COMPACTNESS OF CYCLE SPACES AND DEFINABILITY IN O-MINIMAL EXPANSIONS OF $\mathbb{R}_{\text{an}}$

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Abstract. In this short note we observe that if the cycles in a prime component of the cycle space of a compact complex manifold are uniformly definably in some o-minimal expansion of $\mathbb{R}_{\text{an}}$, then the component is compact.

1. Cycle spaces

For X a reduced complex analytic space, an *n-cycle of X* is a finite linear combination $Z = \sum_i n_i Z_i$ where the Z_i's are distinct n-dimensional irreducible compact analytic subsets of X, and each n_i is a positive integer called the *multiplicity* of Z_i in Z. By $|Z|$ we mean the underlying set or *support* of Z, namely $\bigcup_i Z_i$. We denote the set of all n-cycles of X by $\mathcal{B}_n(X)$, and the set of all cycles of X by $\mathcal{B}(X) := \bigcup_n \mathcal{B}_n(X)$. In [1] Barlet endowed $\mathcal{B}_n(X)$ with a natural structure of a reduced complex analytic space whereby if for $s \in \mathcal{B}_n(X)$ we let Z_s denote the cycle respresented by s, then the set $\{(s,x) : s \in \mathcal{B}_n(X), x \in |Z_s|\}$ is an analytic subset of $\mathcal{B}_n(X) \times X$. Equipped with this complex structure, $\mathcal{B}(X)$ is called the *Barlet space of* X. When X is a projective variety the Barlet space coincides with the Chow scheme.

In [2] it is shown that

$$\mathcal{B}^*(X) := \{s \in \mathcal{B}(X) : Z_s \text{ is irreducible with multiplicity } 1\}$$

is a Zariski open subset of $\mathcal{B}(X)$. An irreducible component of $\mathcal{B}(X)$ is *prime* if it has nonempty intersection with $\mathcal{B}^*(X)$. If S is a prime component of the Barlet space, then the set

$$G_S := \{(s,x) : s \in S, x \in |Z_s|\}$$

is an irreducible analytic subset of $S \times X$, and if $\pi : G_S \to S$ is the projection map then the general fibres of π are reduced and irreducible. We call G_S the *graph* of (the family of cycles parametrised) by S.

2. Volume

Recall that if X is a complex manifold then there is a 1-1 correspondence between hermitian metrics and positive real $(1,1)$-forms on X, given by $h \longmapsto \omega := -\operatorname{Im}(h)$. Moreover, Wirtinger's theorem allows us to compute the volume of a compact complex submanifold with respect to the Riemannian metric $\operatorname{Re}(h)$ by integrating the appropriate exterior power of the assocated $(1,1)$-form over the submanifold:

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if $Z \subseteq X$ is a compact complex submanifold of dimension k, then the volume of Z is given by

$$\operatorname{vol}_h(Z) = \frac{1}{k!}\int_Z \omega^k \tag{2.1}$$

where ω^k is the kth exterior power of ω (cf. Section 3.1 of [7]).

If Z is a possibly singular analytic subset (irreducible, compact, dimension k), then equation (2.1) can serve as the *definition* of volume; it agrees with the volume of the regular locus of Z. More generally, if $Z = \sum_i m_i Z_i$ is a k-cycle of X, then the volume of Z with respect to h is $\operatorname{vol}_h(Z) := \sum_{i=1}^{\ell} m_i \operatorname{vol}_h(Z_i)$.

Now suppose S is an irreducible component of the Barlet space $\mathcal{B}(X)$. Then taking volumes of cycles induces a continuous function $\operatorname{vol}_h : S \to \mathbb{R}$ given by

$$\operatorname{vol}_h(s) := \operatorname{vol}_h(Z_s).$$

Fact 2.1 (Lieberman [3])**.** Suppose X is a compact complex manifold equipped with a hermitian metric h, and S is a prime component of $\mathcal{B}(X)$. If vol_h is bounded on S, then S is compact.

3. Definability of cycle spaces

Let X be a reduced compact complex analytic space. Then X can be interpreted in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$. More precisely, extending slightly the notion of complex $\mathbb{R}_{\mathrm{an}}$-manifold from [4] (cf. also [6]), we may use compactness to give X the structure of a *complex* $\mathbb{R}_{\mathrm{an}}$*-space* by equipping it with a *holomorphic* $\mathbb{R}_{\mathrm{an}}$*-atlas*: a finite atlas $\{g_0, \dots, g_\ell\}$ such that

- each chart $g_i : V_i \to U_i$ is a biholomorphism of V_i with an $\mathbb{R}_{\mathrm{an}}$-definable anlaytic subset U_i of an open n_i-ball D_{n_i} in $\mathbb{C}^{n_i}$ (for some n_i),
- for each i and j, $g_i(V_i \cap V_j)$ is an $\mathbb{R}_{\mathrm{an}}$-definable open subset of U_i, and
- the transition maps $g_{ij} := g_j \circ g_i^{-1}$ are definable in $\mathbb{R}_{\mathrm{an}}$.

If X is in fact a compact complex manifold of dimension n, then we can take $U_i = D_n$ for all i.

