Model Theory of universal covering spaces of complex algebraic varieties

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Abstract

We ask whether the notion of a homotopy class of a path on a complex algebraic variety admits a purely algebraic characterisation, and reformulate this question as a question of categoricity of the universal covering space of a complex algebraic variety in a natural countable infinitary language. We provide partial positive results towards the question.

Assuming a conjecture of Shafarevich and some assumptions on the fundamental group of the complex algebraic variety, we introduce a Zariski-like topology on the universal covering space of a complex algebraic variety which enjoys properties slightly weaker than those of a Zariski topology: the topology has descending chain condition for irreducible set, the projection of a closed set is closed, and some others, and we prove that a natural countable language is able to define first-order the irreducible closed sets of the topology. Then we axiomatise a class of structures which admit topologies with similar properties; those properties are enough to prove model stability of the class.

Following the programme of Zilber of «logically perfect structures», the paper aims to provide a new class of examples of such structures.

Contents

Chapter 1

Introduction and Motivation

1.1 Introduction

1.1.1 General Framework

The work below is a generalisation of some aspects of the paper of Boris Zilber [Zilb] on group covers, and and should eventually form a part of author's D.Phil. thesis done under his supervision.

Is the notion of homotopy on a complex algebraic variety an algebraic notion ? That is, can the notion of homotopy be characterised in a purely algebraic way, without reference to complex topology? Restrict to 1-dimensional homotopies only; a 1-dimensional homotopy is a path; this question becomes then whether the notion of a path, up to fixed point homotopy, on a complex algebraic manifold, can be characterised in a purely algebraic way? We provide a partial positive answer to the following more precise question. Assume that one has an abstract notion of a path up to homotopy, so that one is able to speak about homotopy classes of paths, their endpoints, liftings along topological coverings, paths lying in a subvariety; can that notion be described without recourse to complex topology? Is that true that one can axiomatise that notion is such a way that any realisation always comes from a choice of an embedding of underlying field into $\mathbb{C}$, or equivalently, a choice of a complete Hausdorff topology on the underlying field ?

Model-theory tools allow one to formulate the question in a mathematical way as a question of categoricity of a structure related to the fundamental groupoid, or equivalently the universal covering space, of a complex algebraic variety; moreover, such categoricity questions are well-studied in model theory, and one may say that they form the core of a substantial part of the subject, as developed by Shelah[She83a, She83b]. Rather straightforward model-theoretic analysis shows that the partial positive answer we are able to provide depends on a conjecture on the complex analytic geometry of the underlying variety, and some properties of its arithmetics which are conjectured for some particular types of varieties. Thus, this question sheds new light on the Diophantine conjectures, and their relation to complex geometry; the understanding that a natural and simply formulated model-theoretic question, via model-theory techniques, can shed light on questions and conjectures in complex geometry and arithmetics may be thought of as one of the most significant achievements of the paper.

It also turns out that the question falls very naturally into model theoretic framework of Zariski geometries and «logically perfect structures» started by [Hrushovski-Zilber] and further developed by Zilber [Zil05a, Zild]. The programme of «logically perfect structures» of Zilber is based on an expectation that many important structures appearing in physics and mathematics are «logically perfect» when considered in an *appropriate* language; «logical perfection» means that those structures posses either stability properties, categoricity in a, perhaps non-first order, $L_{\omega_1\omega}$-context, or are analytic Zariski, or could be obtained via a Hrushovski construction. Thus, in papers and preprints there are preliminary attempts to show that certain structures arising in complex analytic geometry (exponential function [Zilb, Zilc, Zile]), in non-commutative geometry (non-commutative torus [Zil04]), toric geometry (toric varieties, with L.Smith) and string theory physics ([Zil04]) are either *stable*, $L_{\omega_1\omega}$-*categorical*, analytic Zariski, a result of a Hrushovski construction, or otherwise appear naturally in model-theory context. Most of structures mentioned above did not appear in pure model theory earlier, and thus are hoped to provide new examples of structures with nice model-theoretic properties; proving those properties sometimes requires deep arithmetic conjectures (Mordell-Lang and Schanuel conjectures)

Thus, our result may be seen a case where the hopes of the programme are realised, and lead to a success. However, a distinction is that while in many of those questions the choice of an appropriate language is sometimes non-trivial, it is essential for us that the language is the natural one, but the proof depends on rather deep and recent conjectures in complex geometry and number theory.

From the point of view of model theory, we give a new class of examples of non-homogeneous, but conjecturally ω-homogeneous and model homogeneous structures possessing stability properties; namely, structures in the corresponding $L_{\omega_1\omega}$-class are model homogeneous and stable over countable submodels, and conjecturally ω-homogeneous. The examples may be thought of as *groupoids* associated to classical model theory structures, namely algebraic varieties; to be the best of my limited knoweledge, groupoids have not been studied from a model-theory point of view.

Yet another point of view is that the structures in the $L_{\omega_1\omega}$-class can be thought of as related to the inverse limits of classical stable structures; indeed, our structure naturally embeds into an inverse limit of varieties over an algebraically closed field.

Let us first illustrate the question by an explicit example of Zilber[Zilb] of weak exponentiation, to which all our considerations apply.

1.1.2 An explicit example

Motivated by the belief that every structure naturally occurring in mathematics, is «logically perfect»([Zild]), Zilber proves an $L_{\omega_1\omega}$-categoricity statement for two-sorted structure

$$\mathbb{C}^{\text{lin}}_{\text{exp}} = ((\mathbb{C},+),(\mathbb{C}^*\cup\{\mathbf{0}\},+,\times),\exp:\mathbb{C}\to\mathbb{C}^*)$$

describing the exponential map $\exp:\mathbb{C}\to\mathbb{C}^*$; the language $L(\mathbb{C}^{\text{lin}}_{\text{exp}})$ separates the sorts $\mathbb{C}$ and $\mathbb{C}^*$ for the domain and the range of exp; the structure on sort $\mathbb{C}^*$ is that of an algebraic variety, while the structure on sort $\mathbb{C}$ is that of a $\mathbb{Q}$-vector space, together with the pull-back of the structure on sort $\mathbb{C}^*$. He also gives an explicit axiomatisation of the $L_{\omega_1\omega}(L_{\text{exp}})$-theory.

The integral of $\frac{dz}{z}$ over a path in $\mathbb{C}^*$ does not change with a continuous transformation of the path fixing the ends and avoiding the singularity 0 of $\frac{dz}{z}$; in other words, the integral depends only on the homotopy class of the paths in $\mathbb{C}^*$, with homotopy fixing the ends. Thus, the map

$$\{\text{paths } [\gamma] \text{ in } \mathbb{C}^*, \gamma(0)=1\} \longrightarrow \mathbb{C}$$

$$\gamma \mapsto \int_\gamma \frac{dz}{z} = \ln(\gamma(1)) + 2\pi i k, k\in\mathbb{Z}$$

identifies $\mathbb{C}$ with the homotopy classes $\gamma, \gamma(0) = 1$ of paths in $\mathbb{C}^*$.

The multiplication map $m : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ induces a map on paths $m_* : (\gamma, \gamma') \mapsto \gamma \cdot \gamma'$, where $(\gamma \cdot \gamma')(t) = \gamma(t)\gamma'(t)$ is the pointwise product of paths γ and γ' as functions from $[0, 1]$; this allows us to express addition on $\mathbb{C}$ as

$$\int_\gamma \frac{dz}{z} + \int_{\gamma'} \frac{dz}{z} = \int_{\gamma\cdot\gamma'} \frac{dz}{z} = \ln(\gamma(1)) + \ln(\gamma'(1)) + 2\pi i k, k \in \mathbb{Z}. (1.2)$$

The above observations make it natural to think of $\mathbb{C}^{\text{lin}}_{\text{exp}}$ as describing the homotopy classes of paths in $\mathbb{C}^*$, or indeed $\mathbb{C}^{*n}, n > 0$ in a language reflecting the behaviour of paths under morphisms and their concatenation properties. This approach easily generalases the question to arbitrary algebraic variety.

It is natural not to restrict oneself to paths γ starting at $1, \gamma(0) = 1$, and consider integrals over arbitrary paths in $\mathbb{C}^*$.

The set of the homotopy classes of paths with a given starting point forms the universal covering space; this observation allows us to think instead of $\mathbb{C}^{\text{lin}}_{\text{exp}}$ as a structure describing the universal covering space of $\mathbb{C}^*$, and reflecting the property of being a connected component of an algebraic closed subset.

We use the latter point of view to generalise the approach of Zilber[Zilb]; the interpretation of $\mathbb{C}^{\text{lin}}_{\text{exp}}$ in terms of paths allows one to formulate a categoricity question for arbitrary variety defined over $\overline{\mathbb{Q}}$, or even a field of positive characteristic. The proofs of [Zilb] use Kummer theory, as well as some other number-theoretic tools. Here we use similar tools, but in a rather different appearance; in particular, we have to use holomorphic convexity of the complex structure on the universal covering space to prove Kummer-theory type results over an algebraically closed field.

1.1.3 Technical summary of our results

In this paper we define a natural countable language L^A_{top} associated with the universal covering space $p : U \to A(\mathbb{C})$ of a complex projective algebraic variety $A(\mathbb{C})$ defined over $\mathbb{Q}$ or $\overline{\mathbb{Q}}$. Assuming the conjecture of Shafarevich that the universal covering space U is holomorphically convex, we prove that the positively definable sets in L_A form a topology analogous to Zariski topology on the set of geometric points of a variety. The properties of the topology on U are sufficient to imply that the structure $U^{L^A_{\text{top}}}$ is homogenous over countable submodels. We then consider a fragment of $L_{\omega_1\omega}(L_A)$-theory $\text{Theory}^{L^A_{\text{top}}}_{\omega_1\omega}(U)$ of $U^{L^A_{\text{top}}}$ and introduce several natural axioms of geometric, analytic Zariski, flavour implying the properties of the topology on U. Then we show that the class of models defined by those axioms is stable (in a non-elementary context) over countable models, and, moreover, atomic in a natural extension of the language.

These are prerequisites, by Shelah's theory, of categoricity in uncountable cardinals. Notice that some of the properties, e.g. atomicity, could, by Shelah's theory, be obtained just by an $L_{\omega_1\omega}$-definable extension of the language. Yet, essentially for us, we stay in some natural language.

Thus, by Shelah's theory, this is enough to imply $\aleph_1$-categoricity of a class containing with $U^{L^A_{\text{top}}}$, for an arbitrary smooth projective variety A with a homorphically convex universal cover and with some conditions on the fundamental group. (Cf. Definition 3.1.2.6 for the exact definition of the class of algebraic varieties).

However, in general we do not wish to fix a countable submodel; then there rises a question of ω-homogeneity and ω-stability, and in general, of the existence of a prime model for the theory $\mathrm{Theory}_{L^A_{\mathrm{top}}}(U)$. This question depends on arithmetics of variety A; for $\mathbb{C}^*$ it is just Kummer theory. This preprint does not concern these questions. We hope to include in the thesis the results of this kind concerning elliptic curves over defined over a number field, and if possible, some abelian varieties for which corresponding arithmetics results are known.

1.2 Motivations and implications

In this § we discuss the motivations behind our choice of language, its origins, its relation to other mathematical questions, meanwhile we explain our approach in greater detail. In a way, the motivations here are more important than the proofs.

1.2.1 Logic approach: What is an appropriate language to talk about paths ?

The theory of algebraically closed field provides a language appropriate to talk about algebraic varieties; what language would be appropriate to talk about the homotopies on the algebraic varieties, in particular about paths, i.e. 1-dimensional homotopies?

There is no notion of a path in abstract algebraic geometry over an arbitrary field, but there is a strong intuition based on the naıve notion of a path in complex topology; it is a well-known phenomenon that naive arguments based on the notion of a path quite often lead to statements which generalise, in one way or another, to, say, arbitrary schemes, but which are quite difficult to prove. There have been many attempts to develop substitute notions, starting from Grothendieck [SGA2,SGA4$\frac{1}{2}$] who developed for this purpose the notion of a finite *covering* in the category of arbitrary schemes (étale morphism).

Thus, from the point of view of philosophy of mathematics, it is natural to try to understand why is the notion of a path so fruitful and applicable, despite the fact that all attempts to generalise it to non-topological contexts are only partial. A first question to ask is whether this notion is *algebraic*, i.e. the notion of a path (up to homotopy) in complex topology, can be *axiomatised* purely algebraically ?

1.2.2 Categoricity Theory Approach: Can the notion of path be made algebraic, or categorical ?

Model theory provides a framework to formulate the question precisely, in a mathematical fashion. The central model theoretic notion for us is that of *categoricity* (of non-elementary classes). The relevance of this notion has been exposed in [Zilb]; categoricity is a model-theoretic criterion for a formalisation of a notion to be seen as canonical, i.e. for determining when an algebraic formalisation associated to an object of perhaps geometric character, is canonical and reflecting the properties of the object in a complete way.

Thus, in this work we introduce a language L^A_{top} which is appropriate for describing the basic homotopy properties of algebraic varieties in complex topology, and prove some partial results towards categoricity and stability of associated structures in that language. The language L^A_{top} is able to express properties of 1-dimensional homotopies, i.e. the properties of paths, up to homotopies fixing the ends. Those properties relate to paths-lifting along a topological covering, paths lying in closed algebraic subvarieties (i.e. a homotopy class has a representative which lies in the subvariety), paths in direct products and so on; the properties are sufficient to do many basic 1-dimensional homotopy theory constructions. Most notably, following a construction in Mumford [Mum70] one can definably construct a bilinear form $\phi_L : \pi_1(A(\mathbb{C})) \times \pi_1(A(\mathbb{C})) \to \pi_1(\mathbb{C}^*)$ in the second homology group $H^2(A(\mathbb{C}), \mathbb{Z}) \cong \bigwedge^2 H^1(A(\mathbb{C}), \mathbb{Z})$ associated to an algebraic $\mathbb{C}^*$-bundle L over a complex Abelian variety $X(\mathbb{C})$. Thus, generally the language has more expressive power than the one considered originally by Zilber in [Zile]; in particular, some Abelian varieties which may not be categorical in Zilber's language of [Zile] are supposedly categorical in our language. It would be interesting to know whether our language can interpret Hodge decomposition on cohomology groups, using the isomorphism $H^n(A(\mathbb{C}), \mathbb{C}) \cong \bigwedge^n H^1(A(\mathbb{C}), \mathbb{C}) = \bigwedge^n \mathrm{Hom}\,(\pi_1(A(\mathbb{C})), \mathbb{C})$ (cf. [Mum70]).

The results which we prove are partial results and necessary conditions towards the categoricity of the universal covering space considered as an L_A-structure, in the infinitary logic $L_{\omega_1\omega}$; cf. §1.2.8 for a description of the general theory relating to categoricity of $L_{\omega_1\omega}$, and those conditions in particular.

1.2.3 Geometric approach: Analytic Zariski structures

Perhaps one of the simplest analytic structures associated to an algebraic variety and which is more then an algebraic variety itself, is the universal covering of an algebraic variety; the universal covering space inherits all the local structure the base space possesses; and in particular, for a complex algebraic variety it is a complex analytic space. Thus it is natural to try to consider it in the context of Zariski geometries [Zil05a]: one wants to define a Noetherian-type, Zariski-like topology on the universal covering space U of variety $A(\mathbb{C})$ reflecting the connection between U and A, and such that U possesses homogeneity, stability and categoricity properties, perhaps in a non-first order, $L_{\omega_1\omega}$, way, in a countable language related to the chosen topology on U.

Thus consider the universal covering space $p : U \to A(\mathbb{C})$ of an algebraic variety A. It is natural to assume that the covering map p and the full algebraic variety structure on $A(\mathbb{C})$ are definable. Then the analytic subsets of U which are the preimages $p^{-1}(Z(\mathbb{C}))$ of algebraic subvarieties Z of $A(\mathbb{C})$, are definable. It is natural to let the analytic irreducible components of such sets also to be definable; a justification for this might be the desire for an irreducible decomposition.

The above considerations lead us to define a topology on U by proclaiming a set closed iff it is a union of analytic irreducible components of the preimage of an closed algebraic subvariety $W(\mathbb{C})$ of $A(\mathbb{C})$, or a finite union of such sets.

It turns out that this topology is rather nice, (almost) admits quantifier elimination down to the level of closed sets, DCC (descending chain condition) for *irreducible* sets, and can be defined in a countable language (in full generality assuming Shafarevich conjecture on the universal covering spaces of algebraic varieties). Those properties of topology are sufficient to imply model homogeneity of the structure $p : U \to A(\mathbb{C})$, and, more generally, to construct an $L_{\omega_1\omega}$-class containing $p : U \to A(\mathbb{C})$ stable over models and whose models are model homogeneous. It also turns out that the language obtained is the language appropriate for describing the paths, as explained in subsection above. We explain the connection in the next subsection §1.2.5.

1.2.4 Category theory approach: algebraicity of fundamental groupoid

Let us make a side remark that the above considerations of the notion of path admit a reformulation in terms more familiar to algebraic geometers; we do not make the considerations of this subsection precise, and neither do we use them in this paper;

however, see §2.2 to see how to recover groupoid structure from the L_A-structure on $\mathbb{U}$. Also note that here we do not have to choose a language: that has already been done for us by algebraic geometers.

Indeed, all the properties of the paths on a complex algebraic variety we described in the previous §, could be expressed in terms of the fundamental groupoid functor $\pi_1^{\text{top}} : \mathbb{V} \to \text{Groupoids}$ from the category of complex algebraic varieties $\mathbb{V}$ (say defined over $\overline{\mathbb{Q}}$) to (discrete) groupoids Groupoids; the functor $\pi_1^{\text{top}}(V) = \pi_1^{\text{top}}(V(\mathbb{C}))$ sends a variety V into its discrete fundamental groupoid $\pi_1^{\text{top}}(V(\mathbb{C}))$. Then the question of categoricity of the notion of paths in the language L_A could be reformulated as a question of the following kind:

Question 1.2.4.1. Given a functor $\mathbb{F} : \mathbb{V} \to \text{Groupoids}$ satisfying some algebraic properties of functor $\pi_1^{\text{top}} : \mathbb{V} \to \text{Groupoids}$, there exist automorphisms $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ and σ' : Groupoids $\to$ Groupoids of corresponding categories such that the diagramme commutes:

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{\pi_1^{\text{top}}} & \text{Groupoids} \\ \sigma \downarrow & & \sigma' \downarrow \\ \mathbb{V} & \xrightarrow{\mathbb{F}} & \text{Groupoids} \end{array} \tag{1.2.1}$$

The question above may motivate us to ask whether it would be true if we restrict $\mathbb{V}$ to some subcategory, say that of elliptic curves (and their closed subvarieties).

Let us make it clear that we do not expect a positive answer to the question in this generality; what we expect is a positive answer when $\mathbb{V}_{\overline{\mathbb{Q}}}$ is restricted to be a subcategory of subvarieties of a given variety etc.

The most notable of algebraic properties required of $\mathbb{F}$ is that the corresponding fundamental groups coincide.

1.2.5 Equivalence of analytic Zariski and paths approaches

A definition of the universal covering space $p : U \to A(\mathbb{C})$ (see e.g. [Nov86]) says that it is a set of homotopy classes of paths leaving a basepoint, with an induced topology. Thus, it is equivalent to talk about the universal covering spaces instead of homotopy classes of paths; that is easier from the technical point of view.

The above two observations make it natural to consider the universal covering space $p : U \to A(\mathbb{C})$ in the context of analytic Zariski structures.

We define a topology on U by proclaiming a set closed iff it is a union of analytic irreducible components of a preimage of an closed algebraic subvariety $W(\mathbb{C})$ of $A(\mathbb{C})$, or a finite union of such sets. The critical observation is that a connected component of the preimage of a normal subvariety is always irreducible; and, if one thinks of U as a set of paths, then, forgetting technicalities, a connected component is a set of all paths lying in $W(\mathbb{C})$; thus if subvarieties W were always normal, we would recover the interpretation of definable closed sets in terms of paths, i.e. we would see that each closed set is definable via the basic properties of paths, and algebraic subsets of $A(\mathbb{C})$. For W not normal, it turns out that we may still interpret the irreducible component in a similar way, as a set of paths in a subvariety of *covering spaces* of $A(\mathbb{C})$ of *finite* degree, which are known to be always algebraic ([Ser56]). However, the latter is not trivial, and requires a geometric argument based on assumption that U is *holomorphically convex manifold*; $\mathbb{C}^n$ and submanifolds are examples of such a holomorphically convex manifold. (cf. Def. 4.3.1.1). By a conjecture of Shafarevich (cf. Conj. 4.3.1.3 this should hold for arbitrary variety, say smooth and quasi-projective. In fact, we need a slightly weaker statement that a universal covering space satisfies the conditions of Fact 3.1.2.1.

Thus, we see that analytic Zariski approach leads us back to the homotopy interpretation as above.

1.2.6 Countable language

In previous §§ we have carefully avoided discussing the size of language L_A: we have just discussed the definition of a topology on U, but not that of a language L^A_{top}; what we want is that language L^A_{top} is to be able to define the closed sets in the chosen topology, and we want to consider U as an analytic Zariski structure in a *countable* language. This is desirable for several reasons; one is that if language were too big, the notion of an isomorphism would have been too strong, and we cannot hope to have categoricity at all; another is that we want to be able to apply Shelah's theory of excellent classes, and that theory requires a countable language. In case of an algebraic variety, one takes the language to consists of all subvarieties defined over a sufficiently large number field; similarly, we define the language L^A_{top} by adding a predicate for connected components of each closed algebraic subvariety of A^n *defined over* $\mathbb{Q}$, or a finite extension of the field of definition of A. The fact that L^A_{top} is able to define connected components of the preimage of an *arbitrary* subvariety is a geometrically non-trivial result employing Stein factorisation for normal varieties; and again we use holomorphic convexity to reduce the general case to that of normal subvarieties.

1.2.7 Future work

To reach categoricity, we need, by Shelah's criteria, to prove existence of prime model (here it is essential that we work in a natural countable language); this seems to be an arithmetic statement about Kummer theory of A; for A an abelian variety, that is a statement about the largeness of Galois action on sequences of division points of $A(\overline{\mathbb{Q}})$ (Tate module, Mumford-Tate conjecture).

Combined with the results here, it will give us $\aleph_1$-categoricity of our class.

To obtain categoricity in higher cardinals, we need to consider amalgams of models, etc; the number theoretic questions appearing there involve linearly disjoint fields, tensor products of algebraically closed fields, and infinitely divisible points of $A(\bar{K} \otimes_{\bar{k}} \bar{K})$ over such fields, etc; it is not yet clear how difficult those questions are.

1.2.8 Categoricity theory

In this § let us say a few words on the model-theory machinery of categoricity we use.

Recall we say that a class $\mathfrak{K}$ of structures in language L is λ-*categorical* iff any two models $M_1, M_2 \in \mathfrak{K}$ of cardinality λ, are isomorphic. In classical model theory, an important question is when a class of models of a first-order theory is categorical in uncountable cardinality; a theorem of Morley says that the class of models of a countable first-order theory is λ-categorical for some uncountable $\lambda > \aleph_0$ iff it is λ-categorical for any uncountable $\lambda > \aleph_0$.

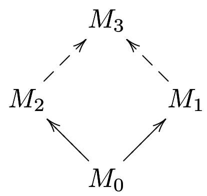
In the more modern approach, created and developed by Shelah, one considers classes $\mathfrak{K}$ of L-structures with a «strong substructure» partial order relation $\preccurlyeq_{\mathfrak{K}}$ which is weaker than the L-substructure relation, i.e. for $M_1, M_2 \in \mathfrak{K}$, the relation $M_1 \preccurlyeq_{\mathfrak{K}} M_2$ implies that M_1 is an L-substructure of M_2. Further assumptions on $\mathfrak{K}$

are formulated in terms of relation $\preccurlyeq_{\Re}$ and are that the relation $\preccurlyeq_{\Re}$ is transitive, is closed under isomorphism, is closed under union of infinite $\preccurlyeq_{\Re}$-chains, and an analogue of Löwenheim-Skolem theorem holds: there exists a «Löwenheim-Skolem» cardinal $\mathrm{LS}(\Re)$ such that if $A \subset M \in \Re$ is an arbitrary subset of a model $M \in \Re$, then there exists a model $A \subset M_0 \preccurlyeq_{\Re} M$ and $\operatorname{card} M_0 \leq \operatorname{card} A + \mathrm{LS}(\Re)$. A class satisfying these and some other properties is called *abstract elementary class*, cf. [Gro02].

The theory of categoricity has been particulary fully developed for *excellent classes*; for us their most most important feature is that the categoricity of such a class depends only on behaviour of its countable models. With some set-theoretic assumptions, Shelah was able to prove Morley's theorem for categorical classes, i.e. that such a class in categorical in uncountable cardinality $\lambda > \operatorname{card} L$ iff it is categorical in all uncountable cardinalities $\lambda > \operatorname{card} L$. Necessary conditions for uncountable categoricity include:

1. existence of a prime model M^{prime}, i.e. a model $M^{\text{prime}} \in \Re$ which is strongly contained in any other model in $\Re$: for any $M \in \Re$ there exists an L-embedding $i : M^{\text{prime}} \to M$ such that $i(M^{\text{prime}}) \preccurlyeq_{\Re} M$. A sufficient condition is that there is an *L-atomic* L-model $M_0 \in \Re$, i.e. such that for every $m \in M$ there exists a formula $\phi(u) \in L$ such that $\phi(m)$ holds in M, and for every other formula $\psi(u) \in L$ if $\psi(m)$ holds in M, then $\forall u(\phi(u) \to \psi(u))$ holds in M.
2. $\Re$ is ω-stable, or model stable: there are not more than countably many types over a *countable model* M realised in an extension $M \preccurlyeq_{\Re} N$, N varies

The model stability of the class allows one to introduce some notion of *independence*; and to define an *excellent class*, one requires some conditions on countability of the set of types over a finite union of «independent n-cubes» of countable models; for $n = 2$, the conditions is that for any two models $M_1, M_2 \in \Re$ *independent* over their intersection $M_0 = M_1 \cap M_2 \preccurlyeq_{\Re} M_1, M_2$, there exists a model $M_1 \preccurlyeq_{\Re} M_3, M_2 \preccurlyeq_{\Re} M_3$ prime over $M_1 \cup M_2$.



What we prove in this paper is the condition 2 of model stability of class $\Re$ required for categoricity; this turns out to be a question of geometric nature. In a forthcoming paper we hope to consider questions relating to the existence of a prime model for some particular examples; these are arithmetic questions.

1.2.9 The structure of the text

The mathematics of what follows is based on model theoretic logic ideas and at the same time heavily relies on some analytic and algebraic geometry. We have chosen to emphasise the logic content and so most of the details on analytic theory we need is shifted to the appendix. So we try to present basic analytic facts and definition in the main body of the text and provide necessarily detailisation and proofs in the Appendix.

