COMPLEX ANALYTIC GEOMETRY AND ANALYTIC-GEOMETRIC CATEGORIES

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Abstract. The notion of a analytic-geometric category was introduced by v.d. Dries and Miller in [4]. It is a category of subsets of real analytic manifolds which extends the category of subanalytic sets. This paper discusses connections between the subanalytic category, or more generally analytic-geometric categories, and complex analytic geometry. The questions are of the following nature: We start with a subset A of a complex analytic manifold M and assume that A is an object of an analytic-geometric category (by viewing M as a real analytic manifold of double dimension). We then formulate conditions under which A, its closure or its image under a holomorphic map is a complex analytic set.

In the second part of the paper we consider the notion of a complex $\mathfrak{S}$-manifold, which generalizes that of a compact complex manifold. We discuss uniformity in parameters, in this context, within families of complex manifolds and their high-order holomorphic tangent bundles. We then prove a result on uniform embeddings of analytic subsets of $\mathfrak{S}$-manifolds into a projective space, which extends theorems of Campana ([1]) and Fujiki ([6]) on compact complex manifolds.

1. Introduction

In a series of papers ([13], [12], [14]) we considered holomorphic manifolds and maps definable in o-minimal structures, over arbitrary real closed fields. A large part of that work was devoted to developing complex analytic tools in nonstandard setting, where the topology on the algebraically closed field is not assumed to be locally compact. Here we focus on the the field of complex numbers and investigate the restrictions which o-minimality puts on subsets of complex manifolds. We work in the more general setting of a geometric-analytic category, introduced by v. d. Dries and Miller in their paper "Geometric categories and o-minimal structures" ([4]).

In that paper the authors presented a category extending the category of subanalytic sets whose objects share many of the properties of the subanalytic category and yet allows a much richer collection of sets and maps. Thus, for example, the subset $\{x, e^{1/x} : x > 0\}$ of $\mathbb{R}^2$, while not subanalytic in $\mathbb{R}^2$, is part of the category in question. Their paper was written in the midst of of intensive work on o-minimality and since it was published new expansions of the field of real numbers were proved to be o-minimal thus providing new examples of analytic-geometric categories (see for example [19], [18], [5], [17]).

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The general problem we consider in the first part of this paper (Sections 2-6) can be viewed as removing singularities from subsets of complex manifolds which are also objects of an analytic-geometric category: We start with a subset A of a complex analytic manifold M and assume that A is an object of an analytic-geometric category in the above sense (by viewing M as a real analytic manifold of double dimension). We then formulate conditions under which A or its closure is a complex analytic subset of M. The proofs make use of the "tameness" features of sets in the analytic-geometric categories. In particular, we make extensive use of the fact that if such a set is relatively compact then its boundary is well-behaved.

Here is a variety of results from the first part of the paper, formulated in the language of subanalytic sets (the results are actually proved in the more general setting of an arbitrary analytic-geometric category).

Theorem 2. *If M is a complex manifold and X is a closed subset of M then X is a complex analytic subset of M if and only if X is subanalytic in M and for every open $U \subseteq M$,*

$$\dim_{\mathbb{R}} Sing_{\mathbb{C}}(U \cap X) \leqslant \dim_{\mathbb{R}}(U \cap X) - 2.$$

(By $Sing_{\mathbb{C}} X$ we mean all points at which the germ of X is not a $\mathbb{C}$-submanifold).

The above theorem follows from a theorem of Shiffman when $Reg_{\mathbb{C}}(X)$ is of pure dimension. It fails in general without the subanalyticity assumption on X.

Theorem 4. *Let M be a complex manifold and $E \subseteq M$ a complex analytic of M (of arbitrary dimension). If A is a complex analytic subset of $M \setminus E$ which is also subanalytic in M then $Cl(A)$ is a complex analytic subset of M.*

Again, the above theorem is the just the Remmert-Stein Theorem when we put an extra dimension assumption on E. It fails without the subanalyticity assumption on A.

Theorem 3. *Assume that A is a closed and subanalytic subset of a complex manifold M such that the set of its complex regular points is dense in A. Assume also that at no point $z_0 \in A$, the germ of A at z_0 is a real manifold with a boundary. Then A is a complex analytic subset of M.*

We also prove the following strong variant of Remmert's Proper Mapping Theorem.

Theorem 1. *Let $f : M \to N$ be a holomorphic map between complex analytic manifolds and A a complex analytic subset of M. If $f(A)$ is closed in N and subanalytic in N then $f(A)$ is complex analytic in N.*

Finally, we prove:

Theorem 4. *Let M, N be complex manifolds, S an irreducible $\mathbb{C}$-analytic subset of M and assume that $L \in \mathcal{C}(M)$ a closed subset of S which contains the set of* *singular points of* S*. Assume that* $f : S \setminus L \to N$ *is a holomorphic map whose graph is in* $\mathcal{C}(M \times N)$*.*

If $\dim_{\mathbb{R}} L \leqslant \dim_{\mathbb{R}} A - 2$ *then the closure of the graph of* f *in* $M \times N$ *is a* $\mathbb{C}$*-analytic subset of* $M \times N$*.*

In the second part of the paper (Sections 7-9) we consider the notion of a complex $\mathfrak{S}$-manifold, a notion extending that of a compact complex manifold. We then formulate several results concerning uniformity in parameters of definable families of analytic subsets of such manifolds. We review some basic notions regarding tangent bundles of high order of a complex manifold and show the definability of these objects for $\mathfrak{S}$-manifolds. Finally, we consider a theorem, proved independently by Campana and Fujiki (see [1], [6]), about a uniform embedding of analytic sets in projective space. This theorem has recently drew the attention of model theorists (see [11] [16] and [10]) because it provides a general tool to establish connections between structures in different model theoretic settings and algebraic varieties.

Here we prove a slight generalization of the original theorem, by replacing compact complex manifolds with $\mathfrak{S}$-manifolds. We prove

Theorem 2. *Let* N, M *be complex* $\mathfrak{S}$*- manifolds, and* S *an irreducible analytic* $\mathfrak{S}$*-subset of* $N \times M$*. Then there is a holomorphic vector* $\mathfrak{S}$*-bundle* $\pi : V \to M$*, a meromorphic* $\mathfrak{S}$*-map* $\lambda : S \to \mathbb{P}(V)$*, and a Zariski open subset* S^0 *of* S *such that* $\sigma(b, a) = \sigma(b', a)$ *if and only if* $S_b = S_{b'}$ *near* a*, for all* $(b, a), (b', a) \in S^0$*, and the following diagram is commutative*

$$\begin{array}{ccc} S_0 & \xrightarrow{\ \sigma\ } & \mathbb{P}(V) \\ & {\scriptstyle \pi_M} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & M & \end{array}$$

A model theoretic remark

Although the paper discusses results for structures over the real and complex fields it is written with an eye for the more general setting of an o-minimal structure over an arbitrary real closed field R and its algebraic closure K. Thus, most proofs can easily be transferred from the $\mathbb{R}$ to an arbitrary real closed field after making proper adjustments, such as replacing the notion of a holomorphic function with a K-holomorphic function (see [13], [12]), and notions such as "connected" with "definably connected".

This however excludes the few places where we use classical results from complex geometry such as Shiffman's Theorem and Chow's Theorem. In unpublished noted we proved analogues of these theorems, as well as other results, in the more general setting, but these will be presented elsewhere.

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2. Preliminaries

2.1. **Analytic-geometric categories.** The following definition is taken from [4]:

Definition 2.1. *An analytic-geometric category* $\mathcal{C}$ is, for every real analytic manifold M, a collection $\mathcal{C}(M)$ of subsets of M, such that:

AG1. $M \in \mathcal{C}(M)$, and $\mathcal{C}(M)$ is closed under complement, finite intersections and finite unions.

AG2. If $A \in \mathcal{C}(M)$ then $A \times \mathbb{R} \in \mathcal{C}(M \times \mathbb{R})$.

AG3. If N is a real analytic manifold, $f : M \to N$ is a proper, real-analytic map and $A \in \mathcal{C}(M)$ then $f(A) \in \mathcal{C}(N)$.

AG4. If $A \subseteq M$ and $\mathcal{U}$ is an open covering of M then $A \in \mathcal{C}(M)$ if and only if $A \cap U \in \mathcal{C}(U)$ for every $U \in \mathcal{U}$.

AG5. Every bounded set in $\mathcal{C}(\mathbb{R})$ has a finite boundary.

An example of such an analytic-geometric category is that of the subanalytic sets.

2.2. **Definable sets.** As pointed out in [4], for every analytic-geometric category $\mathcal{C}$ and every real analytic manifold M, all subanalytic subsets of M are in $\mathcal{C}(M)$. Also, to every analytic-geometric category $\mathcal{C}$ corresponds an o-minimal structure $\mathfrak{S}(\mathcal{C})$ over $\mathbb{R}$ whose definable sets are those subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, which are in $\mathcal{C}(\mathbb{P}^n(\mathbb{R}))$ (under the usual identification of $\mathbb{R}^n$ with an open subset of $\mathbb{P}^n(\mathbb{R}^n)$). As usual, a function is definable in $\mathfrak{S}(\mathcal{C})$ if its graph is in $\mathfrak{S}(\mathcal{C})$.

When $\mathcal{C}$ is the category of subanalytic sets then $\mathfrak{S}(\mathcal{C})$ is the structure denoted by $\mathbb{R}_{an}$, obtained by expanding the real field with all real analytic functions on the closed unit n-cubes. (For more details on o-minimal structures, one may consult v. d. Dries' book [3]).

When M is a real analytic manifold and $U \in \mathcal{C}(M)$ is a relatively compact open subset of M, then U, together with all subsets of U which are in $\mathcal{C}(M)$, can be viewed as definable in $\mathfrak{S}(\mathcal{C})$ (for any analytic-geometric $\mathcal{C}$) as follows: U can be written as the union of finitely many relatively compact open charts, each isomorphic to an open box in $\mathbb{R}^n$, such that the transition maps are real-analytic on the closure of their domain. Furthermore, these boxes can be chosen to be pairwise disjoint. The boxes, and the transition maps are now definable in $\mathfrak{S}(\mathcal{C})$ and every subset of U which is in $\mathcal{C}(M)$ is mapped via these isomorphisms to a definable subset of $\mathbb{R}^n$.

In the opposite direction, every o-minimal structure $\mathfrak{S}$ expanding R_{an}, let $\mathcal{C}(\mathfrak{S})$ be all sets $A \subseteq M$ such that for every $x \in M$ there is an open neighborhood $U \subseteq M$ of x, an open $V \subseteq \mathbb{R}^n$ and a real analytic isomorphism $h : U \to V$ such that $h(U \cap A)$ is definable in $\mathfrak{S}$.

The above transforms, from an analytic-geometric category to an o-minimal structure and vise-versa, are inverse to each other. Thus, when we are given a set A in $\mathcal{C}(M)$ and want to analyze A near a point $z_0 \in M$, we will often consider $U \cap A$ for a relatively compact open neighborhood $U \in \mathcal{C}(M)$ of z_0, together with a proper real analytic isomorphism f between U and an open subset V of $\mathbb{R}^n$. We may then replace U and A by V and $f(A)$, (which are both definable in $\mathfrak{S}(\mathcal{C})$) and assume that U and $U \cap A$ are definable in $\mathfrak{S}(\mathcal{C})$.

We now fix, for the rest of the paper, an analytic-geometric category $\mathcal{C}$ and a corresponding o-minimal structure $\mathfrak{S} = \mathfrak{S}(\mathcal{C})$ containing all restricted analytic functions.

2.3. **Local connectedness.** For topological spaces $X \subseteq Y$, we let $fr_Y(X) = Cl_Y(X) \setminus X$ be the frontier of X in Y. We omit Y when the reference to it is clear from the context.

Definition 2.2. Let X be a subset of $\mathbb{R}^n$. For $U \subseteq \mathbb{R}^n$ an open set containing x, we let $\#(U \cap X)$ be the number of connected components of $U \cap X$ (it can be ∞) and let $\#(U \cap X)_x$ be the number of those components of $U \cap X$ whose closure contains x.

Notice that if $x \in V \subseteq U$ then $\#(U \cap X)_x \leqslant \#(V \cap X)_x$. For $\epsilon > 0$ and $x \in \mathbb{R}^n$, let $B(x;\epsilon)$ be the open ball of radius ϵ centered at x. If $X \subseteq \mathbb{R}^n$ is definable then, by o-minimality, $\lim_{\epsilon \to 0} \#(B(x;\epsilon) \cap X)$ and $\lim_{\epsilon \to 0}(\#B(x;\epsilon) \cap X)_x$ both exist and are finite. Moreover, for all sufficiently small definable open neighborhood V of x, we have $\#(V \cap X)_x = \lim_{\epsilon \to 0} \#(B(x;\epsilon) \cap X)_x$.

Lemma 2.3. *For every definable $X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$*

$$\lim_{\epsilon \to 0} \#(B(x;\epsilon) \cap X) = \lim_{\epsilon \to 0} \#(B(x;\epsilon) \cap X)_x.$$

In particular, there is an $\epsilon > 0$ such that for all open $V \subseteq B(x;\epsilon)$ containing x, $\#(V \cap X)_x$ is the same.

Proof It is immediate that $\lim_{\epsilon \to 0} \#(B(x;\epsilon) \cap X) \geqslant \lim_{\epsilon \to 0} \#(B(x;\epsilon) \cap X)_x$.

Assume that $\lim_{\epsilon \to 0} \#(B(x;\epsilon) \cap X) > \lim_{\epsilon \to 0} \#(B(x;\epsilon) \cap X)_x$. Then for all sufficiently small ϵ there is $x(\epsilon) \in B(x;\epsilon) \cap X$ which is not in any of the components of $B(x;\epsilon) \cap X$ that contain x in their closure.

But, for all sufficiently small ϵ, say $\epsilon < \epsilon_0$, the map $\epsilon \mapsto x(\epsilon)$, from $(0, \epsilon_0)$ into X, is definable and continuous and hence its image is contained in one of the connected components of $B(x;\epsilon_0) \cap X$. This component must have x in its closure, contradicting our choice of $x(\epsilon)$. □

Notice that the above lemma implies that for sufficiently small ϵ every connected component of $B(x;\epsilon) \cap X$ has x in its closure.

Definition 2.4. For X and x as in the last lemma, we call the limit of $\#(B(x;\epsilon) \cap X)$ *the number of connected components of the germ of X at x.*

Let M be a real analytic manifold, $X \in \mathcal{C}(M)$, and $x \in M$. Then *the number of connected components of the germ of X at x* is computed with respect to an open relatively compact chart U containing x and thus assuming that U and $U \cap X$ are definable. (the number we get does not depend on the choice of the chart).

It is not hard to see that the following holds (and therefore X is locally connected at x in the classical sense if and only if the number of components of the germ of X at x, in the above sense, is 1).

Lemma 2.5. *For $X \in \mathcal{C}(M)$ and $x \in M$, the number of connected components of the germ of X at x equals the minimal number n such that every neighborhood U of x contains an open V with $x \in V$ and $\#(V \cap X) = n$.*

Remark By o-minimality, given a definable set $X \subseteq \mathbb{R}^m$, we can partition $\mathbb{R}^m$ into finitely many definable sets $Y_1, \ldots, Y_r$ such that for each i, the number of connected components of the germ of X at every point of Y_i is the same.

