_i,\epsilon_i)\}\right)\cap\widetilde{M}\right).$$

The proposition now follows from the remark preceding this lemma and an argument similar to that used in the proof of Proposition 4.1. $\square$

A similar, but somewhat easier, proof yields the following lemma; we leave the details to the reader.

Lemma 4.3. *Let $C \subseteq \mathbb{R}^n$ be a definable cell. Then there is a definable open subset $\widetilde{M}$ of $M \times \mathbb{R}^n \times (0,\infty)$ such that*

$$\mathrm{cl}(K \cap C) = \bigcup_{q=1}^{p} \Pi_n(K_q),$$

where $K_1, \dots, K_p \subseteq \mathbb{R}^{2n+1}$ are admissible limits obtained from $\widetilde{d}|_{\widetilde{M}}$ with cores among the admissible integral manifolds of $\widetilde{e}|_{\widetilde{M}}$ whose union is $\widetilde{W} \cap \widetilde{M}$. □

Finally, we establish some crucial facts about the dimension of admissible limits.

Lemma 4.4. *Let $K \subseteq \mathbb{R}^n$ be an admissible limit obtained from d. Then K is definable in $\mathcal{P}(\mathcal{R})$ and $\dim(K) \leq \dim(d)$.*

Proof. Write $k = m - \dim(d)$, let also (V_i) be an admissible sequence of integral manifolds of d_k such that $K = \lim_i V_i$, and let V be an admissible integral manifold of e_{k-l} and (B_i) a sequence of leaves of d_l such that $V_i = V \cap B_i$ for each $i \in \mathbb{N}$. Since the family $\{V \cap B :\ B \text{ is a leaf of } d_l\}$ is definable in $\mathcal{P}(\mathcal{R})$, the lemma follows from the main theorem in [12]. □

Definition 4.5. Let $K \subseteq \mathbb{R}^n$ be an admissible limit obtained from d. We say that K is **proper** if $\dim(K) = \dim(d)$.

The next proposition is adapted from Lemma 3.6 in [13]; it may be interpreted as a "fiber cutting" lemma for admissible limits.

Proposition 4.6. *Let $K \subseteq \mathbb{R}^n$ be an admissible limit obtained from d and $m \leq n$. Then there are proper admissible limits $K_1, \dots, K_p \subseteq \mathbb{R}^n$ such that $\Pi_m(K) = \Pi_m(K_1) \cup \cdots \cup \Pi_m(K_p)$, and such that $\deg(K_q) \leq \deg(K)$ and $\dim \Pi_m(K_q) = \dim(K_q)$ for each $q = 1, \dots, p$.*

Proof. We proceed by induction on $r := \dim(M)$; the case $r = 0$ is trivial, so we assume that $r > 0$ and that the proposition holds for lower values of r. Let (V_i) be an admissible sequence of integral manifolds of d_k such that $K = \lim_i V_i$, and let V be an admissible integral manifold of e_{k-l} and (B_i) a sequence of leaves of d_l such that $V_i = V \cap B_i$ for each $i \in \mathbb{N}$. Without loss of generality, we may assume that l is minimal; then by Corollary 3.2, we may also assume that e is separated. If $l = 0$, we are done by Theorem 3.1 in [4] or the Main Theorem of [12], so we assume that $l > 0$. In this situation, the fact that V is an admissible integral manifold of e_l implies that V is a Rolle leaf of e_l.

Choosing a suitable C^2-cell decomposition of M compatible with d and using the inductive hypothesis, we reduce to the case where M is a definable C^2 manifold such that for every $l \leq m$ and every strictly increasing map

$\lambda : \{1, \dots, l\} \longrightarrow \{1, \dots, m\}$, the rank of $\Pi^n_\lambda|_{T_x V_i}$ is constant on V_i and independent of i; we denote this rank by r_λ.

Let $s := \dim(\Pi_m(X))$; if $s = \dim(d)$, then we are done by Lemma 4.4, so we assume that $s < \dim(d)$. Let $\lambda : \{1, \dots, s\} \longrightarrow \{1, \dots, m\}$ be strictly increasing; since $s < \dim(d)$, we have $\dim(V_i \cap (\Pi^n_\lambda)^{-1}(y)) \geq \dim(d) - s > 0$ if $V_i \cap (\Pi^n_\lambda)^{-1}(y) \neq \emptyset$, that is, $r_\lambda < \dim(d)$. Hence by Lemma 2.5, there is a closed, definable set $B_\lambda \subseteq M$ such that $\dim(B_\lambda) < r$ and

- for all $y \in \mathbb{R}^s$ and $i \in \mathbb{N}$, each component of the fiber $V_i \cap (\Pi^n_\lambda)^{-1}(y)$ intersects the fiber $B_\lambda \cap (\Pi^n_\lambda)^{-1}(y)$.

In particular $\Pi^n_\lambda(V_i \cap B_\lambda) = \Pi^n_\lambda(V_i)$ for all i, and for all $y \in \mathbb{R}^s$, every component of $\Pi_m(V_i) \cap (\Pi^m_\lambda)^{-1}(y)$ intersects the fiber $\Pi_m(V_i \cap B_\lambda) \cap (\Pi^m_\lambda)^{-1}(y)$.

Passing to a subsequence if necessary, we may assume that the sequence $(V_i \cap B_\lambda)_i$ converges for every strictly increasing $\lambda : \{1, \dots, s\} \longrightarrow \{1, \dots, m\}$, and we put $X^\lambda := \lim(V_i \cap B_\lambda)$. Choosing a C^2-cell decomposition of B_λ and using the inductive hypothesis, we see that the proposition holds with each X^λ in place of X. It therefore remains to show that $\Pi_m(X) = \bigcup_\lambda \Pi_m(X^\lambda)$. To see this, we fix a strictly increasing $\lambda : \{1, \dots, s\} \longrightarrow \{1, \dots, m\}$; since each $\Pi_m(X^\lambda)$ is closed, it suffices by Remark 3.5 of [13] to establish the following

Claim. Let $y \in \Pi_\lambda(X)$, and let $x \in \Pi_m(X) \cap (\Pi^m_\lambda)^{-1}(y)$ be isolated. Then $x \in \Pi_m(X^\lambda)$.

Proof. Note that $\Pi_m(X) = \lim \Pi_m(V_i)$ since M is bounded. Let $x_i \in \Pi_m(V_i)$ be such that $x_i \to x$, and put $y_i := \Pi^m_\lambda(x_i)$. Let $C_i \subseteq \mathbb{R}^m$ be the component of $\Pi_m(V_i) \cap (\Pi^m_\lambda)^{-1}(y_i)$ containing x_i, and let x'_i belong to $C_i \cap \Pi_m(V_i \cap B_\lambda)$. Since also $\Pi_m(X^\lambda) = \lim \Pi_m(V_i \cap B_\lambda)$, we may assume, after passing to a subsequence if necessary, that $x'_i \to x' \in \Pi_m(X^\lambda)$. We show that $x' = x$, which then proves the claim. Assume for a contradiction that $x' \neq x$, and let $\delta > 0$ be such that $\delta \leq \|x - x'\|$ and

$$B(x, \delta) \cap \Pi_m(X) \cap (\Pi^m_\lambda)^{-1}(y) = \{x\}, \tag{4.1}$$

where $B(x, \delta)$ is the open ball with center x and radius δ. Then for all sufficiently large i, there is an $x''_i \in C_i$ such that $\delta/3 \leq \|x''_i - x_i\| \leq 2\delta/3$, because $x_i, x'_i \in C_i$ and C_i is connected. Passing to a subsequence if necessary, we may assume that $x''_i \to x'' \in \Pi_m(X)$. Then $x'' \in B(x, \delta)$ with $x'' \neq x$, and since $x''_i \in C_i$ implies that $\Pi^m_\lambda(x''_i) = y_i$, we get $\Pi^m_\lambda(x'') = y$, contradicting (4.1). $\square$

5. Blowing-up in jet space

We fix a bounded, definable manifold $M \subseteq \mathbb{R}^n$ of dimension m and a tangent, definable nested distribution $d = (d_1, \dots, d_k)$ on M, and we assume that both are of class C^2. We fix an arbitrary $j \in \{1, \dots, k\}$.

Definition 5.1. Put $n_1 := n+n^2$ and let $\Pi : \mathbb{R}^{n_1} \longrightarrow \mathbb{R}^n$ denote the projection on the first n coordinates. We define

$$M^1 := \mathrm{gr}(d_j) \subseteq \mathbb{R}^{n_1},$$

$$d_l^1 := (\Pi|_{M^1})^* d_l, \text{ the pull-back to } M^1 \text{ of } d_l \text{ via } \Pi, \text{ for } l = 1, \dots, k.$$

We call $d^1 := (d_1^1, \dots, d_k^1)$ the **blow-up of** d **along** d_j (we do not need to explicitely indicate its dependence on j). Finally, for $l \in \{1, \dots, k\}$ and an integral manifold V of d_l, we define

$$V^1 := (\Pi|_{M^1})^{-1}(V),$$

the **lifting of** V **along** d_j. Note that, in this situation, V^1 is an integral manifold of d_l^1, and if $l = j$, then $V^1 = T^1 V$.

Next, we write $M = \bigcup M_{\sigma,2}$, where σ ranges over Σ_n and the $M_{\sigma,2}$ are as in Lemma 1.4 with d there equal to d_j here.

Definition 5.2. For a leaf $V \subseteq M$ of d_j and $\sigma \in \Sigma_n$, we put $V_\sigma := V \cap M_{\sigma,2}$, and we define

$$F^1(V) := \bigcup_{\sigma \in \Sigma_n} \mathrm{fr}\left((V_\sigma)^1\right).$$

For the next proposition, we let $D \subseteq \mathrm{cl}(M^1)$ be a definable cell such that $C := \Pi(D)$ has the same dimension as D and for every $\sigma \in \Sigma_n$, either $C \cap \mathrm{cl}(M_{\sigma,2}) = \emptyset$ or $C \subseteq \mathrm{cl}(M_{\sigma,2})$. Then $D = \mathrm{gr}(g)$, where $g : C \longrightarrow G_n^{m-j}$ is a definable distribution, and we assume that the following hold:

(i) the distribution $g \cap g_C$ has dimension;
(ii) if g is tangent to C, then either g is integrable or g is nowhere integrable.