Chapter 2

Definitions and Examples

2.1 Motivations and Statements

The goal of this chapter is to introduce main objects, state precisely main definitions and motivations, present the main results and give some examples.

2.1.1 The Goal

Let $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ be the universal covering space of an algebraic variety $\mathbb{A}$. As explained in §1.2.3,§1.2.1, we have the following

Aspiration 2.1.1.1. The universal covering space of an algebraic variety is essentially an *algebraic* object, by which we mean that one may

1. it is an *algebraic* object, i.e.
   (a) there is a *countable* set L of distinguished subsets P_i of Cartesian powers of U
   (b) there is a *countable* set of properties of those distinguished set expressible by «formulae of countable length and finite depth»
   (c) the distinguished subsets P_i are enough to describe the structure on U which we are interested in and consider U in
   (d) which characterises U uniquely, up to isomorphism preserving the distinguished relations on U.
2. the distinguished relations P_i on U describe *some analytic or homotopic datum*, say
   (a) the distinguished relations P_i's on U are enough to describe connected components of preimages of algebraic subvarieties of $\mathbb{A}(\mathbb{C})^n$.
   (b) the distinguished relations P_i's on U are enough to define analytic irreducible components of preimages of analytic subsets of $\mathbb{A}(\mathbb{C})^n$

An example of an algebraic object are a field, a ring, a module over a ring, an algebraic variety defined over a countable ring, considered as the set of its points over an algebraically closed field; on the other hand, a priori a topological space or an analytic manifold are not algebraic objects in this sense: the topology consists of uncountably many sets, and a priori there is no way to choose a countable set of relations defining the topology. Thus, what we want is, in particular, to show that this is possible to do for the universal covering space of some varieties which are good enough.

In model-theoretic jargon the above translates to

Aspiration 2.1.1.2. There is a countable language L_A on the universal covering space $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$, describing some analytic or homotopic data on U, such that $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ in L_A admits a *natural* axiomatisation, perhaps non-first order, say in $L_{\omega_1\omega}(L_A)$, which is *uncountably categorical*.

Following considerations of §§ in introduction, the aspiration above may be translated to the following question of model theory; we explain our precise partial results towards Aspiration 2.1.1.2 (Theorem 3.5.4.7)after giving the definition of language L_A in next subsection.

In $3'$ below, $\pi(\boldsymbol{U}) = \pi_1(\mathbb{A}(\mathbb{C}))$ denotes the *group of deck transformations* of covering $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$, otherwise known as the fundamental group of $\mathbb{A}(\mathbb{C})$; $\pi_1(\mathbb{A}(\mathbb{C}))$ consists of the continuous analytic isomorphisms $\tau : \boldsymbol{U} \to \boldsymbol{U}$ which commute with the covering map p, $\tau \circ p = p$; cf. §4.1 for more details and definitions.

Aspiration 2.1.1.3. There exist a countable *language* L_A, i.e. a countable collection $L_A = \{P_i\}$ of predicates (i.e. distinguished subsets of Cartesian powers of $\boldsymbol{U}$), and an *axiomatisation* $\mathfrak{X} = \mathfrak{X}(L_A, p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C}))$ of $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ in $L_{\omega_1\omega}(L_A)$, i.e. a countable collection ϕ_i of sentences in $L_{\omega_1\omega}(L_A)$, such that

1 U admits an $L_{\omega_1\omega}(L_A)$-axiomatisation $\mathfrak{X}$

  $1'$ all sentences $\mathfrak{X} = \{\phi_i\}$ are valid on $\boldsymbol{U}$, in notation $\boldsymbol{U} \models \phi_i$, $\phi_i \in \mathfrak{X}$

2 axiomatisation $\mathfrak{X}$ is uncountably categorical

  $2'$ for any two uncountable models $\boldsymbol{U}_1, \boldsymbol{U}_2$ of the same cardinality, if $\boldsymbol{U}_1, \boldsymbol{U}_2 \models \phi_i$, for all $\phi_i \in \mathfrak{X}$, then $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ are isomorphic as L_A-structures, $\boldsymbol{U}_1 \cong_{L_A} \boldsymbol{U}_2$, i.e. there exists a bijection $\phi : \boldsymbol{U}_1 \to \boldsymbol{U}_2$ preserving distinguished relations P_i's in L_A.

3𝔞 language L_A describes some of analytic structure on $\boldsymbol{U}$,

  $3'$ a closed analytic $\pi_1(\mathbb{A}(\mathbb{C}))$-invariant subset S of $\boldsymbol{U}^n$ are L_A-definable, with parameters, as well as the irreducible analytic components thereof

3𝔟 language L_A describes some of topological, homotopy theory data on $\mathbb{U}$ as the universal covering space $\mathbb{A}(\mathbb{C})$.

  $3''$ the connected components of a preimage of an algebraic closed subset of $\mathbb{A}^n(\mathbb{C})$ are L_A-definable, with parameters

In addition to the precise statements above, we want

  $1''$ the sentences $\phi_i \in \mathfrak{X}$ have a natural geometric meaning; there is an explicit description of the class of models satisfying axiomatisation $\mathfrak{X}$.

Remark 2.1.1.4. The choice of 3𝔞, 3𝔟 is rather arbitrary, and is specific to this work; for example, in [Zilc] Zilber considers $\mathbb{A} = \mathbb{C}^*$ and replaces 3𝔞, 3𝔟

  $3'''$ any analytic set definable by polynomial-exponential equations (i.e. those consisting of $+$, $\times$, exp) is $L_{\mathbb{C}^*}$-definable.

2.1.2 L_A-structure on the universal covering space $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$

Item 3 leads us to introduce the following $\pi(\boldsymbol{U})$-invariant relations.

For a closed subvariety $Z \subset \mathbb{A}(\mathbb{C})^n$, let $\sim_{Z,A}$ denote the relation on $\boldsymbol{U}^n$ given by

  $x' \sim_{Z,A} y' \iff$ points $x' \in \boldsymbol{U}^n$ and $y' \in \boldsymbol{U}^n$ lie in the same (analytic) irreducible component of $p^{-1}(Z(\mathbb{C})) \subset \boldsymbol{U}^n$.

For $Z(\mathbb{C})$ smooth, or, even *normal*, subvariety, the connected components of $p^{-1}(Z(\mathbb{C}))$ are irreducible (as analytic closed sets). Thus, for such varieties, the relation $\sim_{Z,A}$ is an equivalence relation encoding topological data only, and data on $\mathbb{A}(\mathbb{C})$.

For a normal subgroup $H \lhd \pi(\boldsymbol{U})^n$, let $x' \sim_H y'$ say that points $x', y' \in \boldsymbol{U}^n$ are conjugated by action of H:

  $x' \sim_H y' \iff \exists \tau \in \pi(\boldsymbol{U})^n : \tau x' = y'$ and $\tau \in H$.

Definition 2.1.2.1. We consider the structure $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ in the language L_A which has the following symbols:

  the symbols $\sim_{Z,A}$ for Z a closed subvariety defined over number field k, and, the symbols $\sim_H$, for each normal subgroup $H \lhd_{\text{fin}} \pi(\boldsymbol{U})^n$ of finite index

Note that we do not assume Z to be connected; that is important, because Z is defined over a small field.

The group $\pi(\boldsymbol{U})$ acts on $\boldsymbol{U}$ by analytic isomorphisms; and thus $\pi(\boldsymbol{U})$ acts by L_A-automorphisms of $\boldsymbol{U}$ as an L_A-structure, which is a property we have been after.

2.1.3 Results

Theorem 2.1.3.1 (Model Stability of $\mathfrak{X}(\mathbb{A}(\mathbb{C}))$). *Let $\mathbb{A}$ be a smooth projective algebraic variety defined over $\mathbb{Q}$ such that the universal covering space $\boldsymbol{U}$ of $\mathbb{A}$ is Stein (holomorphically convex). Also assume that the fundamental group $\pi_1(\mathbb{A}(\mathbb{C}), x))$ of any connected component of $\mathbb{A}(\mathbb{C})$ is residually finite and lerf.*

Let language L_A be the countable language defined in Def. 2.1.2.1. Then $(\mathfrak{1}', \mathfrak{3a}, \mathfrak{3b})$ hold, and $\mathfrak{2}'$ is weaken to $\mathfrak{2}'_{\aleph_1}$:

$\mathfrak{2}'_{\aleph_0 \to \aleph_1}$ *Any two models $\boldsymbol{U}_1 \models \mathfrak{X}$ and $\boldsymbol{U}_2 \models \mathfrak{X}$ of axiomatisation $\mathfrak{X}$ and of cardinality $\aleph_1$, such that*

there exist a common countable submodel $\boldsymbol{U}_0 \models \mathfrak{X}$, $\boldsymbol{U}_0 \subset \boldsymbol{U}_1$ and $\boldsymbol{U}_0 \subset \boldsymbol{U}_1$

are isomorphic, $\boldsymbol{U}_1 \cong_{L_A} \boldsymbol{U}_2$, and, moreover, the isomorphism ϕ is identity on $\boldsymbol{U}_0$.

According to a conjecture of Shafarevich, the universal covering space of an arbitrary smooth projective variety is Stein (holomorphically convex), and thus Theorem above should apply to arbitrary variety. An example of a Stein space are compact complex spaces, $\mathbb{C}^n$, Gaussian half-plane $\mathbb{H}$, and closed analytic subsets thereof; thus the conjecture holds for $\mathbb{C}^*$, elliptic curves, and arbitrary curves.

The property that the fundamental group is residually finite or even lerf is known ([Sco85]) when variety $\mathbb{A}$ is of dimension 1, $\dim \mathbb{A} = 1$, and also holds $\mathbb{A}$ an Abelian variety.

The general theory of $L_{\omega_1 \omega}$ by Shelah [She83a, She83b] implies that there is exists an $\aleph_1$-categorical countable $L_{\omega_1\omega}$-axiomatisation $\mathfrak{X}' = \mathfrak{X}'(\boldsymbol{U})$ extending $\mathfrak{X}$.

Corollary 2.1.3.2. *The model $\boldsymbol{U}$ belongs to an $\aleph_1$-categorical class.*

In a forthcoming paper we hope to prove an $\aleph_1$-categoricity result for the case of $A = E$ an elliptic curve defined over $\overline{\mathbb{Q}}$.

2.1.4 Linear structure on the universal covering spaces of field multiplicative group $\mathbb{C}^*$ and elliptic curves $E(\mathbb{C})$

Here we present some examples, and, for certain varieties A carrying abelian group structure, make explicit a locally modular, linear structure contained in the L_A-structure on the universal covering space $\boldsymbol{U}$. In general, it is not clear whether L_A-structure on $\mathbb{U}$ is a combination of a locally modular structure and the pull-backs to U of the algebraic variety structure on varieties $A^H(\mathbb{C})$'s. In a few of the examples we present below, it is so; however, the proof requires a geometric argument, and we defer it until the last chapter where we actually have to use it to obtain categoricity.

Example 2.1.4.1 ($A = \mathbb{G}_m$; $\mathbb{A}(\mathbb{C}) = \mathbb{C}^*$, the complex exponential map as the structure $\mathbb{C}^{\mathbf{lin}}_{\exp}$). Take $\mathbb{A} = \mathbb{G}_m$ be the multiplicative group of a field; as a variety, it is defined over $\mathbb{Q}$. The universal covering space is $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$: $\boldsymbol{U}_{\mathbb{G}m} = \mathbb{C}$ and $p = \exp$. Variety $\mathbb{G}_m$ is defined over $\mathbb{Q}$, as well as the morphisms $m : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ and morphism $z^n : \mathbb{C}^* \to \mathbb{C}^*$, and a closed subvariety $\Delta \subset \mathbb{C}^* \times \mathbb{C}^*$. Thus their graphs are also $\mathbb{Q}$-defined, and we get that the following equivalence relations on $\mathbb{C}$ are in L_A:

$$\begin{aligned}(x_1, y_1) \sim_\Delta (x_2, y_2) &\iff x_1 - y_1 = x_2 - y_2 \\ (x_1, y_1) \sim_{\{z^n=y\}} (x_2, y_2) &\iff nx_1 - y_1 = nx_2 - y_2 \\ (x_1, y_1, z_1) \sim_{\{xy=z\}} (x_2, y_2, z_2) &\iff x_1 + y_1 - z_1 = x_2 + y_2 - z_2\end{aligned} \tag{2.1.1}$$

The fundamental group $2\pi i\mathbb{Z}$ acts by translations $z \mapsto z + 2\pi ik$, which is an L_A-automorphism of $\mathbb{C}^{\text{lin}}_{\text{exp}}$.

Complex conjugation $x \to \overline{x}$ provides a continuous L_A-autmorphism of $\mathbb{C}^{\text{lin}}_{\text{exp}}$ which does not come from the fundamental group; it follows from the property

$$\exp(\bar{z}) = \overline{\exp(z)}, z \in \mathbb{C}.$$

In general, there are many such automorphisms, but they are not necessarily continuous or even measurable.

Thus, one can see that any automorphism of the kernel $2\pi i\mathbb{Z}$ *as an abelian group* can be extended to an automorphism of the whole structure. That is a property we want to keep in other examples; it is a property of homogeneity of the L_A-structure.

To conclude, we see that the structure on the $\mathbb{C}$-sort of $\mathbb{C}^{\text{lin}}_{\text{exp}}$ is the pull-back of algebraic structure on $\mathbb{C}^*$ enriched by an affine, locally modular structure on $\mathbb{C}$ itself.

The language L_A is rather robust under change of A; for example, if we take $A = \mathbb{G}_m \times \mathbb{G}_m$, then the structure obtained is essentially equivalent to the structure $\mathbb{C}^{\text{lin}}_{\text{exp}}$.

Example 2.1.4.2 ($A = \mathbb{G}_m \times \mathbb{G}_m = \mathbb{C}^* \times \mathbb{C}^*$). In this case the relations of 2.1.1 concerning $\Delta_{\mathbb{C}^*}$ and $\{z^n = y\}$ are still definable, but as relations with two variables in $\mathbb{C}\times\mathbb{C}$ and not 4 variables in $\mathbb{C}$; the analogues of relation concerning multiplication is also definable; the formulas are as follows; here (x_i, y_i) denote the coordinates of $z_i \in \boldsymbol{U}_{\mathbb{G}_m\times\mathbb{G}_m} \cong \boldsymbol{U}_{\mathbb{G}m} \times \boldsymbol{U}_{\mathbb{G}m}$.

$$\begin{aligned}z_1 \sim_\Delta z_2 &\iff x_1 - y_1 = x_2 - y_2 \\ z_1 \sim_{\{x_1^n=y_1\}} z_2 &\iff nx_1 - y_1 = nx_2 - y_2 \\ (z_1, z_2) \sim_{\{x_1y_1=x_2\}} (z_3, z_4) &\iff x_1 + y_1 - x_2 = x_3 + y_3 - z_4\end{aligned} \tag{2.1.2}$$

We can repeat the previous example word-by-word to get an example about elliptic curves.

Example 2.1.4.3 ($A = E$ an elliptic curve; the Weierstrass function $\mathbb{C} \to E(\mathbb{C})$). Take $\mathbb{A} = E$ be an elliptic curve defined over $\mathbb{Q}$ and possessing a $\mathbb{Q}$-rational point $O \in E(\mathbb{C})$; thus it carries a addition operation defined over $\mathbb{Q}$ where O is the zero point. The universal covering space is $\mathbb{C}$: $\boldsymbol{U}_E = \mathbb{C}$ and $p = \rho$ is the Weierstrass function. Thus, the morphisms $m : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ and morphism $z^n : \mathbb{C}^* \to \mathbb{C}^*$, and a closed subvariety $\Delta \subset \mathbb{C}^* \times \mathbb{C}^*$ are defined over $\mathbb{Q}$, and the following equivalence relations on $\boldsymbol{U}_E = \mathbb{C}$ are in L_A:

$$\begin{aligned}(x_1, y_1) \sim_\Delta (x_2, y_2) &\iff x_1 - y_1 = x_2 - y_2 \\ (x_1, y_1) \sim_{\{z^n=y\}} (x_2, y_2) &\iff nx_1 - y_1 = nx_2 - y_2 \\ (x_1, y_1, z_1) \sim_{\{xy=z\}} (x_2, y_2, z_2) &\iff x_1 + y_1 - z_1 = x_2 + y_2 - z_2\end{aligned} \tag{2.1.3}$$

The fundamental group $\Lambda = \text{Ker } \rho$ is a 2-generated lattice in $\mathbb{C}$ acts by translations $z \mapsto z + \lambda, \lambda \in \Lambda$, which is an L_A-automorphism of $\boldsymbol{U}$.

Similarly to $\mathbb{C}^*$-case, it is also true that any automorphism of the kernel Λ *as an* $\mathit{End}\,E$*-module* can be extended to an automorphism of the whole structure $\boldsymbol{U}$. However, this is a statement about arithmetic of the elliptic curve, and is more difficult to prove. We do it in the last chapter while proving the existence and homogeneity of the prime model.

We will also see that the structure on the $\mathbb{C}$-sort of $\mathbb{C}_{\text{exp}}^{\text{lin}}$ is the pull-back of algebraic structure on $\mathbb{C}^*$ enriched by an affine, locally modular structure on $\mathbb{C}$ itself.

2.1.5 $L_A(x_0')$-definable subgroups $\pi(\boldsymbol{U}, x_0')$ of $\boldsymbol{U}$

By abuse of notation, we will use $\boldsymbol{U}$ to refer to the covering map $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$.

Recall we denote

$$\begin{aligned}\pi(\boldsymbol{U}) = \pi(p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})) = &\quad \text{Gal}(p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})) \\ &= \{\tau : \boldsymbol{U} \to \boldsymbol{U} \text{ continuous } : p \circ \tau = p\}\end{aligned} \tag{2.1.4}$$

and call it *group of deck transformations*, or the *group of Galois transformations*, or *Galois group* of covering $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$. For a point $x_0' \in \boldsymbol{U}$, let the fibre of p above $p(x_0') \in \mathbb{A}(\mathbb{C})$ be denoted by

$$\pi(\boldsymbol{U}, x_0') = \{x' : p(x') = p(x_0')\} = \{\tau x_0' : \tau \in \pi(\boldsymbol{U})\}; \tag{2.1.5}$$

the fibre $\pi(\boldsymbol{U}, x_0')$ acquires the group structure from $\pi(\boldsymbol{U})$ via the bijection

$$\tau x_0' \mapsto \tau.$$

Via that bijection, the group $\pi(\boldsymbol{U}, x_0')$ acts on $\boldsymbol{U}$: in particular, $x_0' \in \pi(\boldsymbol{U}, x_0')$ corresponds to the identity of $\pi(\boldsymbol{U})$ and acts trivially. Actions of $\pi(\boldsymbol{U}, x_0')$ and $\pi(\boldsymbol{U}, x_1')$ are conjugated by the unique element of $\pi(\boldsymbol{U})$ taking x_0' into x_1'.

2.2 The analogue of fundamental groupoid associated to L_A-structure on U

We show that L_A-structure $\mathbb{U}$ can be equivalently thought of as the fundamental groupoid of $\mathbb{A}(\mathbb{C})$; we do so by interpreting a path in $\mathbb{A}(\mathbb{C})$ as an equivalence class of $\boldsymbol{U} \times \boldsymbol{U}/_{\sim_\pi \,\&\, \sim_\Delta}$.

2.2.1 Relation to the fundamental group $\pi_1(\mathbb{A}(\mathbb{C}), x_0)$ of $\mathbb{A}(\mathbb{C})$

For any normal subgroup $H \lhd \pi(\boldsymbol{U})$, the map $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ factors as

$$\boldsymbol{U} \to^{p_H} A^H(\mathbb{C}) \to^{p^H} \mathbb{A}(\mathbb{C}),$$

where both $p_H : \boldsymbol{U} \to A^H(\mathbb{C})$ and $p^H : A^H(\mathbb{C}) \to \mathbb{A}(\mathbb{C})$ are covering maps, and

$$H = \pi(p_H : \boldsymbol{U} \to A^H(\mathbb{C})) \text{ and also } \pi/H \cong \pi(p^H : A^H(\mathbb{C}) \to \mathbb{A}(\mathbb{C})).$$

For a subset $X' \subset \boldsymbol{U}$, let

$$\begin{aligned}\pi(X') &= \{\tau \in \pi(\boldsymbol{U}) : \tau X' \subset X'\} \\ \pi(X', x_0') &= \{\tau \in \pi(\boldsymbol{U}, x_0') : \tau x_0' \text{ and } x_0' \\ &\quad \text{lie in the same connected component of } X'\}.\end{aligned} \tag{2.2.1}$$

For a topological space B, one defines *the fundamental group* $\pi_1(B, b)$ as the group of all loops starting and ending at point b, with the operation of concatenation; the loops are considered up to homotopy, i.e. a continuous transformation of one into another. Given a universal covering space U of B with a distinguished point b' above b, $p(b') = b$, there is a well-defined action of $\pi_1(B, b)$ on U via path-lifting property, as explained below.

Path-lifting property of covering $p : U \to B$ says that for every path $\gamma : [0, 1] \to \mathbb{A}(\mathbb{C})$, and every point $a', p(a') = a = \gamma(0)$, there exist a unique path $\tilde{\gamma} : [0, 1] \to U$ such that $p(\tilde{\gamma}(t)) = \gamma(t), 0 \leq t \leq 1$; intuitively, one should think that we «lift path γ to the covering space piece by piece», using the fact that p is the local isomorphism between the neighbourhoods in U and B, given by the definition of the covering.

In fact it can be shown that if any point in U can be joined by a path, then, given a basepoint $b' \in U$, path lifting property identifies U with the set of all paths in B leaving b. In general, if U is not connected, this identifies the set of points in a connected component of U containing b', and the set of paths in the connected component of B containing b, up to homotopy (continuous transformation) fixing the paths; cf. Appendix 4.1.2 for generalities and §4.1.4 for particulars about the fundamental group and paths-lifting property.

Thus, in our situation there are well-defined isomorphisms

$$\pi(\boldsymbol{U}, x_0') \cong \pi_1(\mathbb{A}(\mathbb{C}), x_0), \;\; x_0' \mapsto x_0 \text{ as the trivial path}$$

and thus we canonically identify

$$\pi(\boldsymbol{U}, x_0') = \pi_1(\mathbb{A}(\mathbb{C}), x_0).$$

Note, however, that the isomorphism $\pi(\boldsymbol{U}) = \pi_1(\mathbb{A}(\mathbb{C}), x_0)$ is defined only up to a conjugation by an element of $\pi(\boldsymbol{U})$; this corresponds to choosing point x_0' above x_0.

With an embedding $\iota : X \to \mathbb{A}(\mathbb{C})$, there is also a topological way to associate a subgroup $\iota_* \pi_1(X, x_0) \subset \pi_1(\mathbb{A}(\mathbb{C}), x_0)$; indeed, a continuous map $f : X \to \mathbb{A}(\mathbb{C})$ induces a well-defined map on homotopy classes of paths taking $\gamma : [0, 1] \to X$ into the composition $f \circ \gamma : [0, 1] \to \mathbb{A}(\mathbb{C})$. It is not hard to check that this induces a group homomorphism

$$f_* \pi_1(X, x_0) \subset \pi_1(\mathbb{A}(\mathbb{C}), f(x_0)).$$

For a subspace $X \subset \mathbb{A}(\mathbb{C})$, there is a natural embedding of fundamental groups $\iota_* : \pi_1(X, x_0) \to \pi_1(\mathbb{A}(\mathbb{C}), x_0)$, which is takes a path into itself. We denote its image by $\iota_* \pi_1(X, x_0)$. Then under identification $\pi(\boldsymbol{U}, x_0') = \pi_1(\mathbb{A}(\mathbb{C}), x_0)$ it holds

$$\iota_* \pi_1(X, x_0) = \pi(X', x_0').$$

2.2.2 Interpretation of U as paths: groupoid structure on $\boldsymbol{U}$

L_A-structure $\boldsymbol{U}$ interprets the paths in $\mathbb{A}(\mathbb{C})$ up homotopy fixing the points.

The Lifting Property says that, given a path $\gamma : [0, 1] \to \mathbb{A}(\mathbb{C})$, there is a path $\gamma' : [0, 1] \to \boldsymbol{U}$ above $\gamma, p \circ \gamma'(t) = \gamma(t), t \in [0, 1]$, starting at any point $\gamma'(0) = a_0', p(a_0') = a_0 = \gamma(0)$ above the starting point of path γ. Moreover, the homotopy class of γ' depends only on the homotopy class of γ. On the other hand, by properties of the universal covering space, the homotopy class of a path in $\boldsymbol{U}$ is determined by its endponts. Thus, we see that a path γ in $\mathbb{A}(\mathbb{C})$ gives rise to a pair of points $\gamma'(0), \gamma'(1)$ defined up to π-action.

This leads us to the following.

We think of an element of $\boldsymbol{U} \times \boldsymbol{U}/_{\sim_\Delta}$ as *a path in* $\mathbb{A}(\mathbb{C})$ *up to loose-endpoint homotopy*; an element of $\boldsymbol{U} \times \boldsymbol{U}/_{\sim_\pi \,\&\, \sim_\Delta}$ is *a path in* $\mathbb{A}(\mathbb{C})$ *up fixed-point homotopy*. Namely, $(x_1, y_1), (x_2, y_2) \in \boldsymbol{U} \times \boldsymbol{U}$ are equivalent iff $x_1 \sim_\pi x_2 \,\&\, y_1 \sim_\pi y_2 \,\&\, (x_1, y_1) \sim_\Delta (x_2, y_2)$. The endpoints of a path (x, y) are the points x and y considered up to $\sim_\pi$, i.e., points $p(x), p(y) \in \mathbb{A}(\mathbb{C}) = \boldsymbol{U}/\sim_\pi$. Thus, two paths (x_1, y_1) and (x_2, y_2) *have the same end-points* iff $x_1 \sim_\pi x_2 \,\&\, y_1 \sim_\pi y_2$.

Let $\pi(\mathbb{A}(\mathbb{C})) = \boldsymbol{U} \times \boldsymbol{U}/_{\sim_\pi \,\&\, \sim_\Delta}$ denote the *set of paths in* $\mathbb{A}(\mathbb{C})$ *up to fixed-point homotopy*. The set $\pi(\mathbb{A}(\mathbb{C}))$ carries a groupoid structure; we concatenate path (x, y) and (y, z) to obtain $(x, z) = (x, y) \cdot (y, z)$. A *loop* is a pair (x, y) of $\sim_\pi$-equivalent points, $x \sim_\pi y$. The set of loops around a given point $a_0/\sim_\pi$ carries a group structure; it is customary to represent loops by pairs (a_0, y) where $a_0 \in \boldsymbol{U}$ is fixed. With this choice, the concatenation is given by the formula

$$\begin{gathered}(a_0, a_1) \circ ... \circ (a_0, a_n) = (a_0, y_n) \iff \\ \exists x_1 ... \exists x_n ((a_0, x_1, x_1, y_1, y_1, ..., x_n, y_n) \sim_\Delta (a_0, a_1, a_0, a_2, ..., a_0, a_n))\end{gathered} \tag{2.2.2}$$

Note the essential use of existential quantifiers in the formula.