A subset $A \subseteq X$ is then said to be *definable in* $\mathbb{R}_{\mathrm{an}}$ if for each i, $g_i(A \cap V_i)$ is an $\mathbb{R}_{\mathrm{an}}$-definable subset of U_i. Similarly one defines what it means for a collection of subsets of X to be *unformly definable in* $\mathbb{R}_{\mathrm{an}}$. Because of compactness, any two $\mathbb{R}_{\mathrm{an}}$-atlases have a common refinement which is also an $\mathbb{R}_{\mathrm{an}}$-atlas. Hence the notion of an $\mathbb{R}_{\mathrm{an}}$-definable subset (or a uniformly $\mathbb{R}_{\mathrm{an}}$-definable collection of subsets) of X, does not depend upon the given atlas.

Every analytic subset of X is definable in $\mathbb{R}_{\mathrm{an}}$.

These definitions can be naturally extended to o-minimal expansions of $\mathbb{R}_{\mathrm{an}}$.

Proposition 3.1. *Suppose X is a compact complex manifold and S is a prime component of $\mathcal{B}(X)$. Let $G \subset S \times X$ be the graph of S. Then S is compact if and only if for some non-empty Zariski open subset $S' \subseteq S$, the collection of fibres $\{G_s : s \in S'\}$ is uniformly definable in some o-minimal expansion of* $\mathbb{R}_{\mathrm{an}}$.

Proof. First assume S is compact. Then we can endow S and $S \times X$ with the structures of complex $\mathbb{R}_{\mathrm{an}}$-spaces. As G is an analytic subset of $S \times X$ it is $\mathbb{R}_{\mathrm{an}}$-definable. It follows that the collection of fibres $\{G_s : s \in S\}$ is uniformly definable in $\mathbb{R}_{\mathrm{an}}$.

For the converse, we will use Fact 2.1. Fixing a hermitian metric h on X, we need to show that $\operatorname{vol}_h(Z_s)$ is uniformly bounded as s varies in S. By the continuity of $\mathrm{vol}_h : S \to \mathbb{R}$, it suffices to show that vol_h is bounded on some dense subset of S. Now, as S is a prime component of the Barlet space, $S^* := S' \cap \mathcal{B}^*(X)$ is a non-empty Zariski open (and hence dense) subset of S such that Z_s is irreducible and of multiplicity 1 for all $s \in S^*$. By definition then, for $s \in S^*$, $G_s = Z_s$. Hence it suffices to show that $\mathrm{vol}_h(G_s)$ is bounded as s varies in S^*.

By compactness, it suffices to show that at every point $p \in X$ there is an open neighbourhood V such that $\mathrm{vol}_h(G_s \cap V)$ is bounded as s varies in S^*. Suppose for the moment that *for every point $p \in X$ we can find a chart $g : V \to D$ at p, where D is an open n-ball in $\mathbb{C}^n$, such that for any analytic set $A \subseteq X$, the volume of $g(A \cap V)$ with respect to the standard Riemannian metric on D is not less than* $\mathrm{vol}_h(A \cap V)$. Now, shrinking the chart if necessary, we may assume that $\{g(G_s \cap V) : s \in S^*\}$ is uniformly definable in some o-minimal expansion of $\mathbb{R}_{\mathrm{an}}$ – this is by our assumption on the uniform definability of $\{G_s : s \in S'\}$. Let k be the dimension of Z_s (for any $s \in S$) – that is, S is a component of $\mathcal{B}_k(X)$. Then the volume of $g(G_s \cap V)$ with respect to the standard Riemannian metric is equal to the k-dimensional Hausdorff measure of $g(G_s \cap V)$ and hence is uniformly bounded for $s \in S^*$ by Proposition 4.1 of [5]. It folows by the italicised assumption above, that $\mathrm{vol}_h(G_s \cap V)$ is bounded as s varies in S^*, as desired.

It suffices therefore to find such a chart about every point $p \in X$. We start with any chart $g : V \to D$ at p, with $g(p) = 0$. Now $\mathrm{Re}(h)|_V$ induces a Riemannian metric f on D such that $\mathrm{vol}_h(A \cap V) = \mathrm{vol}_f(g(A \cap V))$ for any analytic set $A \subseteq X$. For each $d \in D$, let r_d be the maximum of $f_d(v, v)$ for all v in the closed unit sphere in the tangent space $T_d D$ of D at d. By the continuity of the map $d \mapsto r_d$, for a smaller open ball D' compactly contained in D, there is an $r > 0$ such that $r_d \leq r$ for all $d \in D'$. Shrinking the chart if necessary we may assume $D' = D$. Replacing the chart g with $\dfrac{g}{\sqrt{r}}$, we can now assume that for all $d \in D$ and all $v \in T_d D$, $f_p(v, v) \leq \|v\|_d^2$ where $\| \ \|$ denotes the standard norm. In other words, the length of tangent vectors with respect to the Riemannian metric induced on D from $h|_V$ is not greater than the length with respect to the standard metric. It follows that the volume of $g(A \cap V)$ with respect to the standard Riemannian metric on D is not less than $\mathrm{vol}_h(A \cap V) = \mathrm{vol}_f(g(A \cap V))$.

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