Reconstruction of homogeneous relational structures

Silvia Barbina,*
Università di Torino,
via Carlo Alberto 10,
I-10123 Torino,
Italy

Dugald Macpherson,
Department of Pure Mathematics,
University of Leeds,
Leeds LS2 9JT, England,
h.d.macpherson@leeds.ac.uk

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1 Introduction

This paper contains a result on the reconstruction of certain homogeneous transitive ω-categorical structures from their automorphism group. The structures treated are relational. In the proof it is shown that their automorphism group contains a *generic pair* (in a slightly non-standard sense, coming from Baire category).

Reconstruction results give conditions under which the abstract group structure of the automorphism group $\mathrm{Aut}(\mathcal{M})$ of an ω-categorical structure $\mathcal{M}$ determines the topology on $\mathrm{Aut}(\mathcal{M})$, and hence determines $\mathcal{M}$ up to bi-interpretability, by [1]; they can also give conditions under which the abstract group $\mathrm{Aut}(\mathcal{M})$ determines the permutation group $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M} \rangle$, so determines $\mathcal{M}$ up to bi-definability. One such condition has been identified by M. Rubin in [10], and it is related to the definability, in $\mathrm{Aut}(\mathcal{M})$, of point stabilisers. If the condition holds, the structure is said to have a *weak* $\forall\exists$ *interpretation*, and $\mathrm{Aut}(\mathcal{M})$ determines $\mathcal{M}$ up to bi-interpretability or, in some cases, up to bi-definability.

A better-known approach to reconstruction is via the 'small index property': an ω-categorical stucture $\mathcal{M}$ has the *small index property* if any subgroup of $\mathrm{Aut}(\mathcal{M})$ of index less than $2^{\aleph_0}$ is open. This guarantees that the abstract group structure of $\mathrm{Aut}(\mathcal{M})$ determines the topology, so if $\mathcal{N}$ is ω-categorical

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with $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$ then $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable. All the structures handled in this paper are known to have the small index property. However, in unpublished work A. Singerman has shown that there is an ω-categorical structure which has a weak $\forall\exists$-interpretation but does not have the small index property – it is the well-known example whose automorphism group has a quotient which is elementary abelian of rank $2^{\aleph_0}$. There are also familiar examples, the random tournament and the universal homogeneous partial order, which are proved in [10] to have a weak $\forall\exists$-interpretation, but for which the small index property is unknown. On the other hand, there are easy examples with the small index property but no weak $\forall\exists$-interpretation: for example, an equivalence relation with all classes of size two, or indeed, any ω-categorical structure whose automorphism goup has non-trivial centre – see Proposition 1.2.1 of [3].

Our belief, reinforced by the present paper and by [2], is that the existence of weak $\forall\exists$-interpretations is rather easier to prove than the small index property, and that there are many different (slightly *ad hoc*) approaches. The existence of a weak $\forall\exists$-interpretation provides extra information which apparently does not follow from the small index property; namely, that the structure $\mathcal{M}$ is interpretable (with parameters) in $\mathrm{Aut}(\mathcal{M})$.

In this paper, we describe a method for obtaining weak $\forall\exists$ interpretations for a range of relational structures; these include universal homogeneous k-kypergraphs, homogeneous K_m-free graphs and the 'Henson digraphs'. For all of these, the small index property is known, through the proof in [7] together with various extension lemmas for partial isomorphisms proved by Herwig [6]. These lemmas are an analogue of Hrushovski's extension lemma for graphs [8], which is needed to ensure that the argument in [7] applies to the random graph. Hrushovski's lemma states that given any finite graph Γ, there is a finite graph Δ which contains Γ as an induced subgraph, such that any isomorphism between subgraphs of Γ extends to an automorphism of Δ. We shall need suitable versions of Herwig's extension lemmas, relativised to partial isomorphisms having a specific cycle type. It is possible that the method that we give here for obtaining weak $\forall\exists$ interpretations might work where Herwig's method for small index does not. In particular, the extension lemmas required here only involve extending *two* partial isomorphisms.

Section 2 of the paper contains the required theory of 'generic pairs' of automorphisms, and a description of some sufficient conditions for their existence. In Section 3 we show how to derive a weak $\forall\exists$ interpretation from the existence of such a pair. Section 4 contains a description of Herwig's arguments in [6], and of how to modify them so that the construction in Section 2 works.

The theorem we prove is the following. We warn, though, that our definition of $\forall\exists$-interpretation is marginally weaker than that of Rubin (see Definition 1.3 below, and the remark before it). In the theorem below our notation is as follows: if G is a permutation group on a set Y, and $y \in Y$, then G_y denotes the stabiliser of y, and if $g \in G$ then $\mathrm{fix}(g) := \{y \in Y : g(y) = y\}$. In this paper, a *homogeneous structure* is always a countable relational structure such that every isomorphism between finite substructures extends to an automorphism. The transitivity assumption below is to keep the statements as simple as possible. All the examples we have in mind are transitive.