We also assume that there is a definable set $W \subseteq \mathrm{cl}(M^1)$ such that both W and $W \cup D$ are open in $\mathrm{cl}(M^1)$. In this situation, for any sequence (V_i) of leaves of d_j such that $\lim V_i^1$ and $\lim F^1(V_i)$ exist, we put

$$L_{(V_i)} := (D \cap \lim V_i^1) \setminus \left(\lim F^1(V_i) \cup \mathrm{cl}(W \cap \lim V_i^1)\right).$$

Finally, we let $g^1 : D \longrightarrow G_{n_1}^{m-j}$ be the pullback of g to M^1 via Π.

Proposition 5.3 (see also Propositions 2.3 in [13] and 8 in [12]). *Assume that d is admissible, and let D and W be as above. Then exactly one of the following holds:*

(1) $L_{(V_i)} = \emptyset$ *for every admissible sequence (V_i) of leaves of d_i such that $\lim V_i^1$ and $\lim F^1(V_i)$ exist;*
(2) *g is tangent to C and integrable, and for every admissible sequence (V_i) of leaves of d_i such that $\lim V_i^1$ and $\lim F^1(V_i)$ exist, the set $L_{(V_i)}$ is an embedded integral manifold of g^1 and is open in $\lim V_i^1$.*

In particular, if D is open in M^1, then $D \cap \lim V_i^1$ is a union of leaves of $d_j^1|_D$.

Remark. Let $\sigma \in \Sigma_n$. Then the conjugate map

$$g^\sigma := \sigma \circ g \circ \sigma^{-1} : \sigma(C) \longrightarrow G_n^{m-j}$$

satisfies

$$g(x) \subseteq T_x C \quad \text{if and only if} \quad g^\sigma(\sigma(x)) \subseteq T_{\sigma(x)}\sigma(C).$$

Moreover, σ induces a diffeomorphism $\sigma^1 : \mathbb{R}^n \times G_n \longrightarrow \mathbb{R}^n \times G_n$ defined by $\sigma^1(x,y) = (\sigma(x), \sigma y)$, where σy denotes the element of G_n corresponding to the linear subspace of $\mathbb{R}^n$ obtained as the image under σ from the linear subspace of $\mathbb{R}^n$ corresponding to y. (Note that σ^1 is also just a permutation of coordinates.) Then

$$(g^\sigma)^1 = \sigma^1 \circ g^1 \circ (\sigma^1)^{-1},$$

and if (V_i) is a sequence of Rolle leaves in M of d_j such that $\lim V_i^1$ and $\lim F^1(V_i)$ exist, then $\lim \sigma(V_i^1)$ also exists and

$$\sigma(C) \cap \lim \sigma(V_i^1) = \sigma^1(C \cap \lim V_i^1).$$

Proof of Proposition 5.3. By the remark, after replacing M by $\sigma(M_{\sigma,2})$ for each $\sigma \in \Sigma_n$, we may assume for the rest of this proof that d_j is 2-bounded; in particular, for every embedded leaf V of d_j, we have $F^1(V) = \mathrm{fr}(V^1)$.

Let (V_i) be an admissible sequence of leaves of d_j such that $\lim V_i^1$ and $\lim \mathrm{fr}(V_i^1)$ exist, and write $L := L_{(V_i)}$. Let $N \in \mathbb{N}$ be such that for every open box $U \subseteq \mathbb{R}^n$ and every $i \in \mathbb{N}$, the set $U \cap V_i$ has at most N connected components.

Assume that $L \neq \emptyset$, and choose an arbitrary $(x,y) \in L$ with $x \in \mathbb{R}^n$ and $y \in G_n$. Since $W \cup D$ is open in $\mathrm{cl}(M^1)$ and $(x,y) \notin \mathrm{cl}(W \cap \lim V_i^1)$, there is an open box $B \subseteq \mathbb{R}^{n_1}$ such that $(x,y) \in B$ and

$$\mathrm{cl}(B) \cap \lim V_i^1 \subseteq D \setminus \big(\lim \mathrm{fr}(V_i^1) \cup \mathrm{cl}(W \cap \lim V_i^1)\big).$$

Writing $B = B_0 \times B_1$ with $B_0 \subseteq \mathbb{R}^n$ and $B_1 \subseteq \mathbb{R}^{n^2}$, we may also assume that $D \cap \big(\mathrm{cl}(B_0) \times \mathrm{fr}(B_1)\big) = \emptyset$, because D is the graph of the continuous map g and C is locally closed.

On the other hand, $B \cap \lim V_i^1 = B \cap \lim(B \cap V_i^1) = B \cap \lim V_{i,B}^1$, where $V_{i,B} := \{x \in V_i : (x, T_x V_i) \in B\}$. We now claim that $x \notin \lim \mathrm{fr}(V_{i,B})$: in fact, the previous paragraph implies that $\mathrm{fr}(V_i) \cap \mathrm{cl}(V_{i,B}) = \emptyset$ for all sufficiently large i, and hence $\mathrm{fr}(V_{i,B}) \subseteq \mathrm{fr}(B_0)$ for all sufficiently large i, which proves the claim.

Since each V_i is an embedded, closed submanifold of M, we now apply Lemma 1.2 with $V_{i,B}$ in place of V_i and $\eta = 2$, to obtain a corresponding open neighbourhood $U \subseteq B_0$ of x and $f_1, \dots, f_N : \Pi_{m-j}(U) \longrightarrow \mathbb{R}^{n-m+j}$. We let $l \in \{1, \dots, N\}$ be such that $x \in \mathrm{gr}(f_l)$. We claim that f_l is differentiable at $z := \Pi_{m-j}(x)$ with $T_x \,\mathrm{gr}(f_l) = g(x)$; since x is arbitrary, this then implies that each $\mathrm{gr}(f_l)$ is an embedded, connected integral manifold of g. Assumption (ii) and [13, Lemma 1.6] now imply that g is tangent to C and integrable. Since

$(x, y) \in L$ was arbitrary, it follows that L is an embedded integral manifold of g^1, as desired.

To prove the claim, let $f_{l,i} : \Pi_{m-j}(U) \longrightarrow \mathbb{R}^{n-m+j}$ be the functions corresponding to f_l as in the proof of Lemma 1.2. After a linear change of coordinates if necessary, we may assume that $g(x) = \mathbb{R}^{m-j} \times \{0\}$ (the subspace spanned by the first $m - j$ coordinates). It now suffices to show that f_l is η-Lipschitz at x for every $\eta > 0$, since then $T_x \operatorname{gr}(f_l) = \mathbb{R}^{m-j} \times \{0\}$. So let $\eta > 0$; since $\lim V^1_{i,B} \subseteq D = \operatorname{gr}(g)$ and $x \in C$, and because C is locally closed and g is continuous, there is a neighborhood $U' \subseteq U$ of x such that $\operatorname{gr}(f_{l,i}) \cap U'$ is $\frac{\eta}{m-j}$-bounded for all sufficiently large i. Thus by Lemma 1.2 again, f_l is η-Lipschitz at x, as required.

Finally, if D is open in M^1, then we can take $W := \emptyset$, and by assumption we have $C \cap M_{\sigma,2} = \emptyset$ or $D \subseteq M_{\sigma,2}$ for each $\sigma \in \Sigma_n$. Hence $F^1(V_i) \cap D = \emptyset$ for every $\sigma \in \Sigma_n$, and it follows that $L_{(V_i)} = D \cap \lim V_i$ in this case. $\square$

6. Rewriting admissible limits

We fix a $p \in \mathbb{N} \cup \{\infty, \omega\}$ such that $p \geq 1$. Let $N \subseteq \mathbb{R}^m$ be a definable C^p cell and $f = (f_0, \dots, f_l)$ a definable nested C^p distribution on N. Let also $n \leq m$ and $D \subseteq \mathbb{R}^n$ a definable C^p cell such that $\Pi_n(N) \subseteq D$, and let $h : D \longrightarrow G^\nu_n$ be a tangent, definable, integrable C^p distribution on D, with $\nu \leq \dim(D)$. We assume that for all $d \leq n$ and all $\lambda : \{1, \dots, d\} \longrightarrow \{1, \dots, n\}$, the dimension of the spaces

$$F_\lambda(y) := \Pi^m_\lambda(f_l(y)) \quad \text{and} \quad F^h_\lambda(y) := \Pi^n_\lambda\big(\Pi^m_n(f_l(y)) \cap h(\Pi^m_n(y))\big)$$

is constant as y ranges over N; we denote these dimensions below by $\dim(F_\lambda)$ and $\dim(F^h_\lambda)$, respectively. For the identity map $\lambda : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$, we write $F := F_\lambda$ and $F^h := F^h_\lambda$ and put $d := \dim(F^h) \leq \nu$. Finally, we assume that $\dim(f_l) = \dim(F) = \nu + 1$.

Definition 6.1. Our assumptions imply that there is a $\lambda : \{1, \dots, d+1\} \longrightarrow \{1, \dots, n\}$ such that $\dim(F_\lambda) = d + 1$ and $\dim(F^h_\lambda) = d$. We let $f_{l+1} : N \longrightarrow G^\nu_m$ be the distribution defined by

$$f_{l+1}(y) := f_l(y) \cap (\Pi^m_\lambda)^{-1}\left(F^h_\lambda(y)\right).$$

Clearly, the distribution f_{l+1} is definable, $f_{l+1} \subseteq f_l$ and $\dim(\Pi^m_n(f_{l+1}(y)) = \dim(f_{l+1}(y)) = \nu$ for all $y \in N$.

Next, let $Z \subseteq N$ be an (embedded) integral manifold of f_l, and let $L \subseteq D$ be an (embedded) integral manifold of h. By our assumption on f_l, $\Pi^m_n|_Z$ is an immersion; we assume here in addition that $\Pi^m_n(Z)$ is a submanifold of $\mathbb{R}^n$. Then by our assumptions and the Rank Theorem, $\Pi^m_n(Z) \cap L$ is either empty or a submanifold of D of dimension d. Let also $L' \subseteq \Pi^m_n(Z) \cap L$ be a submanifold of dimension d. Again by our assumption on F^h_λ, $\Pi^n_\lambda|_{L'}$ is an immersion; we also assume here that $\Pi^n_\lambda(L')$ is a submanifold of $\mathbb{R}^{d+1}$. In this situation, we define

$$Z(L') := Z \cap (\Pi^m_\lambda)^{-1}(\Pi^n_\lambda(L')).$$

Note that, by the Rank Theorem, the set $Z(L')$ is an integral manifold of f_{l+1}.