Let γ in a subset Z of $\mathbb{A}(\mathbb{C})$; then any lifting γ' of γ to $\boldsymbol{U}$ lie in $p^{-1}(Z) \subset \boldsymbol{U}$; in particular, each path γ' has to lie in a connected component of $p^{-1}(Z)$. Thus, we say that *path* $(a, b) \in \pi(\mathbb{A}(\mathbb{C}))$ *lies in* Z iff $p(a), p(b) \in Z$ and $a \sim_Z b$. Then, we may think of a connected component of $p^{-1}(Z)$ as the set of end-points, or fixed homotopy classes, of all paths lying in Z lifted from a particular fixed point in Z.

2.2.3 Topology on the universal covering space $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$

It is not hard to notice that all basic L_A-definable sets have that property that their π-invariant closure is closed in analytic topology. Provided A is Shafarevich, we may introduce the following topology on U:

Definition 2.2.3.1. A subset of U is $\mathcal{T}$*-closed* iff either of the following two equivalent conditions holds

1. it is a union, possibly infinite, of irreducible components of a $\pi(\boldsymbol{U})$-invariant analytic closed subset of $\boldsymbol{U}$,
2. it is a union, possibly infinite, of connected components of an analytic closed subset of $\boldsymbol{U}$ invariant under action of a finite index subgroup of $\pi(\boldsymbol{U})$,

or a finite union of such sets.

We will see in those conditions are equivalent Lemma 3.1.4.1 and Corollary 3.5.3.4, and that $\mathcal{T}$-closed sets do form a topology, provided A is Shafarevich. We use this fact freely in this chapter to describe examples.

There is another characterisation of the $\mathcal{T}$-topology in terms of the structure on factor-space $\mathbb{A}^H(\mathbb{C}) = \boldsymbol{U}/H$; indeed, a closed π-invariant subset of $\boldsymbol{U}$ is the preimage of its image in $\boldsymbol{U}/H$. The action of H on $\boldsymbol{U}$ is free and discrete, and thus factor-space $\mathbb{A}^H(\mathbb{C}) = \boldsymbol{U}/H$ inherits the analytic structure from $\boldsymbol{U}$; similarly the image of a π-invariant closed analytic set is an analytic subset of $\boldsymbol{U}$. Now, the analytic space morphism $\boldsymbol{U}/H \to \mathbb{A}(\mathbb{C})$ is a covering, and by a result of Serre [GAGA], $\boldsymbol{U}/H = \mathbb{A}^H(\mathbb{C})$ carries the structure of a algebraic variety defined over the algebraic closure of k. Since A is assumed projective, $\mathbb{A}^H(\mathbb{C})$ is projective also. By Chow lemma, any analytic subset of $\mathbb{A}^H(\mathbb{C})$ is an algebraic subset of it, and thus we come to the following equivalent definition.

Definition 2.2.3.2. A subset of $\boldsymbol{U}$ is called $\mathcal{T}$-closed iff it is a union of the connected components of a preimage of a closed algebraic subvariety of $\mathbb{A}^H(\mathbb{C}), H \triangleleft_{\text{fin}} \pi$ (defined over $\mathbb{C}$), or a finite union of such sets.

The advantage of this as a definition is that it could be generalised to other fields instead of $\mathbb{C}$.

An important property of $\mathcal{T}$-topology we are after, is that any an analytic irreducible component of a $\mathcal{T}$-closed set is $\mathcal{T}$-closed and L_A-definable.

The $\mathcal{T}$-topology is not compact: the fibre $p^{-1}(x_0) = \pi(\boldsymbol{U}, x'_0)$ is a countable discrete subset of U. The discrete group $\pi(\boldsymbol{U})$ acts on $\boldsymbol{U}$ by continuous transformations.

2.3 Examples of subvarieties and their associated relations $\sim_Z$

The previous subsections provide us with the definition of the object we want to consider. Let us see the basic constructions which are possible in L_A.

2.3.1 Basic Examples of L_A-definable sets associated to normal subvarieties

In this §, to give explicit examples, we use the principles that for a large class of analytic sets defined by a local property, a connected component is always irreducible. Namely, if Z is a normal, say smooth, analytic set, then a connected component of Z is irreducible. Thus, if one considers relations $\sim_Z$ for Z normal, it has to be an equivalence relation.

We give examples of interpretations of basic L_A-predicates:

Example 2.3.1.1 (A arbitrary, Z normal). For normal Z, the connected components of $p^{-1}(Z)$ are analytically irreducible. Let Z' denote a connected component of $p^{-1}(Z)$, then

$$x \sim_Z y \iff \exists \gamma \in \pi(\boldsymbol{U}) : x, y \in \gamma Z'$$

Thus, $\sim_Z$ is an equivalence relation.

We will later see that for arbitrary Z, $\sim_Z$ is a finite number of several equivalence relations (cf. Lemma 3.1.4.1, 3.5.3.2).

Example 2.3.1.2 ($\mathbb{A}$ arbitrary, $Z = \Delta \subset \mathbb{A}^2$; L_A-definability of action of $\pi(\boldsymbol{U})$). Let $Z = \Delta = \{(x,x) : x \in X\}$ be the *diagonal* subvariety of $\mathbb{A} \times \mathbb{A}$. Then $p^{-1}(\Delta) = \{(x', \gamma x') : x' \in U, \gamma \in \pi(\boldsymbol{U})\}$, and the connected components of $p^{-1}(\Delta)$ have form $\Delta_\gamma = \{(x', \gamma x') : x' \in U\}, \gamma \in \pi(\boldsymbol{U})$. Thus,

$$(x,y) \sim_\Delta (z,t) \iff \exists \gamma \in \pi(\boldsymbol{U})\, (x = \gamma y \;\&\; z = \gamma t)\,. \tag{2.3.1}$$

In particular, for any point $x_0 \in U$ the formula $(x_0, x_0) \sim_\Delta (z,t)$ defines the diagonal $\Delta' = \{(x', x') : x' \in U\}$, and, for a point $z_0 \in U$, the formula $(x_0, x_0) \sim_\Delta (z_0, t)$ defines the fibre $p^{-1}(p(z_0)) = \pi(\boldsymbol{U}) z_0$.

In general, given a point $x'_0 \in U$, the predicate $\sim_Z$ defines the action $\tau : p^{-1}(x_0) \times U \to U$ of $p^{-1}(x_0), x_0 = p(x'_0)$ on U by defining $\tau_y(z)$ as the unique element such that

$$\begin{aligned} &\tau : p^{-1}(x_0) \times U \to U \\ &(x'_0, y) \sim_Z (z, \tau_y(z)). \end{aligned} \tag{2.3.2}$$

Thus, we have a $\mathcal{T}$-continuous $L_A(x'_0)$-definable action

$$\pi(\boldsymbol{U}, x'_0) \times \boldsymbol{U} \to \boldsymbol{U}$$

and a $L_A(x'_0)$-definable group $\pi(\boldsymbol{U}, x'_0)$.

Example 2.3.1.3 ($\mathbb{A}$ arbitrary, $Z = \Gamma_f$ is a graph of a morphism $f : A \to A$). A connected component of $\Gamma_f = \{(x, f(x)) : x \in \mathbb{A}(\mathbb{C})\}$ has form $\{(x, f'(x)) : x \in U$, for some function $f' : U \to U$, and we have $p^{-1}(\Gamma_f) = \{(x, \gamma f'(x)) : x \in U, \gamma \in \pi\}$. Thus, the function $f' : U \to U$ is L_A-definable with parameters.

Analogously, the induced homomorphism $f_* : \pi(\boldsymbol{U}, x_0) \to \pi(\boldsymbol{U}, f(x_0))$ of fundamental groups $\pi(\boldsymbol{U})$ is also definable with parameters.

An important observation is that the function f' is quite often bijective while f is not; an example is $\mathbb{A}(\mathbb{C}) = \mathbb{C}^*$, $U = \mathbb{C}$, $f = z^2$, then f' is just multiplication by 2 (and an additive constant). In a way, this hints that the structure upstairs in $\boldsymbol{U}$ is simpler then the structure on $\mathbb{A}(\mathbb{C})$, at least in some respects.

Example 2.3.1.4 (A arbitrary, $X, Y, Z \subset A$, $Z \cong X \times Y$). Let $\phi : X \times Y \to Z$ be the isomorphism between $X \times Y$ and Z; then Γ_ϕ embeds into $A \times A$.

The next example hints that to get *all* the extra structure on U, we may restrict ourselves by considering only maximal subvarieties with a given fundamental group (as a subgroup of π^n). Sometimes such varieties admit a very clear description and form a locally modular geometry; for example, in case of $\mathbb{C}^{*n}$ such varieties are given by $\mathbb{Z}$-linear equations; in case of elliptic curves they are given by End E-linear equations. In case of abelian varieties, this class consists of abelian subvarieties; we will prove that in a forthcoming chapter of thesis; in case $\pi_1(A(\mathbb{C}))$ is abelian, the fibres of Shafarevich morphisms should be enough (cf. [Kol95] for relevant definitions and results).

Example 2.3.1.5 (A arbitrary, $Z \subset Y$ normal, $\pi(Z, z') = \pi(Y, z')$). For $Z \subset Y$, $\pi(Z, z') = \pi(Y, z')$, we have $\gamma Z' \cap Y' \neq \emptyset$ implies $\gamma Y' \cap Y' \neq \emptyset$; Y' is a connected component of $p^{-1}(Y) = \pi Y'$, and thus $\gamma Y' = Y'$, i.e. $\gamma \in \pi(Y', z') = \pi(Z', z')$. Thus we see $Z' = \pi Z' \cap Y' = p^{-1}(Z) \cap Y'$.

In general, copies of Z' in Y' are indexed by $\pi(Y')/\pi(X')$.

Are connected components of $p^{-1}(Z)$, Z is not necessarily defined over $\mathbf{k}$, are L_A-definable ?

The following example shows a geometric condition implying that fibres of a L_A-definable irreducible set are L_A-definable with parameters; that condition always holds for generic fibres, cf. Corollary 3.2.2.2, Proposition 3.2.1.1 for exact statement.

Example 2.3.1.6 (A arbitrary, $Z = Y(x, \alpha)$ connected normal, $Y \subset A^2$ is normal and defined over k, α arbitrary). Take $Z = Y(x, \alpha) = Y_\alpha$, Y/k, and consider fibre $Y'_{\alpha'} = Y' \cap U \times \{\alpha'\}$ where $p(\alpha') = \alpha$. Then, $Y'_{\alpha'}$ is a union of π-translates of Z', which do not intersect due to normality of Z, and thus it is connected iff $Y'_{\alpha'}$ is in fact just Z' itself. That is, $Y'_{\alpha'} = Z'$ is connected iff for any $\gamma \notin \pi(Z')$, it holds $(\gamma, \mathrm{id}) \notin \pi(Y')$. Thus, we conclude tat $Y'_{\alpha'} = Z'$ iff $\pi(Y') \cap \pi \times \mathrm{id} = \pi(Z') \times \mathrm{id}$.

In general, similar considerations show that the translates of $Z' \times \{\alpha\}$ within Y are indexed by $\pi(Y')/\pi(Z') \times \mathrm{id}$.

In fact, we will later see in Lemma 3.2.2.2, Lemma 3.2.1.1 that the condition $\pi(Y') \cap \pi \times \mathrm{id} = \pi(Z') \times \mathrm{id}$ always holds for $\alpha \in \mathrm{pr}\, Y(\mathbb{C})$ generic, provided Y irreducible (and modulo some mild assumptions on ambient variety A); see the Lemma for exact formulation. This will imply that L_A is able to define generic fibres of closed sets, and in fact all $\mathcal{T}$-closed sets.

Example 2.3.1.7 ($A = X \times X$, $\boldsymbol{U}_A \cong \boldsymbol{U}_X \times \boldsymbol{U}_X$). For $A = X \times X$, $Z = \Delta = \{(x, x) : x \in X\}$ the diagonal $\Delta' = \{(x', x') : x' \in U\}$ is a connected component of $p^{-1}\Delta$; other connected components are $p^{-1}(\Delta) = \{(x', \gamma x') : x' \in U\}$.

Thus we see that in L_A, the decomposition $\boldsymbol{U}_A \cong \boldsymbol{U}_X \times \boldsymbol{U}_X$ is definable over a parameter.

Example 2.3.1.8 (L^* is a homogenous $\mathbb{C}^*$-bundle over A, and $\pi_1(A(\mathbb{C}))$ is Abelian). Let $U \to A$ be the universal covering space of A, let $\mathrm{pr} : L \to A$ be a homogeneous $\mathbb{C}^*$-bundle over A, and let $U_L \to L$ be the universal covering space of $\mathbb{C}^*$-bundle L. We want to define *Chern class* of L in $H^2(\pi_1(A), \mathbb{Z})$.

Take $\lambda'_1, \lambda'_2 \in \pi(U_L)$, and consider $\tau = [\lambda_1, \lambda_2] = \lambda_1\lambda_2\lambda_1^{-1}\lambda_2^{-1}$. Then $\mathrm{pr}\,[\lambda_1, \lambda_2] = [\mathrm{pr}\,\lambda'_1, \mathrm{pr}\,\lambda'_2] = 0$, and thus, for each fibre F_a of L, we have $[\lambda_1, \lambda_2] \in \pi(F_y) \cong \mathbb{Z}$. Thus we have a map $\pi(U_L) \times \pi(U_L) \to \pi(F_y)$. Moveover, a simple topological argument gives that $\gamma\lambda = \lambda\gamma$ for $\gamma \in \pi(F_y)$ and $\lambda \in \pi(L)$ arbitrary; using paths interpretation, this is because one can shift γ along λ in L^* by multiplicating it by $\gamma(t)$. Thus, $\pi(F_y)$ is central in $\pi(L)$, and the commutator map on $\pi(U_{L^*})$ descends to a map

$$\pi(U_A) \times \pi(U_A) \to \pi(F_y), \tag{2.3.3}$$

i.e. a map $\Lambda \times \Lambda \to \mathbb{Z}$. This map could be checked to be bilinear and may be considered as an element of

$$\bigwedge H^1(A(\mathbb{C}), \mathbb{Z}) \cong H^2(\Lambda, \mathbb{Z}).$$

The above construction corresponds to constructing group cohomology associated to short exact sequence of $\mathbb{Z}$-modules

$$0 \to \pi(F_y) \to \pi(U_{L^*}) \to \pi(U_A) \to 0$$

Cf. [Mum70, p.239] for more details on line bundles and their bilinear forms associated to Abelian varieties.

To conclude, we see that in the case $A = L^*$ there is a bilinear form definable of fibres of the covering map. Moreover, if the form is non-degenerate, it allows one to interpret the ring of integers with addition and multiplication; this makes first-order theory of the structure unstable. Despite this, as was said before, we prove that it is model stable in $L_{\omega_1\omega}$; and in general conjecture it to be uncountably categorical in $L_{\omega_1\omega}$.

Example 2.3.1.9 (A arbitrary, $\mathrm{pr}\, Z$). We will see later that for every Z' irreducible in $\mathbb{U}$, the projection $\mathrm{pr}\, Z$ is $\mathcal{T}$-closed, and there exists a finite index subgroup H such that $\pi(\mathrm{pr}\, Z') \cap H = \mathrm{pr}\,(\pi(Z') \cap [H \times H])$

2.4 More examples of universal covering spaces

2.4.1 Examples of 1-dimensional universal covering spaces

Here we give a complete classification of the universal covering spaces of 1-dim complex algebraic curves. Such curves are also known as *Riemann surfaces*; they have two real dimensions.

The following are all simply connected (i.e. $\pi(X) = 0$ is trivial):

$\mathbb{CP}^1$ is the Riemann sphere
$\mathbb{C}$ is the Gaussian plane (i.e. the complex field viewed as a complex analytic space)
$\mathbb{H} = \{Im\ z > 0\}$ is the upper half-plane, or Lobachevsky plane
$\mathbb{D} = \{|z| < 1\}$ is the unit disk

Their group of automorphisms as complex analytic spaces are:

$\mathrm{Aut}\,\mathbb{C} = \{z \mapsto az + b\}$

$\mathrm{Aut}\,\mathbb{CP}^1 = \mathrm{GL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$??
$\mathrm{Aut}\,\mathbb{H} = SL(2,\mathbb{R})/\pm I$

Thus, $\Gamma = \pi_1(X(\mathbb{C}))$ is a discrete subgroup of $\mathrm{Aut}\,S$.

The unit disk $\mathbb{D}$ and upper-half plane $\mathbb{H}$ are isomorphic as Riemann surfaces.

2.4.2 Classification of 1-dim Riemann surfaces and complete algebraic curves

Theorem 2.4.2.1. *Any Riemann surface X (in particular, a complex algebraic curve) is isomorphic to a quotient $X = {}^{S}/_{\Gamma}$, where $S = \mathbb{CP}^1, \mathbb{C}, \mathbb{H}$ is either of simply connected* canonical regions $\mathbb{CP}^1, \mathbb{C}, \mathbb{H}$*, and the group Γ acts on S freely and discretely by automorphisms.*

Thus, $\Gamma = \pi_1(X)$ is the fundamental group of the Riemann surface X; for X a Riemann surface, such groups admit an explicit description in terms of generators and relations.

Γ_g : For $X(\mathbb{C})$ a complete algebraic curve, i.e. a compact Riemann surface, then the fundamental group of a complete algebraic curve $X(\mathbb{C})$ is

$$\Gamma_g =< a_1, b_1, ..., a_g, b_g : a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1} > .$$

$\Gamma_{g,n}$: The fundamental group of a *punctured* compact Riemann surface $X(\mathbb{C}) - \{p_1, ..., p_n\}$ without n points, is

$$\Gamma_{g,n} =< a_1, b_1, ..., a_g, b_g, c_1, ..., c_n : a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1} c_1 ... c_n > .$$

For $g = 0$, Γ_g is trivial, and $X = \mathbb{CP}^1$. For $g = 1$, $X = {}^{\mathbb{C}}/_{\Lambda}$ is an elliptic curve, and $\Gamma_1 = \mathbb{Z} \times \mathbb{Z}$. For $n > 1$, the groups Γ_n are non-commutative, hyperbolic.

2.4.3 Some other examples of covering spaces

Here are the basic examples of the universal covering spaces of complex manifolds. Incidently, this lists all universal covering spaces of 1-dim Riemann surfaces.

$\mathbb{CP}^1 \to \mathbb{CP}^1$ is the universal covering space of itself

$\mathbb{C} \to \mathbb{C}$ is the universal covering space of itself

$\mathbb{C} \xrightarrow{\exp} \mathbb{C}/2\pi i\mathbb{Z} = \mathbb{C}^*$ is the universal covering space of the multiplicative group $\mathbb{C}^*$

$\mathbb{C} \to \mathbb{C}/\Lambda = E_\Lambda(\mathbb{C})$ is the universal covering space of an elliptic curve, where Λ is a 2-dim discrete lattice in $\mathbb{C}$

$\mathbb{C}^{2g} \to \mathbb{C}^{2g}/\Lambda$ is the universal covering space of a complex torus $\mathbb{C}^{2g}/\Lambda$; for some choices of a $2g$-dimensional lattice Λ the complex torus $\mathbb{C}^{2g}/\Lambda = A(\mathbb{C})$ has the structure of an algebraic variety (*abelian variety*)

$\mathbb{H} \to \mathbb{H}/\Gamma$ is the universal covering space of the quotient $\mathbb{H}/\Gamma$ where $\Gamma \subset SL(2,\mathbb{R})/\pm I$ is a discrete subgroup of $SL(2,\mathbb{R})/\pm I$; for some choices of $\Gamma \cong \Gamma_g$ one obtains the complete algebraic curves of genius $g > 1$.

The universal covering space of a direct product is a direct product of the universal covering spaces.

Chapter 3

Geometric Case: model stability of the universal covering space of a variety

In this chapter we study the universal covering space $\boldsymbol{U}$ of a smooth projective algebraic variety $\mathbb{A}(\mathbb{C})$; we assume that $\boldsymbol{U}$ is holomorphically convex, or rather, that complex analytic space $\boldsymbol{U}$ satisfies the conclusions of Fact 3.1.2.1. We then introduce a topology on $\boldsymbol{U}$ which has the property that projection of a closed set is closed; it also posses a property of more technical character relating the topology and the action of the fundamental group $\pi(\boldsymbol{U})$, cf. Proposition 3.2.1.1.

The properties of $\mathcal{T}$-topology allow us to introduce a countable language L_A such that any $\mathcal{T}$-irreducible closed subset of $\boldsymbol{U}$ is L_A-definable. Then we study $\boldsymbol{U}$ as an L_A-structure and prove that it is model homogeneous.

In the second half of the chapter we introduce an $L_{\omega_1\omega}(L_A)$-axiomatisation for $\boldsymbol{U}$ and prove that it describes a class of model homogeneous models which satisfies conclusion $2_{\aleph_0 \to \aleph_1}$ of Theorem 3.5.4.7.

3.1 A Zariski-type topology on the universal covering space

We define here a topology $\mathcal{T}$ on the holomorphically convex universal covering space U of a projective complex algebraic variety $A(\mathbb{C})$ which is an analogue of Zariski topology on $A(\mathbb{C})$. The topology $\mathcal{T}$ is substantially weaker than the analytic Zariski topology on U, i.e. the topology given by closed analytic subsets of complex space U.

An important feature of topology $\mathcal{T}$ is that an analogue of Chevalley lemma holds for $\mathcal{T}$; recall that Chevalley Lemma for a compact complex algebraic variety says that a projection of a closed set is closed; model-theoretically it is quantifier elimination to the level of closed sets. We prove the analogous property of $\mathcal{T}$ in §3.2.2. Note that this properties fails in general analytic context: there is an example of a closed analytic subset in $\mathbb{C}^3$ whose projection on $\mathbb{C}$ is contained in an open disk.

3.1.1 Definition of $\mathcal{T}$-topology on holomorphically convex universal covering space U

Recall that $p : U \to A(\mathbb{C})$ is the universal covering space of a projective algebraic variety $A(\mathbb{C})$, and that we assume U to be a holomorphically convex (as a complex analytic space). We need the latter assumption in order for $\mathcal{T}$ to be a topology indeed.

Definition 3.1.1.1 (Topology $\mathcal{T}$ on holomorphically convex universal covering space U of an algebraic variety $A(\mathbb{C})$). An analytic subset of $\boldsymbol{U}$ is called *$\mathcal{T}$-closed* iff it is a union, possibly infinite, of the irreducible analytic components of an π-invariant closed analytic subset of $\boldsymbol{U}$, or a finite union of such sets.

Similarly we extend the definition to $\boldsymbol{U}^n$, for any n.

A π-invariant closed sets of $\boldsymbol{U}$ covers a closed subset of $\mathbb{A}(\mathbb{C})$. The covering map $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ being local isomorphism and analyticity being a local property, it implies that it covers a closed analytic subset. The ambient variety $\mathbb{A}(\mathbb{C})$ is assumed projective, and by Chow Lemma, it is a Zariski closed algebraic subset of $\mathbb{A}(\mathbb{C})$.

It would be natural to consider the unions of *connected components* of such sets and not that of irreducible ones; this is natural if we try not to use the analytic structure of $\boldsymbol{U}$ but only the topological structure of it as of the covering space of $\mathbb{A}(\mathbb{C})$ as an algebraic variety. This seems plausible because for analytic sets good enough (smooth or even normal), the notions of a connected component and an irreducible component coincide (cf. §4.2.2), and indeed, that is possible, with a price:

> A $\mathcal{T}$-closed set is a union of connected components of an H-invariant set, for a finite index subgroup $H \lhd_{\mathrm{fin}} \pi$, or a finite union of such sets

We prove the equivalence of these two definitions in Decomposition Lemma 3.1.4.1.

3.1.2 Normalisation and Local-to-global principles

The proof of Decomposition Lemma 3.1.3.1 essentially uses the various local-to-global properties implied by homomorphic convexity: Local Identity Principle(Uniqueness of Analytic Continuation), and others. It also uses the properties of normalisation of algebraic varieties; we state them here, too.

We choose to list the properties here to put an emphasis on the properties of complex analytic space U which we use in the proof of Decomposition Lemma. The exact formulation of those properties may be useful as a property one would expect from analytic Zariski structures.

Fact 3.1.2.1. *Let U be a holomorphically convex space, and let $Y, Z \subset U$ be closed analytic sets in U. Then*

1. *(analyticity is a local property) a set $X \subset U$ is analytic iff for all $x \in X$, there exists an open neighbourhood $x \in V_x$ such that $X \cap V_x$ is an analytic subset of V_x*
2. *(local identity principle) for an open neighbourhood $V \subset U$, if Y is irreducible and $Y \cap V \subset Z \cap V$ then $Y \subset Z$*
3. *(local identity principle) for an open neighbourhood $V \subset U$, if Y and Z are irreducible, and $Y \cap V$ and $Z \cap V$ have a common irreducible component, then $Y = Z$*
4. *(density of smooth points) for an open neighbourhood $V \subset U$, if $Z_0 \subset Z \cap V$ is an irreducible component of $Z \cap V$, then there exist a point $z_0 \in Z_0$ and an open neighbourhood $z_0 \in V_0 \subset V$ such that $V_0 \cap Z \subset Z_0$.*
5. *(local finiteness) a compact set $C \subset U$ intersect only finitely many of irreducible components of closed analytic set Z*
6. *(analyticity of a union of irreducible components) a union of, possibly infinitely many, irreducible components of an analytic set is analytic.*
7. *(irreducible decomposition) For Y, Z closed analytic subsets of $\boldsymbol{U}$, if $Y \subset Z$ and Y is irreducible, then Y is contained in an irreducible component of Z*

Proof. Those are well-known properties of holomorphically convex spaces.