Theorem 1.1 *Let $\mathcal{M}$ be an ω-categorical transitive homogeneous relational structure, let $d \in \mathcal{M}$ and $f_1, f_2 \in \mathrm{Aut}(\mathcal{M})$ be such that*

1. $\mathrm{fix}(f_1) = \mathrm{fix}(f_2) = \{d\}$,
2. *the conjugacy class* $(f_1, f_2)^{\mathrm{Aut}(\mathcal{M})_d}$ *is comeagre in* $X_d \times X_d$,

where $X_d = \{g \in \mathrm{Aut}(\mathcal{M}) : \mathrm{fix}(g) = \{d\}\}$.
Then $\mathcal{M}$ has a weak $\forall\exists$ interpretation.

From the above, via Theorem 1.6 below, we obtain the following, which also follows from the work of Herwig. The 'Henson digraphs' are the family of size continuum of countable homogeneous digraphs described by Henson in [5].

Corollary 1.2 *Let $\mathcal{M}$ be a universal homogeneous K_m-free graph, a universal homogeneous k-hypergraph or a Henson digraph, and let $\mathcal{N}$ be ω-categorical and such that* $\mathrm{Aut}(\mathcal{N}) \cong \mathrm{Aut}(\mathcal{M})$. *Then $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable.*

We now give the definition of a weak $\forall\exists$ interpretation (to be found in [10]). In fact, we work (throughout this paper) with a slightly weaker notion of $\forall\exists$ equivalence formula than that of Rubin, since in the final clause below we do not require that the formula defines an equivalence relation which is invariant under conjugation in *all groups*. Inspection of Rubin's proofs shows that this does not affect his applications to bi-interpretability and bi-definability.

Definition 1.3 *Let G be a group, and let $\bar{g} = \langle g_1, \ldots, g_n \rangle \in G^n$. Let $\phi(\bar{g}, x, y)$ be a formula in the language of groups with parameters $\bar{g}$. Let $C := g_1^G$. We say that ϕ is an $\forall\exists$* ***equivalence formula*** *for G if:*

- *ϕ is $\forall\exists$;*
- *Group theory $\vdash \forall\bar{u}(\phi(\bar{u}, x, y)$ is an equivalence relation on the conjugacy class of $u_1)$;*

- *$\phi(\bar{g}, x, y)$ defines a conjugacy invariant equivalence relation on C.*

We shall write E^ϕ for the equivalence relation defined by ϕ.

Definition 1.4 (Weak $\forall\exists$ interpretation, transitive case) *Let $\mathcal{M}$ be ω-categorical, and such that $\mathrm{Aut}(\mathcal{M})$ acts transitively on $\mathcal{M}$. A **weak $\forall\exists$ interpretation** for $\mathcal{M}$ is a triple $\langle \phi, \vec{g}, \tau \rangle$, where ϕ is an $\forall\exists$-equivalence formula, $\vec{g} \in \mathrm{Aut}(\mathcal{M})^n$, τ is an isomorphism between the permutation groups $\langle \mathrm{Aut}(\mathcal{M}), C/E^\phi \rangle$ and $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M} \rangle$, that is, $\tau : C/E^\phi \to \mathcal{M}$ is a bijection such that for all $g, h \in \mathrm{Aut}(\mathcal{M})$*

$$[\tau(h/E^\phi)]^g = \tau(h^g/E^\phi).$$

By the Ryll-Nardzewski theorem, $\mathrm{Aut}(\mathcal{M})$ has finitely many orbits on $\mathcal{M}$. We can thus extend the definition of a weak $\forall\exists$ interpretation to the general case when $\mathcal{M}$ is not transitive.

Definition 1.5 (Weak $\forall\exists$ interpretation) *Let $\mathcal{M}$ be an ω-categorical structure with 1-types $P_1, \ldots, P_n$. A **weak $\forall\exists$ interpretation** for $\mathcal{M}$ is a tuple $\langle \vec{\phi}, \vec{g}, \vec{\tau} \rangle$, where $\vec{\phi} = (\phi_1, \ldots, \phi_n)$ are $\forall\exists$ equivalence formulae, $\vec{g} = (\vec{g}^1, \ldots, \vec{g}^n)$ are tuples of elements of $\mathrm{Aut}(\mathcal{M})$, $\vec{\tau} = (\tau_1, \ldots, \tau_n)$ are maps such that each triple $\langle \phi_i, \vec{g}^i, \tau_i \rangle$ is a weak $\forall\exists$ interpretation for the structure induced on P_i.*

We can now state Rubin's main result. A structure $\mathcal{M}$ is *without algebraicity* if $\mathrm{acl}(A) = A$ for all $A \subset M$.