Lemma 6.2. *The tuple $f' := (f_0, \dots, f_l, f_{l+1})$ is a definable nested distribution on N.*

Proof. It remains to show that f_{l+1} is integrable. The integrability of f_l and h and our assumptions imply that for every $y \in N$, there are an integral manifold Z of f_l containing y and an integral manifold L of h containing $\Pi^m_n(y)$ such that $L' := \Pi^m_n(Z) \cap L$ and $\Pi^n_\lambda(L')$ are embedded manifolds. So by the remarks in Definition 6.1, the corresponding $Z(L')$ is an integral manifold of f_{l+1} containing y. $\square$

Next, we let Z be an integral manifold of f_l and $L \subseteq D$ an integral manifold of h, and we assume that both Z and L are definable in $\mathcal{P}(\mathcal{R})$.

Lemma 6.3. *There are integral manifolds $Z'_1, \dots, Z'_q$ of f_{l+1}, contained in Z and definable in $\mathcal{P}(\mathcal{R})$, such that*

$$\Pi^m_n(Z) \cap L \subseteq \bigcup_{p=1}^{q} \Pi^m_n(Z'_p). \tag{6.1}$$

Proof. By Lemma 1.1, we may assume that $\Pi^m_n(Z)$ is a submanifold of N. Again by Lemma 1.1, we have $\Pi^m_n(Z) \cap L = L'_1 \cup \cdots \cup L'_q$, where each L'_p is an open subset of $\Pi^m_n(Z) \cap L$ such that $\Pi^n_\lambda(L'_p)$ is a submanifold of $\mathbb{R}^{d+1}$. Now we take $Z'_p := Z(L'_p)$, and we claim that these Z'_p work. To see this, it remains to prove (6.1). Let $x \in \Pi^m_n(Z) \cap L$, and let $p \in \{1, \dots q\}$ be such that $x \in L'_p$; we show that $x \in \Pi^m_n(Z'_p)$. Then $Z \cap (\Pi^m_\lambda)^{-1}(\Pi^n_\lambda(x)) \subseteq Z'_p$, and since $(\Pi^m_n)^{-1}(x) \subseteq (\Pi^m_\lambda)^{-1}(\Pi^n_\lambda(x))$, it follows that $Z \cap (\Pi^m_n)^{-1}(x) \subseteq Z'_p$, as required. $\square$

We are now ready to prove Proposition 1 of the introduction.

Theorem 6.4 (see also Theorem 5.1 in [13]**).** *Let $X \subseteq \mathbb{R}^n$ be an admissible limit. Then there exists a $q \in \mathbb{N}$, and for each $p = 1, \dots, q$ there exist $n_p \geq n$, a definable manifold $N_p \subseteq \mathbb{R}^{n_p}$, a definable nested distribution $f_p = (f_{p,1}, \dots, f_{p,k(p)})$ on N_p and a nested integral manifold $U_p = (U_{p,1}, \dots, U_{p,k(p)})$ of f_p such that*

$$X \subseteq \bigcup_{p=1}^{q} \Pi_n\left(U_{p,k(p)}\right)$$

and for each p, the tuple U_p is definable in $\mathcal{P}(\mathcal{R})$, $\dim(U_{p,k(p)}) \leq \dim(X)$ and $\Pi_n|_{U_{p,k(p)}}$ is an immersion.

Proof. By induction on the pair $(\deg(X), \dim(X))$ (simultaneously for all n), where we consider $\mathbb{N}^2$ with its lexicographic ordering. If $\deg(X) = 0$, then X is definable in $\mathcal{R}$ by Theorem 3.1 in [4] or the Main Theorem in [12], so the proposition follows from cell decomposition and Remark 3.3. If $\dim(X) = 0$, then X is finite and the proposition follows again from Remark 3.3. Therefore, we assume that $\deg(X) > 0$ and $\dim(X) > 0$ and that the proposition holds for admissible limits Y such that $(\deg(Y), \dim(Y)) < (\deg(X), \dim(X))$. Moreover, by Proposition 4.6, we may assume that X is a proper admissible limit.

Let $M \subseteq \mathbb{R}^n$ be a definable manifold of dimension m, $d = (d_1, \dots, d_k)$ an admissible nested distribution on M and (V_i) an admissible sequence of leaves of d_k such that $X = \lim V_i$, $\deg(X) = \deg(d)$ and $\dim(X) = \dim(d)$. By C^{m+2} cell decomposition, we may assume that M and d are of class C^{m+2}; we adopt all the corresponding notions introduced before this proposition. Passing to a subsequence if necessary, we may assume that $X^j := \lim_i V_i^j$ and $\lim_i F_j(V_i)$ exist for $j = 0, \dots, m$ (so $X^0 = X$). Since X is proper, each X^j is also proper. By o-minimality, Proposition 4.1(2), Lemma 4.4 and the inductive hypothesis,

(6.2) for every $j = 0, \dots, m$,

the proposition holds with $\lim\limits_i F_j(V_i)$ in place of X.

For each $j = 0, \dots, m+1$ the nested distribution $e^j := (d_1^j, \dots, d_{k-1}^j)$ is admissible and satisfies $\deg(e^j) < \deg(d)$. Moreover, by assumption there is an admissible sequence (U_i) of leaves of d_{k-1}^0 such that $V_i \subseteq U_i$ for all $i \in \mathbb{N}$. Passing to a subsequence if necessary, we may assume that $Y^j := \lim_i U_i^j$ exists for all j, where

$$U_i^0 := U_i \text{ and for } j = 1, \dots, m, \; U_i^j := \operatorname{gr}\left(d_k^{j-1}|_{U_i^{j-1}}\right).$$

By the inductive hypothesis, there is a $q_j \in \mathbb{N}$ and for each $p = 1, \dots, q_j$, there exist $m_{j,p} \geq n_j$, a definable manifold $N_{j,p} \subseteq \mathbb{R}^{m_{j,p}}$, a definable nested distribution $f_{j,p} = \left(f_{j,p,1}, \dots, f_{j,p,k(j,p)}\right)$ on $N_{j,p}$ and a nested integral manifold $Z_{j,p} = \left(Z_{j,p,1}, \dots, Z_{j,p,k(j,p)}\right)$ of $f_{j,p}$ such that

$$Y^j \subseteq \bigcup_{p=1}^{q_j} \Pi_{n_j}\left(Z_{j,p,k(j,p)}\right)$$

and for each j and p, the tuple $Z_{j,p}$ is definable in $\mathcal{P}(\mathcal{R})$, $\dim(Z_{j,p,k(j,p)}) \leq \dim(X) + 1$ and $\Pi_{n_j}|_{Z_{j,p,k(j,p)}}$ is an immersion.

For $j = 0, \dots, m+1$, we let $\mathcal{C}^j$ be a C^2 cell decomposition of $\mathbb{R}^{n_j}$ compatible with M, $\mathrm{fr}(M^m)$, $\{M^j_\sigma : \sigma \in \Sigma_{n_j}\}$ and $\{\mathrm{fr}(M^j_\sigma) : \sigma \in \Sigma_{n_j}\}$, and we put

$$\mathcal{C}^j_M := \left\{C \in \mathcal{C}^j : C \subseteq \mathrm{cl}(M^j)\right\}.$$

Refining each C^j if necessary, we may assume for $j = 0, \dots, m$,

(i) $\mathcal{C}^j$ is a stratification compatible with $\Pi^{j+1}_j(\mathcal{C}^{j+1})$;

and for every $D \in \mathcal{C}^{j+1}$ that is the graph of a distribution $g : C \longrightarrow G^{m-k}_{n_j}$, where $C := \Pi^{j+1}_j(D)$,

(ii) the distribution $g \cap g_C$ has dimension;

(iii) if g is tangent to C, then either g is integrable, or g is nowhere integrable.

Refining the collections $\{N_{j,1}, \dots, N_{j,q_j}\}$ if necessary, we may furthermore assume for each j, C and D as above that

(iv) the collection $\{\Pi_{n_{j+1}}(N_{j+1,p}) : p = 1, \dots, q_{j+1}\}$ is compatible with $\mathcal{C}^{j+1}$;

(v) for each $p \in \{1, \dots, q_{j+1}\}$ such that $\Pi_{n_{j+1}}(N_{j+1,p}) \subseteq D$, the dimension of the spaces $\Pi_{n_{j+1}}(f(y))$ and $\Pi_{n_{j+1}}(f(y)) \cap g^1(x))$ is constant as y ranges over $N_{j+1,p}$, where $f := f_{j+1,p,k(j+1,p)}$ and $x := \Pi_{n_{j+1}}(y)$.

By Proposition 4.2, o-minimality, Proposition 4.5 and the inductive hypothesis,

(6.3) for every $j = 0, \dots, m$ and every $E \in \mathcal{C}^j$,
the proposition holds with $\mathrm{fr}(X^j \cap E)$ in place of X.

We now fix a cell $C \in \mathcal{C}^j_M$ for some $j \in \{0, \dots, m\}$ such that $j \le \dim(C)$.

Claim: There exists $X^j_C \subseteq X^j$ such that $X^j \cap C \subseteq X^j_C$ and the proposition holds with X^j_C in place of X.

The proposition follows by applying this claim to each $C \in C^0_M$.

To prove the claim, we proceed by reverse induction on $\dim(C)$. Let

$$\mathcal{D}_C := \left\{D' \cap (\Pi^{j+1}_j)^{-1}(C) : D' \in \mathcal{C}^{j+1}_M,\ C \subseteq \Pi^{j+1}_j(D')\right\},$$

and fix an arbitrary $D \in \mathcal{D}_C$; it clearly suffices to prove the claim with X^{j+1} and D in place of X^j and C. Let $D' \in \mathcal{C}^{j+1}$ be such that $D \subseteq D'$; if $\dim(D') > \dim(C)$, then the claim with X^{j+1} and D in place of X^j and C follows from the inductive hypothesis, so we also assume that $\dim(D') = \dim(C)$. Thus, there is a distribution $g : C \longrightarrow G^{m-k}_{n_j}$ such that $D = \mathrm{gr}(g)$, and D is open in D'. Let

$$W := \bigcup \left\{E \in \mathcal{C}^{j+1}_M : \dim(E) > \dim(C)\right\};$$

since $\mathcal{C}^{j+1}$ is a stratification, both W and $W \cup D'$ are open in $\mathrm{cl}(M^{j+1})$, and since D is open in D', the set $W \cup D$ is also open in $\mathrm{cl}(M^{j+1})$. Hence by Proposition 5.3, the set $(X^{j+1} \cap D) \setminus \big(\lim_i F_{j+1}(V_i) \cup \mathrm{cl}(W \cap X^{j+1})\big)$ is an embedded integral manifold of g^1. But $D \cap \mathrm{cl}(W \cap X^{j+1}) \subseteq F$, where

$$F := \bigcup \left\{ \mathrm{fr}(X^{j+1} \cap E) : \ E \in \mathcal{C}_M^{j+1} \text{ and } \dim(E) > \dim(C) \right\},$$

so the set

$$L := (X^{j+1} \cap D) \setminus \left(\lim_i F_{j+1}(V_i) \cup F \right)$$

is a finite union of connected integral manifolds of g^1 definable in $\mathcal{P}(\mathcal{R})$. Thus by (6.2) and (6.3), to prove the claim with X^{j+1} and D in place of X^j and C, it now suffices to prove the proposition with L in place of X.