By Prop. 5.3 of [Č85], Theorem 5.1 [ibid.] states (6) and (5). Corollary 2 of Prop. 5.3 [ibid.] implies (2) and (3). Theorem 5.4 [ibid.] implies (4). (2,3,4) together imply (7). □

Fact 3.1.2.2 (Chow Lemma). *A closed analytic subset of a complex projective algebraic variety is algebraic.*

Proof. This is a well-known fact in algebraic geometry, cf. Hartshorne[Har77]; there is also a model-theoretic sketch of a proof in Zilber[Zil05a] in the context of Zariski structures. □

Fact 3.1.2.3 (GAGA). *Let $\mathbb{A}(\mathbb{C})$ be an algebraic variety. If $q : T \to A(\mathbb{C})$ is a covering of topological spaces, then T admits a structure of a complex algebraic variety such that $q : T \to \mathbb{A}(\mathbb{C})$ becomes an algebraic morphism, i.e. there exists an algebraic variety $B(\mathbb{C})$ over $\mathbb{C}$, an algebraic morphism $q_{alg} : B(\mathbb{C}) \to \mathbb{A}(\mathbb{C})$, and a homeomorphism $\phi : T \to B(\mathbb{C})$ of topological spaces such that the diagramme of topological spaces commutes*

$$\begin{array}{ccc} T & \xrightarrow{q} & \mathbb{A}(\mathbb{C}) \\ \phi\downarrow & & id\downarrow \\ B(\mathbb{C}) & \xrightarrow{q_{alg}} & \mathbb{A}(\mathbb{C}) \end{array} \tag{3.1.1}$$

Moreover, the homeomorphism $\phi : T \to B(\mathbb{C})$ is well-defined up to an automorphism of B commuting with the covering morphism q_{alg}.

Proof. The existence of an analytic space B with the above properties follows from the fact that we may pull back the local analytic structure of $\mathbb{A}(\mathbb{C})$ onto T; in 1-dim case this already implies that B would be an algebraic variety; the general case is done in [Ser56]. □

Fact 3.1.2.4. *A closed analytic subset of a holomorphically convex set admits a unique decomposition into a countable union of analytic irreducible closed subsets.*

Proof. [Č85, §5.4,Theorem, p.49]. □

Fact 3.1.2.5. *A connected component of a normal analytic set is irreducible.*

Proof. [Č85] □

We use the following fact as the defining property of an etale covering: the morphism $B(K) \to A(K)$ of varieties over an algebraically closed field K of char 0 is *etale* iff there exists an embedding $i : K \to \mathbb{C}$ of the field of definition of A and B into $\mathbb{C}$ such that the corresponding morphism $i(B)(\mathbb{C}) \to i(A)(\mathbb{C})$ is a covering of topological spaces.

Definition 3.1.2.6. We call a smooth projective algebraic variety $\mathbb{A}(\mathbb{C})$ *Shafarevich* if

1. the universal covering space of $\mathbb{A}(\mathbb{C})$ satisfies the conclusions of Fact 3.1.2.1
2. the fundamental group $\pi_1(\mathbb{A}(\mathbb{C}))$ has the property that, for every finitely generated subgroup $H \subset \pi^n$ and an element $h \notin H$ outside it, there exists a finite index normal subgroup $G \lhd_{\text{fin}} \pi$ such that $h/G\times..\times G \notin H/(G\times...\times G)$.

In particular, second condition implies that the fundamental group is *residually finite*, i.e. for every $h \in \pi_1(\mathbb{A}(\mathbb{C}))$ there exists a homomorphism $\phi : \pi_1(\mathbb{A}(\mathbb{C})) \to F$ into a finite group F such that $\phi(h) \neq \emptyset$. Indeed, take ϕ to be $\pi_1(\mathbb{A}(\mathbb{C})) \to \pi_1(\mathbb{A}(\mathbb{C}))/G$, where G is a finite index normal subgroup from condition 2 where $H = e$.

The property formulated in item (2) is knows as *lerf*; it holds for fundamental groups of surfaces and for $\mathbb{Z}^n$; cf. [Sco85].

3.1.3 A geometric Decomposition Lemma; Noetherian property

The lemma states a finiteness property of the irreducible decomposition of the preimage of an algebraic subvariety in $A(\mathbb{C})$; it may be interpreted as saying that the irreducible components are not too far from being *connected* components of the preimage, *up to finite index*.

For a subset $Z \subset U$, let $\pi Z = \bigcup_{\gamma\in\pi} \gamma Z'$ denote the π-orbit of set Z.

For $H \lhd_{\text{fin}} \pi$, let $p_H : \boldsymbol{U} \to \boldsymbol{U}/\sim_H$ be the factorisation map; by Fact 3.1.2.3, we choose and fix isomorphisms $A^H(\mathbb{C}) \cong \boldsymbol{U}$.

Lemma 3.1.3.1 (First Decomposition lemma; Noetherian property). *Assume A is Shafarevich.*

A π-invariant analytic closed set has an analytic decomposition of the form

$$W' = HZ'_1 \cup ... \cup HZ'_k,$$

where $H \lhd_{\text{fin}} \pi$ is a finite index normal subgroup of π, the analytic closed sets $Z'_1, .., Z'_k$ are irreducible, and for any $\tau \in H$ either $\tau Z'_i = Z'_i$ or $\tau Z'_i \cap Z'_i = \emptyset$.

Such decomposition also exists for closed analytic sets invariant under action of a finite index subgroup of π.

Proof. The proof of the first part is relatively simple, and follows from the Fact 3.1.2.1 in a rather straightforward way; we do it first.

The proof of the second claim uses rather more delicate local analysis of the structure, and several local-to-global properties of analytic subsets of holomorphically convex spaces as well as properties of Zariski geometry of algebraic varieties.

So let us start to prove (a). Let Z' be an irreducible component of $p^{-1}(Z(\mathbb{C}))$; by π-invariance of $p^{-1}(Z(\mathbb{C}))$, for any $\gamma \in \pi$, the set $\gamma Z'$ is also an irreducible component of $p^{-1}(Z(\mathbb{C}))$, and so $\pi Z'$ is a union of irreducible components of $p^{-1}Z(\mathbb{C})$; thus, by Fact 3.1.2.1 above, $\pi Z' \subset p^{-1}(Z(\mathbb{C}))$ is analytic.

The covering morphism $p : U \to A(\mathbb{C})$ is a local isomorphism, and analyticity is a local property; by π-invariance of $\pi Z'$, it implies $p(\pi Z')$ is analytic. For different irreducible components $Z'_1 \neq Z'_2$ of $p^{-1}(Z(\mathbb{C}))$ it can not hold that $p(Z'_1) \subsetneq p(Z'_2)$; indeed, then $\pi Z'_1 = p^{-1}p(Z'_1) \subset \pi Z'_2 = p^{-1}p(Z'_2)$, and so $Z'_1 = \bigcup(Z'_1 \cap \gamma Z'_2), \gamma \in \pi$; thus, Z'_1 can not be irreducible unless $Z'_1 \subset \gamma Z'_2$, for some $\gamma \in \pi$, which is impossible by π-invariance of $p^{-1}Z(\mathbb{C})$. To conclude, closed sets $p(Z')$, Z' vary among irreducible components of an algebraic subvariety $Z(\mathbb{C})$, cover the whole of $Z(\mathbb{C})$; they are also irreducible. Thus they are the analytic irreducible components of Z. By [GR65], the analytic irreducible components of an algebraic set are algebraic and irreducible, and thus they are the algebraic irreducible components; in particular there are only finitely many of them. That gives the required decomposition.

Now let us start to prove (b). First of all, note that we may suppose Z to be irreducible.

Let $Z'^{(n)} = \bigcup Z'_{i_1} \cap ... \cap Z'_{i_n}$ be the union of all intersections of n-tuples of different irreducible components of $p^{-1}(Z(\mathbb{C}))$.

Claim 3.1.3.2. The set $p(Z'^{(n)})$ is an algebraic subset of $Z(\mathbb{C})$, for $n > 0$. For n sufficiently large, $Z'^{(n)}$ is empty.

Proof. By the local finiteness (Fact 3.1.2.1) a compact subset intersects only finitely many of the irreducible components $\gamma Z_i'$'s; thus $Z'^{(n)}$ is locally a finite union of intersections of analytic sets, and therefore is analytic. By the π-invariance of $\gamma Z_i'$'s it is π-invariant, and thus p provides a local isomorphism of $Z'^{(n)}$ and its image; therefore the image $p(Z'^{(n)})$ is analytic. By Chow Lemma 3.1.2.2 this implies it is in fact algebraic. If n is greater then the number of local irreducible components of a point of Z in A, then by Fact 3.1.2.1(local identity principle) $Z'^{(n)}$ has to be empty. □

The claim above implies $Z'^{(n)}$ are $\mathcal{T}$-closed, for any n. By Claim (a) of Lemma, we may choose finitely many points z_i''s so that any irreducible component of $Z'^{(n)}$, $n > 0$, contains a π-translate of one of z_i''s.

By Fact 3.1.2.1(5) there are only finitely many irreducible components of $p^{-1}(Z(\mathbb{C}))$ containing each point z_i', call them Y_j''s. Use residual finiteness of π to see that there exist a finite index subgroup $H \subset \pi$ such that $p_H(Y_i') \neq p_H(Y_j')$; indeed, it is sufficient to take H large enough to distinguish points $y_i' \in Y_i' - \bigcup_{j \neq i} Y_j'$, $p_H(y_i') \neq p_H(y_j')$.

Then it follows that for any two intersecting irreducible component $X' \neq Y'$ of $p^{-1}(Z(\mathbb{C}))$ it holds that $p_H(X') \neq p_H(Y')$. Indeed, there exist $\gamma \in \pi$ and a point z_i' such that $\gamma z_i' \in X' \cap Y'$, i.e. $z_i' \in \gamma^{-1}X' \cap \gamma^{-1}Y'$. Then $p_H(\gamma^{-1}X') \neq p_H(\gamma^{-1}Y')$, by construction of H. And now this implies $p_H(X') \neq p_H(Y')$, as required.

Now, for $\gamma \in H$, it holds $p_H(Z_i') = p_H(\gamma Z_i')$, and thus, by the arguments above, $Z_i' \cap \gamma Z_i' = \emptyset$.

In other words, we have proven that there exists a finite index subgroup $H < \pi(A(\mathbb{C})$ such that Z_i' is a connected component of $p_H^{-1} p_H(Z_i')$, i.e. the connected components of the preimages of the irreducible components of $p_H p^{-1}(Z(\mathbb{C}))$ are irreducible. □

The next corollary allows for an equivalent definition of $\mathcal{T}$-topology.

Notice that the notion of an H-invariant set is essentially algebraic; it is a preimage of a closed algebraic subset of $A^H(\mathbb{C})$. Thus, the meaning of next corollary that in fact $\mathcal{T}$-closed sets encode a mix of algebraic data and *topological, homotopical* data, not of analytic one.

Corollary 3.1.3.3. *A set is $\mathcal{T}$-closed iff it a union of* connected *components of a finite number of H-invariant sets, for some $H \triangleleft_{\text{fin}} \pi$ a finite index subgroup of π.*

Proof. Lemma 3.1.3.1 above implies that each $\mathcal{T}$-closed set can be represented in such a form.

On the other hand, the lemma implies that each H-invariant set is a finite union of sets of the form HZ_i' where Z_i' are irreducible. Then, $\pi Z_i'$ is also closed analytic as a finite union of translates of HZ_i', and moreover, each translate of Z_i' is an irreducible component of $\pi Z_i'$ and thus $\mathcal{T}$-closed. This implies the converse of the corollary. □

The Lemma has the following algebraic consequence. By Lefshetz principle, it holds for any characteristic 0 algebraically closed field instead of $\mathbb{C}$. One may think of this property as a rather weak property of irreducible decomposition for *etale* topology.

Corollary 3.1.3.4. *Let A be Shafarevich. Then for any closed subvariety $Z \subset \mathbb{A}(\mathbb{C})$, there exists a finite etale cover $q : A^H(\mathbb{C}) \to \mathbb{A}(\mathbb{C})$ such that, for any further etale cover $q' : A^G(\mathbb{C}) \to A^H(\mathbb{C})$, the connected components of $q'^{-1}(Z_i) \subset A^G(\mathbb{C})$ are irreducible, where Z_i's are the irreducible components of $q^{-1}(Z)$.*

Proof. Indeed, it is enough to take H as in Decomposition Lemma. □

3.1.4 Decomposition Lemma for topology $\mathcal{T}$

Recall notation $\pi Z' = \bigcup_{\gamma\in\pi} \gamma Z'$. Note we do not yet know that the intersection of two $\mathcal{T}$-closed sets is $\mathcal{T}$-closed.

Corollary 3.1.4.1 (Decomposition Lemma). *Assume A is Shafarevich.*

The collection of $\mathcal{T}$-closed subsets of $\boldsymbol{U}$ forms a topology with a descending chain conditions on irreducible sets. A $\mathcal{T}$-closed set possesses an irreducible decomposition as a union of a finite number of $\mathcal{T}$-closed sets whose $\mathcal{T}$-connected components are $\mathcal{T}$-irreducible. A union of irreducible components of a $\mathcal{T}$-closed set is a $\mathcal{T}$-closed.

That is,

1. *the collection of $\mathcal{T}$-closed subsets on $\boldsymbol{U}^n, n > 0$ forms a topology. The projection and inclusion maps $pr : \boldsymbol{U}^n \to \boldsymbol{U}^m, (x_1, ..., x_n) \mapsto (x_{i_1}, .., x_{i_m})$ and $\iota : \boldsymbol{U}^n \hookrightarrow \boldsymbol{U}^m, (x_1, ..., x_n) \mapsto (x_{i_1}, .., x_{i_{m'}}, c_{m'}, .., c_m)$ are continuous.*
2. *There is no infinite decreasing chain $.. \subsetneq U_{i+1} \subsetneq U_i \subsetneq ... \subsetneq U_0$ of closed $\mathcal{T}$-irreducible sets.*
3. *A union of irreducible components of a $\mathcal{T}$-closed set is $\mathcal{T}$-closed.*
4. *A set is $\mathcal{T}$-closed iff it a union of* connected *components of a finite number of H-invariant sets, for some $H \lhd_{\text{fin}} \pi$ a finite index subgroup of π.*
5. *Each $\mathcal{T}$-closed set is a union of a finite number of $\mathcal{T}$-closed sets whose $\mathcal{T}$-connected components are $\mathcal{T}$-irreducible. Moreover, those sets may be taken so that their connected components within the same set are translates of each other by the action of a finite index subgroup $H \lhd_{\text{fin}} \pi$.*

More specific properties are

Corollary 3.1.4.2. *1. For any $\mathcal{T}$-closed set W', the set $\pi W'$ is $\mathcal{T}$-closed.*

2. *An irreducible $\mathcal{T}$-closed set W' is a connected component of an H-invariant set HW' for some $H \lhd_{\text{fin}} \pi$ a finite index normal subgroup of π.*
3. *For any two irreducible $\mathcal{T}$-closed sets V', W' if for any finite index normal subgroup $H \lhd_{\text{fin}} \pi$ it holds $HW' = HV'$, then $W' = V'$.*

Proof. (1,2) trivially follows from irreducible decomposition; (3) is slightly more difficult and requires an additional assumption on $\mathbb{A}(\mathbb{C})$. We may assume H be so that V' and $W' = hV'$ are connected components of $HV' = HW'$, correspondingly. Take $x \in V'$, $y = hv' \in W'$. With the help of assumption $V = \mathbb{A}(\mathbb{C})$ is Shafarevich, we may assume H sufficiently small of finite index that $Hx \neq Hy = Hhy$, a contradiction.

Note that the latter argument can be done topologically. Consider $p_H(HW') = HW'/H \subset \boldsymbol{U}/H$ the subset of $\boldsymbol{U}/H$ covered by HW'. The set HW' is the preimage of HW'/H under the factorisation map $p_H : \boldsymbol{U} \to \boldsymbol{U}/H$, and thus a connected component of HW' is the preimage of a connected component of $p^H(HW')$. Thus, both $p_H(W')$ and $p_H(V')$ are connected components of $p_H(HW') = p_H(HV')$. By the Shafarevich property of $\mathbb{A}(\mathbb{C})$, we may take H such that for points $x \in W'$, $y \in V'$, $p_H(x) \neq p_H(y')$ and therefore $p_H(W') \neq p_H(V')$. That implies $W' \neq V'$, as required.

In fact, one of the main interests of these paper is to understand *why* such arguments can always be replaced by algebraic «abstract non-sense». □

The following property is «ideologically» important, and is the main property in proving the properties of $\mathcal{T}$ above. An analogue of this property should also hold in other examples, say full exponentiation; there it says that a *definably* irreducible set is analytically irreducible.

Lemma 3.1.4.3. *A $\mathcal{T}$-irreducible closed set is analytically irreducible, i.e. it is irreducible as an analytic subset of $\boldsymbol{U}$.*

Proof of Lemma.. Property (4) is Corollary 3.1.3.3. By definition 3.1.1.1, an $\mathcal{T}$-irreducible $\mathcal{T}$-closed set W' is a countable union of irreducible component of π-invariant closed analytic sets. Those components are $\mathcal{T}$-closed by definition, and thus $\mathcal{T}$-irreducibility implies the union is necessarily trivial. Thus, the set is an analytic irreducible component of a π-invariant set, i.e. in particular irreducible as an analytic set. □

Proof of Corollary 3.1.4.1. We defer the proof of (1) until we prove (4), (2), (3).

By Lemma 3.1.4.3, a decreasing chain of $\mathcal{T}$-irreducible sets is a decreasing sequence of closed analytic irreducible sets, and this implies (2) immediately.

Property (3) is immediate from the same property of the irreducible decomposition of analytic sets (Fact 3.1.2.16).

A finite union of $\mathcal{T}$-closed sets satisfying (4) also satisfies (4), and thus it is enough to prove that a union V' of the irreducible components of a π-invariant set $W' = \pi V'$ satisfies (4). By Decomposition Lemma 3.1.3.1, set $W' = \pi V'$ admits a decomposition

$$W' = HZ'_1 \cup ... \cup HZ'_k,$$

where $H \lhd_{\text{fin}} \pi$ is a finite index normal subgroup of π, the analytic closed sets $Z'_1, .., Z'_k$ are irreducible, and for any $\tau \in H$ either $\tau Z'_i = Z'_i$ or $\tau Z'_i \cap Z'_i = \emptyset$.

First note that by the definition its irreducible components Z'_i's are $\mathcal{T}$-closed. By assumption, the analytic irreducible decomposition of set V' thus has form

$$\begin{aligned} V' = V' \cap W' = (HZ'_1 \cap V') \cup ... \cup (HZ'_k \cap V') = \\ \bigcup_{h \in H: hZ'_1 \subset V'} hZ'_1 \cup ... \cup \bigcup_{h \in H: hZ'_k \subset V'} hZ'_k. \end{aligned} \tag{3.1.2}$$

The sets $\bigcup_{h \in H: hZ'_i \subset V} hZ'_i$'s are $\mathcal{T}$-closed, hZ'_i's are their connected components by the second claim of Decomposition Lemma 3.1.3.1. Thus this is the $\mathcal{T}$-irreducible decomposition required in (4).

Let us prove that $\mathcal{T}$ is a topology, the most difficult property.

A union of a finite number of $\mathcal{T}$-closed set is closed by the definition.

Let us prove the intersection of two $\mathcal{T}$-closed set Z'_i and Y'_i is $\mathcal{T}$-closed.

Assume W' and V' are unions of connected component of H-invariant sets HW' and HV'. The intersection $HW' \cap HV'$ is H-invariant and the set $W' \cap V'$ is a union of the connected components of $HW' \cap HV'$. The intersection $HW' \cap HV'$ is $\mathcal{T}$-closed by Corollary 3.1.3.3, and thus its connected components are also $\mathcal{T}$-closed. This by definition implies $W' \cap V'$ is $\mathcal{T}$-closed.

To prove that an infinite intersection is closed, it is sufficient to prove that the intersection of a decreasing sequence of $\mathcal{T}$-closed sets is $\mathcal{T}$-closed. Use Koenig lemma, Fact 3.1.2.1(7) and the fact that a sequence of decreasing $\mathcal{T}$-irreducible sets stabilizes.

The argument is as follows. Let $... \subset X_i \subset X_{i-1} \subset ..$ be the decreasing sequence of $\mathcal{T}$-closed sets, and let Z^i_j be the irreducible components of X_i up to $\pi(\boldsymbol{U})$-action.

Let us make a tree whose vertices are sets Z^i_j, and Z^i_j and Z^{i-1}_k are joined by an edge iff $Z^i_j \subset Z^{i-1}_k$. The number of vertices in each level is finite, and thus tree has finite branching; on the other hand, each branch is finite as it consists of irreducible sets. Thus, the tree has to be finite by Koenig's Lemma. This means that for i large, the intersection of first i sets is a union of translates of a fixed finite number of $\mathcal{T}$-irreducible sets. As any such union is $\mathcal{T}$-closed, this concludes the argument.

This completes the proof of the lemma. □

3.1.5 Θ-definable sets, Generic points and Θ-definable closure

Recall that $\boldsymbol{U}/\pi \cong \mathbb{A}(\mathbb{C})$ has the structure of an algebraic variety over $\mathbb{C}$ and that the π-invariant sets are in a bijective correspondence with the algebraic subvarieties of $A^H(\mathbb{C})$. Thus suggests us that we may try to pull back to $\boldsymbol{U}$ the notion of a generic point in $\mathbb{A}(\mathbb{C})$.

The following definition behaves well only for $\Theta \subset \mathbb{C}$ algebraically closed.

Definition 3.1.5.1. We say that a π-invariant $\mathcal{T}$-closed subset $W' \subset \boldsymbol{U}$ is *defined over an algebraically closed subfield* $\Theta \subset \mathbb{C}$ iff $p(W') \subset \mathbb{A}(\mathbb{C})$ is a subvariety defined over Θ.

A $\mathcal{T}$-closed set is *defined over a subfield* $\Theta \subset \mathbb{C}$ iff it is a countable union of irreducible components of π-invariant $\mathcal{T}$-closed subsets defined over Θ.

Definition 3.1.5.2. For a set $V \subset U^n$, let $\mathrm{Cl}_\Theta V$ be the intersection of all closed Θ-definable sets containing V:

$$\mathrm{Cl}_\Theta(V) = \bigcap_{V \subset W, W/\Theta \text{ is } \Theta\text{-definable closed}} W$$

A point $v \in V$ is called *Θ-generic* iff $V = \mathrm{Cl}_\Theta(v)$, i.e. there does not exist a closed Θ-definable proper subset of V containing v.

The following finiteness property will be needed to show that a variant of Chevalley lemma implies a variant of ω-homogeneity.

Lemma 3.1.5.3. *(a)* $\quad$ *$Cl_\Theta(V)$ is Θ-definable*
(b) $\quad$ *$Cl_\Theta(V) = \bigcup_{v \in V} Cl_\Theta(v) = \bigcup_{S \subset_{fin} V} Cl_\Theta(S)$ (union over all finite subsets)*

Proof. (a) : it is immediate that an intersection of Θ-definable sets is Θ-definable.

(b) : This follows from the Decomposition Lemma. If V is irreducible, then $V = \mathrm{Cl}_\Theta(v)$ for v a Θ-generic point of V. If not, by Decomposition Lemma, V decomposes as a union of translates of irreducible sets $V_1, ..., V_n$. Thus the union $\bigcup_{v \in V} \mathrm{Cl}_\Theta(v)$ is the union of corresponding translates of closures $\mathrm{Cl}_\Theta(V_1)$,...,$\mathrm{Cl}_\Theta(V_n)$ of irreducible components $V_1, ..., V_n$. By Lemma 3.1.4.1, $\mathrm{Cl}_\Theta(V_i)$ being closed implies any union of translates of $\mathrm{Cl}_\Theta(V_i)$ is closed; and thus $\bigcup_{v \in V} \mathrm{Cl}_\Theta(v)$ is a finite union of closed sets, therefore closed itself. But obviously $V \subset \bigcup_{v \in V} \mathrm{Cl}_\Theta(v)$ and therefore $\mathrm{Cl}_\Theta(V) \subset \bigcup_{v \in V} \mathrm{Cl}_\Theta(v)$. On the other hand, for any $v \in V$ $\mathrm{Cl}_\Theta(v) \subset \mathrm{Cl}_\Theta(V)$, and thus $\mathrm{Cl}_\Theta(V) \supset \bigcup_{v \in V} \mathrm{Cl}_\Theta(v)$. This implies the lemma. □

Lemma 3.1.5.4. *If a set $W' \subset \boldsymbol{U}$ is defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$ then $W' \subset \boldsymbol{U}$ is L_A-defined with parameters from $p^{-1}(\mathbb{A}(\overline{\mathbb{Q}}))$.*

Proof. An irreducible component of the preimage of algebraic variety $W(\mathbb{C}) \subset \mathbb{A}(\mathbb{C})$ defined over $\overline{\mathbb{Q}}$ is an irreducible component of the preimage of variety

$$\bigcup_{\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})} \sigma W(\mathbb{C}).$$

In order for the union be finite, we use that W is defined over $\overline{\mathbb{Q}}$, i.e. over a finite degree subfield of $\overline{\mathbb{Q}}$.

□

Recall we assume Θ to be algebraically closed.

Lemma 3.1.5.5. *If W' is irreducible $\mathcal{T}$-closed, then $w' \in W'$ is Θ-generic iff $w = p_H(w') \in W = p_H(W')$ is Θ-generic in W.*

Proof. The point $w' \in W'$ is not Θ-generic iff there exists a Θ-defined irreducible set $w' \in V' \subsetneq W'$; by Corollary 3.1.4.2 the latter is equivalent to $p_H(V') \neq p_H(W')$. □

We would rather avoid using this Corollary due to its non-geometric character, but unfortunately we do use it.

Lemma 3.1.5.6. *A connected component of a Θ-generic fibre of a closed irreducible set defined over Θ contains a Θ-generic point.*

Proof. The property holds for algebraic varieties. Let $W'^c_{g'}$ be a connected component of a fibre of W' over a Θ-generic point g' of $\mathrm{Clpr}\, W'$. Then $p(W'^c_{g'})$ is a connected component of fibre W_g, where $W = p_H(W'), g = p(g')$ is such that W' is a connected component of $p_H^{-1}(W)$; this may be seen with the help of path lifting property, for example. Genericity of $g' \in \mathrm{Clpr}\, W'$ implies that the point $g \in \mathrm{Clpr}\, W$ is Θ-generic, and, as a connected component of fibre W_g of an algebraic variety, $p(W'^c_{g'})$ contains a Θ-generic point, and then its preimage in $W'^c_{g'}$ is also Θ-generic. □

3.2 Main property of group action

The following property is specific to algebraic geometry; it fails in a general topological situation. It is this property which allows us to connect the Zariski topology on $A(\mathbb{C})$ and the $\mathcal{T}$-topology on the universal covering space U. This is the property used to imply Chevalley Lemma for $\mathcal{T}$-closed sets.

Roughly, the property of group action is a corollary of what is know as Stein factorisation. Stein factorisation says that every algebraic morphism can be decomposed as a finite morphism and a morphism with connected fibres. Moreover, on a Zariski open subset, over $\mathbb{C}$, any morphism can be decomposed as a finite etale morphism, i.e. topological covering in complex topology, and a fibre bundle in complex topology, cf. Lemma 4.3.4.1. Thus, it has a rather transparent structure.