2.4. **Complex and real analytic manifolds.** We are going to consider in this paper subsets of complex manifolds which are defined via their real and imaginary coordinates. This can be done by viewing every complex manifold M as a real analytic manifold of double dimension.

Recall that a subset $A \subseteq M$ of a complex manifold M is called *locally analytic in* M (to avoid ambiguity we write *locally* $\mathbb{C}$*-analytic in* M) if for every $x \in A$ there exists an open neighborhood $U \subseteq M$ of x such that $A \cap U$ is the zero set of finitely many holomorphic functions on U. $A \subseteq M$ is called *an analytic subset of* M (or *a* $\mathbb{C}$*-analytic subset of* M) if A is locally $\mathbb{C}$-analytic in M and in addition A is closed in M.

Given any set $A \in \mathcal{C}(M)$, we denote by $Reg_{\mathbb{C}}A$ the set of points $z \in A$ such that the germ of A at z is a complex submanifold of M, and by $Sing_{\mathbb{C}}A$ its complement in A.

Notice that every $\mathbb{C}$-analytic subset of M is in $\mathcal{C}(M)$, since it is given near every point in M as the zero set of real analytic functions.

Remark We emphasize that the $\mathbb{C}$-analytic sets we consider are point-subsets of complex manifolds and we do not view them as ringed spaces. Although we treat in this paper only subsets of complex manifolds, we believe that much of this treatment can go through for complex analytic spaces, once we formulate properly what subsets of such spaces belong to the category $\mathcal{C}$.

Fact 2.6. *(i) Let* M, N *be complex manifolds and assume that* $f : M \to N$ *is a continuous function whose graph is in* $\mathcal{C}(M \times N)$*. Then the set of points in* M *at which* f *is holomorphic is in* $\mathcal{C}(M)$*.*
(ii) Let M *be a complex manifold and* $X \subseteq M$ *in* $\mathcal{C}(M)$*. Then the set* $Reg_{\mathbb{C}}X$ *is in* $\mathcal{C}(M)$ *as well.*
(iii) If $A \subseteq M$ *is a locally* $\mathbb{C}$*-analytic subset of a complex manifold* M *and if* A *is in* $\mathcal{C}(M)$*, then for every* $x \in M$ *there is an open neighborhood* U *of* x *such that* $A \cap U$ *is either empty or has finitely many irreducible components (as a* $\mathbb{C}$*-analytic set).*

Proof (i) For $z \in M$, we may replace M and N by definable open sets $U \subseteq \mathbb{C}^n$ containing z and $V \subseteq \mathbb{C}^m$ containing $f(z)$ such that $f : U \to V$ is definable. We now use the fact that a complex function is holomorphic, as a function of several variables, if and only if it is continuous, and holomorphic in each variable separately. Holomorphicity in one variable and continuity are defined using an ϵ-δ definition, therefore the set of points where f is holomorphic is in $\mathcal{C}(M)$ (by arguments similar to B.8 in [4]).

(ii) Here we just point out that a set $X \subseteq \mathbb{C}^n$ is a d-dimensional complex submanifold of $\mathbb{C}^n$ near a point $z \in X$ if and only if after a permutation of coordinates, the set X near z, is the graph of a holomorphic function from $\mathbb{C}^d$ into $\mathbb{C}^{n-d}$ Working in charts and using (i), this set itself is in $\mathcal{C}(M)$

(iii) Given $x \in M$, we may find a relatively compact open $U \subseteq M$ and assume that U is a definable subset of $\mathbb{C}^n$ and $U \cap A$ is definable. Since $Reg_{\mathbb{C}}(A \cap U)$ has finitely many connected components and every irreducible component of A is the closure of such a connected component, A has only finitely many irreducible components. □

2.5. **Good $\mathbb{C}$-Direction.** The following theorem is a complex version of the Good Direction Lemma for definable sets of $\mathbb{R}^n$ and $\mathbb{R}$-linear subspaces (see Theorem 7.4.2 in [3]).

Theorem 2.7. *Let A be a definable subset of $\mathbb{C}^{n+1}$ of real-dimension at most $2n+1$. Then there is a 1-dimensional complex subspace $\ell \subseteq \mathbb{C}^n$ such that for any $p \in \mathbb{C}^n$ the intersection of A with the affine $\mathbb{C}$-line $\begin{bmatrix} 0 \\ p \end{bmatrix} + z \begin{bmatrix} 1 \\ \ell \end{bmatrix}$, $z \in \mathbb{C}$, has real-dimension at most 1. Moreover, the set of all such ℓ's is definable and dense in $\mathbb{P}^{n-1}(\mathbb{C})$.*

Proof We will follow the idea of the proof of the Good Direction Lemma from [3].

Assume that the theorem fails. Then for every ℓ in an open set $W \subseteq \mathbb{P}^{n-1}(\mathbb{C})$ there is $p(\ell) \in \mathbb{C}^n$ such that the set $B(\ell) = \{z \in \mathbb{C}^{n+1} : \begin{bmatrix} 0 \\ p(\ell) \end{bmatrix} + z \begin{bmatrix} 1 \\ \ell \end{bmatrix} \in A\}$ has real-dimension 2. By dimension considerations, there is a fixed open ball $B \subseteq \mathbb{C}$ and an open set $V \subseteq \mathbb{C}^n$ such that for every $u \in V$ there is $p(u) \in \mathbb{C}^n$ with $\begin{bmatrix} 0 \\ p(u) \end{bmatrix} + z \begin{bmatrix} 1 \\ u \end{bmatrix} \in A$ for all $z \in B$. Using definable choice, we can assume that the function $u \mapsto p(u)$ is definable.

Consider the function $F(u, z)$ from $V \times B$ into $\mathbb{C}^n$ defined as

$$F : (u, z) \mapsto \begin{bmatrix} 0 \\ p(u) \end{bmatrix} + z \begin{bmatrix} 1 \\ u \end{bmatrix}$$

To obtain a contradiction we will show that the image of $V \times B$ under F has real-dimension $2n + 2$.

Considering V as an open subset of $\mathbb{R}^{2n}$ and $p(u)$ as a function into $\mathbb{R}^{2n}$, we obtain that there is a nonempty open definable set $V_0 \subseteq V$ such that p is C^1 on V_0, and hence F is C^1 on $V_0 \times B$.

The following claim, combining with the Inverse Function Theorem, finishes the proof.

Claim 2.8. *For any $w \in V_0$ the set*

$$\Lambda_w = \{\lambda \in B : \textit{the } \mathbb{R}\textit{-differential of } F \textit{ at } (w, \lambda) \textit{ is not invertible } \}$$

has real-dimension at most 1.

Proof Let $w \in V_0$ and $\lambda \in B$. The $\mathbb{R}$-differential of F at (w, λ) is the $\mathbb{R}$-linear map from $\mathbb{C}^{n+1}$ into $\mathbb{C}^{n+1}$ given by

$$(u, z) \mapsto \begin{bmatrix} 0 \\ L(u) \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ u \end{bmatrix} + z \begin{bmatrix} 1 \\ w \end{bmatrix}$$

where L is the $\mathbb{R}$-differential of p at w (It is an $\mathbb{R}$-linear map from $\mathbb{C}^n$ into $\mathbb{C}^n$.)

This $\mathbb{R}$-linear map is not invertible if and only if $L(u) + \lambda u = 0$ for some nonzero $u \in \mathbb{C}^n$. Thus the claim reduces to the following statement:
If $L : \mathbb{C}^n \to \mathbb{C}^n$ is an $\mathbb{R}$-linear map then the set

$$\Lambda = \{\lambda \in \mathbb{C} : L(u) + \lambda u = 0 \text{ for some nonzero } u \in \mathbb{C}^n\}$$

has real-dimension at most 1.

Considering the determinant of the $\mathbb{R}$-linear map $L + \lambda \mathrm{Id}$, we obtain that $\Lambda = \{a + ib \in \mathbb{C} : Q(a, b) = 0\}$ for some $Q \in \mathbb{R}[x, y]$. Thus if the real-dimension of Λ is greater than 1 then $Q \equiv 0$ and $\Lambda = \mathbb{C}$. However, if $|\lambda| > \|L\|$ then $|\lambda u| > |L(u)|$ for any nonzero u, and λ can not be in Λ. □

Remarks 1. We could not claim, in the above theorem, that the intersection of A with the affine $\mathbb{C}$-line $\begin{bmatrix} 0 \\ p \end{bmatrix} + z \begin{bmatrix} 1 \\ \ell \end{bmatrix}$, $z \in \mathbb{C}$, is finite even if we assumed that A has real-dimension at most $2n$. An example is the set $A \subseteq \mathbb{C}^2$ consisting of all (u, v) such that $|u| = 1$ and $v = \bar{z} + uz$ for some $z \in \mathbb{C}$. We have $\dim_{\mathbb{R}} A = 2$ but for every ℓ there is p (the complex conjugate of ℓ) such that $\left(\begin{bmatrix} 0 \\ p \end{bmatrix} + z \begin{bmatrix} 1 \\ \ell \end{bmatrix} \right) \cap M$ has real dimension 1.

2. Notice that a corresponding theorem on good $\mathbb{C}$-direction, for A in $\mathcal{C}(\mathbb{C}^n)$, follows from the above. Indeed, by viewing A as a countable union of definable sets, we may find a set of complex lines, all intersecting A in a set of real-dimension one, whose complement in projective space is a countable union of definable nowhere dense sets.

Corollary 2.9. *Assume that A is a definable, locally $\mathbb{C}$-analytic subset of $\mathbb{C}^{n+1}$ of complex-dimension d, $d \leqslant n$ and take $m \leqslant (n+1) - d$. Then for any generic $\mathbb{C}$-linear subspace Π of $\mathbb{C}^{n+1}$ of dimension m and for any $p \in \mathbb{C}^{n+1}$, the intersection of A with $p + \Pi$ is finite. (By a "generic subspace" we mean a subspace outside a definable nowhere dense subset of the suitable Grassmannian)*

Proof By induction on m.

If $m = 1$ then, by the above theorem, the intersection has real-dimension at most 1. However, the intersection of a complex line with a locally $\mathbb{C}$-analytic subset is again locally $\mathbb{C}$-analytic and hence, by o-minimality, must be finite.

Let $m = m' + 1$ and Π be a generic subspace of complex-dimension m. Assume that for some $p \in \mathbb{C}^{n+1}$ the intersection of A with $p + \Pi$ is infinite. Since A is locally $\mathbb{C}$-analytic set this intersection has real-dimension at least 2. We can write Π as $\Pi' + l$, where Π' is a generic plane of complex-dimension m' and ℓ is a $\mathbb{C}$-line generic over Π'. Since the complex-dimension of $A + \Pi'$ is at most n, by the above theorem, the intersection of $A + \Pi'$ with $p + \ell$ has complex-dimension at most one. It is not hard to see that then there is $a \in \ell$ such that the intersection of A and $p + a + \Pi'$ is infinite. □

We say that *every generic projection* π of $\mathbb{C}^n$ onto a d-dimensional $\mathbb{C}$-linear subspace has a certain property P if there is a definable nowhere dense subset D of the appropriate Grassmannian, such for every d-dimensional linear subspace $L \subseteq \mathbb{C}^n$ outside of D, the orthogonal projection onto L has property P.

In contrast to the failure of the corresponding global statement (as shown in the above remark), its local version turns out to be true:

Lemma 2.10. *Assume that $X \subseteq \mathbb{C}^n$ is a a definable real C^1-submanifold, $\dim_{\mathbb{R}} X \leqslant 2d < 2n$, $z_0 \in X$. Then for every generic projection π of $\mathbb{C}^n$ onto a complex linear subspace L of complex-dimension d, there is a neighborhood U of z_0 such that $\pi|(U \cap X)$ is a C^1-embedding into L.*

Proof We consider the real tangent space T of X at z_0 (of real dimension $\leqslant 2d$) and use the fact that every generic $\mathbb{C}$-linear subspace $L \subseteq \mathbb{C}^n$ of complex dimension $n-d$ (i.e. of real-dimension $2n-2d$) intersects T exactly at 0. If we now project X orthogonally onto $L^{\perp}$ then we obtain locally, near z_0, an embedding of X into $L^{\perp}$. □

3. Boundary behavior

The following is a generalization of Theorem 2.13(1) from [12].

Theorem 3.1. *Let M be a connected complex submanifold of $\mathbb{C}^n$ and let $f : M \to \mathbb{C}$ be a definable holomorphic function. Assume that Z is the set of all $z_0 \in Cl_{\mathbb{C}^n}(M)$ such that the limit of $f(z)$ exists and equals 0 as z approaches z_0 in M. Then Z is definable and if $\dim_{\mathbb{R}} Z \geqslant \dim_{\mathbb{R}} M - 1$ then f is the constant zero function on M.*

Proof The existence of a limit to f at a point $z \in M$ is described via an $\epsilon - \delta$ statement and therefore Z is definable. Let $\dim_{\mathbb{C}} M = d$ and assume that $\dim_{\mathbb{R}} Z \geqslant 2d-1$. Since $\dim_{\mathbb{R}} fr(M) \leqslant 2d-1$, by o-minimality, we may assume that $\dim_{\mathbb{R}}(Z) = 2d-1$. By o-minimality, we may further shrink M and assume that Z is a real submanifold of $\mathbb{C}^n$ of dimension $2d-1$.

Fix $z_0 \in Z$ and let π be a generic projection of $\mathbb{C}^n$ onto a d-dimensional $\mathbb{C}$-linear subspace $L \subseteq \mathbb{C}^n$. By Lemma 10 we may shrink M further and assume that $\pi|Z$ is an embedding of Z into L, considered as a real manifold. Finally, we may assume that π is the projection onto the first d coordinates. Notice that $\pi|M$ is a local biholomorphism outside a a $\mathbb{C}$-analytic set of complex dimension at most $d-1$.

By o-minimality, there are finitely many pairwise disjoint, definable connected open sets $U_1, \ldots, U_r \subseteq \mathbb{C}^n$ with the following properties:
(i) $\dim_{\mathbb{R}}(M \setminus (\bigcup_i U_i)) \leqslant 2d-1$.
(ii) For every $i = 1, \ldots, r$, $\pi|U_i$ is a biholomorphism, call it ϕ_i, between $U_i \cap M$ and an open definable $V_i \subseteq \mathbb{C}^d$.

(Clearly, we cannot do any better in (i), as is seen by the example of $M = \{(z,w) \in (\mathbb{C}^*)^2 : w = z^2\}$ and the projection onto w).

Since the union of the U_i's is necessarily dense in M, there is an $i_0 \in \{1, \ldots, r\}$ such that $\dim_{\mathbb{R}}(Cl(U_{i_0} \cap M) \cap Z) = 2d-1$. It follows that $\dim_{\mathbb{R}}(Cl(V_{i_0}) \cap \pi(Z)) = 2d-1$.

Consider the map $\Psi : V_{i_0} \to \mathbb{C}$ defined by $\Psi(z) = f(\phi_{i_0}^{-1}(z))$. This is a holomorphic map, which tends to 0 whenever z tends to an element of $\pi(Z)$ in V_{i_0}.