Let $p \in \{1, \ldots, q_{j+1}\}$ be such that $\Pi_{n_{j+1}}(N_{j+1,p}) \subseteq D$. If the dimension of $Z_{j+1,p,k(j+1,p)}$ is at most that of $\dim(X)$, we let $Z_{j+1,p}$ be one of the U_p we are looking for. Otherwise, by (iv) and (v) we can apply Lemma 6.3 with $m := m_{j+1,p}$, $N := N_{j+1,p}$, $f := f_{j+1,p}$, $Z := Z_{j+1,p}$, $l := k(j+1,p)$ and $h := g^1$ to obtain finitely many new U_p. This finishes the proof of the proposition. $\square$

7. Some properties of analytic o-minimal structures

We assume from now on that $\mathcal{R}$ admits analytic cell decomposition.

Definition 7.1. An open set $U \subseteq \mathbb{R}^n$ is **$\mathcal{R}$-normal** if there exists an analytic carpeting function on U.

Throughout this paper, since $\mathcal{R}$ is fixed, we shall simply say "normal" instead of "$\mathcal{R}$-normal".

Remark. If $U, V \subseteq \mathbb{R}^n$ are normal, then so are $U \cap V$ and $U \times V$.

Example 7.2. Every open, analytic and definable cell is normal.

Proposition 7.3. *Let $A \subseteq \mathbb{R}^n$ be open and definable. Then A can be covered by finitely many normal sets.*

Proof. By induction on n; we may assume that $A \neq \mathbb{R}^n$. The case $n = 0$ is trivial, so we assume that $n > 0$ and that the proposition holds for lower values of n. By analytic cell decomposition, it suffices to show that every analytic cell contained in A is in turn contained in a finite union of normal subsets of A.

So we let $C \subseteq A$ be an analytic cell; we proceed by induction on the dimension d of C. If $d = 0$, then C is a singleton and any ball centered at C and contained in A will do. So we assume that $d > 0$ and that every analytic cell of dimension less than d contained in A is in turn contained in a finite union of normal subsets of A. If $d = n$, then C is open and hence normal by Example 7.2; so we also assume that $d < n$.

After permuting coordinates if necessary, there is an open, analytic cell $D \subseteq \mathbb{R}^d$ and a definable, analytic map $g : D \longrightarrow \mathbb{R}^{n-d}$ such that $C = \mathrm{gr}(g)$. For $x \in \mathbb{R}^n$, we write $x = (y, z)$ with $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^{n-d}$. Define $\beta : D \longrightarrow (0, \infty)$ by

$$\beta(y) := \mathrm{dist}\big((y, g(y)), \mathbb{R}^n \setminus A\big).$$

By the inductive hypothesis and analytic cell decomposition, we may assume that β is analytic. Now we put

$$U := \left\{(y, z) \in A : \ y \in D \text{ and } \|z - g(y)\|^2 < \beta^2(y)\right\}.$$

This U is normal: given an analytic carpeting function $\gamma : D \longrightarrow (0, \infty)$, we define $\phi : U \longrightarrow (0, \infty)$ by

$$\phi(y, z) := \gamma(y)\big(\beta^2(y) - \|z - g(y)\|^2\big),$$

which is easily seen to be an analytic carpeting function on U. $\square$

Definition 7.4. Let $U \subseteq \mathbb{R}^n$ be normal and $A \subseteq U$. We say that A is **normal in** U if A is a finite union of sets of the form

$$\{x \in U : \ g(x) = 0, h(x) > 0\},$$

where $g : U \longrightarrow \mathbb{R}^q$ and $h : U \longrightarrow \mathbb{R}^r$ are definable and analytic. A **normal leaflet in** U (**of codimension** p) is a set of the form

$$A = \{x \in U : \ f(x) = g(x) = 0, h(x) > 0\},$$

where $f : U \longrightarrow \mathbb{R}^p$, $g : U \longrightarrow \mathbb{R}^q$ and $h : U \longrightarrow \mathbb{R}^r$ are analytic and definable and for all $x \in A$, the rank of f at x is p and $\ker df(x) \subseteq \ker dg(x)$.

Remark. Let $A \subseteq U$ be a normal leaflet in U of codimension p, say $A = \{x \in U : \ f(x) = g(x) = 0, h(x) > 0\}$ as in the previous definition; then $T_x A = \ker df(x)$ for all $x \in A$. Moreover, the restriction of $\phi := \delta \cdot \prod_{s=1}^r h_s : U \longrightarrow \mathbb{R}$ to A takes values in $(0, \infty)$, and $1/\phi|_A$ is a proper map. Thus, we call ϕ an **analytic carpeting function for** A **in** U.

Example 7.5. Let $U \subseteq \mathbb{R}^n$ be normal; then U is a normal leaflet in U. Let also $f : U \longrightarrow \mathbb{R}^p$ and $h : U \longrightarrow \mathbb{R}^r$ be analytic and definable; then the set

$$\begin{aligned}\{x \in U : \ f(x) = 0, h(x) > 0, \text{ and } f \text{ has rank } p \text{ at } x\}& \\ = \big\{x \in U : \ f(x) = 0, h(x) > 0, |df|^2(x) > 0\big\}&\end{aligned}$$

is a normal leaflet in U.

The following lemma is elementary, and its proof is left to the reader.

Lemma 7.6. *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be normal.*

(1) *If A and B are normal in U, then so are $A \cup B$, $A \cap B$ and $A \setminus B$.*

(2) *Let A be normal in U and B be normal in V. Then $A \times B$ is normal in $U \times V$. Moreover, if A and B are normal leaflets in U and V, respectively, then $A \times B$ is a normal leaflet in $U \times V$.*
(3) *Let $\phi : U \longrightarrow V$ be definable and analytic, and let B be normal in V. Then $\phi^{-1}(B)$ is normal in U.*
(4) *Let A be a normal leaflet in U of codimension p, and assume that there is a definable, analytic embedding $\phi : A \longrightarrow \mathbb{R}^{n-p}$. Then $\phi(A)$ is normal. Moreover, if $B \subseteq A$ is normal (resp., a normal leaflet) in U, then $\phi(B)$ is normal (resp., a normal leaflet) in $\phi(A)$.* $\square$

For the next proposition, we need a lemma (used again in Section 9):

Lemma 7.7. *Let $\eta : X \longrightarrow (0, \infty)$ and be definable in $\mathcal{P}(\mathcal{R})$ and put*

$$Y := \{(x, t) : \ x \in X \text{ and } 0 < t < \eta(x)\}.$$

Let also $\alpha : Y \longrightarrow [0, \infty)$ be definable in $\mathcal{P}(\mathcal{R})$, and assume that for every $x \in X$, the function $\alpha_x : (0, \eta(x)) \longrightarrow [0, \infty)$ defined by $\alpha_x(t) := \alpha(x, t)$ is semianalytic. Then there exists an $N \in \mathbb{N}$ such that for all $x \in X$, either ultimately $\alpha_x(t) = 0$ or ultimately $\alpha_x(t) > t^N$ (where "ultimately" abbreviates "for all sufficiently small $t > 0$").

Proof. By cell decomposition, for every $x \in X$ the function α_x is ultimately of constant sign. By Puiseux's Theorem, for every $x \in X$ such that α_x is ultimately positive, there are $c_x > 0$ and $r_x \in \mathbb{Q}$ such that ultimately $\alpha_x(t) = c_x t^{r_x} + o(t^{r_x})$. However, the set $R_X := \{r_x \in \mathbb{R} : \ x \in X\}$ is definable, since for all $x \in X$ we have $r_x = \lim_{t \to 0^+} t\alpha'_x(t)/\alpha_x(t)$. Since each r_x is rational, it follows that R_X is finite, so any $N > \max R_X$ will do. $\square$

Proposition 7.8 (Gabrielov [8]). *Let $U \subseteq \mathbb{R}^n$ be normal and A be normal in U. Then A is a finite union of normal leaflets in U.*

Proof. Let $g : U \longrightarrow \mathbb{R}^q$ and $h : U \longrightarrow \mathbb{R}^r$ be definable and analytic such that $A = \{x \in U : \ g(x) = 0, h(x) > 0\}$; we proceed by induction on $d := \dim(A)$. If $d = 0$, the proposition is trivial, so we assume that $d > 0$ and the proposition holds for lower values of d.

Let $\mathcal{C}$ be a finite decomposition of A into analytic cells; we shall show that for each $C \in \mathcal{C}$, there is a normal leaflet $A_C \subseteq A$ such that $\dim(C \setminus A_C) < d$. The proposition then follows from the inductive hypothesis, since $A \setminus \bigcup_{C \in \mathcal{C}} A_C$ is a normal subset of U of dimension less than d.

Fix a $C \in \mathcal{C}$, and let $\mathcal{G}$ be the set of all partial derivatives (of all orders) of $g_1, \dots, g_q$. Let M be the set of all natural numbers m for which there exist $f_1, \dots, f_m \in \mathcal{G}$ such that $C \subseteq \{x \in U : \ f_1(x) = \cdots = f_m(x) = 0\}$ and $df_1(a) \wedge \cdots \wedge df_m(a) \neq 0$ for some $a \in C$.

Put $p := \sup M \leq n - d$; we claim that $p = n - d$. To see this, let $f_1, \dots, f_p \in \mathcal{H}$, $a \in C$ and an open ball B centered at a be such that $C \cap B$ is a connected submanifold of

$$\Gamma := \{x \in U : \ f_1(x) = \cdots = f_p(x) = 0\} \cap B,$$

and such that Γ is a connected, analytic submanifold of codimension p contained in

$$\{x \in U : \ h(x) > 0, \ df_1(x) \wedge \cdots \wedge df_p(x) \neq 0\}.$$

The maximality of p now implies that $g(x) = 0$ for all $x \in \Gamma$, that is, $C \cap B = \Gamma$, which proves the claim.