Holomorphic convexity is used deal with non-normal case.

3.2.1 Stabilisers of irreducible closed subsets

Recall notation $\pi(V') = \{\gamma \in \pi : \gamma V' \subset V'\}$.

Proposition 3.2.1.1 (Action of $\pi(U)$ on U). *Let W' and $V' = \mathrm{Clpr} W'$ be $\mathcal{T}$-irreducible closed sets. Then there is a finite index subgroup $H \lhd_{\mathrm{fin}} \pi$ such that*

1. $\pi(W') \cap H = \{\gamma \in H : \gamma W' \subset W'\} = \{\gamma \in H : \gamma W' \cap W' \neq \emptyset\} = \{\gamma \in H : \gamma W' = W'\}$
2. $pr\, \pi(W') \cap H = \pi(V') \cap H$.

3. *for an open subset $V^{0\prime} \subset V'$ it holds that for arbitrary connected component $W'^c_{g'}$ of fibre $W'_{g'}$ over $g' \in V^0$ it holds*

$$\pi(W') \approx \pi(V') \rtimes \pi(W'^c_{g'}),$$

i.e. there exists a finite index subgroup $H \lhd_{\text{fin}} \pi$ such that

$$\pi(W') \cap [H \times H] = \pi(V') \rtimes \pi(W'^c_{g'}) \cap [H \times H].$$

Moreover, if W' and V' are defined over an algebraically closed field Θ, so is $V - V^0$. In particular, the above equality holds for g a Θ-generic point of $V' = \mathit{Clpr}\, W'$.

In Proposition, $\rtimes$ denotes skew product.

Proof of Proposition. To prove (1), apply Decomposition Lemma to $\mathcal{T}$-closed set $\pi W'$; by Decomposition Lemma, take $H \lhd_{\text{fin}} \pi$ to be such that the set $\pi W'$ decomposes as a union of a finite number of H-invariant sets whose connected components are irreducible, and therefore they are translates of W'. This implies (1). The item (2) is implied by (3).

Let us now prove item (3). Let H be such that W' and V' are connected components of $p_H^{-1} W(\mathbb{C})$, $p_H^{-1}(V(\mathbb{C}))$, respectively. where $W(\mathbb{C}) = p_H(W'), V(\mathbb{C}) = p_H(V')$. Consider projection morphism $\mathrm{pr} : A \times A \to A$; it induces a morphism $\mathrm{pr} : W(\mathbb{C}) \to V(\mathbb{C})$. By Lemma 4.3.4.1 it gives rise to a sequence exact up to finite index:

$$\iota_*\pi_1(W^c_g(\mathbb{C}), w) \to \iota_*\pi_1(W(\mathbb{C}), w) \to \iota_*\pi_1(V(\mathbb{C}), \mathrm{pr}\, w) \to 0$$

where W^c_g is a connected component of a fibre of W over $g \in V$, and g varies among an open subset V^0 of V, and w varies among an open subset of W.

Recall by §2.2.1 we may identify $\pi(W')$ and $\iota_*\pi_1(W(\mathbb{C}), w)$, and $\iota_*\pi_1(W^c_g(\mathbb{C}), w)$ and $\pi(W'^{c'}_{g'})$, etc.

The existence of such a sequence by definition mean that

$$\iota_*\pi_1(W(\mathbb{C}), w) \approx \iota_*\pi_1(W^c_g(\mathbb{C}), w) \rtimes \iota_*\pi_1(V(\mathbb{C}), \mathrm{pr}\, w)$$

That implies (recall we have assumed that π is strongly residually finite)

$$\pi(W') \cap [H' \times H'] = \pi(W'^c_{g'}) \rtimes \pi(V') \cap [H' \times H'],$$

for some finite index subgroup $H' \lhd_{\text{fin}} \pi$. □

3.2.2 Corollaries: Chevalley Lemma and Finiteness of Generic Fibres

Corollary 3.2.2.1 (Chevalley Lemma). *For $\mathcal{T}$-topology, it holds:*

1. *Projections of closed sets are closed.*
2. *Projection of a set open in its closure is a set open in its closure.*

Proof. The projection of an H-invariant closed set is closed; indeed, say $H = \pi$, then note $\mathrm{pr}\, p(\pi W') = p \mathrm{pr}\, (W')$, and thus $\mathrm{pr}\, \pi W' = p^{-1}p(\mathrm{pr}\, W') = p^{-1}p(V)$, where $V = \mathrm{pr}\, p(W')$. For $\mathbb{A}(\mathbb{C})$ is projective, V is closed algebraic subset of $\mathbb{A}(\mathbb{C})$, and thus $p^{-1}p(V)$ is a π-invariant closed subset of $\boldsymbol{U}$. By definition of $\mathcal{T}$, it is $\mathcal{T}$-closed.

Let now W' be a $\mathcal{T}$-irreducible closed set which is a connected component of HW'. As in lemma of preceeding §, let V' be the closure of $\mathrm{pr}\, W'$.

The set pr HW' is closed, and thus $V' \subset \mathrm{pr}\, HW'$. The set V' is closed, and thus it is contained in a connected component V_1' of pr HW'.

Take $v' \in V' \subset V_1'$, and find $w' \in W'$ such that $\mathrm{pr}\,(hw') = v'$, this is possible due to $V' \subset \mathrm{pr}\, HW'$. Also $\mathrm{pr}\, W' \subset V'$, and thus $\mathrm{pr}\,(w') \in V', \mathrm{pr}\,(h)\mathrm{pr}\,(w') = v' \in V'$. Then $v' \in \mathrm{pr}\,(h)V_1' \cap V_1'$. We may further take H is sufficiently small so that

$$\pi(V_1') \cap H = \{\tau \in \pi : \tau(V_1') \cap V_1' \neq \emptyset\} = \{\tau \in \pi : \tau V_1' = V_1'\}.$$

Then $\mathrm{pr}\,(h) \in \pi(V_1')$, and Proposition 3.2.1.1(2) implies there exists an element $h_1 \in \pi(W') \cap H \times H$ such that $\mathrm{pr}\,(h) = \mathrm{pr}\, h_1$. Then, $h_1 W' = W'$, and thus $\mathrm{pr}\,(h_1 w') = \mathrm{pr}\,(h)\mathrm{pr}\, w' = v'$, as required.

Again this argument can be given topologically. We prove the second claim topologically.

First, we may assume that W' is a connected component of $p_H^{-1} p_H(W') = HW'$, and By Chevalley Lemma for algebraic varieties there is a set $V^0 \subset \mathrm{pr}\, p_H(W') \subset V$ such that $V^0 \subsetneq V$ is open in V. Let V' be the connected component of $p_H^{-1}(V)$ containing $\mathrm{pr}\, W'$. Take $V^{0\prime} = V' \cap p_H^{-1}(V^0)$; then $V^{0\prime} \subset V'$ is open in V' as an intersection with an open set.

Take $v' \in V^{0\prime}$, and take $w' \in W'$, $\mathrm{pr}\, p_H(w') = p_H(v') \in V^0 \subset \mathrm{pr}\, W$; such a point w' in W' exists by what we call Covering Property of connected components. Now, $\mathrm{pr}\, w' \in V'$, and thus $\gamma_0 \in \pi(V')$ where γ_0 is defined by $v' = \gamma_0 \mathrm{pr}\, w'$. Condition $\mathrm{pr}\, p_H(w') = p_H(v') \in A^H(K)$ implies $\gamma_0 \in H$. Thus the inclusion $\mathrm{pr}\, \pi(W') \cap H = \pi(V') \cap H$ implies there exists $\gamma_1 \in \pi(W')$, $\mathrm{pr}\, \gamma_1 = \gamma_0$, and thus $v' = \gamma_0 \mathrm{pr}\, w' = \mathrm{pr}\,(\gamma_1 w')$, and the Chevalley lemma is proven. □

Let $\pi_0(W')$ denote the set of irreducible components of W'.

Corollary 3.2.2.2 (Generic Fibres). *A generic fibre of a fibre of an irreducible closed set has finitely many connected components. A connected component of a «generic» fibre of an irreducible set can be represented as the intersection of the fibre with an H-invariant closed set, for some $H \triangleleft_{\mathrm{fin}} \pi$ a finite index normal subgroup.*

In notation of Proposition above, $W'_{g'}$ has finitely many connected components and for any connected component $W'^c_{g'}$ of $W'_{g'}$, it holds

$$W' \cap g' \times HW'^c_{g'} = g' \times W'^c_{g'},$$

$$W' \cap g' \times HW'_{g'} = g' \times W'_{g'}.$$

The Proposition implies that the set of $g' \in V$ such that the fibre over g' has an infinite number of connected components, is contained in a closed proper subset of V:

there exits a closed subset $V_0' \subsetneq V'$ such that

$\{\, g' \in V :$ *the fibre $W'_{g'}$ has an infinite number of connected components* $\} \subset V_0' \subsetneq V'$.

Proof. Let H be as in Proposition 3.2.1.1. The fibre $W'_{g'}$ is the intersection of W_g with a coordinate plane, and therefore is $\mathcal{T}$-closed. By Decomposition Lemma, the fibre $W'_{g'}$ is a union of H-translates of a finite number of irreducible sets $Z_1', .., Z_k'$. We claim that in fact it is the union of $\pi(W'^c_{g'})$-translates of $Z_1', ..., Z_k'$. This implies that the number of connected components of $W'_{g'}$ does not exceed k: indeed, the action of $\pi(W'^c_{g'})$ leaves each connected component of $W'_{g'}$ invariant, and thus two different components may not contain H-translates of the same Z_i'.

To prove the claim, take $h \in H$ such that $Z'_i, hZ'_i \subset W'_{g'}$. Then $(\mathrm{id}, h) \in H \times H$, and $(\mathrm{id}, h^{-1})W' \cap W' \supset g' \times Z'_i \neq \emptyset$, and by Proposition 3.2.1.1(1) this implies $(\mathrm{id}, h^{-1})W' = W'$ and $(\mathrm{id}, h^{-1}) \in \pi(W')$. However, by Proposition 3.2.1.1(2) $\pi(W'^c_{g'}) = \mathrm{pr}\,\pi(W') \cap [H \times H]$, and thus and $hW'^c_{g'} = W'^c_{g'}, h \in \pi(W'^c_{g'})$ for any connected component $W'^c_{g'}$ of fibre $W'_{g'}$.

To prove $W' \cap g' \times HW'^c_{g'} = g' \times W'^c_{g'}$, take $h \in H$ such that $g' \times hW'_{g'} \cap W \neq \emptyset$. Then $(\mathrm{id}, h) \in H \times H$ and

$$(\mathrm{id}, h)W' \cap W' \supset g' \times hW'_{g'} \cap W_g \neq \emptyset, \tag{3.2.1}$$

by Proposition 3.2.1.1(1) this implies $(\mathrm{id}, h)W' = W'$ i.e. $(\mathrm{id}, h) \in \pi(W')$. Now Proposition 3.2.1.1(2), $\pi(W'_{g'}) \cap H = \mathrm{pr}\,\pi(W') \cap H \times H$ gives $hW'_{g'} = W'_{g'}$, i.e $h \in \pi(W'_{g'})$, as required.

In particular, $W' \cap HW'_{g'} = W'_{g'}$ and $W' \cap W'^c_{g'} = g' \times W'^c_{g'}$ □

3.3 Language for topology $\mathcal{T}$: $\mathbb{Q}$-definable sets

So far we have defined a topology on U (and its Cartesian powers U^n's) whose closed sets are rather easy to understand. Now, to put the considerations above in a framework of model-theory, we want to define a *language* able to define closed sets in topology $\mathcal{T}$. From an algebraic point of view, that corresponds to defining the automorphism group of U with respect to topology $\mathcal{T}$.

Let us draw an analogue of the action of Galois group on an algebraic variety $\overline{\mathbb{Q}}$ defined over $\mathbb{Q}$ with Zariski topology. The Galois group may not be defined as the group of bijections continuous in Zariski topology: for example, all polynomial maps are continuous automorphisms of Zariski topology in this sense; linear and affine maps $x \to ax + b$ are such continuous bijections.

Thus we distinguish certain $\mathbb{Q}$-definable subsets among Zariski closed subsets of $\overline{\mathbb{Q}}^3$, and then define Galois group as the group of transformation (of $\overline{\mathbb{Q}}$) *preserving the distinguished $\mathbb{Q}$-defined subsets (of $\overline{\mathbb{Q}}^3$)*; in this case the graphs of addition and multiplication. It is *then* derived, rather trivially, that this implies that Galois group acts by transformation continuous in Zariski topology.

Recall the way this is derived: the $\mathbb{Q}$-definable subsets are given *names*, in this case addition and multiplication, and then each closed set (subvariety) is given a name by the equations defining the set of its points; in fact, in algebraic geometry the word variety means rather the *name*, the set of equations, rather that the set of points the equations define.

In order to define a useful automorphism group of topology $\mathcal{T}$, we follow the same pattern.

Model theory provides us with means to give precise meaning to the argument above, and to define mathematically what is it exactly that we want. In these terms, the distinguished subsets form a *language*, and the Galois group is the group of automorphisms of *the structure in that language*. Model theory studies that group via the study of the structure.

3.3.1 Definition of a language L_A for universal covers in $\mathcal{T}$-topology

In this §, it becomes essential that A is defined over an algebraic field $\mathbf{k} \subset \overline{\mathbb{Q}} \subset \mathbb{C}$ embedded in $\mathbb{C}$.

We consider $p : U \to A(\mathbb{C})$ as a structure in the following language.

Definition 3.3.1.1. We consider the structure $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ as a one-sorted structure $\boldsymbol{U}$, in the language L_A which has the following symbols:

the symbols $\sim_{Z,A}$ for Z a closed subvariety of $\mathbb{A}(\mathbb{C})^n$ defined over number field k, and,

the symbols $\sim_H$, for each normal subgroup $H \lhd_{\text{fin}} \pi(\boldsymbol{U})^n$ of finite index

The symbols are interpreted as follows:

$x' \sim_{Z,A} y' \iff$ points $x' \in \boldsymbol{U}^n$ and $y' \in \boldsymbol{U}^n$ lie in the same (analytic) irreducible component of the π-invariant closed analytic set $p^{-1}(Z(\mathbb{C})) \subset \boldsymbol{U}^n$.

$x' \sim_H y' \iff \exists \tau \in \pi(\boldsymbol{U})^n : \tau x' = y'$ and $\tau \in H$.

Note that we do not assume Z to be connected.

Note that by Decomposition Lemma it is enough to introduce predicates for *connected* components of closed analytic sets which are invariant under the action of *finite index* subgroup of the fundamental group. To such sets, Chow Lemma still applies.

Thus we may use an alternative definition by considering predicates for each $Z \subset \mathbb{A}(\mathbb{C})$ defined over $\mathbb{Q}$.

$x' \sim^c_{Z,A} y')$ iff x' and y' lie in the same *connected* component of the preimage $p_H^{-1}(Z_i(\mathbb{C}))$, $Z_i \subset A^H(\mathbb{C})^n$ an irreducible component of algebraic variety $p_H^{-1}(Z(\mathbb{C})) \subset A^H(\mathbb{C})^n$.

Note that the language L_A is countable. This is an essential property, from model-theory point of view.

Let us use this opportunity to remind that we use symbols $\sim_Z$ rather abusively to mean «lie in the same irreducible component of» either πZ, $p_H^{-1}(Z)$, etc.

3.3.2 L_A-definability of $\pi(\boldsymbol{U})$-action etc

In next lemma, a closed set means $\mathcal{T}$-closed set.

Lemma 3.3.2.1. *For any normal finite index subgroup $H \lhd_{\text{fin}} \pi$ it holds*

1. *the relation*

$$Aff_H(x, y, z, t) = \exists \gamma \in H : \gamma x = y \ \&\ \gamma s = t$$

is $L_A(\emptyset)$-definable

2. *An H-invariant closed set is L_A-definable with parameters.*
3. *A connected component of a generic fibre of an L_A-definable irreducible closed set is uniformly L_A-definable; the definition is valid over an over an open subset of the projection, definable over the same set of parameters.*
4. *Any $\mathcal{T}$-closed irreducible set is a connected component of a fibre of an L_A-definable set.*
5. *An irreducible closed set is L_A-definable.*

Proof. To prove (1), note that

$$p^{-1}(\Delta(\mathbb{C})) = \bigcup_{\gamma \in \pi} \{(x, \gamma x) : x \in U\}$$

where $\Delta = \{(x, x) : x \in A\}$ is an algebraic closed subvariety defined over $\mathbb{Q}$. The connected components $\{(x, \gamma x) : x \in U\}, \gamma \in \pi$ are the equivalence classes of $\sim_\Delta$, and thus are definable with parameters.

Evidently $\mathrm{Aff}_\pi(x, y, s, t)$ iff $(x, y) \sim_\Delta (s, t)$ lie in the same connected component of $p^{-1}(\Delta(\mathbb{C})) \subset U \times U$.

$\overline{\mathbb{Q}}$-case: An irreducible closed subvariety $Z/\overline{\mathbb{Q}} \subset \mathbb{A}$ defined over $\overline{\mathbb{Q}}$ is an irreducible component of subvariety

$$Z_{\mathbb{Q}} = \bigcup_{\sigma : k_Z \hookrightarrow \mathbb{C}} \sigma(Z)$$

of $\mathbb{A}$, where k_Z is the field of definition of Z of finite degree. The formula implies Z is L_A-definable with parameters with the help of symbol $\sim_{Z_{\mathbb{Q}},A}$; the parameters may be taken to lie in $\mathbb{A}(\overline{\mathbb{Q}})$ but not necessarily in $\mathbb{A}(k_Z)$. A slightly more complicated argument could give a construction defining Z as a connected component.

For an analytic $\mathcal{T}$-closed irreducible set $Z' \subset \boldsymbol{U}$, it holds that Z' is an irreducible component of $\pi Z'$, i.e. it is an irreducible component of $Z = p(Z')$. Thus the above argument gives that every $\mathcal{T}$-irreducible subset of $\boldsymbol{U}$ defined over $\overline{\mathbb{Q}}$ is L_A-definable with parameters.

$\overline{\mathbb{Q}}(t_1, ..., t_n)$-case: Thus we have to deal with the case when $p(Z)$ is not $\overline{\mathbb{Q}}$-definable. Our strategy is to show that any such set is a connected component of a $\overline{\mathbb{Q}}$-generic fibre of a $\overline{\mathbb{Q}}$-definable set, and then show that such connected components are uniformly definable. Uniformity will be important for us later in axiomatising $\boldsymbol{U}$.

Let us see first that each $\mathcal{T}$-closed irreducible set is a connected component of a fibre of a $\mathcal{T}$-closed irreducible? set defined over $\overline{\mathbb{Q}}$.

Take a $\mathcal{T}$-irreducible set Z' and take $H \lhd_{\text{fin}} \pi$ such that Z' is a connected component of $HZ' = p_H^{-1}(Z)$, for an irreducible algebraic closed set $Z = p_H(Z')$. By the theory of algebraically closed field, we know that Z can be defined as a Boolean combination, necessarily a positive one, of $\overline{\mathbb{Q}}$-definable closed subsets and their fibres; by passing to a smaller subset if necessary, we see that the irreducibility of Z implies that algebraic subset $Z \subset \mathbb{A}(\mathbb{C})$ is a connected component of a $\overline{\mathbb{Q}}$-generic fibre of a $\overline{\mathbb{Q}}$-definable closed subset $W \subset \mathbb{A}(\mathbb{C})^n$. Then HZ' is the corresponding fibre of $p_H^{-1}(W)$. Z' is a union of corresponding fibres of the irreducible components of $p_H^{-1}(W)$, and irreducibly of Z' implies that union is necessarily trivial. Thus, we have that Z' is a connected component of a fibre of an irreducible $\mathcal{T}$-closed set. We may also ensure that Z' is a connected component of a $\overline{\mathbb{Q}}$-generic fibre of W' by intersecting W' with the preimage of an irreducible $\overline{\mathbb{Q}}$-definable set containing $\operatorname{pr} Z'$, and repeating the process if necessary.

Let us now prove that the connected components of the $\overline{\mathbb{Q}}$-generic fibres of an irreducible $\overline{\mathbb{Q}}$-definable set are $\overline{\mathbb{Q}}$-definable.

Let $W' \subset \mathbb{A}(\mathbb{C})^2$, and let $V' = \operatorname{Clpr} W'$ be as in Proposition 3.2.1.1 and Corollary 3.2.2.2. The morphism $\operatorname{pr} : W \to V$ admits a Stein factorisation (Fact 4.3.3.4) $\operatorname{pr} = f_0 \circ f_1$ as a composition of a finite morphism $f_0 : W \to V_1$ and a morphism with connected fibres $f_1 : V_1 \to V$. In particular, two points $x_1, x_2 \in W_g$ lie in the same connected component of fibre W_g iff $f_0(x_1) = f_0(x_2)$.

Now set

$$x' \sim^c_{W_g} y' \iff x' \sim_W y' \;\&\; \& \operatorname{pr} x' = \operatorname{pr} y' \;\&\; f_0(p_H(x')) = f_0(p_H(y')) \qquad (3.3.1)$$

In notation of Corollary 3.2.2.2, we have

Claim 3.3.2.2. If $g' \in V'^0$, then the formula $x' \sim^c_{W_g} y'$ holds iff $x' \sim_H y'$ and x' and y' lie in the same connected component of fibre $W'_{g'}$ of W. If W, V are Θ-definable, so is V'^0.

Proof. This is a reformulation of the formula $W' \cap g' \times HW'^c_{g'} = g' \times W'^c_{g'}$. Indeed, $\operatorname{pr} x' = \operatorname{pr} y' \;\&\; f_0(p_H(x')) = f_0(p_H(y'))$ holds iff $x', y' \in g' \times HW'^c_{g'}$ for $g' = \operatorname{pr} x' = \operatorname{pr} y'$ and some $W'^c_{g'}$ connected component of fibre of W' above g'. The relation of lying in the same connected component of a fibre being translation invariant, we may as well assume $x', y' \in W'$ if $x' \sim_W y' \in W'$ lie in the same connected component of W'. Then the formula means that x', y' lie in the same connected component of fibre $g' \times W'_{g'}$.

The claim that the formula holds for $g' \in V'^0$ in an open subset is Θ-definable is a part of conclusion of Corollary 3.2.2.2. □

The claim above implies (3); (4) and (3) imply (5) and (2). □

Corollary 3.3.2.3. *Let $Aut_{L_A}(U)$ be the group of bijections $\phi : U \to U$ preserving relations $\sim_{Z,A} \in L_A$; then $Aut_{L_A}(U)$ acts by transformations continuous in $\mathcal{T}$-topology.*

Proof. Immediate by previous results. □

The results above justify thinking of $\mathrm{Aut}_{L_A}(U)$ as a *Galois group of* U.

Remark 3.3.2.4. There is a natural inclusion $\mathrm{Aut}_{L_A}(U) \hookrightarrow \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ induced by the action of $\mathrm{Aut}_{L_A}(U)$ on the interpretable set $A(\mathbb{C})$. What can one say about $\mathrm{Aut}_{L_A}(U)$ as a subgroup of the Galois group, or rather a conjugacy class of such subgroups ? Is there any relations of $\mathrm{Aut}_{L_A}(\boldsymbol{U})$ with the Grothendieck's fundamental group $\hat{\pi}_1(\mathbb{A}_{\mathbb{Q}})$?

3.4 Model homogeneity; an analogue of n-transitivity of $\mathrm{Aut}_{L_A}(U)$-action.

Now we want to study the action of $\mathrm{Aut}_{L_A}(U)$ on U, and analyse orbits of its action on U and $U^n, n > 1$. In model theory one would hope that aforementioned orbits can be analysed in terms of language; in presence of a nice topology with a Chevalley property we may hope to analyse orbits in terms of closed sets.

The situation when this is possible is called homogeneity; in this § we state and prove *model homogeneity* of U. Model homogeneity says, roughly, that two tuples of points lie in the same orbit (of action fixing an algebraically closed subfield) iff there are no obvious obstructions, i.e. iff they lie in the same closed sets (defined over an algebraically closed subfield which we assume fixed).

Definition 3.4.0.5. We say that W is a Θ-*constructible set* iff

1. the closure $\mathrm{Cl}W$ is defined over Θ
2. W contains all Θ-generic points of the irreducible components of $\mathrm{Cl}W$.

An *irreducible* constructible set is a set whose closure is irreducible.

Lemma 3.4.0.6. *A projection of a Θ-constructible set is Θ-constructible.*

Proof. Let $W \subset U \times U$ be an irreducible set defined over Θ, and let W_0 be the set of all Θ-generic points of W; generally speaking, W_0 is not definable. We need to prove that $\mathrm{pr}\, W$ is also Θ-constructible. Let $g \in \mathrm{pr}\, W$ be a Θ-generic point of the closure of $\mathrm{pr}\, W$; we know $g \in \mathrm{pr}\, W$ by Chevalley Lemma. By Lemma 3.2.2.2 we know that the (non-empty) fibre W_g contains a Θ-generic point of W, and thus $g \in \mathrm{pr}\, W$, as required. □

The set of realisations of a complete quantifier-free syntactic type p/Θ with parameter set Θ is Θ-constructible; and conversely, every Θ-constructible set can be represented in this form.

Thus, the above lemma is equivalent to ω-homogeneity for such types.

Definition 3.4.0.7. We say that U is *homogeneous for closed sets over* Θ, *or syntactic quantifier-free complete types over* Θ, or *model homogeneous* iff either of the following equivalent conditions holds

1. the projection of a Θ-constructible set is Θ-constructible;
2. for any tuples $a, b \in U^n$ and $c \in U^m$ if $\mathrm{qftp}(a/\Theta) = \mathrm{qftp}(b/\Theta)$ then there exists $d \in U^m$ such that $\mathrm{qftp}(a, c/\Theta) = \mathrm{qftp}(b, d/\Theta)$

To see that the conditions are equivalent, note that the set of realisations of quantifier-free type $\mathrm{qftp}(a, c/\Theta)$ is Θ-constructible; its projection contains a and also is Θ-constructible; a is its Θ-generic point; then $\mathrm{tp}(a/\Theta) = \mathrm{tp}(b/\Theta)$ implies b is also Θ-generic, i.e. belongs to it.

The above proves the following result.

Property 3.4.0.8. The standard model $p : U \to A(\mathbb{C})$ in language L_A is model homogenous, i.e. ω-homogeneous for closed sets over arbitrary algebraically closed subfield $\Theta \subset \mathbb{C}$.