Theorem 1 (Rubin, 1987) *Let K be the class of ω-categorical structures without algebraicity. Let $\mathcal{M} \in K$ have a weak $\forall\exists$-interpretation, and let $\mathcal{N} \in K$ be such that $\mathcal{M} \cong \mathcal{N}$ as pure groups. Then $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M} \rangle \cong \langle \mathrm{Aut}(\mathcal{N}), \mathcal{N} \rangle$, that is, $\mathcal{M}$ and $\mathcal{N}$ are bi-definable.*

We also state the following consequence of Rubin's work, noted in [10] and proved in [3] (Proposition 1.1.10).

Theorem 1.6 *Let $\mathcal{M}$ and $\mathcal{N}$ be ω-categorical structures with isomorphic automorphism groups, and suppose that $\mathcal{M}$ has a weak $\forall\exists$-interpretation. Then $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable.*

The existence of weak $\forall\exists$-interpretations also yields the following (which is not terribly surprising, given that the theory of the automorphism group is likely to be far wilder than that of the structure).

Proposition 1.7 *Suppose that the ω-categorical structure $\mathcal{M}$ has a weak $\forall\exists$-interpretation. Then the structure $\mathcal{M}$ is interpretable with parameters in $\mathrm{Aut}(\mathcal{M})$.*

Proof If $\mathcal{M}$ has a weak $\forall\exists$-interpretation then there are conjugacy classes $C_1, \ldots, C_n$ and for each i a $G := \mathrm{Aut}(\mathcal{M})$-invariant equivalence relation E_i on C_i such that the domain of $\mathcal{M}$ can be identified with the disjoint union of the C_i/E_i, with G acting by conjugation (we can use a formal device to ensure that the C_i/E_i are disjoint, e.g. replacing a conjugacy class by a conjugacy class of pairs $(g, 1)$). Every $\emptyset$-definable relation of $\mathcal{M}$ is a finite union of G-orbits, and G-orbits on $(C_1/E_1)^{r_1} \times \ldots \times (C_t/E_t)^{r_t}$ are definable in the group language. □

In the second part of [10], Rubin showed that many binary relational ω-categorical structures have weak $\forall\exists$-interpretations. In a further paper [2] the first author has exhibited weak $\forall\exists$-interpretations (again, with a slightly different version of the notion) for $\aleph_0$-dimensional projective geometries over a finite field, possibly equipped with a non-degenerate sequilinear form.

2 Structures with a generic pair of automorphisms

Let $\mathcal{M}$ be a transitive ω-categorical structure, $G = \mathrm{Aut}(\mathcal{M})$, $d \in \mathcal{M}$ and $X_d \subseteq G$ be the set of automorphisms fixing only d:

$$X_d := \{p \in G : \mathrm{fix}(p) = \{d\}\}.$$

It is well known that G can be given the structure of a Polish space (i.e. a completely metrisable space which is also separable): indeed, let $\{x_i : i \in \omega\}$ list the domain of $\mathcal{M}$, and put $d(g, h) = 1/2^n$ where n is least such that $g(x_n) \neq h(x_n)$ or $g^{-1}(x_n) \neq h^{-1}(x_n)$. Clearly, X_d is closed in G. Hence X_d is a Polish space in its own right, and so are the stabiliser G_d and the product space $X_d \times X_d$.

Definition 2.1 *Let $X \subseteq \mathrm{Aut}(\mathcal{M})$ be closed in $\mathrm{Aut}(\mathcal{M})$, so that X is a Polish space with the inherited topology. Suppose $H \leq \mathrm{Aut}(\mathcal{M})$ is a subgroup such that $X^H := \{x^h : x \in X, h \in H\} \subseteq X$, so that H acts on X by conjugation. A tuple $(g_1, \ldots, g_n) \in X^n$ is an H-**generic tuple** in X if the orbit $(g_1, \ldots, g_n)^H$ of H on X^n is comeagre in the Polish space X^n.*

Fact 2.2 *Any two H-generic n-tuples are conjugate in X^n under H.*

Proof This follows from the fact that orbits of H on X^n are either disjoint or equal. □

We shall be concerned with relational structures whose automorphism group contains a pair (f_1, f_2) of automorphisms such that $\mathrm{fix}(f_1) = \mathrm{fix}(f_2) = \{d\}$ and the pair (f_1, f_2) is G_d-generic in $X_d \times X_d$ (i.e. $(f_1, f_2)^{G_d}$ is comeagre in $X_d \times X_d$).