Put $f := 0$ if $M = \emptyset$ and $f := (f_1, \dots, f_p)$ otherwise, where $f_1, \dots, f_p \in \mathcal{H}$ are as in the previous paragraph. Let

$$X := \{x \in U : \ f(x) = 0, \ h(x) > 0, \ \text{and } f \text{ has rank } p \text{ at } x\},$$

a normal leaflet in U. For $d \in \mathbb{N}$, we also let $S(f, g, d)$ be the set of all $\phi : U \longrightarrow \mathbb{R}$ for which there exist $d' \leq d$ and functions $\phi_0, \dots, \phi_{d'} : U \longrightarrow \mathbb{R}$ such that

(i) $\phi_0 \in \{g_1, \dots, g_q\}$;
(ii) for $i \in \{0, \dots, d'\}$, ϕ_{i+1} is one of the coefficient functions of $d\phi_i \wedge df$ if $M \neq \emptyset$, or of $d\phi_i$ if $M = \emptyset$, respectively;
(iii) $\phi = \phi_{d'}$,

and we put $X_d := \{x \in X : \ \phi(x) = 0 \text{ for all } \phi \in S(f, g, d)\}$.

Next, we let $\eta : X \longrightarrow (0, \infty)$ be a definable function such that for all $x \in X$,

$$B(x, 2\eta(x)) \subseteq \{x \in U : \ h(x) > 0, \ \text{and } f \text{ has rank } p \text{ at } x\}.$$

For $x \in X$ and $t \in (0, \eta(x))$, we put

$$\alpha(x, t) := \max\{|h|(y) : \ y \in X, \ \|y - x\| \leq t\}.$$

Note that α is definable, and for each $x \in X$ the function $\alpha_x : (0, \eta(x)) \longrightarrow [0, \infty)$ is semianalytic. Hence by Lemma 7.7, there is an $N \in \mathbb{N}$ such that either ultimately $\alpha_x = 0$ or ultimately $\alpha_x(t) > t^N$.

On the other hand, for all $x \in X$ we have that $x \in X_N$ if and only if α_x is ultimately positive. Hence X_N is the union of all those connected components of X on which g is equal to 0. Since X_N is a normal leaflet in U, we can take $X_C := X_N$. $\square$

8. Fiber cutting using normal leaflets

We describe in this section stratifications by leaflets adapted to nested distributions, building on the techniques found in Moussu and Roche [14], Lion and Rolin [11] and [16]. The goal is to obtain a corresponding fiber cutting lemma for nested distributions. We proceed along the lines of Section 2.

Let $U \subseteq \mathbb{R}^n$ be normal and $\mathcal{D}$ be a finite collection of definable, analytic distributions on U. Using Lemma 7.6 and Proposition 7.8 in place of cell decomposition, we obtain:

Lemma 8.1. *Let A be a normal subset of U. Then there is a finite partition $\mathcal{P}$ of A into normal leaflets in U such that $\mathcal{P}$ is compatible with $\mathcal{D}$.* $\square$

For the rest of this section, we let $\Delta = \{d^1, \dots, d^p\}$ be a set of definable, analytic nested distributions on U; we write $d^p = (d_0^p, \dots, d_{k(p)}^p)$ for $p = 1, \dots, q$ and associate $\mathcal{D}_\Delta$ to Δ as in Section 2. Let also A be a normal subset of U.

Lemma 8.2. *Assume that A is a normal leaflet in U compatible with $\mathcal{D}_\Delta$, and suppose that* $\dim\left(d_{k(\Delta,A)}^{\Delta,A}\right) > 0$. *Then there is an analytic carpeting function ϕ on A in U such that the definable set*

$$B := \left\{a \in A : \ \nabla_A \phi(a) \textit{ is orthogonal to } d_{k(\Delta,A)}^{\Delta,A}(a) \textit{ in } T_a A\right\}$$

has dimension less than $\dim(A)$.

Proof. Let ψ be an analytic carpeting function on A in U. For $u \in (0,\infty)^n$, we define $\psi_u : A \longrightarrow (0,\infty)$ by $\psi_u(x) := \psi(x)\phi_u(x)$, where

$$\phi_u(x) := (u_1 x_1^2 + \cdots + u_n x_n^2);$$

note that ψ_u is an analytic carpeting function on A in U. Now consider the definable set

$$D := \left\{(u,a) \in \mathbb{R}^n \times A : \ \nabla_A \psi_u(a) \text{ is orthogonal to } d_{k(\Delta,A)}^{\Delta,A}(a) \text{ in } T_a A\right\}.$$

If $\dim(D_u) < \dim(A)$ for some $u \in (0,\infty)^n$, we take $\phi := \psi_u$; so we assume for a contradiction that $\dim(D_u) = \dim(A)$ for all $u \in (0,\infty)^n$. Then by o-minimality, there are a nonempty, open $V \subseteq (0,\infty)^n$ and a nonempty, open subset W of A such that $V \times W \subseteq D$. Since W is a manifold of dimension $m := \dim(A)$, there are $1 \le j_1 < \cdots < j_m \le n$ and $a = (a_1, \dots, a_n) \in W$ such that $a_{j_l} \neq 0$ for $l = 1, \dots, m$. On the other hand,

$$\nabla_A \psi_u(a) = \phi_u(a) \nabla_A \psi(a) + 2\psi(a) \nabla_A(\phi_u|_A)(a);$$

hence an elementary calculation shows that the subset $\{\nabla_A \psi_u(a) : \ u \in V\}$ of $T_a A$ generates $T_a A$ as an $\mathbb{R}$-vector space, which contradicts $\dim\left(d_{k(\Delta,A)}^{\Delta,A}\right) > 0$. $\square$

Next, for $I \subseteq \{1, \dots, q\}$ we put $\Delta(I) := \{d^p : \ p \in I\}$. Combining Lemmas 8.1 and 8.2, we obtain

Proposition 8.3. *Let $I \subseteq \{1, \dots, q\}$. Then there is a finite collection $\mathcal{P}$ of normal leaflets in U contained in A such that $\mathcal{P}$ is compatible with $\mathcal{D}_{\Delta(J)}$ for* *every $J \subseteq \{1, \dots, q\}$, and such that whenever V_p is a Rolle leaf of $d^p_{k(p)}$ for $p = 1, \dots, q$, we have*

(i) $\dim\left(d^{\Delta(I),N}_{k(\Delta(I),N)}\right) = 0$ *for every* $N \in \mathcal{P}$;

(ii) *every component of* $A \cap \bigcap_{p \in I} V_p$ *intersects some leaflet in* $\mathcal{P}$.

Proof. By induction on $m := \dim(A)$; if $m = 0$, the proposition is trivial, so we assume that $m > 0$ and the proposition holds for lower values of m. Let $\mathcal{P}$ be a partition obtained from Lemma 8.1; replacing A by any of the leaflets in $\mathcal{P}$, we may assume by the inductive hypothesis that A is a normal leaflet in U compatible with $\mathcal{D}_\Delta$. If $\dim\left(d^{\Delta(I),A}_{k(\Delta(I),A)}\right) = 0$, we are done; otherwise, let ϕ and B be as in Lemma 8.2 with Δ_I in place of Δ.

Let V_p be a Rolle leaf of $d^p_{k(p)}$ for $p = 1, \dots, q$. By Lemma 8.1 and the inductive hypothesis, it now suffices to show that every component of $X := A \cap \bigcap_{p \in I} V_p$ intersects B. However, since $d^{\Delta(I),A}_{k(\Delta(I),A)}$ has dimension, X is a closed, embedded submanifold of A. Thus, ϕ attains a maximum on every component of X, and any point in X where ϕ attains a local maximum belongs to B. ☐

Corollary 8.4. *Let $d = (d_0, \dots, d_k)$ be a definable, analytic nested distribution on U and $m \le n$. Then there is a finite collection $\mathcal{P}$ of normal leaflets in U contained in A such that for every Rolle leaf V of d_k, we have*

$$\Pi_m(A \cap V) = \bigcup_{N \in \mathcal{P}} \Pi_m(N \cap V)$$

and for every $N \in \mathcal{P}$, the set $N \cap V$ is an analytic submanifold of U such that for every $n' \le n$, the projection $\Pi_{n'}|_{(N \cap V)}$ has constant rank, and such that $\Pi_m|_{(N \cap V)}$ is an immersion.

Proof. Apply Proposition 8.3 with d_p the nested distribution associated to $\Omega_p := \{dx_1, \dots, dx_p\}$ as in Example 2.2, for $p = 1, \dots, n$, and with $d_{n+1} := d$ and $I := \{1, \dots, m, n+1\}$. ☐

9. Regular closure

Let $U \subseteq \mathbb{R}^n$ be normal and A a normal subset of U. Let also $d = (d_0, \dots, d_k)$ be a definable, analytic nested distribution on U and V a Rolle leaf of d_k in U. Following [8], we study in this section the *closure in U* of $A \cap V$.