It follows that the set

$$\{z_0 \in Cl(V_{i_0}) : \lim_{z \to z_0} \Psi(z) \text{ exists and equals } 0\}$$

has real-dimension not less than $2d-1$.

We now use Theorem 2.13 (1) from [12] and conclude that Ψ is identically zero on V_{i_0}. It follows that F is identically zero on U_{i_0} and therefore on all of M. $\square$

We can now deduce an important technical tool:

Theorem 3.2. *Let $A_1 \in \mathcal{C}(M)$ be an irreducible, locally $\mathbb{C}$-analytic subset of a complex manifold M and assume that $\dim_{\mathbb{C}} A_1 = d$. Assume that A_2 is a locally $\mathbb{C}$-analytic subset of N. Then either $A_1 \subseteq A_2$ or $\dim_{\mathbb{R}}(Cl(A_1) \cap A_2) \leqslant 2d-2$.*

Proof Assume that $\dim_{\mathbb{R}}(Cl(A_1) \cap A_2) \geqslant 2d-1$. We may replace A_1 by the set of $\mathbb{C}$-regular points of A_1, $Reg_{\mathbb{C}}(A_1)$ (since $Reg_{\mathbb{C}} A_1$ is in $\mathcal{C}(M)$ and dense in A_1), so we may assume that A_1 is a connected complex submanifold of M. Consider $z_0 \in Cl(A_1) \cap A_2$ such that $\dim_{\mathbb{R}}(Cl(A_1)) \cap A_2 = 2d-1$ at z_0. Since $z_0 \in A_2$ there exist an open neighborhood U of z_0, which we may assume to be a definable subset of $\mathbb{C}^n$, and holomorphic definable $f_1, \ldots, f_t : U \to \mathbb{C}$ holomorphic such that $A_2 \cap U = Z(f_1, \ldots, f_t)$. We may also assume that $A_1 \cap U$ is definable. If $A_1 \cap U$ is not connected, we may replace it with one of its connected components.

Now consider the restriction of each f_i to A_1 and notice that the set of points in $Cl(A_1)$ at which the limit of f_i exists and equals zero has dimension not less than $2d-1$ (all the points in $Cl(A_1)\cap A_2$). By Theorem 1, $(A_1\cap U)\subseteq A_2$ and therefore $A_1\cap A_2$. □

Notice that we do not claim, in the above theorem, that $\dim_{\mathbb{R}}(Cl(A_1)\cap Cl(A_2))\leqslant 2d-2$. This is of course false because A_1 and A_2 could be for example open boxes in $\mathbb{C}$ with $\dim_{\mathbb{R}}(Cl(A_1\cap Cl(A_2))=1$.

4. Variations on the Remmert-Stein theorem

Theorem 4.1. *Let M be a complex manifold and assume that A is a locally $\mathbb{C}$-analytic subset of M, which is also in $\mathcal{C}(M)$.*

Assume that for every open $U\subseteq M$, $\dim_{\mathbb{R}} fr_U(A\cap U)\leqslant \dim_{\mathbb{R}}(A\cap U)-2$. Then $Cl(A)$ is a $\mathbb{C}$-analytic subset of M.

Notice that since the frontier of A in M is piecewise a C^1-manifold, the theorem is an immediate corollary of Shiffman's theorem in the case that A has pure dimension in M. However, it is false as stated without the assumption on that A is in $\mathcal{C}$: Take $M=\mathbb{C}^3$ and $A=\{(x,e^{1/x},1):x\neq 0\}\cup\{(0,y,z):z\neq 1, y\in\mathbb{C}\}$.

Proof By working in a neighborhood of a particular point of $Cl(A)$, we may assume that M is a definable open set $U\subseteq\mathbb{C}^n$ and that A is a definable subset of U. We take the closure and frontier relative to U. Assume that $M_1,\ldots,M_r$ are the connected components of $Reg_{\mathbb{C}}A$, ordered by $\dim_{\mathbb{C}} M_1\leqslant \dim_{\mathbb{C}} M_2\leqslant\cdots\leqslant\dim_{\mathbb{C}} M_r$. Since $Reg_{\mathbb{C}}A$ is dense in A, we have $Cl(A)=\bigcup_i Cl(M_i)$.

Claim*For every $i=1,\ldots,r$ we have*

$$\dim_{\mathbb{R}} fr(M_i)\leqslant \dim_{\mathbb{R}} M_i-2.$$

We prove the claim by induction on r, with the case $r=1$ just the assumption of the theorem. Since each M_i is relatively open in A and the M_i's are pairwise disjoint, for every $i\neq j$, we have $Cl(M_i)\cap M_j=\emptyset$. In particular, $Cl(M_r)\subseteq Cl(M)\setminus(M_1\cup\cdots\cup M_{r-1})$. It follows that $fr(M_r)\subseteq fr(A)\cup Sing_{\mathbb{C}}(A)$, and therefore, by our assumption, $\dim_{\mathbb{R}} fr(M_r)\leqslant\dim_{\mathbb{R}} A-2=\dim_{\mathbb{R}} M_r-2$. Now, by Shiffman's theorem, $Cl(M_r)$ is a $\mathbb{C}$-analytic subset of M and we may repeat the same argument for all components of maximal dimension, $M_{t+1},\ldots,M_r$. Let B the union of the closures of all these components, thus B is a $\mathbb{C}$-analytic subset of M as well.

Consider $A'=\cup_{i=1}^t M_i$ the union of all components of $Reg_{\mathbb{C}}A$ of dimension smaller than $\dim A$. In order to use the induction we show now that the assumption of the theorem holds for A'. Namely, for all open sets $V\subseteq M$, we have $\dim_{\mathbb{R}} fr_V(A'\cap V)\leqslant\dim_{\mathbb{R}}(A'\cap V)-2$.

Indeed, without loss of generality, $V=M$ and

$$fr(A')=(fr(A')\cap B)\cup(fr(A')\setminus B)\subseteq(Cl(A')\cap B)\cup(fr(A')\setminus B).$$

By Theorem 2, the real dimension of $Cl(A')\cap B$ is at most $\dim_{\mathbb{R}} A'-2$. Consider the open set $W=M\setminus B$ and notice that $fr(A')\setminus B\subseteq fr_W(A\cap W)\cup Sing_{\mathbb{C}}(A\cap W)$. By our assumption on A, we have

$$\dim_{\mathbb{R}} fr_W(A\cap W)\leqslant\dim_{\mathbb{R}}(A\cap W)-2=\dim_{\mathbb{R}} A'-2,$$

and

$$\dim_{\mathbb{R}} Sing(A \cap W) \leqslant \dim_{\mathbb{R}}(A \cap W) - 2 = \dim_{\mathbb{R}} A' - 2.$$

We therefore showed that A' indeed satisfies the assumption of the theorem. The number of connected components of $Reg_{\mathbb{C}}(A')$ is $t < r$ and therefore, by induction, for every $i = 1, \dots, t$ we have $\dim_{\mathbb{R}} fr(M_i) \leqslant \dim_{\mathbb{R}} M_i - 2$.

Now, by the minimality of r, for every $i \leqslant t$, we have $\dim_{\mathbb{R}} frM_i \leqslant \dim M_i - 2$. But now for $i = 1, \dots, r$, we have $\dim_{\mathbb{R}} frM_i \leqslant \dim_{\mathbb{R}} M_i - 2$ thus proving the claim.

Using the claim, we may now apply Shiffman's theorem to each M_i and conclude that $Cl(A) = \bigcup_i Cl(A_i)$ is a $\mathbb{C}$-analytic subset of M. □

Corollary 4.2. *Assume that M is complex manifold and A is a closed subset of M. Then A is a $\mathbb{C}$-analytic subset of M if and only if A is in $\mathcal{C}(M)$ and for every open $U \subseteq M$, we have*

$$\dim_{\mathbb{R}} Sing_{\mathbb{C}}(U \cap A) \leqslant \dim_{\mathbb{R}}(U \cap A) - 2.$$

Proof The only-if direction follows from the fact that every complex analytic subset of M is subanalytic in M. For the converse, we apply Theorem 1 to $Reg_{\mathbb{C}}A$ instead of A. □

Compare the following result to Piekozs ([15]), where a similar type of theorems are proved in the real analytic setting.

Corollary 4.3. *(1) Let $D \subseteq \mathbb{R}^n$ be a definable set, $W \subseteq \mathbb{C}^m$ a definable open set and let X be a definable subset of $D \times W$. Then, the set of all $a \in D$ such that $X_a = \{y \in W : (a, y) \in X\}$ is a (locally) complex analytic subset of W, is definable. (2) Let A be a subset of a complex manifold M that is in $\mathcal{C}(M)$. Then the set of all points $z \in M$ such that the germ of A at z is a $\mathbb{C}$-analytic in M is in $\mathcal{C}(M)$ as well.*

Proof By Corollary 2, for every $a \in D$, X_a is locally $\mathbb{C}$-analytic in W if and only if:

For every $x \in W$ and for every rectangular open $x \in W_1 \subseteq W$, $\dim_{\mathbb{R}} Reg_{\mathbb{C}}(W_1 \cap X_a) \leqslant \dim_{\mathbb{R}}(W_1 \cap X_a) - 2$.

Because dimension is uniformly definable in parameters, and because the set $Reg_{\mathbb{C}} X_a$ is definable in parameters (see the proof of Fact 6), the set of points $a \in D$ such that X_a is locally $\mathbb{C}$-analytic is definable. X_a is $\mathbb{C}$-analytic in W if in addition it is closed in W.

(2) is done similarly. □

The theorem below is an immediate corollary of the Remmert-Stein Theorem when A is assumed to be of pure dimension and $\dim E < \dim A$. However, it is easy to see that the theorem fails in general if we omit the assumption that A is in $\mathcal{C}(M)$.

Theorem 4.4. *Let M be a complex manifold and $E \subseteq M$ a $\mathbb{C}$-analytic subset of M (of arbitrary dimension). If A is a $\mathbb{C}$-analytic subset of $M \setminus E$ which is also in $\mathcal{C}(M)$ then $Cl(A)$ is a $\mathbb{C}$-analytic subset of M.*

Proof By Theorem 1, it is enough to see that for every open $U \subseteq M$ we have $\dim_{\mathbb{R}} fr_U(U \cap A) \leqslant \dim_{\mathbb{R}}(U \cap A) - 2$. By working locally we may assume that A is definable.

Let $M_1, \dots, M_r$ be the connected components of $Reg_{\mathbb{C}}(A \cap U)$. By Theorem 2 (applied with M_i and A for A_1 and A_2, respectively), for every i we have $M_i \subseteq E$

or $\dim_{\mathbb{R}}(Cl_U(M_i \cap E)) \leqslant \dim_{\mathbb{R}} M_i - 2$. Since $A \cap E = \emptyset$ the latter must hold and therefore $\dim_{\mathbb{R}} fr(A \cap U) \leqslant \dim_{\mathbb{R}}(A \cap U) - 2$. □

One corollary of the above theorem is:

Corollary 4.5. *Let X be a definable $\mathbb{C}$-analytic subset of $\mathbb{C}^n$. Then X is an algebraic subset of $\mathbb{C}^n$.*

Proof Since $\mathbb{C}^n$ is obtained from $\mathbb{P}^n(\mathbb{C})$ by removing a $\mathbb{C}$-analytic set, it follows from the above theorem that the closure of X in $\mathbb{P}^n(\mathbb{C})$ is a $\mathbb{C}$-analytic subset of $\mathbb{P}^n(\mathbb{C})$. Now apply Chow's Theorem. □

Theorem 4.6. *Let M be a complex manifold and $\{A_n : n \in \mathbb{N}\}$ a family of locally complex analytic subsets of M. If $A = \bigcup_{n\in\mathbb{N}} A_n$ is a closed subset of M which is also in $\mathcal{C}(M)$ then A is a complex analytic subset of M.*

Proof We use the following corollary of the Baire Category Theorem: If X is in $\mathcal{C}(M)$ with $\dim_{\mathbb{R}} X = k$, and $X \subseteq \bigcup_{n\in\mathbb{N}} X_n$ a countable union of either sets in $\mathcal{C}(M)$ or sets which are locally complex analytic in M, then $\max\{\dim_{\mathbb{R}} X_n : n \in \mathbb{N}\} \geqslant k$. We also use the fact that every complex manifold can, by definition, be written as a countable and increasing union of open sets whose closure in M is compact.

First notice that since A is in $\mathcal{C}(M)$ the set of $\mathbb{R}$-regular points of A is dense in A. Now, by the above observation, the set of $\mathbb{R}$-regular points of A must contain a complex submanifold of M of the same real-dimension (namely, the $\mathbb{C}$-regular points of one of the A_n's). The same is true for some open subset of M, and therefore the set of $\mathbb{C}$-regular points of A is dense in the set of its $\mathbb{R}$-regular points. It easily follows that these sets are equal. We now need the following technical lemma:

Lemma 4.7. *Assume that X is a closed subset of a complex manifold M, $X \in \mathcal{C}(M)$, and that $Reg_{\mathbb{C}}X$ is dense in X. Assume that Y is a subset of X that is locally $\mathbb{C}$-analytic in M. Then $\dim_{\mathbb{R}}(Sing_{\mathbb{C}}(X) \cap Y) \leqslant \dim_{\mathbb{R}} X - 2$.*

Proof It is enough to prove the result in every open relatively compact subset of M, thus we may assume, by shrinking M if needed, that X is definable and M an open definable subset of $\mathbb{C}^n$. Let $X_1, \ldots, X_r$ be the connected components of $Reg_{\mathbb{C}}X$. First observe that by the density assumption, for each $z \in Y$, either the germs at z of X and Y are equal, or there is an i such that $z \in Cl(X_i)$ and the germ of X_i at z is not contained in Y.

By working locally, at points of Y, we may further assume Y is a definable complex analytic subset of M. In particular, if the germ of X_i at some $z \in M$ is contained in Y then $X_i \subseteq Y$.

We may assume then that for some $s \leqslant r$, $X_1, \ldots, X_s$ are those components of $Reg_{\mathbb{C}}X$ that are not contained in Y. We now claim that

$$Sing_{\mathbb{C}}(X) \cap Y \subseteq Sing_{\mathbb{C}}(Y) \cup \bigcup_{i=1}^{s} (Cl(X_i) \cap Y).$$

Indeed, consider $z \in Sing_{\mathbb{C}}(X) \cap Y$. If $z \notin Cl(X_i) \cap Y$ for every $i = 1, \ldots, s$ then, by our previous observation, the germs of X and Y agree near z and therefore $Sing_{\mathbb{C}}X = Sing_{\mathbb{C}}Y$ near z.

By Theorem 2, the dimension of each $Cl(X_i) \cap Y$ is at most $\dim_{\mathbb{R}} X_i - 2$, and clearly, $\dim_{\mathbb{R}} Sing_{\mathbb{C}}Y \leqslant \dim_{\mathbb{R}} Y - 2 \leqslant \dim_{\mathbb{R}} X - 2$, ending the proof of the lemma. □

We now return to the proof of Theorem 6.

Since A is closed and in $\mathcal{C}(M)$ it is sufficient, by Corollary 2, to prove that $\dim_{\mathbb{R}} Sing_{\mathbb{C}}(A) \leqslant \dim_{\mathbb{R}} A - 2$ and that the same is true inside every open subset of M.