Proof. Follows directly from Def. 3.4.0.7 and Lemma 3.4.0.6. □

Corollary 3.4.0.9. *The set of realisations of a quantifier-free type qftp*(x/Θ) *over* $p^{-1}(A(\Theta))$ *consists of* Θ*-generic points of some* $\mathcal{T}$*-irreducible closed subset of* $\boldsymbol{U}$.

Proof. Follows from the previous statements. □

3.5 An $L_{\omega_1\omega}$-axiomatisation $\mathfrak{X}(\mathbb{A}(\mathbb{C}))$ and stability of the corresponding $L_{\omega_1\omega}$-class.

In this § we introduce an axiomatisation $\mathfrak{X}(\mathbb{A}(\mathbb{C}))$ for $L_{\omega_1\omega}(L_A)$-class which contains the standard model $p : U \to A(\mathbb{C})$, and is stable over models and all models in it are model homogeneous. We then show that the class of models satisfies $2_{\aleph_0 \to \aleph_1}$ of Theorem 3.5.4.7.

3.5.1 Algebraic L_A-structures

We know that $\boldsymbol{U}/G = A^G(\mathbb{C})$ carries the structure of an algebraic variety over field $\mathbb{C}$. The covering $A^G(\mathbb{C}) \to \mathbb{A}(\mathbb{C})$ carries a structure in a reduct $L_A(G)$ of language L_A. In fact, similar interpretation works for an arbitrary algebraically closed field K instead of $K = \mathbb{C}$.

For every finite index subgroup $G \lhd_{\mathrm{fin}} \pi$, there is a well-defined covering $A^G \to \mathbb{A}$ of finite degree. The space $\mathbb{A}(\mathbb{C})$ is projective, and thus $A^G(\mathbb{C})$ is also a complex projective manifold. By Fact 3.1.2.3, A^G has the structure of an algebraic variety.

Recall that we use the following fact as the defining property of etale covering: the morphism $B(K) \to A(K)$ of varieties over an algebraically closed field K of char 0 is *etale* iff there exists an embedding $i : K \to \mathbb{C}$ of the field of definition of A and B into $\mathbb{C}$ such that the corresponding morphism $i(B)(\mathbb{C}) \to i(A)(\mathbb{C})$ is a covering of topological spaces.

Definition 3.5.1.1 (Finitary reducts of L_A). Let $p_G : A^G(K) \to \mathbb{A}(K)$ be a finite etale morphism. Let $L_A(G) \subset L_A$ be the language consisting of all predicates of L_A of form $\sim_Z$ and those $\sim_H$, $G \subset H$. Then, $A^G(K) \to \mathbb{A}(K)$ carries a $L_A(G)$-structure as follows:

1. $x' \sim_Z y' \iff$ points $x', y' \in A^G(K)^n$ lie in the same irreducible component of algebraic closed subset $p_G^{-1}(Z(K))$ of $A^G(K)^n$.

2. $x' \sim_H y' \iff$ there exist an algebraic morphism $\tau : A^G \to A^G$ and an etale covering morphism $q : A^G \to A^H$ such that $\tau(x') = y'$ and $\tau \circ q = q$:

$$\begin{array}{ccc} A^G & \xrightarrow{\ \tau\ } & A^G \\ \Big\downarrow q & \text{étale cover} & \Big\downarrow q \\ A^H & \xrightarrow{\ \text{id}\ } & A^H \end{array} \tag{3.5.1}$$

For $G = e$ the trivial group and $K = \mathbb{C}$, the construction above degenerates into the interpretation of $\boldsymbol{U} \to \mathbb{A}$ if it were well-defined.

For $G = \pi$, $A^G = \mathbb{A}$, and thus $L_A(\pi)$ is just a form of the language for algebraic variety $\mathbb{A}$; here the point is that we have predicates for $\mathbb{Q}$-definable closed subsets only.

In general, the above is just a variation of an ACF structure on $\mathbb{A}$. In particular, all closed subsets of A^{Gn} are $L_A(G)$-definable.

3.5.2 Axiomatisation $\mathfrak{X}$

Basic Axioms

These axiom describe quotations $U/\sim_H$ for $H \lhd_{\text{fin}} \pi$, and some properties of $U \to \mathbb{U}/\sim_H$.

Axiom 3.5.2.1. *All first-order statements valid in $\boldsymbol{U}$ and expressible in terms of L_A-interpretable relations*

$$x' \sim_{Z,A^G} y' := \exists x'' \exists y'' (x'' \sim_Z y'' \,\&\, x'' \sim_G x' \,\&\, y'' \sim_G y'), G \lhd_{\text{fin}} \pi$$

and $\sim_G, G \lhd_{\text{fin}} \pi$.

Note that we do not allow $\sim_{Z,A}$ by itself, the Axioms essentially describe $\boldsymbol{U}/G$, which is an algebraic variety.

Lifting Property Axiom, or Covering Property Axiom

Let $\Re = \Re(A, \pi) = \Re(A, \pi_1(A(\mathbb{C}))$ be a class of substructures $\boldsymbol{U}$ of $\mathbb{U}(K)$, for some K, satisfying the following properties:

Axiom 3.5.2.2 (Lifting Property for W). *For all L_A-predicates $\sim_W$ and all $G \lhd_{\text{fin}} \pi$ sufficiently small, we have an axiom*

$$x' \sim_{W,A^G} y' \Longrightarrow \exists y'' (y'' \sim_G y' \,\&\, x' \sim_W y'') \tag{3.5.2}$$

We also have a stronger axiom for *fibres* of W; here we use that the relation «to lie in the same connected component of a fibre of a variety» is algebraic and therefore the corresponding G-invariant relation is L_A-definable.

Axiom 3.5.2.3 (Lifting Property for fibres). *For all $G \lhd_{\text{fin}} \pi$ sufficiently small, we have an axiom*

$$(x'_0, x'_1) \sim^c_{W_g, A^G} (y'_0, y'_1) \Longrightarrow \exists y''_1 [y'_0 \sim_G x'_0 \,\&\, y''_1 \sim_G y'_1 \,\&\, (x'_0, x'_1) \sim_W (x'_0, y''_1)] \tag{3.5.3}$$

in a slightly different notation

$$x' \sim^c_{W_g, A^G} y' \Longrightarrow \exists y'' (y'' \sim_G y' \,\&\, pr\, x' = pr\, y'' \,\&\, x' \sim_W y'') \tag{3.5.4}$$

The relation $x' \sim^c_{W_g, A^G} y'$ is defined by the formula (3.3.1) (cf. Claim 3.3.2.2).

Axiom 3.5.2.4 (Fundamental group is residually finite).

$$\forall x' \forall y' (x' = y' \iff \bigwedge_{H \triangleleft_{\text{fin}} \pi} x' \sim_H y') \tag{3.5.5}$$

Thus, it says that two elements $\sim_H$-close, have to be equal.

The next property is screeching of this one; namely, if an element b is $\sim_H$-equivalent to an element of a group generated by $a_1, .., a_n$, then it is actually in the group. In terms of paths, this has the following interpretation: take loops $\gamma_1, ..., \gamma_n$ and a loop λ. If for every $H \triangleleft_{\text{fin}} \pi$ it holds that λ is $\sim_H$-equivalent to some concatenation of paths $\gamma_1, ..., \gamma_n$, then it is actually a concatenation of these paths.

Axiom 3.5.2.5 (Translations have finite length). *For all $N \in \mathbb{N}$ we have an $L_{\omega_1 \omega}$-axiom*

$$\forall b \forall a_1 ... \forall a_N$$

$$\bigwedge_{H \triangleleft_{\text{fin}} \pi} \bigvee_{n \in \mathbb{N}} \exists h_1 ... h_n \left(b \sim_H h_n \,\&\, h_1 = a_1 \,\&\, \bigwedge_{1 \le i \le n} \bigvee_{1 \le j < N} (h_i, h_{i+1}) \sim_\Delta (a_j, a_{j+1}) \right)$$

$$\Longrightarrow \bigvee_{n \in \mathbb{N}} \exists h_1 ... h_n \left(b = h_n \,\&\, h_1 = a_1 \,\&\, \bigwedge_{1 \le i \le n} \bigvee_{1 \le j < N} (h_i, h_{i+1}) \sim_\Delta (a_j, a_{j+1}) \right) \tag{3.5.6}$$

The next axiom is needed to apply the principles above. It reflects the fact that the fundamental groups of varieties are finitely generated, cf. Arapura [Ara95]; this can also be obtained from the fact that topologically a variety can be split into finitely many contractible pieces nicely glued together (CW-complex).

Axiom 3.5.2.6 (Groups $\pi(W_g)$ are finitely generated). *For all symbols $\sim_W$ and for each H sufficiently small we have an $L_{\omega_1 \omega}$-axiom:*

$$\bigvee_{N \in \mathbb{N}} \exists a_1 ... \exists a_N \forall b$$

$$\Bigg(\bigwedge_{1 \le i \le N} (b \sim_W a_i \,\&\, pr\, b = pr\, a_i) \,\&\, \bigwedge_{1 \le i \ne j \le N} (a_i \sim_W a_j \,\&\, a_i \sim_H a_j \,\&\, pr\, a_i = pr\, a_j) \Longrightarrow$$

$$\bigvee_{n \in \mathbb{N}} \exists h_1 ... h_n \left(b = h_n \,\&\, h_1 = a_1 \,\&\, \bigwedge_{1 \le i \le n} \bigvee_{1 \le j < N} (h_i, h_{i+1}) \sim_\Delta (a_j, a_{j+1}) \,\&\, pr\, h_i = pr\, h_{i+1} \right) \Bigg) \tag{3.5.7}$$

Standard model $\boldsymbol{U}$ is a model of $\mathfrak{X}$

The universal covering space $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ satisfies the axioms Axiom 3.5.2.1 by definition.

To prove $\boldsymbol{U}$ satisfies Axiom 3.5.2.2, note that for $G \triangleleft_{\text{fin}} \pi$ small enough, the relations $x' \sim_{W,G} y'$ means that $p_G(x')$ and $p_G(y')$ lie in the same irreducible component W_i of the preimage of $W \subset \mathbb{A}(\mathbb{C})^n$ in $A^G(\mathbb{C})^n$. Take a path γ connecting $\gamma(0) = p_G(x')$ and $\gamma(1) = p_G(y')$ lying in W_i; by the Lifting Property it lifts to a path $\gamma', \gamma'(0) = x'$ such that $p_G(\gamma'(t)) = \gamma(t)$, $0 \le t \le 1$. Then, $p_G(\gamma'(1)) = p_G(y')$, and thus $\gamma'(1) \sim_G y'$. On the other hand, $\gamma'(1)$ and x' lie in the same connected component of the preimage of the irreducible component W_i in $\boldsymbol{U}$. Now note that by Decomposition Lemma 3.1.4.1 for G small enough such a connected component has to be irreducible, and thus Axiom 3.5.2.2 holds.

The Axiom 3.5.2.3 has a similar geometric meaning as Axiom 3.5.2.2; the assumption is that $p_G(x')$ and $p_G(y')$ lie in the same connected component of a fibre W_g; it is enough to take γ to lie in fibre W_g to arrive to the conclusion of Axiom 3.5.2.2.

Let us see Axiom 3.5.2.4 follows from the condition 2 of the definition of a Shafarevich variety.

Axioms 3.5.2.5 is condition 2 from the definition of a Shafarevich variety.

The geometric meaning of $(h_i, h_{i+1}) \sim_\Delta (a_i, a_{i+1})$ is as follows. The pair of points a_i, a_{i+1} determines a path γ in $\mathbb{A}(\mathbb{C})$, $\gamma(0) = \gamma(1) = p(a_i) = p(a_{i+1})$. For points h_i, h_{i+1} such that $p(h_i) = p(h_{i+1})$, they can be joined by a lifting of γ iff $(h_i, h_{i+1}) \sim_\Delta (a_i, a_{i+1})$. ... Thus the assumption in the axiom says that if any two points of fibre above $p(b) = p(a_1)$ can be joined by a concatenation of liftings of finitely many paths γ_i's in $\mathbb{A}(\mathbb{C})$, *up to a translate by an element of* H, then they can in fact be just joined by such a sequence. In a way, this can be though of as disallowing paths of infinite length.

On the other hand, the condition $(h_i, h_{i+1}) \sim_\Delta (a_i, a_{i+1})$ can be interpreted as $h_{i+1} = \tau_i h_i$ where τ_i is the deck transformation taking a_i into a_{i+1}, $\tau_i a_i = a_{i+1}$. Then, the assumption says that if $b \in \pi(\boldsymbol{U})$ belongs to the group generated by τ_i's, up to $\sim_H$, then b does belong to the subgroup generated by τ_i's.

The last remaining Axiom 3.5.2.6 means that the fundamental groups $\pi(W_g)$ is finitely generated, which is a well-known fact, see for example Arapura [Ara95].

3.5.3 Analysis of models of $\mathfrak{X}$

Models $U/_{\sim_H}$ *as algebraic varieties*

Let $U \models \mathfrak{X}$ be a L_A-structure modelling axiomatisation $\mathfrak{X}(L_A, \mathbb{A})$, and let $\boldsymbol{U}$ be the standard model, i.e. the universal covering space of $\mathbb{A}(\mathbb{C})$ considered as an L_A-structure.

We know that $\boldsymbol{U}/\sim_H \cong A^H(\mathbb{C})$ for some algebraic varieties $A^H(\mathbb{C})$ defined over $\mathbb{C}$. The relations $\sim_H, \sim_{Z,H}$ are essentially relations on $U/\sim_H$, and thus Axiom 0 says that the first-order theories of $U/\sim_H$ and standard model $\boldsymbol{U}/\sim_H$ in the languages $L_A(H) = \{\sim_H, \sim_{Z,H}\colon$ Z varies$\}$ coincide. We know by properties of analytic covering maps that an irreducible $\mathcal{T}$-closed subset of $\boldsymbol{U}$ covers an irreducible Zariski closed subset of $A^H(\mathbb{C})$, and thus the relation $\sim_{Z,H}$ on $\boldsymbol{U}/\sim_H$ interpreters as $x, y \in A^H(K)$ lie in the same (Zariski) irreducible component of the preimage of $Z(K)$ in $A^H(K)$.

Thus, by Lemma 3.1.5.4 any algebraic subvariety defined over $\overline{\mathbb{Q}}$ of $A^H(\mathbb{C})$ is $L_A(H)$-definable. Thus, full theory of an algebraically closed field is reconstructible in $L_A(H)$ on $U/\sim_H$; and thus, there is an algebraically closed field $K = \bar{K}$, $\mathrm{char} K = 0$ such that $U/\sim_H \cong A^H(K)$; here $A^H(K)$ corresponds to $A^H(\mathbb{C})$ with a different ground field.

Fix these isomorphisms $U/\sim_H \cong A^H(K)$, and let $p_H : U \to A^H(K)$ be the projection morphism. Then the above considerations say

> $x' \sim_{W,H} y' \iff p_H(x') \sim_{W,H} p_H(y') \iff x'$ and y' lie the same (Zariski) irreducible component of the preimage of $Z(K)$ in $A^H(K)$.
>
> $x \sim_G y' \iff$ there exist an algebraic morphism $\tau : A^G \to A^H$ and an etale covering morphism $q : A^H \to A^G$ such that $\tau(x') = y'$ and $\tau \circ q = q$:

$$\begin{array}{ccc} A^H & \xrightarrow{\tau} & A^H \\ \Big\downarrow q \text{ étale cover} & & \Big\downarrow q \\ A^G & \xrightarrow{\text{id}} & A^G \end{array} \qquad (3.5.8)$$

An important corollary of above considerations is that any set of form $p_H^{-1}(Z(K)), Z(K) \subset A^H(K)$ is L_A-definable.

Notation 3.5.3.1. Let us introduce new relations on U; eventually we will prove that they are first-order definable. We introduce the relations below for every closed subvariety of $\mathbb{A}(K)$, not necessarily defined over $\mathbb{Q}$ (those would be in L_A)

$$x' \sim_W y' \iff p_H(x') \sim_{W,H} p_H(y') \text{ for all } H \lhd_{\text{fin}} \pi.$$

An *irreducible component* of relation $\sim_W$ is a maximal set of points in U pairwise $\sim_W$-related. A subset of U is *closed* iff it is a union of irreducible components of relations $\sim_{W_1}, ..., \sim_{W_n}$, for some $W_1, ..., W_n$. An *irreducible closed set* is an irreducible component of a relation $\sim_W$ for some closed subvariety W. Let us call a subset of U *$\mathcal{T}$-closed* iff it is a union of irreducible components of a finite number of relations $\sim_{W_1}, ..., \sim_{W_n}$. This defines an analogue of topology $\mathcal{T}$ on $\boldsymbol{U}$.

Group action of fibres of $p : U \to \mathbb{A}(K)$ on U

For a point $x_0 \in U$, let $\pi(U, x_0') = \{y : y \sim_\pi x_0'\} = p^{-1}p(x_0')$ be the fibre of $p : U \to \mathbb{A}(K)$. For every point $z' \in U$ and every point $y' \sim_\pi x_0'$, there exists a point $z'' \in U$ such that $p_G(z', z') \sim_\Delta p_G(x_0', y')$; this follows from Axiom 3.5.2.1. Then, by Lifting Property for $\Delta \subset \mathbb{A}^2(K)$, there exists $z''' \in U$ such that $z''' \sim_G z''$ and $(z', z''') \sim_\Delta (x_0', y')$. Moreover, such a point z''' is unique. Indeed, by Axiom 0 the conditions $p_H(z''') \sim_H p_H(z'')$ and $(z', z''') \sim_{\Delta,H} (x_0', y')$ determine $p_H(z''')$ uniquely for every $H \lhd_{\text{fin}} G$. This implies that z''' is unique by Axiom 3.5.2.4.

The above construction defines an action σ of $\pi(U, x_0') = \{y : y \sim_\pi x_0'\} = p^{-1}p(x_0')$ on U: a point $y' \sim_\pi x_0'$ sends z' into z''', $\sigma_{y'} z' = z'''$. Axiom 0 and Minimality Property 1 imply that it is in fact a group action.

Let $\pi(U)$ be the group of transformations of U induced by $\pi(U, x_0')$; the group does not depend on the choice of x_0'. We refer to $\pi(U)$ as the *group of deck transformations*, or *the fundamental group* of U. This terminology is justified by the fact that $\tau \circ p = p$, for $p : U \to \mathbb{A}(K)$ the covering map.

For a subset $W \subset U^n$, let $\pi(W) = \{\tau : U^n \to U^n : \tau(W) \subset W, \tau \in \pi(U)^n\}$.

In terms of terminology above, the Axiom 3.5.2.5 says that for every subgroup finitely generated H of $\pi(U)$ and every $h \in \pi(U)$, if there exists $h_G \in H$ such that $h \sim_G h_G$ for every $G \lhd_{\text{fin}} H$, then $h \in H$. Since G ranges through all finite index subgroups of $\pi_1(\mathbb{A}(\mathbb{C}))$, this means that if for a finitely generated subgroup of $\pi(U)$, if $\iota(h) \in \hat{\pi}_1(\mathbb{A}(\mathbb{C}))$ belongs to $\iota(H) = \hat{H} \subset \hat{\pi}_1(\mathbb{A}(\mathbb{C}))$, then in fact $h \in H$.

Decomposition Lemma for U

We use a Corollary to Lemma 3.1.3.1.

Lemma 3.5.3.2 (Decomposition lemma; Noetherian property). *Assume A is Shafarevich.*

A subset $p^{-1}(W), W \subset \mathbb{A}(K)$ has an decomposition of the form

$$W' = HZ_1' \cup ... \cup HZ_k',$$

where $H \lhd_{\text{fin}} \pi$ is a finite index normal subgroup of π, the $\mathcal{T}$-closed sets $Z_1', .., Z_k'$ are irreducible components of relations $\sim_{Z_i}$, for some algebraic subvarieties Z_i of $\mathbb{A}(K)$, and for any $\tau \in H$ either $\tau Z_i' = Z_i'$ or $\tau Z_i' \cap Z_i' = \emptyset$.

Proof. By a corollary to Decomposition Lemma 3.1.3.1 we may choose $H \lhd_{\text{fin}} \pi$ with the following property.

Let $Z_i \subset A^H(K)$'s be the irreducible components of $p_H p^{-1}(W)$. Then, they have the property that the connected components of $p_G p_H^{-1}(Z_i) \subset A^G(K)$ are irreducible.

Choose Z_i' to be an irreducible components of relations $\sim_{Z_i}$, or equivalently of the closed sets $p_H^{-1}(Z_i)$. We claim that these Z_i''s give rise to a decomposition as above.

Before we are able to prove this, let us prove the *lifting property for* $\sim_{Z_i}$, namely that the map $p_H : Z_i' \to Z_i(K)$ is surjective. For convenience, we drop the index i below.

By passing to a smaller subgroup if necessary we may find a variety $W \subset A^H(K)^n$ such that for some $g \in \mathbb{A}^n(K)$, Z_i is a connected component of fibre W_g of W over g, and it holds that if points x', y' are such that $p_H(x'), p_H(y') \in Z_i$ and $x' \sim_W y', p_H(\mathrm{pr}\, x') = p_H(\mathrm{pr}\, y') = g'$ lie in the same connected component of W' over $g, p_H(g') = g$, then in fact x' and y' lie in the same connected component of the preimage of $g \times Z_i$, $x' \sim_Z y'$.

Consider Axiom 3.5.2.3 for all $G \lhd_{\mathrm{fin}} \pi$ sufficiently small

$$x' \sim^c_{W_g, A^G} y' \Longrightarrow \exists y''(y'' \sim_G y' \,\&\, \mathrm{pr}\, x' = \mathrm{pr}\, y'' \,\&\, x' \sim_W y'') \tag{3.5.9}$$

Now take any point $z' \in Z' \subset U$ and a point $y \in Z(K)$. We want to prove $p_H(Z') \supset Z(K)$, and thus it is enough to prove there exists $y_1 \in U$, $p_H(y_1) = y, z' \sim_Z y_1$. We know that there exist $y_2 \in U$, $z' \sim^c_{Z,A^G} y_2$, due to Axiom 3.5.2.1. Since $Z = W_g$ for some $g \in U^{n-1}$, we also have $(g', z') \sim^c_{W_g, A^G} (g', y_2)$, and taking $p_H(g') = g$, $x' = (g', z'), y' = (g', y_2)$, Axiom 3.5.2.3 gives the conclusion

$$\exists y''(y'' \sim_G y' \,\&\, \mathrm{pr}\, x' = \mathrm{pr}\, y'' \,\&\, x' \sim_W y''). \tag{3.5.10}$$

The conclusion says points $x', y'' \in U^n, p_H(x'), p_H(y') \in Z_i$ lie in the same connected component of $p_H^{-1}(W)$, are $\sim_G$-equivalent, and lie above the same point $g', p_H(g') = g$. Then by Lemma 3.3.2.2 we know that $p_H(x'), p_H(y')$ lie in the same connected component of the corresponding preimage of Z_i. By definition of Z', this means $\mathrm{pr}_2 y' \in Z'$. Thus, we have proved that $p_H(Z') = Z(K)$ is surjective. The property that p_H is surjective from Z' to $Z(K)$, we call *covering property, or lifting property* for $Z(K)$.

Now the following by now standard argument concludes the proof.

The Covering Property implies that

$$p_H^{-1}(Z(K)) = \bigcup_{h \in H} hZ' = HZ';$$

indeed, by properties of Z we know that the relations $x' \sim_{Z,G} y'$ are equivalence relations for all $G \lhd_{\mathrm{fin}} H$. Moreover, we know that any two equivalence classes are conjugated by the action of an element of H; this is so because Covering Property implies that there is an element of each of the classes above each element of $Z(K)$. This implies the lemma. □

We single out the following part of the proof as a corollary.

Recall that $\sim^c$ means «to lie in the same connected component of».

Corollary 3.5.3.3 (Covering Property). $x' \sim^c_{Z,G} y' \Longrightarrow \exists y''(y'' \sim_G y' \,\&\, x' \sim^c_Z y'')$.

Proof. The proof of lifting property above proves the corollary for $Z \subset A^H(K)$ if the relations $\sim^c_Z$ and $\sim_Z$ are equivalent. However, by Decomposition Lemma any set $p_H^{-1}(Z)$ can be decomposed into a union of such sets; then going from one irreducible component to another one intersecting it gives the corollary. □

Corollary 3.5.3.4 (Topology on U). *The collection of $\mathcal{T}$-closed subsets of U forms a topology with a descending chain conditions on irreducible sets. A $\mathcal{T}$-closed set possesses an irreducible decomposition as a union of a finite number of $\mathcal{T}$-closed sets whose $\mathcal{T}$-connected components are $\mathcal{T}$-irreducible. A union of irreducible components of a $\mathcal{T}$-closed set is a $\mathcal{T}$-closed.*

That is,

1. *the collection of $\mathcal{T}$-closed subsets on $\boldsymbol{U}^n, n > 0$ forms a topology. The projection and inclusion maps $pr : \boldsymbol{U}^n \to \boldsymbol{U}^m, (x_1, ..., x_n) \mapsto (x_{i_1}, .., x_{i_m})$ and $\iota : \boldsymbol{U}^n \hookrightarrow \boldsymbol{U}^m, (x_1, ..., x_n) \mapsto (x_{i_1}, .., x_{i_{m'}}, c_{m'}, .., c_m)$ are continuous.*
2. *There is no infinite decreasing chain $.. \subsetneq U_{i+1} \subsetneq U_i \subsetneq ... \subsetneq U_0$ of closed $\mathcal{T}$-irreducible sets.*
3. *A union of irreducible components of a $\mathcal{T}$-closed set is $\mathcal{T}$-closed.*
4. *A set is $\mathcal{T}$-closed iff it a union of* connected *components of a finite number of H-invariant sets, for some $H \lhd_{\text{fin}} \pi$ a finite index subgroup of π.*
5. *Each $\mathcal{T}$-closed set is a union of a finite number of $\mathcal{T}$-closed sets whose $\mathcal{T}$-connected components are $\mathcal{T}$-irreducible. Moreover, those sets may be taken so that their connected components within the same set are translates of each other by the action of a finite index subgroup $H \lhd_{\text{fin}} \pi$.*

Proof. The last item is just a reformulation of Decomposition Lemma. All the items trivially follow from it but (1).

Let us prove the intersection of two $\mathcal{T}$-closed set Z'_i and Y'_i is $\mathcal{T}$-closed.

Assume W' and V' are unions of connected component of H-invariant sets HW' and HV'. The intersection $HW' \cap HV'$ is H-invariant and the set $W' \cap V'$ is a union of the connected components of $HW' \cap HV'$. The intersection $HW' \cap HV' = p_H^{-1}(p_H(W') \cap p_H(V'))$ is $\mathcal{T}$-closed by definition, and thus its connected components are also $\mathcal{T}$-closed. This by definition implies $W' \cap V'$ is $\mathcal{T}$-closed.