Proposition 9.1. *There are normal sets B and C in U such that*

$$U \cap \mathrm{cl}(A \cap V) = B \cap V \quad \text{and} \quad U \cap \mathrm{fr}(A \cap V) = C \cap V.$$

For the proof of Proposition 9.1, we need the following preliminary observations. Let Σ_n be the finite set of all permutations of $\{1, \dots, n\}$, considered as a definable subset of $\mathbb{R}^{2n}$. For every $\sigma \in \Sigma_n$, we let U_σ be the set of all $x \in U$ such that $\sigma(d_k(x))$ is the graph of a linear map $L : \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^k$ satisfying

$\|L\| < 2$. Then $U = \bigcup_{\sigma\in\Sigma_n} U_\sigma$ by Lemma 1.4, and by definable choice there is a definable map $x \mapsto \sigma_x : U \longrightarrow \Sigma_n$ such that $x \in U_{\sigma_x}$ for all $x \in U$. Since each U_σ is open, there is a definable map $x \mapsto \eta_x : U \longrightarrow (0,\infty)$ such that $y \in U_{\sigma_x}$ for all $y \in B(x, 2\eta_x)$; we put

$$G_0 := \{(x,y) \in U \times U : \ y \in B(0, 2\eta_x)\}\,.$$

Finally, for $x \in \mathbb{R}^n$, we write $x_- := (x_1, \dots, x_{n-k})$ and $x_+ := (x_{n-k+1}, \dots, x_n)$, and for $\epsilon > 0$ we put

$$B^k(x,\epsilon) := B(x_-,\epsilon) \times B(x_+, 2\epsilon).$$

Lemma 9.2. *For each $\nu \in \mathbb{N}$ there is a definable map $P_\nu : G_0 \longrightarrow \mathbb{R}^k$ such that for all $x \in U$,*

(i) *the map $P_{\nu,x} : B(0, 2\eta_x) \longrightarrow \mathbb{R}^k$ defined by $P_{\nu,x}(y) := P_\nu(x,y)$ is a homogeneous polynomial of degree ν;*
(ii) *the leaf V_x of $d_k|_{B^k(x,\eta_x)}$ is an analytic submanifold of $B^k(x,\eta_x)$;*
(iii) *the sum $\sum_{l=0}^\infty P_{\nu,x}(y-x)$ converges to analytic map $\phi_x : B^k(x,\eta_x) \longrightarrow \mathbb{R}^k$ definable in $\mathcal{P}(\mathcal{R})$ such that $\sigma_x(V_x) = \big\{y \in B^k(x,\eta_x) : \ \phi_x(y) = 0\big\}$.*

Proof. Let $G := \{(x,y) \in U \times U : \ y \in B(x, 2\eta_x)\}$, and define $L : G \longrightarrow GL(\mathbb{R}^{n-k}, \mathbb{R}^k)$ such that $\sigma_x(d_k(y))$ is the graph of $L(x,y)$ for all $(x,y) \in G$. Then L is definable and for every $x \in U$, the map $L_x : B(x, 2\eta_x) \longrightarrow GL(\mathbb{R}^{n-k}, \mathbb{R}^k)$ defined by $L_x(y) := L(x,y)$ is analytic.

By the proof of Lemma 5 of [12] and the definition of V_x, for every $x \in U$ the set $\sigma_x(V_x)$ is the graph of a Lipschitz map $F_x : B(x_-, \eta_x) \longrightarrow B(x_+, 2\eta_x)$ such that

$$dF_x(z) = L_x(z, F_x(z)) \quad \text{for all } z \in B(x_-, \eta_x).$$

Differentiating with respect to z (as in ...), one finds by induction on $|\alpha| = \alpha_1 + \cdots + \alpha_{n-k}$ for $\alpha \in \mathbb{N}^{n-k}$ that there is a definable function $L^\alpha : G \longrightarrow \mathbb{R}$ such that for all $x \in U$ and $z \in B(x_-, \eta_x)$,

$$\frac{\partial^\alpha}{\partial z^\alpha} F_x(z) = L^\alpha(x, z, F_x(z)),$$

and such that for all $x \in U$, the function $L_x^\alpha : B(x, 2\eta_x) \longrightarrow \mathbb{R}$ defined by $L_x^\alpha(y) := L^\alpha(x,y)$ is analytic. For $x \in U$, we now define $\phi_x : B^k(x,\eta_x) \longrightarrow \mathbb{R}^k$ by $\phi_x(y) := y_+ - F_x(y_-)$; then ϕ_x is analytic and definable in $\mathcal{P}(\mathcal{R})$, and $\sigma_x(V_x) = \big\{y \in B^k(x,\eta_x) : \ \phi_x(y) = 0\big\}$. Moreover, from the computation above we get $\phi_x(y) = \sum_{q=0}^\infty P_{q,x}(y-x)$, where $P_{0,x}(y) := x_+ - F_x(x_-)$, $P_{1.x}(y) := y^+ - L_x(x_-) \cdot y_-$ and

$$P_{\nu,x}(y) := \sum_{|\alpha|=\nu} L_x^\alpha(x_-, F_x(x_-)) \cdot (y_-)^\alpha \quad \text{for } \nu > 1;$$

hence $P_\nu(x,y) := P_{\nu,x}(y)$ will do. $\square$

Given an analytic map $h = (h_1, \dots, h_l) : U \longrightarrow \mathbb{R}^l$, $\nu \in \mathbb{N}$ and $x \in U$, we denote by $h_x^\nu : U - x \longrightarrow \mathbb{R}^l$ the Taylor expansion of order ν of h at x.

Proof of Proposition 9.1. Assume that

$$A = \{x \in U : \ g(x) = 0, \ h(x) > 0\},$$

with $g : U \longrightarrow \mathbb{R}^q$ and $h : U \longrightarrow \mathbb{R}^r$ definable and analytic, and put $Z(h) := \{x \in U : \ h(x) = 0\}$. It suffices to find a normal set $C \subseteq Z(h)$ in U such that $U \cap \mathrm{fr}(A \cap V) = C \cap V$, since then $B := A \cup C$ will do. Below we work with the notations from Lemma 9.2 and the paragraph preceding it.

Let $Y := \{(x,t) \in U \times (0,\infty) : \ x \in V \cap Z(h), \ 0 < t < \eta_x\}$. First, we define $\alpha : Y \longrightarrow [0,\infty)$ by

$$\alpha(x,t) := \max\big(\{h_{\min}(y) : \ y \in V, \ g(y) = 0, \ \|y - x\| \leq t\} \cup \{0\}\big).$$

By Lemma 7.7, there exists an $N \in \mathbb{N}$ such that for all $x \in V \cap Z(h)$, either ultimately $\alpha_x(t) = 0$ or ultimately $\alpha_x(t) > t^N$.

Fix an arbitrary $x \in V \cap Z(h)$. Then $x \in \mathrm{fr}(A \cap V)$ if and only if ultimately $\alpha_x(t) > t^N$. However, we have ultimately $\alpha_x(t) > t^N$ if and only if x belongs to the closure of $\big\{y \in V : \ g(y) = 0, \ h(y) > \|y - x\|^N > 0\big\}$, and the latter clearly holds if and only if x belongs to the closure of $\{y \in D_x : \ g(y) = 0\}$, where

$$D_x := \big\{y \in V : \ 2h_x^N(y - x) \geq \|y - x\|^N > 0\big\}.$$

Second, we define $\beta : Y \longrightarrow [0,\infty)$ by

$$\beta(x,t) := \min\big(\big\{|(g,\phi_x)|(y) : \ 2h_x^N(y-x) \geq \|y - x\|^N, \ \|y - x\| = t\big\} \cup \{1\}\big).$$

Again by Lemma 7.7, there exists an $M \in \mathbb{N}$ such that for all $x \in V \cap Z(h)$, either ultimately $\beta_x(t) = 0$ or ultimately $\beta_x(t) > t^M$.

Fix again an arbitrary $x \in V \cap Z(h)$. Then x is *not* in the closure of $\{y \in D_x : \ g(y) = 0\}$ if and only if ultimately $\beta_x(t) > t^M$. However, if ultimately $\beta_x = 0$, then x is in the closure of

$$\big\{y \in D_x : \ 4|(g,\phi_x)|(y) < \|y - x\|^M\big\},$$

which implies that x is in the closure of

$$E_x := \big\{y \in D_x : \ 2|(g_x^M, (\phi_x)_x^M)|(y - x) < \|y - x\|^M\big\}.$$

Conversely, if $x \in \mathrm{cl}(E_x)$, then x is in the closure of

$$\big\{y \in D_x : \ |(g,\phi_x)|(y) < \|y - x\|^M\big\},$$

which implies that ultimately $\beta_x = 0$. It follows from the above that

($*$) for all $x \in V \cap Z(h)$, $x \in \mathrm{fr}(A \cap V)$ if and only if $x \in \mathrm{cl}(E_x)$.

Let $G \in \mathbb{R}[a,y]^q$ be the general q-tuple of polynomials in y of degree M and coefficients $a \in \mathbb{R}^{m_1}$, $H \in \mathbb{R}[b,y]^r$ the general r-tuple of plynomials in y of degree N and coefficients $b \in \mathbb{R}^{m_2}$ and $\Phi \in \mathbb{R}[c,y]^k$ the general k-tuple of polynomials in y of degree M and coefficients $c \in \mathbb{R}^{m_3}$. Let $S \subseteq \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3} \times \mathbb{R}^n$ be the semialgebraic set

$$S := \big\{(x,a,b,c,y):\ 2H(b,y-x) \geq \|y-x\|^N > 0,\\ 2|(G(a,y-x),\Phi(c,y-x)\| < \|y-x\|^M\big\}.$$

Then there are definable, analytic functions $a : U \longrightarrow \mathbb{R}^{m_1}$, $b : U \longrightarrow \mathbb{R}^{m_2}$ and $c : U \longrightarrow \mathbb{R}^{m-3}$ such that for all $x \in U$,

$$E_x = \{y \in V :\ (x, a(x), b(x), c(x), y) \in S\}.$$

Thus by $(*)$,

$(**)$ for all $x \in V \cap Z(h)$, $x \in \mathrm{fr}(A\cap V)$ if and only if $x \in \mathrm{cl}(S_{(x,a(x),b(x),c(x))})$.

By Tarski's Theorem, there is a semialgebraic set $T \subseteq \mathbb{R}^{n+m_1+m_2+m_3+n}$ such that for all $(x,a,b,c) \in \mathbb{R}^{n+m_1+m_2+m_3}$, we have $T_{(x,a,b,c)} = \mathrm{cl}(S_{(x,a,b,c)})$. Therefore, the set

$$C := \{x \in U :\ (x, a(x), b(x), c(x), x) \in T\}$$

is normal in U and satisfies $\mathrm{fr}(A \cap V) = C \cap V$. □

Combining Corollary 8.4 with Proposition 9.1, we obtain

Corollary 9.3. *Let $m \leq n$. Then there is a finite collection $\mathcal{P}$ of normal leaflets in U contained in A such that*

(i) $\Pi_m(U \cap \mathrm{cl}(A\cap V)) = \bigcup_{N\in\mathcal{P}} \Pi_m(N \cap V)$;
(ii) $\Pi_m(A\cap V)$ *and* $\Pi_m(U\cap \mathrm{fr}(A\cap V))$ *are unions of some of the* $\Pi_m(N\cap V)$ *with* $N \in \mathcal{P}$;
(iii) *for every $N \in \mathcal{P}$, the set $N \cap V$ is an analytic submanifold of U, the restriction of Π_m to $N \cap V$ is an immersion, and for every $m' \leq m$ the restriction of $\Pi_{m'}$ to $N \cap V$ has constant rank.* □

10. Proper nested sub-Pfaffian sets

In this section, we put $I := [-1,1]$ and $I' := I \setminus \{0\}$.