By the lemma we just proved, for each n, $\dim_{\mathbb{R}}(Sing_K(A) \cap A_n) \leqslant \dim_{\mathbb{R}} A - 2$. But $Sing_{\mathbb{C}}(A)$ is a countable union of sets of this form and hence we must have $\dim_{\mathbb{R}}(Sing_{\mathbb{C}}(A)) \leqslant \dim_{\mathbb{R}} A - 2$.

This remains true, via the same reasoning, when restricted to any open subset of M, thus by Corollary 2, A is $\mathbb{C}$-analytic in M. □

Note that the last theorem fails without the assumption that $A \in \mathcal{C}(M)$, even for a union of two locally $\mathbb{C}$-analytic sets: Take A_1 to be the graph of the function $e^{1/z}$ in $\mathbb{C}^* \times \mathbb{C}$ and $A_2 = \{0\} \times \mathbb{C}$.

5. Frontiers of locally analytic sets

Our goal here is to discuss the possible behavior of the frontier of a locally $\mathbb{C}$-analytic $A \subseteq M$ such that $A \in \mathcal{C}(M)$. We do it up to a definable nowhere dense subset of $fr(A)$.

Assume that M is a complex manifold, $A \in \mathcal{C}(M)$ is a subset of M whose real dimension is k. Then properties of the analytic-geometric category imply that the frontier of A has real dimension at most $k-1$. As we show in this section, if A is also a locally $\mathbb{C}$-analytic subset of M then outside a definable subset of $fr(A)$ of dimension $k-2$, the closure of A is well-behaved.

Theorem 5.1. *Let M be a complex manifold, $A \in \mathcal{C}(M)$ a locally $\mathbb{C}$-analytic subset of M. Then there is a closed set D in $\mathcal{C}(M)$, $D \subseteq fr(A)$ with $\dim_{\mathbb{R}} D \leqslant \dim_{\mathbb{R}} A - 2$, such that for every $z_0 \in fr(A) \setminus D$ one of the following two possibilities must occur:*

(i) A is locally connected at z_0, in which case $Cl(A)$ near z_0 is an $\mathbb{C}$-submanifold whose boundary is C^1.

(ii) A has locally two connected components at z_0, in which case $Cl(A)$ is a $\mathbb{C}$-analytic submanifold of M in some neighborhood of z_0.

Notice that the analogous theorem is false in the category of real analytic sets. Consider for example, the two-dimensional real manifold

$$A = \{(x, y, z) \in \mathbb{R}^3 : x = 0 \text{ or } y = 0\} \setminus \{(0, 0, z) : z \in \mathbb{R}\}.$$

The frontier of A in $\mathbb{R}^3$ has dimension $1 = \dim_{\mathbb{R}} A - 1$, but the germ of A at every point in $fr(A)$ has four components.

Proof We need first the following technical lemma:

Lemma 5.2. *Assume that A is a definable, locally complex analytic subset of an open set $U \subseteq \mathbb{C}^n$, such that A is of pure complex dimension d. Let B be the set of all $z \in U \cap fr_U(A)$ with the following property: For every generic projection π of $\mathbb{C}^n$ onto a d-dimensional $\mathbb{C}$-linear subspace L, there in an open U containing z such that π is a biholomorphism of $U \cap A$ with some open subset of L. Then B is definable and $\dim_{\mathbb{R}}(fr_U(A) \setminus B) \leqslant 2d - 2$.*

(Note that the analogous statement is false if A is a real analytic subset of $\mathbb{R}^n$, as the last example shows).

Proof of Lemma By removing from A its $\mathbb{C}$-singular points we may assume that A is a $\mathbb{C}$-submanifold of M. We assume that the lemma fails and proceed by a sequence of reductions until we reach a contradiction.

We may assume, after possibly shrinking A and U, that for every $z_0 \in fr_U(A)$ and for every d-dimensional subspace L in some open subset W of the suitable Grassmannian, the orthogonal projection π_L of $\mathbb{C}^n$ onto L, when restricted to A, is not a local biholomorphism near z_0 (notice that z_0 is a point *outside* of A). By the "good $\mathbb{C}$-direction Lemma", we may also assume that $\pi|A$ is everywhere finite-to-one, for every $\pi \in W$.

We may assume, after further shrinking U, that $fr_U(A)$ is a C^1-submanifold of $\mathbb{C}^n$ and that for all $L \in W$, the restriction of π to $fr_U(A)$ is an embedding of $fr_U(A)$ into L (see Lemma 10). Furthermore, using standard arguments, we may also assume that for every such π there is a neighborhood $U_1 \subseteq U$ intersecting $fr(A)$ non-trivially, such that $\pi_L : U_1 \cap Cl(A) \to \pi(U_1)$ is a proper map. We fix one such L, set $\pi = \pi_L$ and replace U by the corresponding U_1 and call it U again.

By removing the set of points where the Jacobian of $\pi|A$ has small rank, we may also assume, shrinking U if needed, that $\pi|A$ is a local biholomorphism at every point of A.

Thus the only reason that $\pi|A$ might not be a local biholomorphism near $z \in fr_U(A)$ is that it is not injective in any neighborhood of z. As we now show, this leads to a contradiction.

Since $\pi|A$ is a local biholomorphism, the set $\pi(A)$ is an open subset of $\pi(U) \subseteq L$. After shrinking again A and U we may also assume that the boundary of $\pi(A)$ in $\pi(U)$ is the C^1-manifold $\pi(fr_U(A))$.

For every $v \in \pi(A)$ let $m(v)$ be the number of pre-images of v in A under π. Since we assume that $\pi|A$ is not a locally injective near any point of $fr_U(A)$, there is a number $m > 1$ and an open set $U_1 \subseteq U$ intersecting non-trivially $fr_U(A)$, such that $m(v) = m$ for all $v \in \pi(U_1 \cap A)$. We denote $\pi(U_1 \cap A)$ by V.

To simplify notation, we will assume from now on that $L = \mathbb{C}^d$, identified with the first d coordinates of $\mathbb{C}^n$, and for $z \in \mathbb{C}^n$ write $z = (z_1, z_2) \in \mathbb{C}^d \times \mathbb{C}^{n-d}$.

Let $\phi_1, \dots, \phi_m : V \to \mathbb{C}^{n-d}$ be definable functions which give the branches of $U_1 \cap A$ over V (we use definable choice here). By o-minimality, the ϕ_i's are continuous outside a set $D \subseteq V$ whose real-dimension is at most $2d-1$. Since the intersection of $Cl(D)$ with $fr(V)$ is not dense in $fr(V)$, we may assume, after possibly shrinking U, A and V, that the ϕ_i's are continuous on all of V. But since $\pi|A$ was a local biholomorphism, it is easy to verify that each ϕ_i is actually a holomorphic map on V (see for example Theorem 2.14 in [12]).

For each $i \neq j \in \{1, \dots, m\}$, let $\Psi_{i,j} = \phi_i - \phi_j$ (the subtraction is done coordinate-wise). Each $\Psi_{i,j}$ is a holomorphic map from V into $\mathbb{C}^{n-d}$.

Let $z_0 = (z_1, z_2)$ be in $fr_U(A)$. Since $z_1 \in fr_{\pi(U)}(V)$ and $\pi : Cl(A) \to \pi(U)$ is a proper map and $\pi|fr_U(A)$ is injective, for each $i = 1, \dots, m$, the map $\phi_i(z)$ tends to z_2, as z tends to z_1 in V. Therefore, the limit of each $\Psi_{i,j}(z')$ is zero as z' tends to z_1 in V.

But then, the set

$$\{z_1' \in fr(V) : \lim_{z' \to z_1} \Psi_{i,j}(z') = 0\}$$

has dimension at least $2d-1$.

By Theorem 1, this implies that each $\Psi_{i,j}$ is identically zero on V. This in turn implies that $\phi_i = \phi_j$ in some neighborhood of z_0 and therefore $m = 1$, contradicting out assumption, and thus ending the proof of the lemma. □

We now return to the proof of the Theorem 1. Since the statement is local, we may assume, without loss of generality, $M = \mathbb{C}^n$ and A a definable subset of $\mathbb{C}^n$.

Let $d = \dim_{\mathbb{C}} A$. We fix $D = fr(A) \setminus B$, for B as in the last lemma. We will show that this D indeed works for the theorem.

Take $z_0 \in fr(A) \setminus D$ and choose a projection $\pi : \mathbb{C}^n \to L$ onto a d-dimensional subspace L such that $\pi|(U \cap A)$ is a $\mathbb{C}$-biholomorphism in some neighborhood U of z_0. We may assume that M equals U. Moreover, as we saw in the above proof, we may choose π so that $\dim_{\mathbb{R}} \pi(fr_U(A)) = 2d-1$ and $\pi|fr_U(A)$ is an embedding of a real submanifold into L. By restricting ourselves to a smaller neighborhood U_1 of z_0, we may assume that the complement of this submanifold in $\pi(U_1)$ is a union of two disjoint, connected open sets V_1 and V_2.

As before, we assume that $L = \mathbb{C}^d$, and write $z_0 = (z_1, z_2)$.

As we pointed out at the end of Section 2.3, we may partition $fr_U(A)$ into finitely many sets on each of which the number of local components of the germ of A at every point is constant. We only need to analyze what happens on these sets when their dimension equals to $2d-1$, so we assume that the number of local components at every point is constant on $fr_U(A)$.

Case 1 A is locally connected at points in $fr_U(A)$.

Fix $z_0 \in fr_U(A)$. In this case $\pi(A)$, in some neighborhood of z_0, is contained in one of the two components V_1 and V_2. If $\pi(A) \subseteq V_1$ then $\pi(z_0)$ belongs to the boundary of V_1. In this case the inverse map from V_1 into A extends continuously to the boundary of V_1 near $\pi(z_0)$ and thus A is, near z_0, a manifold with a boundary.

Case 2 A is not locally connected at $fr_U(A)$.

The image of every local component, under π, is either contained in V_1 or in V_2 and $\pi(fr_U(A))$ is the joint boundary of all of these images. Since $\pi|A$ is injective, there must be exactly two such local components of A at $z \in fr_U(A)$, one projecting homeomorphically onto V_1 and the other onto V_2 (in some neighborhood of every point in $fr_U(A)$).

Let $z_0 = (z_1, z_2)$ be a point in $fr_U(A)$ and consider the holomorphic map $\phi : V_1 \cup V_2 \to \mathbb{C}^{n-d}$, whose graph is A. As we already saw before, ϕ extends continuously to z_1 (i.e, $\phi(z_1) = z_2$). But then, ϕ extends continuously to some neighborhood of z_1 in $fr_{\pi(U)}(V_1 \cup V_2)$, and we call this extension $\tilde{\phi}$. Since $\tilde{\phi}$ is continuous, and holomorphic outside a set of dimension $2d-1$, it is holomorphic in some neighborhood of z_0, and its graph is $Cl(A)$ there (see Theorem 2.14 in [12]). We therefore showed that $Cl(A)$ is a $\mathbb{C}$-submanifold in a neighborhood of z_0. □

Theorem 5.3. *Assume that X is a closed subset of a complex manifold M, $X \in \mathcal{C}(M)$, such that $Reg_{\mathbb{C}}(X)$ is dense in X. Assume also that at no point z_0 in X, the germ of X at z_0 is a real manifold with a boundary. Then X is a complex analytic subset of M.*

Proof Assume that $\dim_{\mathbb{R}} X = 2d$ (the dimension is even by the density of $Reg_{\mathbb{C}}(X)$).

By Corollary 2, we need to show that $\dim_{\mathbb{R}} Sing_{\mathbb{C}}(X) \leqslant 2d-2$. (If we show it for an arbitrary X with the above property then the same is true inside any open subset of M.)

Assume toward contradiction that $\dim_{\mathbb{R}}(X \setminus Reg_{\mathbb{C}}(X)) = 2d-1$. We now apply Theorem 1 to $A = Reg_{\mathbb{C}}(X)$. Notice that since $Reg_{\mathbb{C}}(X)$ is dense in X, $fr(A) = X \setminus A$. By Theorem 1, there is a point z_0 in $X \setminus A$ such that X is an $\mathbb{R}$-manifold with a boundary near z_0 (since $z_0 \notin Reg_{\mathbb{C}}(X)$ only (i) of the theorem can hold for z_0), contradicting our assumption. It follows that X is a $\mathbb{C}$-analytic subset of M. □

We have been told that the following corollary is due to P. Milman.

Corollary 5.4. *Let M be a complex analytic manifold and assume that $X \subseteq M$ is a real-analytic subset of M. Assume also that every $\mathbb{R}$-regular point of X is also $\mathbb{C}$-regular. Then X is a complex analytic subset of $\mathbb{C}^n$.*

Proof To apply the last theorem we just need to point out that a real-analytic set is not a real manifold with a boundary at any of its points. □

6. Holomorphic and meromorphic maps

6.1. **A variant on Remmert's proper mapping Theorem.** We now prove a strong version of Remmert's classical theorem on proper holomorphic maps.

Theorem 6.1. *Let $f : M \to N$ be a holomorphic map between complex analytic manifolds, $A \subseteq M$ a complex analytic subset of M. If $f(A)$ is a closed subset of N and belongs to $\mathcal{C}(N)$ then $f(A)$ is a complex analytic subset of N.*

Remarks
1. Notice that Remmert's Proper Mapping Theorem follows from the theorem since, if $f : M \to N$ is assumed to be proper then $f(A)$ is necessarily a subanalytic closed subset of N. However, the theorem still applies to cases where f is not proper and yet $f(A)$ is a complex analytic subset of N.
2. The theorem is false without the assumption that $f(A)$ is in $\mathcal{C}(M)$. For example, the projection of $\{(n, 1/n) \in \mathbb{C} \times \mathbb{C} : n \geqslant 1\} \cup \{(0,0)\}$ onto the first coordinate is a closed subset of $\mathbb{C}$ which is clearly not $\mathbb{C}$-analytic in $\mathbb{C}$. (We could not find such an example with A being irreducible).

Proof We first claim that it is enough to prove that $\dim_{\mathbb{R}} Sing_{\mathbb{C}} f(A) \leqslant \dim_{\mathbb{R}} f(A) - 2$. Indeed, if we can show it in general then, by replacing N with an open set $U \in \mathcal{C}(N)$ and M by $f^{-1}(U)$ we can derive the same dimension inequality locally as well and use Corollary 2.

Let A_i, $i \in I$, be the (possibly countably many) irreducible components of A. For each $i \in I$ let f_i be the restriction of f to the submanifold $Reg_{\mathbb{C}} A_i$ and let k_i be the generic rank of $D_z(f_i)$. Since $\dim_{\mathbb{R}} f(A) \leqslant \dim_{\mathbb{R}} N$, the k_i's attain a maximum, call it k.

Claim 1
(i) $\dim_{\mathbb{R}} f(A) = 2k$.
(ii) $Reg_{\mathbb{C}} f(A)$ is dense in $f(A)$,
Proof of Claim: For each A_i let

$$X_i = \{z \in Reg_{\mathbb{C}} A_i : Rank(D_z f) = k_i\}.$$

X_i is a submanifold of M and every $z \in X_i$ has a neighborhood in X_i whose image under f is a complex submanifold of N of dimension k_i.