To prove that an infinite intersection is closed, it is sufficient to prove that the intersection of a decreasing sequence of $\mathcal{T}$-closed sets is $\mathcal{T}$-closed. Use Koenig lemma and the fact that a sequence of decreasing $\mathcal{T}$-irreducible sets stabilises.

D.C.C. follows from the fact that an irreducible subset of an irreducible set necessarily have lesser dimension. □

Chevalley Lemma holds for $\Re$-structures

Let $W' \subset U$ be an *irreducible closed subset* of U, i.e. a subset of U defined by

$$x \sim_W a_1 \,\&\, ... \,\&\, x \sim_W a_n$$

where $a_1, .., a_n \in U$ are such that

$$\forall y \forall z (\bigwedge_{1 \le i \le n} y \sim a_i \,\&\, \bigwedge_{1 \le i \le n} z \sim a_i \Longrightarrow y \sim_W z.$$

Such a set W' we call *an irreducible component of closed set defined by $x \sim_W x$*, or simply *an irreducible component of relation $\sim_W$*.

Lemma 3.5.3.5 (Chevalley Lemma). *A projection of an irreducible closed set is closed.*

Proof. Let W' be such an irreducible set, and let $V' = \operatorname{Clpr} W'$ be the least closed set containing its closure. By definition of V' $p_H(\operatorname{pr} W') \subset p_H(V')$; and by definition of closure $V' \subset \operatorname{pr} HW' = p_H^{-1}(\operatorname{pr} p_H(W'))$; the set $\operatorname{pr} p_H(W')$ is closed by Chevalley Lemma for projective algebraic varieties. The inequalities imply $p_H(\mathrm{pr}\, W') = p_H(V')$ for every subgroup $H \lhd_{\text{fin}} \pi$.

A deck transformation leaving W' invariant, also leaves V' invariant, i.e. $\mathrm{pr}\, \pi(W') \subset \pi(V')$. On the other hand, the equality $p_H(\mathrm{pr}\, W') = p_H(V')$ implies for any $H \lhd_{\text{fin}} \pi$, $\mathrm{pr}\, \pi(W')/H = \pi(V')/H$.

Let us now use Axiom 3.5.2.5 to show that this implies that $\mathrm{pr}\, (\pi(W) \cap [H \times H]) = \pi(V') \cap H$.

Let us now prove that $\pi(W') \cap H \times H$ is finitely generated for some $H \lhd_{\text{fin}} \pi$.

We know by Corollary to Lemma 3.2.2.2 that $W' = Y'_{g'}$ is a fibre of a $\overline{\mathbb{Q}}$-defined set Y' over a point g' such that $p_H(g') \in \mathrm{pr}\, p_H(Y')$ $\overline{\mathbb{Q}}$-generic.

We know that for every $G \lhd_{\text{fin}} H$, for a connected component Y_G of $p_G p_H^{-1}(Y)$, the intersection $Y_G \cap g' \times p_G p_H^{-1}(Y_g)$ is connected; geometrically, that means that a lifting of $W = Y_g \subset Y$ along the covering map $Y_G \to Y$ is a fibre of Y. This holding for every $G \lhd_{\text{fin}} H$, it implies that for Y' a connected component of $p_H^{-1}(Y)$, the intersection $Y'^{\,c}_{g'} = Y' \cap g' \times p_H^{-1}(Y_g)$ is connected, and therefore it coincides with a connected component of $p_H^{-1}(Y_g) = p_H^{-1}(W)$. Moreover, this implies that if $h \in H$ is such that $hY'^{\,c}_{g'} \subset p_H^{-1}(Y_g)$ then $hY'^{\,c}_{g'} \subset Y'^{\,c}_{g'}$, i.e. $h \in \pi(Y'^{\,c}_{g'}) \cap H = \pi(Y'_{g'}) \cap H$. Thus, to prove that $\pi(W) \cap H = \pi(Y'^{\,c}_{g'}) \cap H$ is finitely generated, it is enough to prove that $\pi(Y'_{g'}) \cap H$ is finitely generated. However, the latter is claimed by Axiom 5 for every variety Y defined over $\overline{\mathbb{Q}}$.

Let $g_1, .., g_n$ be the generators of $\pi(W') \cap H \times H$. Now take $\tau \in \pi(V') \cap H, \tau(V') = V'$. We know that $\tau/G \in \mathrm{pr}\, \pi(W')/G$, for every $G \lhd_{\text{fin}} H$, and therefore τ, up to $\sim_G$, is expressible as a product of $g_1, ..., g_n$. In other words, that means that x' and $\tau x'$ can be joined by a sequence of points $x' = h_1, h_2, .., h_n = \tau x'$ such that $h_{i+1} = g_{j_i} h_i$ for all $1 \le i \le n$, and here $n = n(G)$ depends on subgroup G. By Axiom 5 there is a uniform bound on such $n = n(G)$, and τ is expressible as a product of $g_1, ..., g_n$, and therefore belongs to $\mathrm{pr}\, \pi(W')$.

Now we finish the proof by a Covering Property argument similar to the topological proof of Chevalley Lemma in complex case.

Let $V_0 \subset \mathrm{pr}\, p_H(W') \subset V$ where $V_0 \subsetneq V$ is open in V; then V is irreducible. Recall $V' = \mathrm{Clpr}\, W'$ and take $V'_0 = V' \cap p_H^{-1}(V_0)$; we know $V'_0 \subset V'$ is open in V'. We also know $V'_0 \subset \mathrm{Clpr}\, W'$.

Take $v' \in V'_0$, and take $w' \in W'$, $\mathrm{pr}\, p_H(w') = p_H(v') \in V_0 \subset \mathrm{pr}\, W$; such a point w' in W' exists by Covering Property. Now, $\mathrm{pr}\, w' \in V'$, and thus $\gamma_0 \in \pi(V')$ where γ_0 is defined by $v' = \gamma_0 \mathrm{pr}\, w'$. Condition $\mathrm{pr}\, p_H(w') = p_H(v') \in A^H(K)$ implies $\gamma_0 \in H$. Thus the inclusion $\mathrm{pr}\, \pi(W') \cap H = \pi(V') \cap H$ implies there exists $\gamma_1 \in \pi(W')$, $\mathrm{pr}\, \gamma_1 = \gamma_0$, and thus $v' = \gamma_0 \mathrm{pr}\, w' = \mathrm{pr}\, (\gamma_1 w')$, and the Chevalley lemma is proven. □

Corollary 3.5.3.6. *A projection of a set open in its irreducible closure contains an open subset of the closure of the projection.*

Proof. Let $\emptyset \neq W^{0\prime} \subset W'$ be an open subset of an irreducible closed set W'. Then $W'_1 = W' \setminus W^{0\prime} \subsetneq W'$ is a closed set. Consider $W' \setminus HW'$. If $W' \subset HW'_1$, then by irreducibility of W' it holds $W' \subset hW'_1 \subset hW'_1$, for some $h \in H$. This forces $hW' = W'$, and also $W'_1 = W'$, which constricts the assumption $\emptyset \neq W^{0\prime}$. Thus $W' \setminus HW'_1 \neq \emptyset$, and $\mathrm{pr}\, p_H(W') \supsetneq p_H(W')$. Now, $p_H(W')$ is an irreducible closed subset of $A^H(K)$, and therefore $p_H(W')$ is of smaller dimension then $p_H(W')$. Now we may apply Chevalley Lemma for algebraic varieties to get the conclusion. □

3.5.4 Homogeneity and stability over models

In the §§ above we have established the main properties of topology $\mathcal{T}(U)$ on U (and its Cartesian powers U^n). That allows us to define and prove the basic properties of Θ-generic points, for Θ an algebraically closed subfield of K.

The notion of a Θ-generic point extends to U in a natural way. Recall that for a closed Θ-defined set V', the set $\mathrm{Cl}_\Theta V'$ is the set of all Θ-generic points of V'. Recall also that a set of Θ-generic points of a Θ-defined set is called Θ-constructible.

Lemma 3.5.4.1 (Homogeneity). *Any structure $\boldsymbol{U} \in \Re$ is model homogeneous, i.e. the projection of a Θ-constructible set is Θ-constructible, for any algebraically closed subfield Θ of the ground field.*

Proof. First note that a point $w' \in W'$ in an irreducible set W' is Θ-generic iff $p(w') \in p(W')$ is Θ-generic. By Chevalley Lemma, the fibre $W'_{g'}$ is non-empty for $g' \in \mathrm{pr}\, W'$ Θ-generic. Moreover, by Lemma 3.1.5.5 a connected component of fibre $W_g, g = p(g')$ always contains a Θ-generic point $w \in W$ of W. The lifting $w', p(w') = w$ is always Θ-generic, and we may find such a lifting in any connected component of a fibre over a generic point. This implies the lemma. □

Definition 3.5.4.2. Let $\boldsymbol{U}, \boldsymbol{U}_1, \boldsymbol{U}_2 \in \Re$ be L_A-models of $\mathfrak{X}(\mathbb{A}(\mathbb{C}))$.

We say that tuples $a \in \boldsymbol{U}_1^n$ and $b \in \boldsymbol{U}_2^n$ *have the same syntactic quantifier-free type over $\boldsymbol{U}$ i n class* $\Re$ if a and b satisfy the same quantifier-free L_A-formulae with parameters in $\boldsymbol{U}$.

Definition 3.5.4.3. A class $\Re$ of L_A-structures is *syntactically stable over countable submodels* iff for any structure $\boldsymbol{U} \in \Re$, the set of complete L_A-types over a structure $\boldsymbol{U}$ realised in a structure $\boldsymbol{U}' \in \Re$ is at most countable.

Definition 3.5.4.4. A class $\Re$ of L_A-structures is *quantifier-free syntactically stable over countable submodels* iff there are only countably many quantifier-free syntactic types in class $\Re$ over any countable model $\boldsymbol{U} \in \Re$.

Lemma 3.5.4.5 (Stability over submodels). *Assume A is Shafarevich. The class of L_A-models of $\mathfrak{X}(\mathbb{A}(\mathbb{C}))$ is quantifier-free syntactically stable over submodels.*

Proof. If $\boldsymbol{U} \prec \boldsymbol{U}'$ is an elementary substructure, then $\boldsymbol{U} = \boldsymbol{U}'(\Theta) = \{u \in \boldsymbol{U}' : p(u) \in A(\Theta)\}$, for some algebraically closed subfield Θ.

Every positive quantifier-free L_A-formula over $\boldsymbol{U}$ determines a closed set defined over Θ. For every tuple $v' \in \boldsymbol{U}'$, there is a least closed set $V' = \mathrm{Cl}_\Theta(v')$ containing v' and defined over Θ; it is irreducible, and is a connected component of an algebraic subvariety V/Θ of A^H defined over Θ, for some $H \lhd_{\mathrm{fin}} \pi$. Moreover, $\mathrm{Cl}_\Theta(v')$ has a Θ-point v'_Θ. Thus, the quantifier-free L_A-type of tuple v' is determined by the point $v'_\Theta \in \boldsymbol{U}$ and a subvariety V/Θ. Therefore, there are only countable number of such types, which implies that class $\Re$ is quantifier-free syntactically stable over submodels. □

Property 3.5.4.6 (Homogeneity and Stability of class $\Re$). Assume A is Shafarevich.

All structures L_A-models of $\mathfrak{X}(\mathbb{A}(\mathbb{C}))$ are model homogeneous. The class of L_A-models of $\mathfrak{X}(\mathbb{A}(\mathbb{C}))$ is syntactically quantifier-free stable over countable submodels.

Proof. Implied by preceeding two lemmata. □

Finally, we may state Theorem 3.5.4.7, which was the goal of the chapter.

Theorem 3.5.4.7 (Model Stability of $\mathfrak{X}(\boldsymbol{U})$). *Let $\mathbb{A}$ be a smooth projective algebraic variety such that the universal covering space $\boldsymbol{U}$ of $\mathbb{A}$ is Stein (holomorphically convex). and such that the fundamental groups of connected components of $\mathbb{A}(\mathbb{C})$ are lerf. Let language L_A be the countable language defined in Def. 2.1.2.1. Then $(\mathfrak{1}', \mathfrak{3a}, \mathfrak{3b})$ hold, and $\mathfrak{2}'$ is weaken to $\mathfrak{2}'_{\aleph_1}$:*

$\mathfrak{2}'_{\aleph_0\to\aleph_1}$ *Any two models $\boldsymbol{U}_1 \models \mathfrak{X}$ and $\boldsymbol{U}_2 \models \mathfrak{X}$ of axiomatisation $\mathfrak{X}$ and of cardinality $\aleph_1$, such that*

there exist a common countable submodel $\boldsymbol{U}_0 \models \mathfrak{X}$, $\boldsymbol{U}_0 \subset \boldsymbol{U}_1$ and $\boldsymbol{U}_0 \subset \boldsymbol{U}_1$

are isomorphic, $\boldsymbol{U}_1 \cong_{L_A} \boldsymbol{U}_2$, and, moreover, the isomorphism ϕ is identity on $\boldsymbol{U}_0$.

Proof. This is closely related to Proposition 3.5.4.6; however, let us prove this directly in an explicit manner; in this argument we try to pun an emphasis the properties of topology, although this could also be treated as a very common model-theoretic argument.

We will prove that every partial L-isomorphism $f : \boldsymbol{U}_1 \dashrightarrow \boldsymbol{U}_2, f(a) = b, a \in \boldsymbol{U}_1^n, f_{|\boldsymbol{U}_0} = \mathrm{id}_{|\boldsymbol{U}_0}$ defined on $\boldsymbol{U}_0 \cup \{a_1, ..., a_n\}$, can be extended to $\boldsymbol{U}_0 \cup \{a_1, ..., a_n\} \cup \{c\}$, $f(c) \in \boldsymbol{U}_2$ for any element $c \in \boldsymbol{U}_1$. This allows to extend a partial L-isomorphism from a countable model to its countable extension. This is enough: by taking unions of chains of countable submodels we get isomorphism between models of cardinality $\aleph_1$. Note that one cannot get isomorphism between models of cardinality $\aleph_2$ in this way.

Let $V_1 = \mathrm{Cl}_{\boldsymbol{U}_0}(a), W_1 = \mathrm{Cl}_{\boldsymbol{U}_0}(a, c)$ be the minimal closed irreducible subsets containing points $a \in \boldsymbol{U}_1^n$ and $(a, c) \in \boldsymbol{U}_1^{n+1}$; let $V_2 = \mathrm{Cl}_{\boldsymbol{U}_0}(f(a))$ be the corresponding subset of $\boldsymbol{U}_2$. Since f is an L-isomorphism, sets V_1 and V_2 are defined by the same L-formulae with parameters in $\boldsymbol{U}_0$.

Take a subgroup $H \triangleleft_{\mathrm{fin}} \pi$ sufficiently small such that V_1, V_2, W_1, W_2 are connected components of $p_H^{-1}p_H(V_1), p_H^{-1}p_H(V_2), p_H^{-1}p_H(W_1), p_H^{-1}p_H(W_2)$, respectively. Pick points $v_1, w_1 \in \boldsymbol{U}_0$ such that $v_1 \in V_1, V_2$ and $w_2 \in W_1, W_2$.

Now, by definition of W_2 we have $\mathrm{pr}\, p_H W_2 = p_H V_2$, and also $\mathrm{pr}\, w_2 \in V_2$; choose $c' \in \boldsymbol{U}_2$ such that $(p_H(b), p_H(c')) \in p_H(W_2)$ is a $\boldsymbol{U}_0$-generic point of $p_H(W_2)$. Then by the lifting property for W_2 there exists a point $(b', c'') \in W_2$ such that $p_H(b') = p_H(b), p_H(c'') = p_H(c')$. However, this implies that $b' \in \mathrm{pr}\, W_2 \subset V_2$ is a $\boldsymbol{U}_0$-generic point of V_2. Therefore by homogeneity properties in Lemma 3.5.4.1, or equivalently because the projection $\mathrm{pr}\, W_2$ is a closed set definable over $\boldsymbol{U}_0$, this implies $V_2 \subset \mathrm{pr}\, W_2$, and, in particular, there exists $d \in \boldsymbol{U}_1$ such that $(b, d) \in W_2$ is a $\boldsymbol{U}_0$-generic point. Now set $f(c) = d$. By constructions, the points $(a, c) \in \boldsymbol{U}_1$ and $(b, d) \in \boldsymbol{U}_2$ lie in the same $\boldsymbol{U}_0$-definable closed sets, and, since every basic relation of L defines a closed set, this implies that f is indeed an L-isomorphism, as required. □

Chapter 4

Appendices

4.1 Appendix A: Basic notions of homotopy theory

Here we introduce basic notions of homotopy theory—that of a homotopy, of a fibration, a covering, a path, covering homotopy property and paths lifting property, and the notion of a fundamental group and a universal covering space.

4.1.1 Homotopy Theory: Fundamental Groups, Universal Covering Spaces

4.1.2 Homotopy and Coverings

We introduce basic notions of homotopy theory, and some analogous notions of complex algebraic geometry. Exposition follows [Nov86, Ch.4,§§2-4].

In this § we assume that all topological spaces are sufficiently nice, i.e. Hausdorff, locally connected and locally linearly connected.

Homotopies and paths

Let X, Y be topological spaces. A *continuous homotopy* $f : X \to Y$, or simply a *homotopy*, is a continuous map

$$F(x,t) : X \times I \to Y, x \in X, a \leq t \leq b$$

of a *cylinder* $X \times I$, where $I = [a,b] = \{t : a \leq t \leq b\}$ is the closed interval of real line from $a \in \mathbb{R}$ to $b \in \mathbb{R}$, and which coincides with f on *boundary* $X \times \{a\}$

$$F(x,a) = f(x), x \in X.$$

Two maps $f, g : X \to Y$ are *homotopic* if there is a continuous homotopy F such that

$$F(x,a) = f(x),$$
$$F(x,b) = g(x).$$

Being homotopic is obviously an equivalence relation; a *homotopy class* is a class of maps $f : X \to Y$ homotopic to each other.

If maps f and g coincide on a point, $f(x_0) = g(x_0)$, then one often requires *connecting homotopy to fix* $f(x_0)$, that is, $F(x_0,t) = f(x_0) = g(x_0)$; this is called a *homotopy fixing* $f(x_0)$.

A homotopy between two points is called a *path*; thus, a *path* γ *in* Y is just a continuous map $\gamma : [0,1] \to Y$; a path γ is *trivial* iff $\gamma(t) = y_0, 0 \leq t \leq 1$ for all t

and some point $y_0 \in A$. Thus, two points are homotopic iff they can be joined by a path. A path is always homotopic to its endpoint, the connecting homotopy just contracts the path by itself. To get a non-trivial notion of homotopy of paths, one usually considers only homotopies of paths fixing the ends. Thus, we say *two paths are fixed point homotopic* iff there is a connecting homotopy fixing their endpoints.

Covering homotopies and fibrations

Consider continuous maps $p : X \to Y$ and an arbitrary map $f : Z \to Y$. The map f is *covered by a map* $g : Z \to X$ iff $p \circ g = f$.

Definition 4.1.2.1. A map $p : X \to Y$ is called a *fibration* iff for any space Z any homotopy $F : Z \times I \to Y$ covered at the initial time $t = a$, can be covered at all times $a \leq t \leq b$ by some homotopy $G : Z \times I \to X$ so that $p \circ G(z, t) = F(z, t), G(z, a) = g(z)$. That is, if map $f(z) = F(z, a) : Z \to Y$ is covered by a map $g : Z \to X$, $f(z) = F(z, a) = p \circ g(z), z \in Z$, then there exist a homotopy $G : Z \times I \to X$ covering $F : Z \times I \to X$,

$$G(z, a) = g(z)$$

$$F(z, t) = p \circ G(z, t).$$

Homotopy G is called a *covering homotopy with initial condition* g.

Quite often one weakens the definition by restricting Z to a subclass of spaces; an example of an important notion of this type is when $Z = I^n$ is required to be a direct product of intervals.

In all most important cases a covering homotopy can be constructed with the help of a *homotopical connection*, i.e. a unique recipe to cover an arbitrary homotopy of a point $y \in Y$ (i.e. a path γ in Y, $\gamma(a) = y$) by a path in X starting from an arbitrary point $x_0 = \gamma(a) \in X, y = p(x_0)$. The recipe should continuously depend on the path in Y and on the starting point of the covering path in X. Continuity on these variables ensures that the covering homotopy property for paths can be extended to arbitrary (in some reasonable sense) spaces Z.

If $p : X \to Y$ is a fibration, then the map p is called a *projection*, X the *total space*, Y *the base*, and $F_y = p^{-1}(y), y \in Y$ a *fibre* of the fibration p. Using the existence of a connection, one can prove that all the fibres F_y of a fibration of a base space Y are homotopically equivalent provided any two points of Y are homotopic, i.e. if space Y is linearly connected.

Given a connection and a path γ in the base of a fibration $p : X \to Y$, we get a map from fibre F_y to $F_{y'}$ above the ends of the path γ: a point $x \in F_y$ goes to the endpoint of the unique lifting $\tilde{\gamma}_x$ of path γ to X starting at x:

$$x \in F_y \mapsto \tilde{x}' = \gamma_x(1) \in F_{y'}, \text{ where } \tilde{\gamma}_x(0) = x.$$

The point x' varies continuously with x, and in fact, properties of a connection ensure that $x \mapsto x'$ is a homotopy equivalence between fibres F_y and $F_{y'}$. The map from F_y to $F_{y'}$ may depend on the path γ and not only its homotopy class; if the correspondence $x \to x'$ depends only on homotopy class of the path γ between points x and x', the fibration is called *flat*. An important case of a flat fibration we define next.

Definition 4.1.2.2. A *covering* $p : X \to Y$ is a fibration with a discrete fibre F, i.e. F is a space such that all its subsets are open and any point $y \in Y$ has a open neighbourhood $U, y \in U \subset Y$ such that the full preimage of U is homeomorphic to a direct product of $U \times F$, where $F = \bigcup_\alpha \{x_\alpha\}$:

$$p^{-1}(U) = \bigcup_\alpha U_\alpha \cong U \times F$$

and each U_α is a homeomorphic copy of U.
Subsets $U_\alpha \subset X$ are open and pairwise non-intersecting; on each of them p is a homeomorphism with U. The homotopic connection is given by the following recipe. Given a point $x_0 \in X$, to lift a path $\gamma : [0,1] \to Y$ in Y which is small enough to fit in a neighbourhood U, we just take path $\tilde{\gamma}(t) = (\gamma(t), x_\alpha) \subset U_\alpha$, where $x_0 = (y_0, x_\alpha) \in U_\alpha \subset U \times F$. If γ is not small enough, we split it in pieces which are small enough, and lift it piece by piece.

If one drops the requirement that fibre F is discrete, then one get the notion of a *locally trivial fibration*; the proof of the covering homotopy property requires an argument, and is not a priori clear; such a fibration has homeomorphic fibres. In more detail, a map $p : X \to Y$ is called *locally trivial fibration* iff any point $y \in Y$ of the base is contained in a neighbourhood U_α such that $p^{-1}(U_\alpha) \subset X$ is homeomorphic to a direct product $U_\alpha \times F$ via a homeomorphism $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ compatible with the projection $\mathrm{pr} \circ \phi_\alpha = p$. There is also a compatibility condition on the behaviour of ϕ_α's on the intersection $V_{\alpha\beta} = U_\alpha \cap U_\beta$; there are two homeomorphisms corresponding to the intersection $V_{\alpha\beta} = U_\alpha \cap U_\beta$:

$$\phi_\alpha : p^{-1}(V_{\alpha\beta}) \to V_{\alpha\beta} \times F,$$

$$\phi_\beta : p^{-1}(V_{\alpha\beta}) \to V_{\alpha\beta} \times F.$$

The map $\lambda_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : V_{\alpha\beta} \times F \to V_{\alpha\beta} \times F$ leaves the fibres invariant setwise, and thus has form

$$\lambda_{\alpha\beta}(\omega, f) = (\omega, \hat{\lambda}_{\alpha\beta}(\omega)(f)), f \in F, w \in V_{\alpha\beta},$$

where $\hat{\lambda}_{\alpha\beta}(\omega) : F \to F$ is a homeomorphism of the fibre continuously depending on point ω. In a neighbourhood $W = U_1 \cap U_2 \cap U_3$ we have

$$\lambda_1 \circ \lambda_2 \circ \lambda_3 \equiv 1.$$

The maps λ_{ij} are called *gluing functions*. If we know the gluing functions $\lambda_{\alpha\beta}$ for some covering of a space Y by neighbouhroods U_α's, and the gluing functions satisfy the condition above, then we can uniquely reconstruct the fibration $p : X \to Y$. Obviously, we require that above each neighbourhood U_α the fibration decomposes into the direct product.

The notions of a locally trivial fibration and the general notion of a fibration are fundamental for the theory of manifolds, differential topology, geometry and their applications.

4.1.3 Fundamental Group, Functoriality and Long Exact Sequence of a Fibration

Definition of homotopy groups $\pi_n(X, x_0), n \geq 0$

Let S^n be the circle in $\mathbb{R}^n$ defined by

$$x_1^2 + x_2^2 + ... + x_n^2 = 1,$$

with a distinguished point $s_0 = (1, 0, .., 0)$. Then, S^1 is a circle with a basepoint, and S^0 is just a point.

Definition 4.1.3.1. For $n > 0$, the set of homotopy classes of maps of basepoint spaces $(S^n, s_0) \to (X, x_0)$ is called *n-th homotopy group* and is denoted $\pi_n(X, x_0)$. We describe group operation only for $n = 1$. For $n = 1$, the group operation is given by concatenation of paths, i.e. by the path which first follows the first path, and then goes along the second path:

$$\gamma(e^{\pi it}) = \gamma_1(e^{2\pi it}), 0 \leq t \leq 1,$$
$$\gamma(e^{\pi it}) = \gamma_1(e^{2\pi i(t-1)}), 1 \leq t \leq 2.$$

For $n = 0$, the «0-th homotopy group» $\pi_0(X, x_0)$ is not a group, but is just a set with a distinguished element; it is the set of all the connected components of X. It is customary to call it a group, although it is an abuse of language.

Thus, the fundamental group $\pi_1(X, x_0)$ consists of fixed homotopy classes of loops based at the point x_0; and the group operation is just concatenation of paths; it is well-defined on the homotopy classes.

The fundamental group is *a covariant functor on the category of basepoint topological spaces*. That means that each map of basepoint spaces

$$f : X \to Y, x_0 \to y_0$$

gives rise to a map of fundamental groups

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0).$$

The homomorphisms are defined sending each map $\gamma : (S^n, s_0) \to (X, x_0)$ into the composition $f \circ \gamma : (S^n, s_0) \to (Y, y_0)$.