Definition 10.1. Let $Y \subseteq \mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n \setminus Y$ be normal such that $I^n \setminus Y \subseteq U$, $d = (d_0,\dots,d_k)$ a definable, analytic nested distribution on U, $V \subseteq U$ a Rolle leaf of d_k and A a normal subset of U. In this situation, we say that the nested Pfaffian set $V \cap A \cap I^n$ is **restricted off** Y.

Example 10.2. Let $X \subseteq I^n$ be restricted nested Pfaffian off $\{0\}$. Then $X \setminus (\{0\} \times \mathbb{R}^{n-1})$ is restricted nested Pfaffian off $\{0\} \times \mathbb{R}^{n-1}$ and the fiber X_0 is restricted nested Pfaffian off $\{0\}$.

Let $Z \subseteq \mathbb{R}^m$. A nested sub-Pfaffian set $W \subseteq I^m$ is **proper off** Z if W is a finite union of sets of the form $\Pi^n_m(X)$, where $X \subseteq I^n$ is restricted nested Pfaffian off $Z \times \mathbb{R}^{n-m}$.

Notation. In this section, we are only interested in nested sub-Pfaffian sets $W \subseteq I^m$ that are proper off $\{0\} \times \mathbb{R}^{m-1}$; thus, we shall simply call such a W **proper**.

Theorem 10.3. *Let $W_1, \dots, W_q \subseteq I^m$ be proper nested sub-Pfaffian sets. Then there is a finite partition $\mathcal{C}$ of $I' \times I^{m-1}$ into analytic cells definable in $\mathcal{P}(\mathcal{R})$ such that each $C \in \mathcal{C}$ is proper nested sub-Pfaffian and for every $C \in \mathcal{C}$ and $p \in \{1, \dots, q\}$, either $C \subseteq W_p$ or $C \cap W_p = \emptyset$.*

To prove Theorem 10.3, we need certain closure properties for proper nested sub-Pfaffian sets.

Lemma 10.4. *The collection of all proper nested sub-Pfaffian sets is closed with respect to taking finite unions, coordinate projections and topological closure inside $I^m \setminus (\{0\} \times \mathbb{R}^{m-1})$.*

Proof. Closure with respect to taking finite unions and coordinate projections is obvious; closure with respect to taking topological closure inside $I^m \setminus (\{0\} \times \mathbb{R}^{m-1})$ follows from Proposition 9.1. $\square$

Unfortunately, the collection of all proper nested sub-Pfaffian sets is obviously not closed with respect to Cartesian products. However, we have the following weaker statement:

Lemma 10.5.
1. *Let $W \subseteq I^m$ be proper nested sub-Pfaffian. Then $W \times I$ is proper nested sub-Pfaffian.*
2. *Let $W \subseteq I^m$ and $W' \subseteq I^{m'}$ be proper nested sub-Pfaffian and $1 \le k \le \min\{m, m'\}$. Write (x, y) and (x, y') for the elements of $\mathbb{R}^m$ and $\mathbb{R}^{m'}$, respectively, where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{m-k}$ and $y' \in \mathbb{R}^{m'-k}$. Then the fiber product*
$$W \times_k W' := \left\{ (x, y, y') \in \mathbb{R}^{m+m'-k} : \ (x, y) \in W, (x, y') \in W' \right\}$$
is proper nested sub-Pfaffian.

Sketch of proof. (1) It suffices to consider the case where $W = \Pi_m(X)$ for some Pfaffian set $X \subseteq I^n$ that is restricted off $\{0\} \times I^{n-1}$. But in this case, the set
$$Y := \left\{ (x, t, y) \in I^{n+1} : \ x \in I^m, y \in I^{n-m}, t \in I \text{ and } (x, y) \in X \right\}$$
is restricted nested Pfaffian off $\{0\} \times I^n$ by Corollary 2.7(1), and $W \times I = \Pi_{m+1}(Y)$.

(2) It suffices to consider the case where $W = \Pi_m(X)$ for some nested Pfaffian set $X \subseteq I^n$ that is restricted off $\{0\} \times I^{n-1}$ and $W' = \Pi_{m'}(X')$ for some nested Pfaffian set $X' \subseteq I^{n'}$ that is restricted off $\{0\} \times I^{n'-1}$. Below, we let z range over I^{n-m} and z' range over $I^{n'-m'}$. Since $k \geq 1$, the set

$$Y := \left\{(x, y, y', z, z') \in \mathcal{I}^{n+n'-k} : \ (x, y, z) \in X \text{ and } (x, y', z') \in X'\right\}$$

is restricted nested Pfaffian off $\{0\} \times I^{n+n'-k-1}$ by Corollary 2.7(1), and $W \times_k W' = \Pi_{m+m'-k}(Y)$. ☐

Corollary 10.6. *Let $W, W' \subseteq I^m$ be proper nested sub-Pfaffian. Then $W \cap W'$ is proper nested sub-Pfaffian.* ☐

Also using Lemma 10.5, we obtain the following lemmas; we leave the details to the reader.

Lemma 10.7. *Let $W_1, \dots, W_q \subseteq I^m$ be proper nested sub-Pfaffian. Then the following subsets of I^{m-1} are proper nested sub-Pfaffian:*

(1) *the set*

$$W' := \left\{x' \in I^{m-1} : \ \exists y_1 < \cdots < y_q, \ (x', y_p) \in W_p, \ p = 1, \dots, q\right\};$$

(2) *for each $p < q$ the set*

$$W := \{(x', y) \in I^m : \ \exists y_1 < \cdots < y_p < y < y_{p+1} < \cdots < y_q, \\ (x', y_l) \in W_l, \ l = 1, \dots, q\};$$

(3) *for each $p \leq q$ the set*

$$W := \{(x', y) \in I^m : \ \exists y_1 < \cdots < y_p = y < \cdots < y_q, \\ (x', y_l) \in W_l, \ l = 1, \dots, q\}.$$ ☐

Proof of Theorem 10.3. By induction on m; the case $m = 0$ is trivial and the case $m = 1$ follows from the o-minimality of $\mathcal{P}(\mathcal{R})$, so we assume that $m > 1$ and the theorem holds for lower values of m. Increasing q if necessary, we may assume that the singleton set $\{0\}$ and the sets $I^{m-1} \times \{-1\}$ and $I^{m-1} \times \{1\}$ are among the W_i. Decomposing each W_i if necessary, we may also assume that each W_i is proper nested sub-Pfaffian. Thus, for each $p \in \{1, \dots, q\}$ there are $n_p \geq m$, a normal set $U_p \subseteq \mathbb{R}^{n_p} \setminus (\{0\} \times \mathbb{R}^{n_p-1})$ containing $I^{n_p} \setminus (\{0\} \times I^{n_p-1})$, a definable, analytic nested distribution $d^p = (d_0^p, \dots, d_{k(p)}^p)$ on U_p, a Rolle leaf V_p of $d_{k(p)}^p$ and a set A_p normal in U_p such that $W_p = \Pi_m^{n_p}(A_p \cap V_p)$.

For each $p \in \{1, \dots, q\}$, we now apply Corollary 9.3 with n_p, U_p, d^p, V_p and A_p in place of n, U, d, V and A. (Here we use the fact that the collection of all proper nested sub-Pfaffian subsets of I^m is closed with respect to taking topological closure inside $I^m \setminus (\{0\} \times I^{m-1})$.) We let $\mathcal{P}_p$ be the corresponding collection of normal leaflets in U_p obtained for m and $\mathcal{P}'_p$ be the corresponding collection of normal leaflets in U_p obtained with $m-1$ in place of m, and we put

$$\mathcal{Q} := \{\Pi^{n_p}_m(N \cap V_p) : \ p \in \{1, \dots, q\}, \ N \in \mathcal{P}_p, \ \dim(N \cap V_p) < m\}$$

and

$$\mathcal{Q}' := \left\{\Pi^{n_p}_{m-1}(N \cap V_p) : \ p \in \{1, \dots, q\}, \ N \in \mathcal{P}'_p, \ \dim(N \cap V_p) < m-1\right\}.$$

By definition, the elements of $\mathcal{Q}$ and $\mathcal{Q}'$ are proper nested sub-Pfaffian sets. Moreover, each $Z \in \mathcal{Q}$ is an immersed, analytic manifold in $\mathbb{R}^m$ with empty interior such that the restriction of Π^m_{m-1} to Z has constant rank; we let F be the union of all sets in $\mathcal{Q}$. Similarly, each $Z' \in \mathcal{Q}'$ is an immersed, analytic manifold in $\mathbb{R}^{m-1}$ with empty interior.

Let $N \in \mathbb{N}$ be such that $Z_{x'}$ has at most N components for every $x' \in \mathbb{R}^{m-1}$ and every $Z \in \mathcal{Q}$. For $k \le N|\mathcal{Q}|$ and $Z_1, \dots, Z_k \in \mathcal{Q}$, we put

$$Z'(Z_1, \dots, Z_k) := \big\{x' \in I^{m-1} : \\ \exists y_1 < \cdots < y_k \text{ such that } (x', y_j) \in Z_j \text{ for } j = 1, \dots, k\big\},$$

and we denote by $\mathcal{Q}''$ the collection of these sets. By Lemma 10.7, each set in $\mathcal{Q}''$ is proper nested sub-Pfaffian. Hence by the inductive hypothesis applied to the collection $\mathcal{Q}' \cup \mathcal{Q}''$, there is a finite partition $\mathcal{C}'$ of $I' \times I^{m-2}$ into analytic cells definable in $\mathcal{P}(\mathcal{R})$ such that $\mathcal{C}'$ is compatible with $\mathcal{Q}' \cup \mathcal{Q}''$ and each $C' \in \mathcal{C}'$ is proper nested sub-Pfaffian.

Fix now a $C' \in \mathcal{C}'$; it suffices to show that $C' \times I$ admits a finite partition $\mathcal{C}$ into analytic cells definable in $\mathcal{P}(\mathcal{R})$ such that $\mathcal{C}$ is compatible with $\{W_1, \dots, W_q\}$ and each $C \in \mathcal{C}$ is proper nested sub-Pfaffian. However, for each $p \in \{1, \dots, q\}$, the set $W_p \cap (C' \times I)$ is the union of some of the sets $Z \cap (C' \times I)$ with $Z \in \mathcal{Q}$ and some of the components of $(C' \times I) \setminus F$. Therefore, it suffices to show that $C' \times I$ admits a finite partition $\mathcal{C}$ into analytic cells definable in $\mathcal{P}(\mathcal{R})$ such that $\mathcal{C}$ is compatible with $\mathcal{Q}$ and each $C \in \mathcal{C}$ is proper nested sub-Pfaffian.