Since A_i is a complex analytic subset of M it can be written as a countable union of definable sets $A_{i,j}$, $j \in \mathbb{N}$, each of which is a definable complex analytic subset of a definable open set $U_{i,j}$, such that $f|U_{i,j}$ is definable. If $X_{i,j} = X_i \cap U_{i,j}$ then the definable set $f(X_{i,j})$ is a countable union of $\mathbb{C}$-submanifolds of N of dimension k_i. In particular, its real-dimension is $2k_i$. Since $X_{i,j}$ is dense in $A_{i,j}$, it follows that $\dim_{\mathbb{R}} f(A_{i,j}) = 2k_i$ and therefore, by our earlier observation, $\dim_{\mathbb{R}} f(A) = 2k$.

(ii) If $Y \subseteq f(A)$ is a definable set that is open in $f(A)$ then it is a countable union of definable sets of the form $f(A_{i,j})$. In particular, $\dim_{\mathbb{R}} Y = 2k_i$ for some i and at least one of these $f(A_{i,j})'$s equals Y in some open set. By the same dimension argument, there is a definable subset X of $X_{i,j}$ such that $f(X) \subseteq Y$ is a complex submanifold of N of dimension k_i and hence $f(X)$ equals to Y on some open set. In particular, $Reg_{\mathbb{C}} f(A)$ is dense in Y. End of Claim 1.

Claim 2

(i) $f(A) \setminus \bigcup_i f(X_i)$ is contained in a countable union of sets in $\mathcal{C}(N)$, each of real-dimension not greater than $2k-2$.

(ii) For each $i \in I$, $Sing_{\mathbb{C}}(f(A)) \cap \bigcup_i f(X_i)$ is contained in a countable union of sets in $\mathcal{C}(N)$, each of real-dimension not greater than $2k-2$.

Taken together, (i) and (ii) imply that $Sing_{\mathbb{C}} f(A)$ is contained in a countable union of sets in $\mathcal{C}(N)$, each of dimension not greater than $2k-2$. Hence we must have $\dim_{\mathbb{R}} Sing_{\mathbb{C}} f(A) \leqslant 2k-2$, and therefore $f(A)$ is $\mathbb{C}$-analytic in N. It is thus sufficient to prove Claim 2.

Proof of Claim 2: For the purpose of (i) it is sufficient to prove that for every i, j we have

$$\dim_{\mathbb{R}}(f(A_{i,j}) \setminus f(X_{i,j})) \leqslant 2k_i - 2.$$

This is exactly the content of the following lemma:

Lemma 6.2. *Let $U \subseteq \mathbb{C}^n$ be an open definable set, $A \subseteq U$ a definable complex analytic subset which is irreducible and of complex dimension d. Let $f : U \to \mathbb{C}^m$ be a definable holomorphic map. Assume that the generic rank of f on $Reg_{\mathbb{C}} A$ is k . Let $X = \{x \in Reg_{\mathbb{C}} A : rank(f|Reg_{\mathbb{C}} A)_x = k\}$. Then, $\dim_{\mathbb{R}}(f(A) \setminus f(X)) \leqslant \dim_{\mathbb{R}} f(A) - 2 = 2k - 2$.*

Proof First note that $\dim_{\mathbb{R}}(A \setminus X) \leqslant 2d-2$. Now, for every $x \in X$, the complex dimension of $(f^{-1}f(x))_x$ is $d-k$. Since the function $x \mapsto \dim(f^{-1}f(x))_x$ is upper semi-continuous (see 1.3.8 in [2]), for every $x \in A$ we have $dim f^{-1}(f(x))_x \geqslant d-k$.

Let $X' = \{x \in A : f(x) \notin f(X)\}$. We have $f(X') = f(A) \setminus f(X)$, and for every $x \in X'$, $f^{-1}(f(x)) \subseteq X'$. Moreover, X' is a subset of $A \setminus X$, hence $\dim_{\mathbb{R}} X' \leqslant 2d-2$. The restriction of f to X' gives a surjection on $f(A) \setminus f(X)$ whose fibres have real-dimension not less than $2d-2k$. It follows that $\dim_{\mathbb{R}} f(A) \setminus f(X) \leqslant 2k-2$. End of Lemma □

To prove (ii) it is sufficient to prove, in the above notation, that for every i, j, $\dim_{\mathbb{R}}(Sing f(A) \cap f(X_{i,j})) \leqslant 2k_i - 2$. Since the rank of f at each $z \in X_{i,j}$ is k_i, we may replace each $U_{i,j}$ by countably many smaller ones, if needed, and assume that $f(X_{i,j})$ is a complex submanifold of N whose complex dimension is k_i. We may now apply Lemma 7 to $f(A)$ and $f(X_{i,j})$ (in place of X and Y) and conclude that $\dim_{\mathbb{R}}(Sing f(A) \cap f(X_{i,j})) \leqslant 2k_i - 2$. We thus proved Claim 2 and therefore the theorem. □

Corollary 6.3. *Let $f : M \to N$ be a holomorphic map between complex manifolds, A an irreducible $\mathbb{C}$-analytic subset of M, and assume that $f(A)$ is in $\mathcal{C}(M)$ (but not necessarily closed).*

Then there is a closed set $E \in \mathcal{C}(N)$, with $\dim_{\mathbb{R}} E \leqslant \dim_{\mathbb{R}} f(A) - 2$ and $\dim_{\mathbb{R}}(f^{-1}(E) \cap A) \leqslant \dim_{\mathbb{R}} A - 2$, such that $f(A) \setminus E$ is locally $\mathbb{C}$-analytic in N .

Proof Let E be the set of all points in $f(A)$ such that $f(A)$ is not locally closed at z. Notice that $E = fr(fr(f(A)))$ and therefore E is in $\mathcal{C}(N)$ and $\dim_{\mathbb{R}} E \leqslant \dim_{\mathbb{R}} f(A) - 2$. Let $f_1 = f|Reg_{\mathbb{C}}(A)$ and let k be the generic rank of $D_z(f_1)$. As we already saw in the proof of the last theorem, $\dim_{\mathbb{R}} f(A) = 2k$ and therefore $\dim_{\mathbb{R}} E \leqslant 2k-2$.

The set $D = f^{-1}(E) \cap A$ is in $\mathcal{C}(M)$. We claim that $\dim_{\mathbb{R}} D \leqslant \dim_{\mathbb{R}} A - 2$. Indeed, assume that $\dim_K(A) = n$ and $\dim_{\mathbb{R}} D \geqslant 2n-1$. Then we may find a set $X \subseteq (D \cap Reg_{\mathbb{C}}(A))$ in $\mathcal{C}(M)$ such that $\dim_{\mathbb{R}} X \geqslant 2n-1$ and the rank of $D_z(f_1)$ equals to k for every $z \in X$ (because the set of points where this rank is not k has co-dimension 2 in $Reg_{\mathbb{C}}(A)$). It follows that for every $z \in X$, we have $\dim_{\mathbb{R}} f_1^{-1}(f_1(x))_x = 2n-2k$ and therefore $\dim_{\mathbb{R}} f(X) \geqslant 2k-1$, contradicting the fact that $f(X) \subseteq D$.

It is left to see that $B = f(A) \setminus E$ is locally analytic in N. Since it is locally closed and in $\mathcal{C}(N)$ there is an open set $N_1 \subseteq N$ containing B such that B is closed in N_1. Let $M_1 = f^{-1}(N_1)$ and $A_1 = f^{-1}(B)$. The set A_1 is $\mathbb{C}$-analytic in M_1 and $f(A_1) = B$ is in $\mathcal{C}(N_1)$ and closed in N_1. By Theorem 1, $f(A_1)$ is $\mathbb{C}$-analytic in N_1. □

6.2. **Meromorphic maps.** Our goal in this section is to prove the following theorem (which again fails without the assumption that the graph of f is in $\mathcal{C}(N \times M)$.).

Theorem 6.4. *Let M, N be complex manifolds, S an irreducible $\mathbb{C}$-analytic subset of M and assume that $L \in \mathcal{C}(M)$ a closed subset of S which contains $Sing_{\mathbb{C}}(S)$. Assume that $f : S \setminus L \to N$ is a holomorphic map whose graph is in $\mathcal{C}(M \times N)$.*

If $\dim_{\mathbb{R}} L \leqslant \dim_{\mathbb{R}} A - 2$ then the closure of the graph of f in $M \times N$, call it Y, is a $\mathbb{C}$-analytic subset of $M \times N$.

If in addition the projection $\pi : Y \to S$ is a proper map then Y is a meromorphic map (namely, in addition to the above there is a $\mathbb{C}$-analytic proper subset D of S such that restriction of Y to $(S \setminus D) \times N$ is a holomorphic map).

We first use a result of Kurdyka and Parusinski to prove a technical statement about definable real functions.

Proposition 6.5. *Let $U \subseteq \mathbb{R}^m$ be a definable open set, $F : U \to \mathbb{R}$ a definable function which is differentiable on U, and a in the closure of U. Assume that U is locally connected at a and $\|\nabla F\|$ is bounded on U. Then the limit*

$$\lim_{x \to a, x \in U} F(x)$$

exists in $\mathbb{R}$.

Proof We will assume $a = 0 \in Cl(U)$ and that for all $u \in U$, $\|\nabla F(u)\| \leqslant M$ on U.

For $r \in \mathbb{R}^+$, we denote by B_r the open ball in $\mathbb{R}^m$ of radius r centered at $\vec{0}$, and by U_r the set $U \cap B_r$. Proposition 5 follows from the following claim.

Claim 6.6. *Under the above assumptions there is $C \in \mathbb{R}$ such that for every sufficiently small positive r and all $x, y \in U_r$*

$$|F(x) - F(y)| < Cr$$

Proof Since U is locally connected at 0 there is $\varepsilon > 0$ such that every U_r is connected for $r < \varepsilon$. By [7, Corollary 1.3], there is a constant $c > 0$ such that for every $r < \varepsilon$ and $x, y \in U_r$ there is a definable continuous curve ξ joining x and y such that length$(\xi) < cr$. Since $\|\nabla F(u)\| \leqslant M$ on U, we obtain $|F(x) - F(y)| < Mcr$. □

By considering the real and imaginary parts of a complex valued function, we obtain a corresponding result to Proposition 5 for holomorphic functions into $\mathbb{C}$ (where $\nabla(F)$ is the gradient of F with respect to its complex variables).

For $f : U \to \mathbb{C}$ and $z \in Cl(U)$, we let $Lim_z f$ be the set of all possible limit points of $f(z')$ as z' tends to z in U. We denote by $|Lim_z f|$ the number of elements in this set (possibly ∞). If U is locally connected at $z \in Cl(U)$ then $Lim_z f$ is a connected subset of $\mathbb{C}$. Moreover, in this case the set $|Lim_z f| = 1$ if and only if f can be extended continuously to z.

Lemma 6.7. *(i) Let $U \subseteq \mathbb{C}^n$ be a definable open set and let $f : U \to \mathbb{C}^m$ be a holomorphic map whose graph is in $\mathcal{C}(U \times \mathbb{C}^n)$. Let*

$$B = \{(z,w) \in \partial U \times \mathbb{C}^m : w \in Lim_z f \& |Lim_z f| = \infty\}.$$

Then $\dim_{\mathbb{R}} B \leqslant \dim_{\mathbb{R}} U - 2 = 2n - 2$.

(ii) Let M_1, N be complex manifolds and $M \subseteq M_1$ a complex submanifold of M_1 which is in $\mathcal{C}(M_1)$. Let $f : M \to N$ be a holomorphic map into N whose graph is in $\mathcal{C}(M_1 \times N)$ and let

$$B = \{(z,w) \in fr_{M_1}(M) \times N : w \in Lim_z f \& |Lim_z f| = \infty\}.$$

Then $\dim_{\mathbb{R}} B \leqslant \dim_{\mathbb{R}} M - 2$.

Notice that the analogous theorem is false for real-analytic functions: Consider for example the function x/y on $U = \mathbb{R}^2 \setminus \{(0,0)\}$. Since the frontier of its graph is infinite above $(0,0)$, this set has dimension $1 = \dim_{\mathbb{R}} U - 1$ and not $\dim_{\mathbb{R}} U - 2$.
Proof (i) Assume toward contradiction that $\dim_{\mathbb{R}} B = 2n - 1$.

By standard o-minimal methods, there is a definable open set $V \subseteq U \times \mathbb{C}^m$ such that the graph of f, restricted to V, is definable, $\dim_{\mathbb{R}}(V \cap B) = 2n - 1$ and for every $(z_0, w_0) \in B \cap V$ the set $(\{z_0\} \times \mathbb{C}^m) \cap (V \cap B)$ is infinite.

Take $U_1 = \{z \in U : (z, f(z)) \in V\}$. Notice that for $(z,w) \in V$ we have

$$(z,w) \in B \cap V \;\Leftrightarrow\; w \in Lim_z f|U_1 \;\&\; |Lim_z f|U_1| = \infty.$$

Therefore, we may replace U by U_1, replace f by $f|U_1$, and assume that f is definable on U.

Write $f(z) = (f_1(z), \ldots, f_m(z))$. Since the set $\{z \in U : Lim_z f_i \text{ is infinite }\}$ is definable, we may assume, after possibly partitioning B and re-ordering the f_i's, that for all $(z,w) \in B$, the set $Lim_z f_1$ is infinite.

By the complex analogue of Proposition 5, at least one of the first partial derivatives of f_1 is unbounded at z, for every $(z,w) \in B$. Without loss of generality, we may therefore also assume that $\partial f_1/\partial z_1$ is unbounded near z for all $(z,w) \in B$ (where z_1 is the first variable of z). For simplicity, let us denote this partial derivative by $h(z)$.

Since h is holomorphic on U, and unbounded near points of B, its zero set has dimension at most $2n - 2$. Therefore, by shrinking U if needed, we may assume that h never vanishes on U.

Let $M \subseteq U \times \mathbb{C}^m$ be the graph of the function f. Notice that M is a complex submanifold of $\mathbb{C}^{n+m}$ of complex-dimension n and that $(z,w) \in Cl(M)$ if and only if $w \in Lim_z f$. In particular, $B \subseteq fr(M)$.

Consider the function g on M given by: $g(z,w) = 1/h(z)$. Since h was definable, holomorphic and non-vanishing on U the function g is definable and holomorphic on M. Since h is unbounded at every z such that $(z,w) \in B$, zero is a limit point of g at every $(z,w) \in B$.

However, since $\dim_{\mathbb{R}} M = 2n$, we may assume, after removing from $fr(M)$ a set of dimension smaller than $2n-1$, that for every $(z,w) \in fr(M)$, the function g has at most finitely many limits at (z,w). (This number is no larger than the number of the connected components of the germ of M at (z,w)).

We may now replace U by an open subset U' and fix $M' \subseteq M$ of dimension $2n$ such that M' is locally connected at every $(z,w) \in B \cap U'$ and 0 is still in $Lim_z(f|M')$. It follows that for every such (z,w), the set $Lim_{(z,w)} f|M'$ contains a single element 0.

If we now let

$$Z = \{z \in Cl(M') : \lim_{z' \to z} g(z) = 0\}$$

then $\dim_{\mathbb{R}}(Z) = 2n-1$. By Theorem 1, g is constantly zero on M', contradicting our assumptions on h.

(ii) We first consider the case where $N = \mathbb{C}^m$. In this case, the proof of (ii) follows from (i) just like Theorem 1 follows from the corresponding "affine" statement in Theorem 2.13(i) [12].