The correspondence $f \mapsto f_*$ is *natural*, which means that a composition of maps of basepoint spaces gives rise to the composition of corresponding maps; that is, for maps $f : X \to Y, g : Y \to Z, f(x_0) = y_0, g(y_0) = z_0$ of basepoint spaces we have

$$(f \circ g)_* = f_* \circ g_*.$$

This correspondence is well-defined for any n, including $n = 0$.

The presence of a basepoint is essential for naturality.

The short exact sequence of fibration

The following observation is very important as for development of the theory of homotopy groups. It allows one to calculate the fundamental group of a fibration; the fact that this is possible is very important for the methods of the paper.

Property 4.1.3.2. To each fibration

$$p : X \to Y, p^{-1}(y_0) = F_0, x_0 \in F_0$$

there corresponds a long exact sequence of homotopy groups

$$\to \pi_n(F_0, x_0) \to \pi_n(X, x_0) \to^{p_*} \pi_n(Y, y_0) \to^{\partial} \pi_{n-1}(F_0, x_0) \to \pi_{n-1}(X, x_0) \to .$$

In particular, for $n = 0, 1$ the end of the sequence looks like

$$\to \pi_1(F_0, x_0) \to \pi_1(X, x_0) \to^{p_*} \pi_1(Y, y_0) \to^{\partial} \pi_0(F_0, x_0).$$

For a fibration with a connected fibre we have $\pi_0(F_0, x_0) = 0$, and thus we get an exact sequence

$$\pi_1(F_0, x_0) \to \pi_1(X, x_0) \to^{p_*} \pi_1(Y, y_0) \to 0$$

This fibration sequence is essential for us to prove the geometry properties of a Zariski-type topology $\mathcal{T}$ on the universal covering space U of an algebraic variety $A(\mathbb{C})$.

4.1.4 Regular Coverings and Universal Covering Spaces

Let $p : X \to Y$ be a covering. A *regular covering* is a covering such that $p_*\pi(X) \subset \pi_1(Y)$ is a normal subgroup.

The biggest regular covering is called *universal covering space* X of base Y; it is a covering such that $\pi_1(X)$ is a trivial group. It can be shown that the universal covering space exists for many spaces, in particular it exists for all complex algebraic varieties (in complex topology). The group $\Gamma = \pi_1(X, x_0)$ acts freely and discretely on X, and its orbits coincide with the fibres $p^{-1}(y)$.

Deck Transformations of a covering

Let $p : X \to Y$ be a covering. We say that a continuous map $g : X \to Y$ is a *deck transformation* iff $p \circ g(x) = p(x), x \in X$. If X is connected, then a deck transforation g is determined by the image of a point; indeed, if two continuous deck transformations $g_1, g_2 : X \to Y$ coincide on a point x, $g_1(x) = g_2(x)$, then they coincide in a admissible open neighbourhood $U_\alpha \subset X$ of x (notation of Def. 4.1.2.2). The space X can be covered by such neighbourhoods, and this proves g_1 and g_2 coincide on a connected component of X containing x.

On the other hand, the path lifting property allows one to define a natural action of the fundamental group $\pi_1(Y, y_0)$ on the fibre $p^{-1}(y_0)$: for a loop $\lambda \in \pi_1(Y, y_0), \lambda(0) = \lambda(1) = y_0$, we set

$$\lambda \cdot x = \tilde{\lambda}(1), \tilde{\lambda}(0) = x, p(\tilde{\lambda}) = \lambda.$$

where $\tilde{\lambda}$ is the lifting of λ starting at point $x, p(x) = y_0$.

Therefore we get an action of the fundamental group $\pi_1(Y, y_0)$ on a covering space of a basepoint base space Y.

In fact, the covering is completely characterized by the fundamental group, as the following important theorem shows:

Fact 4.1.4.1 (Galois correspondence between coverings and subgroups). *There is a bijective correspondence between subgroups of $\pi_1(Y, y_0)$ and basepoint covering spaces of (Y, y_0).*

For a subgroup $H < \pi_1(Y, y_0)$, the corresponding covering is denoted $p^H : \tilde{Y}^H \to Y$ with a basepoint $y^H \in \tilde{Y}^H$. The correspondence is natural, i.e. if $H_1 < H_2$, then there is a well-defined covering

$$p^{H_1,H_2} : Y^{H_1} \to Y^{H_2}, y^{H_1} \to y^{H_2}.$$

The choice of basepoint is important for the functoriality of the interdependence; otherwise there is no unique way to choose a covering corresponding to embedding $H_1 < H_2$.

In particular, there is a covering corresponding to the trivial subgroup $H = 0$; it is called *the universal covering space* of X, and in next § we give an explicit construction for the universal covering space in terms of paths. We also give an explicit construction for deck transformation.

A covering $\tilde{Y}^H$ corresponding to a normal subgroup $H \lhd \pi_1(Y, y_0)$ is called *regular*. The regular covering have the property that any two points of a fibre are conjugated by a deck transformation. For H normal, the group of deck transformations in this case is the group $\pi(Y, y_0)/H$, and it acts transitively on the fibres of covering $p : \tilde{Y}^H \to Y$.

Universal Covering Spaces and Deck Transformation

Definition 4.1.4.2. A covering $p : X \to Y$ is called the *universal covering* iff the space X is *simply connected*, i.e. its fundamental group $\pi_1(X, x) = 0$ is trivial.
The universal covering space is usually denoted by $\tilde{Y}$.

Given a basepoint $y_0 \in Y$, we can construct the universal covering space as the set of homotopy classes of paths leaving the basepoint:

$$\tilde{Y} = \{\gamma : [0, 1] \to Y : \gamma(0) = y_0\}/\{\text{homotopy fixing } \gamma(0)\}$$

with a basepoint $\tilde{y}_0$ being the trivial path in Y

$$\tilde{y}_0(t) = y_0, \text{ for all } t.$$

Each continuous transforation of basepoint spaces $f : (Y_1, y_1) \to (Y_2, y_2)$ induces a transformation on the covering spaces $\widetilde{f} : (\tilde{Y}_1, \tilde{y}_1) \to (\tilde{Y}_2, \tilde{y}_2), \widetilde{f}(\tilde{y}_1) = \tilde{y}_2$.

The dependance is natural, and thus we get a functor from the category of basepoint topological spaces to itself

$$(Y, y_0) \to (\tilde{Y}, \tilde{y}_0)$$

which sends a space with a basepoint to its universal covering space with a basepoint.

The fundamental group $\pi_1(Y, y_0)$ acts naturally on the space $\tilde{Y}, \tilde{y}_0$ by prexing a path in $\tilde{Y}$ with a loop from $\pi_1(Y, y_0)$

$$(\lambda \in \pi_1(Y, y_0), \gamma \in \tilde{Y}) \mapsto \lambda \circ \gamma$$

where $\circ$ denotes the concatenation of paths. The concatenation is well-defined as $\lambda(0) = y_0 = \lambda(1) = \gamma(0)$.

4.2 Some notions of algebraic geometry: normal varieties and étale morphisms

4.2.1 Étale morphisms.

In our context the following definition of an *étale morphism* is most useful; however, it applies only in characteristic 0. The equivalence of this definition to the usual one is given in [DS98].

Definition 4.2.1.1. A morphism $f : Y \to X$ defined over a characteristic 0 field k is *étale* iff, for an embedding $k \hookrightarrow \mathbb{C}$, the induced map $f : Y(\mathbb{C}) \to X(\mathbb{C})$ is a topological covering map, with respect to the complex topology on $Y(\mathbb{C})$ and $X(\mathbb{C})$, i.e. f induces an isomorphism of topological covering spaces of $Y(\mathbb{C})$ and $X(\mathbb{C})$. The morphism $f : Y \to X$ is called étale at a point $y \in Y$ if it is an isomorphism of an neighborhood of y in Y open in the complex topology onto an open neighbourhood of x in X open in the complex topology.

The definition above does not depend on the embedding of k to $\mathbb{C}$; this fact and the equivalence of this definition to the usual one is via an invariant local characterisation of an étale morphism [Mil80, Ch.1,Th.3.14,p.26].

Definition 4.2.1.2. A morphism $f : Y \to X$ of affine varieties defined over an algebraically closed field is called *standard etale* morphism at a point $x \in X(k)$ iff there exist functions $a_1, .., a_r : X \to k$ such that Y is locally described by the equation $P(x, t) = t^r + a_1(x)t^{r-1} + .. + a_{r-1}(x)t + a_r(x) = 0$ i.e.

$$Y(k) \cong \{(x, t); P(t) = 0, x \in X(k)\}$$

and all the roots of the polynomials $P_x(t) = P(x,t)$ are simple at any geometric point $x \in X(k)$. A morphism $f : Y \to X$ of affine varieties defined over an algebraically closed field is called standard étale morphism if it is standard étale at any geometric point $x \in X(k)$.

It is evident that a standard étale morphism induces a covering map in the complex topology.

By [Mil80, Ch.1,Th.3.14,p.26], an étale morphism is locally standard étale;

Fact 4.2.1.3. *Assume $f : Y \to X$ is étale in some (Zariski) open neighbourhood of y in Y. Then there are Zariski open affine neighbourhoods V and U of y and $x = f(y)$, respectively, such that $f|V : V \to U$ is a standard étale morphism.*

We remark that the notion of an étale morphism is in fact defined over arbitrary rings, and is defined via the exactness properties of the functor $h_Y = \mathrm{Hom}_X(-,Y)$ from the category of X-schemes to Y-schemes induced by morphism $f : Y \to X$.

4.2.2 Normal closed analytic sets

Definition 4.2.2.1. A closed analytic subset X of a Stein space is *normal* if any bounded meromorphic function on X is holomorphic.

A *normalisation morphism* $\mathbf{n}$ *of variety* Y is a morphism $\mathbf{n} : X \to Y$ from a normal variety X such that any dominant (surjective on an open subset) morphism $f : Z \to Y$ lifts up to a unique morphism $\tilde{f} : Z \to X$ such that $f = \tilde{f} \circ \mathbf{n}$.

Any smooth closed analytic set is normal.

We only the following two properties of a normal variety:

Fact 4.2.2.2. *A normalisation morphism exists for any variety, and is functorial.*

The following express that normality is a local notion:

Fact 4.2.2.3 (normality is a local notion). *1. if $p : X(\mathbb{C}) \to Y(\mathbb{C})$ is a local isomorphism in complex topology, and $Y(\mathbb{C})$ is normal, so is $X(\mathbb{C})$. In particular,*

2. if $p : X \to Y$ is étale, then X is normal iff Y is normal.

Fact 4.2.2.4. *If X is smooth, then X is normal.*

Fact 4.2.2.5. *Let X be a closed analytic subset of a Stein manifold, or let X be an algebraic variety. If X is connected and normal, then X is irreducible.*

4.3 Appendix: Geometric Conjectures on the convexity of the universal covering space of a complex algebraic variety

4.3.1 Shafarevich conjecture on holomorphic convexity of universal covering spaces

In [Sha94, IX§4.3] Shafarevich proposed a conjecture that the universal covering space of an algebraic variety is holomorphically convex; thus «it has many holomorphic function».

Recall the definition of a holomorphically convexity and separability:

Definition 4.3.1.1. A complex space is called *holomorphically separable* if for every $x_0 \in U$ there are holomorphic functions $f_1, ..., f_l$ on U such that x_0 is isolated in the set $\{u \in U : f_1(u) = ... = f_n(u) = 0\}$.

A complex space U is called *holomorphically convex* iff either of the two equivalent conditions holds:

(i) for any compact subset $K \subset U$ the set

$$\hat{K} = \{u \in U : |f(u)| \leq \sup |f(K)| \text{ for every holomorphic function } f \text{ on } U\}$$

is compact.

(ii) for any infinite discrete subset S of U there exist a holomorphic function $f : U \to \mathbb{C}$ unbounded on S.

A complex space U is called *Stein* iff it is both holomorphically separable and holomorphically convex.

Another characterisation of a Stein manifold is that

Lemma 4.3.1.2. *A manifold which is biholomorphic to a closed analytic set in Euclidean space $\mathbb{C}^n$; in particular the Euclidean space $\mathbb{C}^n$ itself is Stein.*

Proof. [Č85] □

Conjecture 4.3.1.3 (Shafarevich[Sha94, IX§4.3]). The universal cover of a projective variety is holomorphically convex, or even Stein.

Another characterisation of a Stein manifold is that a manifold which is biholomorphic to a closed analytic set in Euclidean space $\mathbb{C}^n$; in particular the Euclidean space $\mathbb{C}^n$ itself is Stein.

4.3.2 Equivalence of isomorphisms of topological and analytic vector bundles over a Stein manifold

On Stein manifolds, analytical properties are often determined by pure topology; in particular, one of the properties of Stein spaces which might turn out to be relevant is that analytic vector bundles over a Stein manifold are isomorphic analytically iff they are so topologically.

Theorem 4.3.2.1 (Oka's principle). *Let X be a Stein manifold. Then*

1. *Every topological fibre bundle over X has an analytic structure*
2. *If two analytic fibre bundles over X are topologically equivalent, then they are also analytically equivalent.*

Also relevant may be Theorems A and B of H. Cartan [Gra91].

Theorem 4.3.2.2 (Theorem A.). *Let $V \to X$ be an analytic vector bundle over a Stein manifold X. Then for any $x_0 \in X$ there are global holomorphic sections $s_1, .., s_n \in \Gamma(X, V)$ of V such that:*

> *If $U = U(x_0) \subset X$ is an open neighbourhood of x_0 and $s \in \Gamma(U, V)$ is a local section of V on $U(x_0)$, then there exist an open neighbourhood $V = V(x_0) \subset U$ of x_0 and holomorphic functions $f_1, .., f_N$ on V such that*
>
> $$s_{|V} = f_1 s_1 + .. + f_N s_N.$$

Theorem 4.3.2.3 (Theorem B.). *Let $\pi : V \to X$ be an analytic vector bundle over a Stein manifold X. Then $H^1(X, V) = 0$.*

Moreover, if $A \subset X$ is an analytic set, U is an open covering of X and a cochain $\xi \in Z^1(U, V)$ such $\xi_{\nu\mu}|_{U_{\nu\mu} \cap A} = 0$, then we can find a cochain $\eta \in C^0(U, V)$ such that $d\eta = \xi$ and $\eta|_{U_\nu} \cap A = 0$.

4.3.3 Stein factorisation and Lefschetz-type properties of algebraic varieties

In this section we use state several somewhat unexpected results about fundamental groups of algebraic varieties; arguably one may call such properties *rigidity properties*, or *Lefschetz-type properties*, or *positivity properties*.

A Zariski open subset has real codimension at least two; the following fact should not seem surprising:

Fact 4.3.3.1. *Let Y be a connected* normal *complex space and $Z \subset Y$ a Zariski closed subspace. Then $\pi_1(Y - Z) \to \pi_1(Y)$ is surjective.*

Proof. Kollar, Prop.2.10.1 □

Let us state first two facts which say that morphisms of complex algebraic *normal* varieties have rather easy and well-understood *topological* structure almost everywhere, i.e. on a dense Zariski open set. It is critical for us that these topological structure allow us to understand the corresponding morphism of covering spaces.

Fact 4.3.3.2. *Let $f : X \to Y$ be a morphism of irreducible* normal *algebraic complex varieties.*

Then there exist an open subset $Y^0 \subset Y$ and $X^0 = f^{-1}(Y^0)$, and a variety Z^0 such that f factorises as follows:

$$X^0 \to^{f^0} Z^0 \to^{f^{et}} Y^0$$

where

1. $Z^0 \to Y^0$ *is a finite étale morphism*
2. $X^0 \to Z^0$ *is a topological fibre bundle (in complex topology) with connected fibres*

Moreover, if $f : X \to Y$ is dominant, then $Z^0 \to Y^0$ is surjective.

Proof. Kollar, Propositon 2.8.1. □

Note that while $f^0 : X^0 \to Y^0$ is interpretable in the theory of algebraic varieties and in L_A, as indeed any morphism of algebraic varieties is, the theory may not say anything about the induced morphism $(f^0)_* : U(X^0) \to U(Y^0)$ of the universal covering spaces of $X^0(\mathbb{C})$ and $Y^0(\mathbb{C})$.

Indeed, the language allows us to speak about the *liftings*, or *induced morphisms* only for morphisms between closed subvarieties; and even then, we lift those only to the cover U_A which generally speaking is much smaller then the univesal covering space of subvarieties concerned.

We find the following useful.

Corollary 4.3.3.3. *Let $f : X \to Y$ be a morphism of irreducible algebraic varieties over Θ, and let $g \in Y(\mathbb{C})$ be a Θ-generic point. Then any open Θ-definable set intersects all connected components of generic fibre $W_g = f^{-1}(g)$.*

Proof. Such a set intersects all generic fibres of a morphism. Factoring projection through a morphism with connected fibres, which is possible by Stein factirisation, the result follows. □

Fact 4.3.3.4. *Any morphism $f : Y \to X$ of algebraic varieties admits a factorisation $f = f_0 \circ f_1$ as a product of a finite morphism $f_0 : Y \to Y'$ and a morphism f_1 with connected fibres.*

Proof. [Har77] □

Corollary 4.3.3.5. *If $f : X \to Y$ is a Θ-definable morphism of Θ-definable irreducible algebraic varieties, then there exist an open Θ-definable set $Y^0 \subset Y$ and $X^0 = f^{-1}(Y^0)$ such that the relation*

> *$x, y \in X^0$ lie in the same connected component of $X_g = f^{-1}(g)$, for some $g \in Y^0$*

is Θ-definable.

Proof. Indeed, the relation is defined by $f(x) = f(y)$ and $f^0(x) = f^0(y)$, in notation of previous lemma. □

The Fact 4.3.3.2 above leads to

Fact 4.3.3.6. *Let $f : X \to Y$ be a morphism of normal algebraic connected complex varieties; let $X_g = f^{-1}(g), g \in X$ be a generic fibre of f over a generic point $g \in Y(\mathbb{C})$.*

Then sequence

$$f_* : \pi_1(X_g(\mathbb{C})) \to \pi_1(X(\mathbb{C})) \to \pi_1(Y(\mathbb{C}))$$

is exact up to finite index.

Moreover, if $f : X \to Y$ is dominant, then

$$f_* : \pi_1(X(\mathbb{C})) \to \pi_1(Y(\mathbb{C}))$$

is surjective up to finite index.

If X, Y and morphism $f : X \to Y$ are defined over a field Θ, then there exists an open subset $Y_0 \subset Y$ defined over Θ such that the above conclusions hold for $g \in V_0$ not necessarily Θ-generic.

Proof. Follows from Facts 4.3.3.1 and 4.3.3.2 and the exact sequence of the fundamental groups of a fibration, from Kollar, Proposition 2.8.1 and Kollar, Proposition 2.10.1. □

Recall $p : U \to A(\mathbb{C})$ is the universal covering of an algebraic variety A.

Recall for a subset W' of U^n, we denote $\pi(W') = \{\gamma \in \pi^n : \gamma W' \subset W'\}$. In next lemma we will drop the assumption on normality using the assumption that the universal covering space U is holomorphically convex.

Corollary 4.3.3.7. *Let W' be a $\mathcal{T}$-irreducible closed set, and let $V' = Cl pr W'$. Assume that $p(W')$ and $p(V')$ are both normal. Then $\pi(pr W')$ is a finite index subgroup of $\pi(V')$, i.e. $\pi(V') \lesssim pr\, \pi(W')$.*

Proof. By Decomposition Lemma 3.1.4.1 we may assume that W' and V' are connected components of $p_H^{-1}(W(\mathbb{C}))$, $p_H^{-1}(V(\mathbb{C}))$ for some normal algebraic varieties W and V, respectively; then by properties in §4.1.1 $\pi(W') = \pi_1(W(\mathbb{C})), w)$, $\pi(V') = \pi_1(V(\mathbb{C}), v)$, for some points $w \in W(\mathbb{C}), v \in V(\mathbb{C})$.

Furthermore, we may still assume normality because it is preserved under taking preimage under an étale map. By the previous lemma $\mathrm{pr}\, \pi_1(W(\mathbb{C})) \lesssim \pi_1(V(\mathbb{C}))$, as required. □

4.3.4 Extending to the case of a non-normal subvariety

The above provides an explicit description of morphisms topologically, between normal algebraic varieties.

However, it is very important for us to deal with an *arbitrary* subvarieties, not necessarily normal. We do that by considering the image of the fundamental groups in the big ambient variety which is normal.

Fact 4.3.4.1. *Assume A is Shafarevich.*

Let $p : U \to A(\mathbb{C})$ be the universal covering space, let $\iota : X \to A \times A$ be a closed subvariety, and let $Y = Cl pr\, X$.

Assume that

connected components of $p^{-1}(X(\mathbb{C}))$ and $p^{-1}(Y(\mathbb{C}))$ are irreducible

Then there is a sequence of subgroups of $\pi_1(A(\mathbb{C}))^2$

$$\iota_*\pi_1(X_g(\mathbb{C})) \to \iota_*\pi_1(X(\mathbb{C})) \to \iota_*\pi_1(Y(\mathbb{C})) \to 0$$

which is exact up to finite index, and the homomorphisms are those of subgroups of $\pi_1(A(\mathbb{C}))^2$.

In particular, the sequence splits, and

1. $pr_* : pr\,\iota_*\pi_1(X(\mathbb{C})) \to \iota_*\pi_1(Cl(pr\,X)(\mathbb{C}))$ *is surjective up to finite index*
2. $\iota_*\pi_1(X(\mathbb{C})) \lesssim \iota_*\pi_1(X_g(\mathbb{C})) \times \iota_*\pi_1(Cl(pr\,X(\mathbb{C})))$, *where $X_g = f^{-1}(g)$ is a generic fibre of X over a generic point $g \in X(\mathbb{C})$.*

Moreover, if X is Θ-definable ($\Theta = \bar{\Theta}$)) then there exist a Θ-definable open subset Z^0 of $Cl pr\, X(\mathbb{C})$ such that the above holds for $g \in Z^0(\mathbb{C})$.

Proof. The first conclusions implies the second, so we only prove the first one.

We prove this by passing to the normalisation of varieties W and $Z = \mathrm{Clpr}\, W$. The assumption about the irreducibility of connected components implies that the composite maps of fundamental groups $\pi_1(\hat{W}) \to \pi(W) \to \iota_*\pi_1(W)$ and $\pi_1(\hat{Z}) \to \pi_1(Z) \to \iota_*\pi_1(Z)$ are surjective.

To show this, first note that the universal covering spaces $\tilde{\hat{W}}(\mathbb{C})$ and $\tilde{\hat{Z}}(\mathbb{C})$ are irreducible as analytic spaces; indeed, normality is a local property, and so they are normal as analytic spaces; they are obviously connected, and for normal analytic spaces connectness implies irreducibility.

By properties of covering maps, a morphism between analytic spaces lifts up to a morphism between their universal covering spaces (as analytic spaces); thus the normalisation map $\mathbf{n}_W : \hat{W} \to W$ lifts up to a morphism $\tilde{\mathbf{n}}_W : \tilde{\hat{W}} \to U$. The normasitation morphism $\mathbf{n}_W$ is finite and closed by Hartshorne[Ref!!]; therefore $\tilde{\mathbf{n}}_W$ is also, and the image of an irreducible set is irreducible. Therefore $\tilde{\mathbf{n}}_W(\tilde{\hat{W}})$ is an irreducible subset of a connected component of $p^{-1}(W(\mathbb{C}))$. Moreover, if we choose different liftings $\tilde{\mathbf{n}}_W$, we may cover $p^{-1}(W(\mathbb{C}))$ by a countable number of such sets. Now, we use the assumption that a connected component of $p^{-1}(W(\mathbb{C}))$ is irreducible to conclude that the image $\tilde{\mathbf{n}}_W(\tilde{\hat{W}})$ coincides with a connected component of $p^{-1}(W(\mathbb{C}))$. This implies that the map of fundamental groups is surjective; this may be easily seen if one thinks of a fundamental group as the group of deck transformations.

Let $\mathbf{n}_W : \hat{W} \to W$, $\mathbf{n}_{W_g} : \hat{W}_g \to W_g$ and $\mathbf{n}_Z : \hat{Z} \to Z$ be the normalisation of varieties W,W_g and Z.

By the universality property of normalisation in §4.2.2 we may lift the normalisation morphism $\mathbf{n}_{W_g} : \hat{W}_g \to W_g$ to construct a commutative diagram:

$$\begin{array}{ccccc} \hat{W}_g & \to & \hat{W} & \to & \hat{Z} \\ \downarrow & & \downarrow & & \downarrow \\ W_g & \to & W & \to & Z \end{array}$$

By functoriality of π_1, this diagram and embedding $\iota : W \to A \times A$ gives us

$$\begin{array}{ccccc} \pi_1(\hat{W}_g) & \to & \pi_1(\hat{W}) & \to & \pi_1(\hat{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(W_g) & \to & \pi_1(W) & \to & \pi_1(Z) \\ \downarrow & & \downarrow & & \downarrow \\ \iota_*\pi_1(W_g) & \to & \iota_*\pi_1(W) & \to & \iota_*\pi_1(Z) \end{array}$$

Now, g' is Θ-generic in $\hat{W}'_{g'}$; We are almost finished now. By 4.3.3.6 the upper row of the diagram is exact up to finite index, and $\pi_1(\hat{W}) \to \pi_1(\hat{Z})$ are surjective, up to finite index; by assumptions on W and Z, the composite morphisms $\pi_1(\hat{Z}) \to \iota_*\pi_1(Z)$ and $\pi_1(\hat{W}) \to \iota_*\pi_1(W)$ are surjective. Diagram chasing now proves that the bottom row is also exact up to finite index, and the map $\iota_*\pi_1(\hat{W}) \to \iota_*\pi_1(\hat{Z})$ is surjective up to finite index.

Finally, recall that $\iota_*\pi_1(Z) \subset \{e\} \times \pi_1(A(\mathbb{C}))$ and $\iota_*\pi_1(W_g(\mathbb{C})) \subset \pi_1(A(\mathbb{C})) \times \{e\}$, and the maps are induced by projection and embedding, correspondingly. That proves the conclusion $\iota_*\pi_1(W(\mathbb{C})) \lesssim \iota_*\pi_1(W_g(\mathbb{C})) \times \iota_*\pi_1(\mathrm{Cl}(\mathrm{pr}\, W(\mathbb{C})))$. Surjectivity has been noted already. □

Corollary 4.3.4.2. *Let W' be a $\mathcal{T}$-irreducible closed set, and let $V' = Cl pr W'$. Then $\pi(pr W')$ is a finite index subgroup of $\pi(V')$, i.e. $\pi(V') \lesssim pr\, \pi(W')$.*

Proof. The proof of the analogous corollary 4.3.3.7 carries on verbatim, except that now we do not need the assumption of normality. □

Recall we use Lemma 3.2.2.2.

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