By construction, Lemma 10.5 and Corollary 10.6, if $Z \in \mathcal{Q}$ then the set $Z \cap (C' \times I)$ is proper nested sub-Pfaffian and an analytic submanifold of $I' \times I^{m-1}$, and each of its components is the graph of an analytic function from C' to $\mathbb{R}$. In particular, $F \cap (C' \times I)$ is a closed subset of $C' \times I$.

Moreover, if $Y \in \mathcal{Q}$ also, then $(Z \cap Y) \cap (C' \times I)$ is the union of some of the components of $Z \cap (C' \times I)$. On the other hand, each component of $Z \cap (C' \times I)$ is of the form

$$\big\{(x', y) \in C' \times I : \ \exists y_1 < \cdots < y_k \text{ such that} \\ y = y_l \text{ and } (x', y_j) \in Z_j \text{ for } j = 1, \dots, k\big\},$$

where $k \leq N|\mathcal{Q}|$, $l \leq k$ and $Z_1, \dots, Z_k \in \mathcal{Q}$. Hence by Lemma 10.7, each such component is proper nested sub-Pfaffian and an anlytic cell definable in $\mathcal{P}(\mathcal{R})$. It follows that each component of $(C' \times I) \setminus F$ is an open analytic cell definable in $\mathcal{P}(\mathcal{R})$, and each such component is proper nested sub-Pfaffian by Lemma 10.7 again, because it is of the form

$$\big\{(x', y) \in C' \times I : \ \exists y_1 < \cdots < y_k \text{ such that} \\ y_l < y < y_{l+1} \text{ and } (x', y_j) \in Z_j \text{ for } j = 1, \dots, k\big\}$$

with $k \leq N|\mathcal{Q}|$, $l < k$ and $Z_1, \dots, Z_k \in \mathcal{Q}$. □

11. Proof of the Main Theorem

Assume that $\mathcal{R}$ admits analytic cell decomposition. Let $M \subseteq \mathbb{R}^n$ be an analytic, definable manifold, and let $d = (d_0, \dots, d_k)$ be an analytic, definable nested distribution on M. Let also $A \subseteq M$ be definable.

Proposition 11.1. *There are $n_1, \dots, n_s \in \mathbb{N}$, and for each $j = 1, \dots, s$ there exist an analytic, definable nested distribution $e_j = (e_{j,0}, \dots, e_{j,k(j)})$ on $C_j := \{y \in \mathbb{R}^{n_j} : \ 0 < \|y\| < 2\}$ and a definable, analytic embedding $\psi_j : C_j \longrightarrow M$ such that, with $B_j := \{y \in \mathbb{R}^{n_j} : \ 0 < |y_i| < 1$ for $i = 1, \dots, n_j\}$,*

(i) *$\psi_j(C_j) \subseteq A$ for each j, and $\{\psi_j(B_j) : j = 1, \dots, s\}$ covers A;*
(ii) *for every Rolle leaf V of d_k, we have $A \cap V = \bigcup_{j=1}^s \psi_j(B_j \cap V_j)$, where each V_j is either empty or a Rolle leaf of $e_{j,k(j)}$.*

In particular, each $B_j \cap V_j$ is a restricted nested Pfaffian set off $\{0\}$.

Proof. By Proposition 2.8, we may assume that $A = M = \mathbb{R}^n$. If d_k has no Rolle leaves, the proposition is now trivial. So we also assume that d_k has a Rolle leaf; in particular, d_1 has a Rolle leaf V_1, say. Then V_1 is embedded, closed and of codimension 1 in $\mathbb{R}^n$, so V_1 separates $\mathbb{R}^n$. Let D_1 and D_2 be two closed balls in $\mathbb{R}^n \setminus V_1$ of positive radius and contained in different components of $\mathbb{R}^n \setminus V_1$, and denote by c_1 and c_2 their centers and by U_1 and U_2 their complements in $\mathbb{R}^n$. For $j = 1, 2$, we let $\phi_j : \mathbb{R}^n \setminus \{c_j\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ be a definable, analytic diffeomorphism such that $\phi_j(U_j) = C := \{x \in \mathbb{R}^n : \ 0 < \|x\| < 2\}$, and we let e_j be the push-forward of d via ϕ_j and put $\psi_j := \phi_j^{-1}$. With $B := \{x \in \mathbb{R}^n : \ 0 < |x_i| < 1$ for $i = 1, \dots, n\}$, we may assume that each $\psi_j(B)$ does not intersect V_1.

By our choice of ϕ_1 and ϕ_2, any Rolle leaf of d_k that intersects U_1 does not intersect $\psi_2(B)$, and any Rolle leaf of d_k that intersects U_2 does not intersect $\psi_1(B)$. On the other hand, by Corollary 2.7(2), there is an $N \in \mathbb{N}$ such that for every Rolle leaf V of d_k and each $j = 1, 2$, the set $V \cap U_j$ has at most N connected components. The corollary now follows. □

Corollary 11.2. *Let V be a Rolle leaf of d_k. Then $A \cap V$ is a finite union of simply connected nested sub-Pfaffian sets that are analytic manifolds definable in $\mathcal{P}(\mathcal{R})$.*

Proof. Let C_j, ψ_j, e_j, etc., be as in the previous proposition; it suffices to prove the corollary with each e_j and C_j in place of d and A. In other words, we may assume that $M = \{x \in \mathbb{R}^n : 0 < \|x\| < 2\}$ and $A = \{x \in \mathbb{R}^n : 0 < |x_i| < 1$ for $i = 1, \dots, n\}$, and we let V be a Rolle leaf of d_k. Since $A \cap V$ is a restricted nested Pfaffian set off $\{0\}$, the corollary now follows from Example 10.2 and Theorem 10.3. □

Proposition 11.3. *Let $W = (W_0, \dots, W_k)$ be a nested integral manifold of d definable in $\mathcal{P}(\mathcal{R})$. Then there exists a $q \in \mathbb{N}$, and for each $p = 1, \dots, q$ there exist $n_p \geq n$, a definable manifold $N_p \subseteq \mathbb{R}^{n_p}$, a definable nested distribution $d^p = (d^p_0, \dots, d^p_{k(p)})$ on N_p and a nested Rolle leaf $V^p = (V^p_0, \dots, V^p_{k(p)})$ of d^p such that*

$$W_k \subseteq \bigcup_{p=1}^{q} \Pi_n(V^p_{k(p)}) \quad \textit{and} \quad \dim(V^p_{k(p)}) \leq \dim(W_k) \textit{ for each } p.$$

Proof. By induction on k. If $k = 1$ we let $\mathcal{C}$ be a definable analytic cell decomposition of $\mathbb{R}^n$ such that for each $C \in \mathcal{C}$ with $C \subseteq M$, the distribution $g_C \cap d_1$ has dimension and $d_1|_C$ is analytic. Then by Haefliger's Theorem [14], for every $C \in \mathcal{C}$ such that $C \subseteq M$, every leaf of $d_1|_C$ is a Rolle leaf. The proposition now follows easily in this case.

Assume now that $k > 1$ and the proposition holds for lower values of k. By the inductive hypothesis and Lemma 6.3, we may assume that W_{k-1} is a Rolle leaf of d_{k-1}. Hence by Corollary 11.2, we may actually assume that W_{k-1} is a simply connected nested sub-Pfaffian set that is an analytic manifold definable in $\mathcal{P}(\mathcal{R})$. Hence by Haefliger's Theorem, the leaf of $d_k|_{W_{k-1}}$ containing W_k is a Rolle leaf. The proposition now follows from Corollary 8.4 and Lemma 6.3. □

Combining Lemma 4.1, Theorem 6.4 and Proposition 11.3, we obtain

Theorem 11.4. *Let V be a Rolle leaf of d_k. Then there are nested sub-Pfaffian sets $W_1, \dots, W_l \subseteq \mathbb{R}^n$ such that $\mathrm{fr}(V) \subseteq W_1 \cup \dots \cup W_l$ and $\dim(W_k) < \dim(V)$ for $k = 1, \dots, l$.* □

The main theorem now follows from the above theorem and Section 2 of [6].

12. Conclusion

We conclude by proving the corollary in the introduction; we continue to assume that $\mathcal{R}$ admits analytic cell decomposition. Let $L \subseteq \mathbb{R}^n$ be one of the Rolle leaves added to $\mathcal{R}$ in the construction of $\mathcal{P}(\mathcal{R})$ in [16]; it suffices to establish the following:

Proposition 12.1. *L is a nested sub-Pfaffian set over $\mathcal{R}$.*

Proof. By construction of $\mathcal{P}(\mathcal{R})$ and Example 2.2, there are a $l \in \mathbb{N}$ and a 1-distribution e on $\mathbb{R}^n$ definable in $\mathcal{R}_l$ such that L is a Rolle leaf of e. We proceed by induction on l; if $l = 0$, we are done, so we assume that $l > 0$ and the proposition holds for all lower values of l; in particular, every set definable in $\mathcal{R}_l$ is definable in $\mathcal{N}(\mathcal{R})$. Thus, by analytic cell decomposition and the Main Theorem, we may assume that e is analytic and there are a definable manifold $M \subseteq \mathbb{R}^{n'}$, a definable nested distribution $d = (d_0, \dots, d_k)$ on M and a Rolle leaf V of d_k such that $\mathrm{gr}(e) = \Pi_n(V)$. By Corollary 8.4, we may further assume that $\Pi_n|_V$ is an immersion and $\dim(\Pi_n \circ d_k(y)) = \dim(d_k(y)) = n$ for all $y \in M$. Since $\Pi_n(V) \subseteq \mathbb{R}^n \times G_n^{n-1}$, we may also assume that $\Pi_n(M) \subseteq \mathbb{R}^n \times G_n^{n-1}$. Let now d_{k+1} be the $(n-1)$-distribution on M defined by

$$d_{k+1}(y) := d_k(y) \cap (\Pi_n^{n'})^{-1}(\pi(\Pi_{n+n^2}(y))),$$

where $\pi : \mathbb{R}^{n+n^2} \longrightarrow \mathbb{R}^{n^2}$ is the projection on the last n^2 coordinates and $\pi(\Pi_n(y))$ is identified with the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ that it represents. Then $d' := (d_0, \dots, d_{k+1})$ is a definable nested distribution on M and $V' := V \cap (\Pi_n^{n'})^{-1}(L)$ is a Rolle leaf of d_{k+1}, as required. $\square$

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