Consider now the general case, where M is a d-dimensional $\mathbb{C}$-submanifold of an n-dimensional M_1 and that N is a $\mathbb{C}$-manifold of dimension m. Assume, toward contradiction, that $\dim_{\mathbb{R}} B = 2d-1$. Since B is a subset of $M_1 \times N$, there is a relatively compact open chart $V \subseteq N$ such that $B \cap (M_1 \times V)$ has dimension $2d-1$ and for every $(z,w) \in M_1 \times V$, we have $(z,w) \in B \cap (M_1 \times V)$ if and only $z \in fr(M_1)$, $w \in Lim_z f|f^{-1}(V)$ and $|Lim_z f|f^{-1}(V)| = \infty$.

If we now replace M by $M' = f^{-1}(V)$ then we may assume that the restriction of f to M' is a holomorphic map from M' into $V \subseteq \mathbb{C}^m$. We now reduced the problem to the case we already handled, ending the proof of Lemma 7. □

We can now prove Theorem 4.

Let $M_0 = Reg_{\mathbb{C}}(S) \setminus L$ and let G_f be the graph of $f : M_0 \to N$. Assume that $\dim_{\mathbb{R}} M_0 = 2n$. By Theorem 1 it is enough to see that $\dim_{\mathbb{R}} fr_{M\times N}(G_f) \leqslant 2n-2$ (since G_f is a set of pure dimension $2n$ there is no need to check every open set). Assume toward contradiction that $\dim_{\mathbb{R}}(fr_{M\times N}(G_f)) = 2n-1$. For every $(z,w) \in fr_{M\times N}(G_f)$, we must have $z \in L$. But since $\dim_{\mathbb{R}} L = 2n-2$, the set

$$\{(z,w) \in fr(G_f) : w \in Lim_z f \& |Lim_z f| = \infty\}$$

must have dimension $2n-1$, contradicting Lemma 7 (applied to M_0 and M for M and M_1, respectively).

We therefore showed that the closure of G_f in $M \times N$ is complex analytic in $M \times N$. Denote this closure by Y.

Assume now that the projection map $\pi : Y \to S$ is a proper map. Let

$$D_1 = \{z \in S : \dim_{\mathbb{C}}(\pi^{-1}(z) \cap Y) > 0\}.$$

Since π is proper, D_1 is a Zariski closed subset of M (see Theorem 7.9F in [19], together with Remmert's Theorem). The continuity of f implies that D_1 is contained in L and hence $\dim_{\mathbb{C}}(D_1) < \dim_{\mathbb{C}}(S)$. We let $D = Sing_{\mathbb{C}}(S) \cup D_1$. We claim that $Y \cap ((S \setminus D) \times N)$ is the graph of a holomorphic map from $S \setminus D$ into N. Notice that we only need to check points in $L \cap (S \setminus D)$ (there might be such points).

Indeed, let $z \in S \setminus D$. Since $\dim_{\mathbb{R}}(L) \leqslant 2n-2$, the set $S \setminus L$ is locally connected at z and therefore $\pi^{-1}(z) \cap Y$ is connected. By the properness of π it must also be nonempty and hence (since its dimension is zero) contains a single point. Therefore Y is indeed the graph of a function over the submanifold set $S \setminus D$. To see that it is holomorphic at z it is enough to check continuity at points in $L \cap (S \setminus D)$ (since

f is holomorphic outside L). This follows from the properness of π and the fact that Y is closed. □

7. Complex $\mathfrak{S}$-manifolds and higher order tangent bundles

We now fix $\mathfrak{S} = \mathfrak{S}(\mathcal{C})$ the o-minimal structure associated to the analytic-geometric category $\mathcal{C}$. A definable set in $\mathfrak{S}$ is called *an* $\mathfrak{S}$*-set* and a definable map in $\mathfrak{S}$ is called *an* $\mathfrak{S}$*-map.*

7.1. **Complex $\mathfrak{S}$-manifolds and $\mathfrak{S}$-families of sets.** The following definition is taken from [4] (see p. 507 there). Let M be a real analytic manifold. An $\mathfrak{S}$*-atlas on* M is an atlas $(g_i)_{i\in I}$ with a **finite** set I such that each chart $g_i : V_i \to U_i$ is a real analytic isomorphism from an open $V_i \subseteq M$ onto an open $\mathfrak{S}$-set $U_i \subseteq \mathbb{R}^m$ such that all transition maps $g_{i,j} = g_j \circ g_i^{-1}$ are $\mathfrak{S}$-maps as well. An $\mathfrak{S}$*-manifold* is a manifold M equipped with an $\mathfrak{S}$-atlas. For $M = \mathbb{R}^n$ we just take the trivial atlas $U = \mathbb{R}^n$. Notice that every relatively compact manifold of a real analytic manifold can be equipped with an $\mathfrak{S}$-atlas (see the discussion in Section 2.2.

Let M be a complex manifold. *A holomorphic* $\mathfrak{S}$*-atlas on* M is an $\mathfrak{S}$-atlas as above, with the U_i's open subsets of $\mathbb{C}^n$ and the g_i's biholomorphisms between V_i and U_i. A *complex* $\mathfrak{S}$*-manifold* is a complex manifold equipped with a holomorphic $\mathfrak{S}$-atlas.

Obviously if M, N are complex $\mathfrak{S}$-manifolds with $\mathfrak{S}$-atlas (g_i) and (h_j) respectively then $M \times N$ is a complex $\mathfrak{S}$-manifold given by $\mathfrak{S}$-atlas $(g_i \times h_j)$.

If M is an $\mathfrak{S}$-manifold with an $\mathfrak{S}$-atlas $(g_i)_{i\in I}$ and $A \subseteq M$ then we say that A *is an* $\mathfrak{S}$*-set in* M if $g_i(A \cap V_i) \in \mathfrak{S}$ for all $i \in I$. If M, N are $\mathfrak{S}$-manifolds and A is an $\mathfrak{S}$-subset of M, then a map $f : A \to N$ is an $\mathfrak{S}$-map if its graph is $\mathfrak{S}$-subset of $M \times N$. When $M = \mathbb{R}^n$ then the $\mathfrak{S}$-subsets are just the definable ones.

Let M be an $\mathfrak{S}$-manifold. A family $\mathcal{A}$ of subsets of M is called *an* $\mathfrak{S}$*-family* if there exist another $\mathfrak{S}$-manifold N, an $\mathfrak{S}$-set $Y \subseteq M$ and an $\mathfrak{S}$-set $X \subseteq Y \times M$, such that $\mathcal{A} = \{X_b : b \in Y\}$ (where $X_b = \{z \in M : (b, z) \in X\}$). We say in this case that *the family is parameterized by* Y. Let M, M' be $\mathfrak{S}$-manifolds. A family $\mathcal{F}$ of partial maps from M into M' is called *an* $\mathfrak{S}$*-family of maps* if the family of graphs of the functions is an $\mathfrak{S}$-family. This implies in particular that the family of domains of the functions in $\mathcal{F}$ is an $\mathfrak{S}$-family.

Notice that in above definition, if M is a complex $\mathfrak{S}$-manifold, we still allow the parameter set Y to be a subset of a real manifold N.

7.2. **Uniformity results.**

Theorem 7.1. *Let $\mathcal{F}$ be an $\mathfrak{S}$-family of local holomorphic maps from a $\mathfrak{S}$-complex manifold M into $\mathbb{C}^k$, parameterized by Y. Then there is a natural number r such that for every $b \in Y$ and every $z \in dom(f_b)$, if all partial derivatives of f_b of order less than r vanish at z then f_b vanishes on a neighborhood of z.*

Proof This can be done in several ways. One way is to use model theory as follows: Since $\mathcal{F}$ is an $\mathfrak{S}$-family we may assume that it is definable in $\mathfrak{S}$ and so is M. Assume toward contradiction that for every $r \in \mathbb{N}$ there is a function f_b in $\mathcal{F}$ and a point $z \in M$ such that all partial derivatives of f_b vanish at z and yet the germ of f_b at z is nonzero. Then in an elementary extension we will be able to find a K-analytic nonzero function $f_{b'}$ all of whose partial derivatives vanish at some $z' \in M$. This contradicts Theorem 2.26 (2) in [12].

Chris Miller has pointed out to us that the following stronger statement is actually true: Assume that F is a uniformly definable family of real-analytic functions from open sets in $\mathbb{R}$ into $\mathbb{R}$ (definable in an o-minimal structure). Then there is an r such that for all $f \in F$ and all $z \in dom(f)$, if all the derivatives of f up to order r vanish at z then f is locally zero. Indeed, in the polynomially bounded case this follows from [8], while if the structure is not polynomially bounded one can use the definability of the exponential function (see [9]) to define the set of n's such that some $f \in F$ has zero of order n in $\mathbb{R}$. O-minimality now prohibits arbitrarily large order of vanishing to appear in F. The corresponding result for complex variables now follows. □

As the following theorem shows, if A is a $\mathbb{C}$-analytic $\mathfrak{S}$-subset of a complex $\mathfrak{S}$-manifold M then its defining holomorphic functions, in a neighborhood of every point in A, can be given uniformly, as an $\mathfrak{S}$-family. Moreover, if we are given an $\mathfrak{S}$-family of such $\mathbb{C}$-analytic sets then their locally defining functions can be obtained as an $\mathfrak{S}$-family as well.

Theorem 7.2. *Let M be a $\mathfrak{S}$-manifold and let $\mathcal{A} = \{X_b : b \in Y\}$ be an $\mathfrak{S}$-family of locally analytic subsets of M (with $X \subseteq Y \times M$ as above).*

Then there is natural number k and an $\mathfrak{S}$-family of local maps, parameterized by X, such that the following holds:

1. *For every $(b, z) \in X$, the function $f_{b,z}$ is a holomorphic map from an open neighborhood $V_{b,z}$ of z into $\mathbb{C}^k$, and $Z(f_{b,z}) = V_{b,z} \cap X_b$ (where $Z(f_{b,z})$ is the zero set of $f_{b,z}$ in $U_{b,z}$).*
2. *When z is a regular point of X_b then $f_{b,z}$ is a submersion. In this case there is a biholomorphism $g_{b,z}$ from an open neighborhood of zero in $\mathbb{C}^d$ (where $d = \dim(X_b \cap V_{b,z})$) onto $V_{b,z} \cap X_b$, such that $g(0) = z$ and such that the family $\{g_{b,z} : b \in Y, z \in Reg_{\mathbb{C}}(X_b)\}$ is also an $\mathfrak{S}$-family.*

Proof The proof is done through a sequence of reductions:
Step 1 We may assume that $\mathcal{A}$ is a definable family of subsets of $\mathbb{C}^n$.

Since $\mathcal{A}$ is an $\mathfrak{S}$-family, we may assume, by working by working in charts of an $\mathfrak{S}$-atlas, that $M = V$ is an open definable of $\mathbb{C}^n$, Y a definable subset of $\mathbb{R}^k$, for some k, and X is a definable subset of $Y \times V$.
Step 2 We may assume that for each $b \in Y$ and each $z \in X_b$, the germ of X_b at z is irreducible and all X_b's have the same dimension.

We first consider the family $\{Reg_{\mathbb{C}}(X_b) : b \in Y\}$. Since regularity is definable in a uniform way, this family is also definable. By o-minimality, there is a number m such that the germ of $Reg_{\mathbb{C}}(X_b)$, at every point of its closure, has at most m connected components. Furthermore, there is a uniformly definable family of open subsets of $\mathbb{C}^n$, $\{W_{b,z} : (b, z) \in X\}$, such that for each $(b, z) \in X$, $X_b \cap W_{b,z}$ is analytic in $W_{b,z}$ and the connected components of $W_{b,z} \cap Reg_{\mathbb{C}}(X_b)$ correspond to the local connected components of the germ of $Reg(X_b)$ at z (see Section 2.3). Again, by o-minimality, one can uniformly partition each $W_{b,z} \cap Reg_{\mathbb{C}}(X_b)$ into its connected components. Namely, there are m uniformly definable families of sets $\{R'_{i,b,z} : (b, z) \in X\}$, $i = 1, \ldots, m$, such that for each $(b, z) \in X$, $R'_{1,b,z}, \ldots, R_{m,b,z}$ are the connected components of $W_{b,z} \cap Reg(X_b)$ (where $R'_{i,b,z}$ might be empty if X_b has less than m components at z). Finally, we let $X'_{i,b,z} = Cl(R_{i,b,z} \cap W_{b,z})$ and thus obtain a uniformly definable family of irreducible locally analytic subsets of $\mathbb{C}^n$ with the property that for each $(b, z) \in X$, the germ of X_b at z equals the union of the germs of $X'_{i,b,z}$ at z, for $i = 1, \ldots, m$.

It is enough to prove the theorem for each of the $X'_{i,b,z}$ (since the functions defining the germ of X_b at z can be obtained using products of the functions defining each $X'_{i,b,z}$ at z). We may therefore assume, after possibly enlarging the parameter set, that each X_b is locally irreducible at every point of X_b. Finally, we may partition Y into finitely many sets on each of which all X_b's have the same dimension.

We assume now that for each $b \in Y$, we have $\dim_{\mathbb{C}}(X_b) = d$.

Step 3 There exist a uniformly definable family of open sets $\{V_{b,z} : (b, z) \in X\}$ and a uniformly definable family of d-dimensional linear subspaces $L_{b,z} \subseteq \mathbb{C}^n$ such that: For each $(b, a) \in X$, $v_{b,z}$ is an open neighborhood of z, and if $\pi_{b,z} : \mathbb{C}^n \to L_{b,z}$ is the orthogonal projection then its restriction to $V_{b,z} \cap X_b$ is a proper finite-to-one map.

This follows from the fact that the family of d-dimensional linear subspaces is uniformly definable and that, in this setting, a function is proper if and only if the pre-image of a closed and bounded set is again closed and bounded. This is clearly a definable property. It follows from properness (together with the fact that compact analytic subsets of $\mathbb{C}^n$ are finite) that the projections are finite-to-one.

We can now read off the defining functions for each of the analytic sets from the set itself and the proper projection, using Whitney coordinates:

Consider a d-dimensional irreducible analytic subset A of an open subset $W \in \mathbb{C}^n$, such that the projection map π from A onto the first d coordinates is proper. By properness, there is a number m and an analytic subset B of L such that the projection is m-to-one on $A \setminus \pi^{-1}(B)$. Using definable choice, there are m definable maps $\phi_1, \ldots, \pi_m$ from $\pi(A) \setminus B$ into $\mathbb{C}^{n-d}$, which for every x, give the last $n - d$ coordinates of the pre-images of x in A. Let $V = \pi(A) \setminus B$. There are now finitely many complex polynomials $F_1, \ldots, F_r$ in the variables $y_1, \ldots, y_m, x'$, ($length(x') = length(y_i) = n - d$ for $i = 1, \ldots, m$), such that $\psi_i(x, x') = F_i(\phi_1(x), \ldots, \phi_m(x), x')$ is analytic on $V \times \mathbb{C}^{n-d}$. These functions can continued analytically to $\pi(A) \times \mathbb{C}^{n-d}$ and the intersection of their zero sets in $V_{b,z}$ is exactly $A \cap V_{b,z}$ (see for example, [20] for details). Since the ψ_i's are definable, their analytic continuations (which are unique) are definable as well and hence we have the defining functions for $A \cap W$.

Assume now that z is a $\mathbb{C}$-regular point of X_b. Then, by shrinking $V_{b,z}$ further, we may assume that the projection map π is injective on $X_b \cap V_{b,z}$. In this case the (unique) map $g(x) = (x, \phi(x))$ is a holomorphic immersion of an open subset of $\mathbb{C}^d$ into $V_{b,z} \cap X_b$. The defining map for X_b in W is just $x' - \phi(x)$ and it is necessarily a submersion. Finally, using translation, we may assume that $g(0) = z$.

We can now do all of that uniformly: By o-minimality, there is number m such that all projections $\pi_{b,z}$ are at most m-to-one on each $V_{b,z} \cap X_b$. Since definable choice can be carried out uniformly, the above construction of the Whitney coordinates can be done uniformly, thus obtaining the desired holomorphic maps $f_{b,z} : V_{b,z} \to \mathbb{C}^r$ (where $r = (n - d)m$). When $z \in Reg_{\mathbb{C}} X_b$ we obtain the desired immersions uniformly as well. □

Remark

Although we do not need it here, we can actually improve the last theorem in several different ways:

Under the assumptions of the theorem the following hold: There are natural numbers k and r and there exist a uniformly definable family $\{V_{b,i} : b \in Y, i = 1, \dots, r\}$ of open subsets of M, and a uniformly definable family of holomorphic maps $\{f_{b,i} : a \in Y, i = 1, \dots, r\}$, $f_{b,i} : V_{b,i} \to \mathbb{C}^k$, such that each X_b is contained in $\cup_{i=1}^r V_{i,b}$ and $Z(f_{b,i}) = V_{b,i} \cap X_b$ (namely, each X_b is defined by finitely many open sets and finitely many functions).

Even stronger, by following the standard proof of the Coherence Theorem for the ideal sheaf of an analytic set, one can choose k, r, the $V_{b,i}$'s and the $f_{b,i}$ so that for each $b \in Y$ and each $i = 1, \dots, r$, the coordinate functions of $f_{b,i}$ generate the ideal sheaf of $V_{b,i} \cap X_b$. Namely, for every $z \in V_{b,i} \cap X_b$ the ideal $I(X_b)_z$ of holomorphic germs at z which vanish on X_b is generated, over the ring of holomorphic germs $\mathcal{O}_z(M)$, by the coordinate functions of $f_{b,i}$.

Clearly, finite covers as above may be found for each X_b separately if we assumed that M is a compact manifold and X_b is closed. However, no such assumptions is needed here and compactness is replaced by definability in an o-minimal structure. The proof will appear elsewhere.

8. Higher order Tangent Bundles and Differentials

We review here the definition and basic properties of higher order tangent bundles of a complex manifold. Since the literature contains different approaches to these notions, we go through it in some details, with an emphasis on definability issues. As a reference we used here [11], [10] and [1].

Let M be a complex $\mathfrak{S}$-manifold with $\mathfrak{S}$-atlas $g_i : V_i \to U_i, i \in I$, and $\rho : X \to M$ a holomorphic fiber bundle over M with the typical fiber F and the structure group $G \hookrightarrow \mathrm{Aut}(F)$. We will consider only bundles where G is an algebraic linear group acting algebraically on a smooth algebraic variety F over $\mathbb{C}$. Thus G can be identified with a subset of $k \times k$-matrices and under this identification G is an $\mathfrak{S}$-subset of $\mathbb{C}^{k\times k}$, with its action $G \times F \to F$ as an $\mathfrak{S}$-map.

We say that X is *a holomorphic* $\mathfrak{S}$*-bundle over* M if there are holomorphic trivializations $\varphi_i : \rho^{-1}(V_i) \to U_i \times F$ such that all corresponding transition bundle maps $\lambda_{ij} : U_i \cap U_j \to G$ are holomorphic $\mathfrak{S}$-maps. Such a bundle is called *a holomorphic vector* $\mathfrak{S}$*-bundle* if $F = \mathbb{C}^d$ for some d and $G = GL(d, \mathbb{C})$.

An example of a holomorphic $\mathfrak{S}$-vector bundle is the tangent bundle $T(M) \to M$ of a complex $\mathfrak{S}$-manifold M.

The following is easy to verify: Let $V \to M$ be a holomorphic vector $\mathfrak{S}$-bundle map over a complex $\mathfrak{S}$-manifold M. Then the Grassmannian bundle $\mathrm{Gr}(d, V)$ of d-planes in V, the projectivization $\mathbb{P}(V)$ of V, and the exterior powers $\bigwedge^k V$ are holomorphic $\mathfrak{S}$-bundles.

8.1. **The affine case.** We fix $n \in \mathbb{N}^+$ and will denote by $\bar{x} = (x_1, \dots, x_n)$ the standard coordinate functions on $\mathbb{C}^n$. We also fix $r \in \mathbb{N}^+$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in {\mathbb{N}^+}^n$, we let $|\alpha| = \Sigma_i \alpha_i$.

8.1.1. *A formal definition.* Let $\mathcal{D}^{(r)}_{\bar{x}}$ be the vector space of all differential operators of the form

$$\sum_{0<|\alpha|\leqslant r} c_\alpha \frac{\partial^{|\alpha|}}{\partial \bar{x}^\alpha}, \quad c_\alpha \in \mathbb{C}$$

with obvious vector addition and scalar multiplications. Obviously, $\mathcal{D}^{(r)}_{\bar{x}}$ is finite dimensional.

Let $U \subseteq \mathbb{C}^n$ be open. For $a \in U$, *an r-th tangent vector to U at a* is a pair $(a, \mathcal{D})$ with $\mathcal{D} \in \mathcal{D}^{(r)}_{\bar{x}}$, that will be also denoted by $\mathcal{D}_a$. We define *the r-the tangent space to U at a* to be the set of all tangent vectors at a, and denote it by $T^{(r)}_a(U)$. Thus

$$T^{(r)}_a(U) = \{(a, \mathcal{D}) \colon \mathcal{D} \in \mathcal{D}^{(r)}_{\bar{x}}\}.$$

We will always consider $T^{(r)}_a(U)$ as a $\mathbb{C}$-vector space with vector addition and scalar multiplication induced from $\mathcal{D}^{(r)}_{\bar{x}}$.

If $\mathcal{D}_a = \big(a, \sum c_\alpha \frac{\partial^{|\alpha|}}{\partial \bar{x}^\alpha}\big) \in T^{(r)}_a(U)$ then $\mathcal{D}_a$ defines a linear function from $\mathcal{O}_a(U)$ into $\mathbb{C}$ given by

$$\mathcal{D}_a \colon f \mapsto \sum c_\alpha \frac{\partial^{|\alpha|} f}{\partial \bar{x}^\alpha}|_{\bar{x}=a}$$

We will denote by $\mathcal{D}_a.f$ the result of applying $\mathcal{D}_a$ to f, thus $\mathcal{D}_a.f \in \mathbb{C}$.

Remark 8.1. Obviously $\mathcal{D}_a.f$ depends only on the partial derivatives of f at a of order at most r. Thus for $\mathcal{D}^1, \mathcal{D}^2 \in \mathcal{D}^{(r)}_{\bar{x}}$ we have $\mathcal{D}^1_0 = \mathcal{D}^2_0$ if and only if $\mathcal{D}^1_0.f = \mathcal{D}^0.f$ for all monomials $f \in \{\bar{x}^\alpha \colon 0 < |\alpha| \leqslant r\}$.

We define *the r-th tangent bundle of U* to be $U \times \mathcal{D}^{(r)}_{\bar{x}}$ (i.e., the disjoint union of $T^{(r)}_a(U)$ as a varies in U) and denote it by $T^{(r)}(U)$).

We will denote by $\rho_U \colon T^{(r)}(U) \to U$ the natural projection.

If $V \subseteq U$ are open subsets of $\mathbb{C}^n$, then, according to the definition, $T^{(r)}(V) \subseteq T^{(r)}(U)$ and the following diagram is commutative

$$\begin{array}{ccc} T^{(r)}(V) & \hookrightarrow & T^{(r)}(U) \\ {\scriptstyle \rho_V}\downarrow & & \downarrow{\scriptstyle \rho_U} \\ V & \hookrightarrow & U \end{array}$$

8.1.2. *Holomorphic structure.* We use $\{\frac{\partial^{|\alpha|}}{\partial \bar{x}^\alpha} \colon 0 < |\alpha| \leqslant r\}$ as a standard basis for $\mathcal{D}^{(r)}_{\bar{x}}$ with the lexicographical ordering of $\{0 < |\alpha| \leqslant r\}$. With respect to this standard basis, $\mathcal{D}^{(r)}_{\bar{x}}$ is identified with $\mathbb{C}^d$, $d = |\mathcal{D}^{(r)}_{\bar{x}}|$, and, for an open $U \subseteq \mathbb{C}^n$, the r-th tangent bundle of U is identified with $U \times \mathbb{C}^d \subseteq \mathbb{C}^n \times \mathbb{C}^d$. Thus every $T^{(r)}(U)$ is a holomorphic vector bundle over U. Obviously, if U is an open $\mathfrak{S}$-subset of $\mathbb{C}^n$, then $T^{(r)}(U)$ is an open $\mathfrak{S}$-subset of $\mathbb{C}^n \times \mathbb{C}^d$ and $\rho_u : T^{(r)}(U) \to U$ is a holomorphic vector $\mathfrak{S}$-bundle over U.

8.1.3. *Higher order differentials.* Let $U \subseteq \mathbb{C}^n$ be an open subset. We also fix $m \in \mathbb{N}^+$ and denote by $\bar{y}$ the standard coordinate functions $(y_1, \ldots, y_m)$ on $\mathbb{C}^m$.

Let $f \colon U \to \mathbb{C}^m$ be a holomorphic function, $a \in U$ and $b = f(a)$. *The r-th differential of f at a*, denoted by $D^{(r)}_a f$, is a linear map from $T^{(r)}_a(U)$ into $T^{(r)}_b(\mathbb{C}^m)$ defined as follows:

For $\mathcal{D}_a \in T^{(r)}_a(U)$, $D^{(r)}_a f\,(\mathcal{D}_a)$ is an element $\mathcal{D}^1_b \in T^{(r)}_b(\mathbb{C}^m)$ such that $\mathcal{D}^1_b.h = \mathcal{D}_a.(h \circ f)$ for every $h \in \mathcal{O}_b(\mathbb{C}^m)$. It is not hard to see that such $\mathcal{D}^1_b$ is unique. Hence $D^{(r)}_a f$ is well defined, and also $D^{(r)}_a f$ is a linear map.

If f is a function from an open subset of $a \in \mathbb{C}^n$ into $\mathbb{C}^m$, and we have $f(a) = b$, we write $f : (\mathbb{C}^n, a) \to (\mathbb{C}^m, b)$.

The next two facts follow from the definition of $D^{(r)}_a f$.

Claim 8.2. *Let $f : U \to \mathbb{C}^m$ be a holomorphic map and $a \in U$. The following conditions are equivalent.*

(1) *The r-th differential of f at a is the zero map.*
(2) *All partial derivatives, up to order r, of all component functions of f computed at a, are zeroes.*

Claim 8.3. (1) *Let $(\mathbb{C}^n, a) \xrightarrow{f} (\mathbb{C}^m, b) \xrightarrow{g} (\mathbb{C}^k, c)$ be a sequence of holomorphic maps. Then $D^{(r)}_a(g \circ f) = (D^{(r)}_b g) \circ (D^{(r)}_a f)$.*
(2) *The r-th differential of the identity map at any point is the identity map.*
(3) *If $f : (\mathbb{C}^n, a) \to (\mathbb{C}^n, b)$ is a local biholomorphism then the r-th differential of f at a is an isomorphisms.*

Example. 8.1. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic, $a \in \mathbb{C}$ and $b = f(a)$. Let $h \in \mathcal{O}_b$. Applying the Chain Rule we obtain:

$$\frac{\partial (h \circ f)}{\partial x}|_{x=a} = f'(a)h'(b), \text{ and}$$
$$\frac{\partial^2 h \circ f}{\partial x^2}|_{x=a} = f''(a)h'(b) + \big(f'(a)\big)^2 h''(b)$$

Thus

$$D^{(2)}_a f\big(\tfrac{\partial}{\partial x}|_{x=a}\big) = f'(a)\tfrac{\partial}{\partial y} \text{ and}$$
$$D^{(2)}_a\big(\tfrac{\partial^2}{\partial x^2}\big) = f''(a)\tfrac{\partial}{\partial y} + \big(f'(a)\big)^2 \tfrac{\partial^2}{\partial y^2}.$$

Let $d = \dim\big(\mathfrak{D}^{(r)}_{\bar{x}}\big)$ and $d_1 = \dim\big(\mathfrak{D}^{(r)}_{\bar{y}}\big)$. With respect to the standard bases for $T^{(r)}_a(U)$ and $T^{(r)}_b(\mathbb{C})$, the linear map $D^{(r)}_a F$ has the corresponding $d_1 \times d$ matrix that we will denote by $J^{(r)}_a(f)$ and call it *the r-th Jacobian matrix of f at a.*

Example. 8.2. In the example 8.2, the 2-nd Jacobian matrix of f at a is

$$\begin{pmatrix} f'(a) & 0 \\ f''(a) & f'(a)^2 \end{pmatrix}$$

In general, it is easy to see, by applying the Chain Rule, that all the entries of the matrix $J^{(r)}_a(f)$ are polynomial functions of $\big\{\frac{\partial^{|\alpha|} f_i}{\partial \bar{x}^\alpha}|_a : i \leqslant m, 0 < \alpha \leqslant r\big\}$, where f_i are the component functions of f. Moreover, these polynomials depend only on r, n and m and do not depend on f and a. Thus we obtain the following claim.

Claim 8.4. (1) *If $U \subseteq \mathbb{C}^n$ is open and $f : U \to \mathbb{C}^m$ holomorphic then the map $a \mapsto J^{(r)}_a(f)$ is a holomorphic map from U into $\mathbb{C}^{d_1 \times d}$. Moreover if f is an $\mathfrak{S}$-map then this map is also an $\mathfrak{S}$-map.*
(2) *If $\{f_b : b \in Y\}$, is an $\mathfrak{S}$-family of holomorphic maps, each from an open $\mathfrak{S}$-set $U_b \subseteq \mathbb{C}^n$ into $\mathbb{C}^m$, then the family $\{h_b(a) = J^{(r)}_a(f_b) : b \in Y\}$, is an $\mathfrak{S}$-family of holomorphic maps from U_s into $\mathbb{C}^{d_1 \times d}$*

We will need the following claim.

Claim 8.5. *Assume $n \leqslant m$. Let $f \colon \mathbb{C}^n \to \mathbb{C}^m$ be the embedding $\bar{x} \mapsto (\bar{x}, 0)$, and $g \colon \mathbb{C}^m \to \mathbb{C}^{m-n}$ be the projection onto the last $m - n$ coordinates. Then the map $D^{(r)}_0 f \colon T^{(r)}_0(\mathbb{C}^n) \to T^{(r)}_0(\mathbb{C}^m)$ is injective and the image of $D^{(r)}_0 f$ coincides with the kernel of $D^{(r)}_0(g)$.*

Proof Using definition of the r-th differential and Remark 1, it is not hard to see that for a multi-index α with $|\alpha| \leqslant r$, $D_0^{(r)}f$ acts as

$$D_0^{(r)}f : \frac{\partial^{|\alpha|}}{\partial \bar{x}^{\alpha}} \mapsto \frac{\partial^{|\beta|}}{\partial \bar{y}^{\beta}}, \text{ where } \beta_i = \begin{cases} \alpha_i & i \leqslant n \\ 0 & i > 0. \end{cases}$$

In particular, $D_0^{(r)}f$ maps invectively the standard basis of $T_0^{(r)}(\mathbb{C}^n)$ into the standard basis of $T_0^{(r)}(\mathbb{C}^m)$. Hence $D_0^{(r)}f$ is injective.

The fact that the image of $D_0^{(r)}f$ coincides with the kernel of $D_0^{(r)}f$, also follows from definition of the r-th differential and Remark 1, by direct computations. □

For an open $U \subseteq \mathbb{C}^n$ and a holomorphic function $f\colon \mathbb{C}^n \to \mathbb{C}^m$ we define *the r-th differential of f* to be the map $D^{(r)}f\colon T^{(r)}(U) \to T^{(r)}(\mathbb{C}^m)$ given by $D^{(r)}f \restriction T_a^{(r)}(U) = D_a^{(r)}f$. With respect to the standard coordinates $D^{(r)}f$ has form $(a, u) \mapsto (f(a), J_a^{(r)}(f)u)$.

Claim 8.6. (1) *If $U \subseteq \mathbb{C}^n$ is open and $f : U \to \mathbb{C}^m$ is holomorphic then the map $D^{(r)}f$ is a holomorphic morphism of vector bundles. Moreover if f is an $\mathfrak{S}$-map then $D^{(r)}f$ is also an $\mathfrak{S}$-map.*

(2) *If $\{f_b : b \in Y\}$ is an $\mathfrak{S}$-family of holomorphic maps, each from an open $\mathfrak{S}$-set $U_b \subseteq \mathbb{C}^n$ into $\mathbb{C}^m$, then $\{D^{(r)}f_b : b \in Y\}$, is also an $\mathfrak{S}$-family.*

Proof Follows from Claim 4

8.2. **Manifolds.** For a complex manifold M and $r \in \mathbb{N}^+$ we construct $T^{(r)}(M)$, the r-th tangent bundle of M, in a similar way to the usual tangent bundle. Let $\{g_i\colon V_i \to U_i, i \in I\}$ be a holomorphic atlas on M. Then the r-th tangent bundle of M is obtained by gluing $T^{(r)}(U_i)$ with the transition maps $D^{(r)}(g_j \circ g_i^{-1})$. Thus, $T^{(r)}(M)$ is a holomorphic vector bundle over M, with fiber atlas $g_i, i \in I$, and $x \mapsto J_{\bar{x}}^{(r)}(g_j \circ g_i^{-1})$ as the transition maps of the bundle.

Claim 8.7. *If M is a complex $\mathfrak{S}$-manifold and $r \in \mathbb{N}^+$, then $T^{(r)}(M)$ is a holomorphic vector $\mathfrak{S}$-bundle.*

Let M, N be complex $\mathfrak{S}$-manifolds and $\{f_b : b \in Y\}$ an $\mathfrak{S}$-family of holomorphic maps, each from an open $\mathfrak{S}$-set $U_b \subseteq M$ into N. Then $\{D^{(r)}f_b : b \in Y\}$, is an $\mathfrak{S}$-family of holomorphic maps.

Proof By working in charts, the claim follows from Claim 6. □

8.3. **Submanifolds.**

Claim 8.8. *Let M, N be complex manifolds and $f\colon (M, a) \to (N, b)$ a holomorphic map. If f is an immersion at a then for any $r \in \mathbb{N}^+$ the map $D_a^{(r)}(f)$ is injective.*

Proof Since f is an immersion at a there are local biholomorphisms $g_1\colon (M, a) \to (\mathbb{C}^n, 0)$ and $g_2\colon (N, b) \to (\mathbb{C}^m, 0)$ such that the following diagram is commutative

$$\begin{array}{ccc} (M, a) & \xrightarrow{\ f\ } & (N, b) \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ (\mathbb{C}^n, 0) & \xrightarrow{\ F\ } & (\mathbb{C}^m, 0) \end{array}$$

where $F : \mathbb{C}^n \to \mathbb{C}^m$ is the embedding $\bar{x} \mapsto (\bar{x}, 0)$. The claim now follows form Claim 3 and Claim 5. $\square$

If M is a complex manifold and S is a submanifold of M, then, for any $r \in \mathbb{N}^+$, the r-th differential of the inclusion map $\imath \colon S \hookrightarrow M$ is injective on $T^{(r)}(S)$. We use $D^{(r)}\imath$ to identify $T^{(r)}(S)$ with the image of $T^{(r)}(S)$ under $D^{(r)}\imath$, and we will always consider $T^{(r)}(S)$ as a subset of $T^{(r)}(M)$. It is not hard to see that in fact $T^{(r)}(S)$ is a submanifold of $T^{(r)}(M)$.

Claim 8.9. *Let M be a complex manifold of dimension n, S a submanifold of M , $a \in S$ and $f : M \to \mathbb{C}^{n-k}$ a holomorphic map such that f is a submersion at a and S is the zero locus of f. Then, for all $r \in \mathbb{N}^+$, the kernel of $D^{(r)}_a F$ is exactly $T^{(r)}_a(S)$.*

Proof Since f is submersion, by the same arguments as in Claim 8, we only need to consider the case when $M = \mathbb{C}^n$, $a = 0$, and f is the projection onto the last $n - k$ coordinates. This case is covered by Claim 5. $\square$

Claim 8.10. *Let M be a complex $\mathfrak{S}$-manifold and $N_s, s \in S$, an $\mathfrak{S}$ family of submanifolds of M. Then, for any $r \in \mathbb{N}^+$, $T^{(r)}(S)$ is an $\mathfrak{S}$-family of submanifolds of $T^{(r)}(M)$.*

Proof Follows from Theorem 2 and Claim 7. $\square$

Theorem 8.11. *Let M be a complex $\mathfrak{S}$-manifold and $\{S_b : b \in B\}$ an $\mathfrak{S}$-family of $\mathfrak{S}$-submanifolds of M all of the same dimension n. Then there is $r \in \mathbb{N}^+$ such that for all $b_1, b_2 \in B$ and $a \in S_{b_1} \cap S_{b_2}$ the following conditions are equivalent.*

(1) $T^{(r)}_a(S_{b_1}) = T^{(r)}_a(S_{b_2})$, *as subspaces of* $T^{(r)}_a(M)$.
(2) $S_{b_1} = S_{b_2}$ *near* a.

Proof Obviously (2) always implies (1) for all r.

Let m be the dimension of M. By Theorem 2, there is an $\mathfrak{S}$-family $\{g_{b,a} : (b, a) \in B \times A\}$, of holomorphic functions from open neighborhoods $U_{b,a}$ of 0 in $\mathbb{C}^n$ into M such that $g_{b,a}(0) = a$, $g_{b,a}$ is an immersion at 0 and the image of $U_{b,a}$ under $g_{b,a}$ coincides with S_b near a. In particular, we have $T^{(r)}_a(S_b) = D^{(r)}_0 g_{b,a}(T^{(r)}_0(\mathbb{C}^n))$.

Also, by Theorem 2, there is an $\mathfrak{S}$-family $\{f_{b,a} : (b, a) \in B \times A\}$ of holomorphic functions from open neighborhoods $V_{b,a}$ of a in M into $\mathbb{C}^{m-n}$ such that $f_{b,a}(a) = 0$, $f_{b,a}$ is an submersion at a and the zero locus of $f_{b,a}$ coincides with S_b near a.

Let $H_{b_1,b_2,a} = f_{b_2,a} \circ g_{b_1,a}$. Obviously $H_{b_1,b_2,a}, (b_1, b_2, a) \in B \times B \times M$ is an $\mathfrak{S}$-family of holomorphic maps from open subsets of $\mathbb{C}^n$ into $\mathbb{C}^{m-n}$, and for $a \in S_{b_1} \cap S_{b_2}$, $S_{b_1} \subseteq S_{b_2}$ near a if and only if $H_{b_1,b_2,a}$ vanishes on an open neighborhood of 0.

By Theorem 1 and Claim 2, there is $r \in \mathbb{N}^+$ such that, uniformly in parameters,, $H_{b_1,b_2,a}$ vanishes on an open neighborhood of 0 if and only if the r-th differential of $H_{b_1,b_2,a}$ vanishes on $T^{(r)}_0(\mathbb{C}^n)$. This is the r we choose for the theorem.

Assume now that $T^{(r)}_a(S_{b_1}) = T^{(r)}_a(S_{b_1})$. By Claim 9, $T^{(r)}_a(S_{b_2})$ is the kernel of $D^{(r)}_a f_{a,b_2}$, and by the equality of higher tangent bundles, it is also the image under $D^{(r)}_a g_{a,b_1}$ of $T^{(r)}_0(\mathbb{C}^n)$. It thus follows, by Claim 3 (1), that the r-th differential of $H_{b_1,b_2,a}$ vanishes on $T^{(r)}_0(\mathbb{C}^m)$. By our choice of r, it follows that $H_{b_1,b_2,a}$ vanishes on an open neighborhood of 0, which implies that $S_{b_1} \subseteq S_{b_2}$ in some neighborhood of a. The opposite inclusion is derived similarly by considering $H_{b_2,b_1,a}$. $\square$

9. Embeddings into the Grassmannian bundle

We are now ready to prove an analogous to the theorem of Campana and Fujiki. On the local level, our proof uses the same geometric idea of the original papers. The novelty is that global compactness is replaced by definability in an o-minimal structure (an $\mathfrak{S}$-set), and hence we may use freely Euclidean neighborhoods as long as we remain in the category of $\mathfrak{S}$-sets. Also, the theorems on removal of singularities from the first part of the paper are now used instead of Hironaka's resolution of singularities.

9.1. **The nonsingular case.**

Theorem 9.1. *Let N, M be complex $\mathfrak{S}$-manifolds, $S \subseteq N \times M$ a connected $\mathfrak{S}$-submanifold such that the projection $\pi_N : S \to N$ has constant rank on S.*
Then there is a holomorphic vector $\mathfrak{S}$-bundle $\pi : V \to M$ and a holomorphic $\mathfrak{S}$-map $\mu : S \to \mathbb{P}(V)$ such that the following diagram is commutative

$$\begin{array}{ccc} S & \xrightarrow{\ \mu\ } & \mathbb{P}(V) \\ & {\scriptstyle \pi_M} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & M & \end{array}$$

and $\mu(b, a) = \mu(b', a)$ if and only if $S_b = S_{b'}$ near a. (S_b, as usual, denotes the set $\{a \in M\,;\,(b, a) \in S\}$.)

Proof The idea of the proof is very geometric. For every $(b, a) \in S$, one associates, in the Grassmannian, the tangent space of the submanifold S_b at a. However, since two distinct S_b's through a might have the same tangent space at a, this might not be enough in order to distinguish between the two. This is the reason one needs to consider tangent spaces of higher order.

Let r be as in Theorem 11, and d be the dimension of the vector space $T_a^{(r)}(S_b)$ for any $a \in S_b$.

Let $W = T^{(r)}(M)$ and $\mathrm{Gr}(d, W)$ be the Grassmannian bundle of d-dimensional planes in W. Since $T^{(r)}(M)$ is a holomorphic vector $\mathfrak{S}$-bundle over M, $\mathrm{Gr}(d, T^{(r)}(M))$ is also a holomorphic $\mathfrak{S}$-bundle over M.

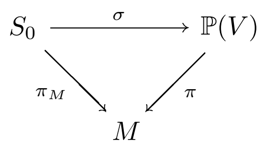
Consider the function $\nu : S \to \mathrm{Gr}(d, W)$ that assigns to (b, a) the element of $\mathrm{Gr}(d, W)$ corresponding to $T_a^{(r)}(S_b)$. Clearly ν is a holomorphic map. By Claim 10, $T^{(r)}(S_b)$ is an $\mathfrak{S}$-family, hence μ is also an $\mathfrak{S}$-map.

By the choice of r we have that $\nu(b, a) = \nu(b', a)$ if and only if $S_b = S_{b'}$ near a.

The required μ is obtained by composing ν with the standard embedding of $\mathrm{Gr}(d, W)$ into the projectivization of the vector bundle $V = \bigwedge^d W$. □

9.2. **The case of analytic sets.**

Theorem 9.2. *Let N, M be complex $\mathfrak{S}$- manifolds, and S an irreducible analytic $\mathfrak{S}$-subset of $N \times M$. Then there is a holomorphic vector $\mathfrak{S}$-bundle $\pi : V \to M$, a meromorphic $\mathfrak{S}$-map $\lambda : S \to \mathbb{P}(V)$, and a Zariski open subset S^0 of S such that $\sigma(b, a) = \sigma(b', a)$ if and only if $S_b = S_{b'}$ near a, for all $(b, a), (b', a) \in S^0$, and the* *following diagram is commutative*



Proof Let π_N denote the projection map from S into N and let k be the generic rank of $D_z(\pi_N|Reg_{\mathbb{C}}(S))$. We define S^0 to be the set of all points z in $Reg_{\mathbb{C}}(S)$ where this rank is atttained. Its complement in $Reg_{\mathbb{C}}(S)$, which we call D', is a $\mathbb{C}$-analytic $\mathfrak{S}$-subset of $Reg_{\mathbb{C}}(S)$ and therefore also of $(N \times M) \setminus Sing_{\mathbb{C}}(S)$. By Theorem 4, the closure of D' in $N \times M$, which we call D, is also $\mathbb{C}$-analytic subset of $N \times M$.

We may now apply Theorem 1 to S^0, N, and M and obtain a complex vector $\mathfrak{S}$-bundle V over M and a holomorphic $\mathfrak{S}$-map $\sigma : S^0 \to \mathbb{P}(V)$ satisying the above requirements.

By Theorem 4, the closure of the graph of σ in $S \times \mathbb{P}(V)$ is an analytic subset of $S \times V$. Moreover, the projection of this closure onto S is a proper map (by the compactness of projective space). Thus σ is a meromorphic $\mathfrak{S}$-map from S into $\mathbb{P}(V)$. □

Remarks (1) If we take, in the last theorem, M and N to be arbitrary compact complex manifolds then they can be equipped with an $\mathfrak{S}$-atlas. Moreover, every $\mathbb{C}$-analytic subset of them is an $\mathfrak{S}$-set. We thus obtain in this case the original theorem of Campana and Fujiki (for complex analytic sets rather than complex analytic spaces).

(2) Take $a \in M$. If $S_b \neq S_{b'}$ at a for all $b \neq b'$ such that $(a, b), (a, b') \in S$ then then map that sends $S^a = \{b \in N : (a, b) \in S\}$ into the projective space $\pi^{-1}(a)$ is injective. If in addition N is compact this image is an algebraic projective set. However, if N is not compact the map may not be proper and therefore the image of S^a might not be an algebraic. We are still not certain about the implications the theorem in this case.

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