Suppose that $\mathcal{M}$ is an ω-categorical, transitive and homogeneous structure in the relational language $L = \{R_1, \ldots, R_n\}$. Let κ be the class of all finite substructures of $\mathcal{M}$. For $\mathcal{A} \in \kappa$, consider an expansion $\mathcal{A}'$ of $\mathcal{A}$ to the language $L' = \{R_1, \ldots, R_n, f_1, f_2, d\}$, where f_1 and f_2 are function symbols and d is a constant. Let κ' be the class consisting of all structures isomorphic to such $\mathcal{A}'$, where we require in addition that f_1, f_2 are automorphisms of the L-reduct $\mathcal{A}$, and $\text{fix}(f_1) = \text{fix}(f_2) = \{d\}$.

Definition 2.3 *Let κ' be the class of structures described above. Let $\mathcal{A}', \mathcal{B}'_1, \mathcal{B}'_2 \in \kappa'$ be such that $\mathcal{A}' \subseteq \mathcal{B}'_i$, $\mathcal{A}' = \mathcal{B}'_1 \cap \mathcal{B}'_2$, and suppose that $f_j^{\mathcal{A}'} \subseteq f_j^{\mathcal{B}'_i}$, for $i, j = 1, 2$. Let $\mathcal{C}'$ be the disjoint union of $\mathcal{B}'_1$ and $\mathcal{B}'_2$ over $\mathcal{A}'$ so that:*
1. $\mathcal{B}'_i \leq \mathcal{C}'$, $i = 1, 2$;
2. $\mathcal{C}' = \mathcal{B}'_1 \cup \mathcal{B}'_2$, $f_i^{\mathcal{C}'} = f_i^{\mathcal{B}'_1} \cup f_i^{\mathcal{B}'_2}$;
3. for all relation symbols $R \in L$ and n-tuples $\bar{a} \in \mathcal{C}'^n$, $\mathcal{C}' \models R\bar{a}$ if and only if $\bar{a} \in \mathcal{B}'_i$ for some $i \in \{1, 2\}$ and $\mathcal{B}'_i \models R\bar{a}$.
*Then $\mathcal{C}'$ is called the **free amalgam** of $\mathcal{B}'_1$ and $\mathcal{B}'_2$.*

*If for all $\mathcal{A}', \mathcal{B}'_1, \mathcal{B}'_2 \in \kappa'$ we have $\mathcal{C}' \in \kappa'$, we say that κ' has the **free amalgamation property**.*

Free amalgamation is generally treated as a property of structures in a relational language. As such, k-hypergraphs, K_n-free graphs and, more generally, the class of structures described by Herwig in [6] all enjoy free amalgamation. The property does not hold, for instance, for the class of all finite tournaments or all finite partial orders. Definition 2.3 may be viewed in this way, if we parse the function symbols as binary relation symbols.

If we assume that κ' has the free amalgamation property, Fraïssé's theorem ensures that κ' has a Fraïssé limit $\mathcal{N}' = (\mathcal{N}, f_1, f_2, d)$, which is countable and homogeneous (meaning that isomorphisms between finite *substructures* extend to automorphisms). The structure will be characterised up to isomorphism by being a union of a countable chain of members of κ', and having the following property, denoted (EMB).

Let $\mathcal{A}', \mathcal{B}' \in \kappa'$, with $\mathcal{A}' \leq \mathcal{B}'$. Let $g : \mathcal{A}' \to \mathcal{N}'$ be an embedding. Then g extends to an embedding $h : \mathcal{B}' \to \mathcal{N}'$.

Our goal is to ensure $\mathcal{M}$ is isomorphic to the L-reduct $\mathcal{N}$ of $\mathcal{N}'$ (so they can be identified), and that the automorphisms f_1, f_2 form a G_d-generic pair in $X_d \times X_d$. The proof is via a Banach-Mazur game, and it requires that the class κ has a certain 'fixed point extension property'.

Definition 2.4 *Let S be a relational language, π a class of finite S-structures. Then π is said to have the **fixed point extension property** (FEP) for finite partial isomorphisms if for all $\mathcal{A} \in \pi$, and $p_1, \ldots, p_n$ finite partial isomorphisms of $\mathcal{A}$ such that* $\text{fix}(p_1) = \cdots = \text{fix}(p_n) = \{d\}$*, there are $\mathcal{B} \in \pi$ such* *that $\mathcal{A} \leq \mathcal{B}$, and $f_1, \ldots, f_n \in \mathrm{Aut}(\mathcal{B})$ with $p_i \subseteq f_i$ and $\mathrm{fix}(f_i) = \{d\}$ for $i = 1, \ldots, n$.*

We shall use the property with $n = 2$, which we call FEP_2. Section 3 below will be devoted to proving FEP for a range of different classes of relational structures.

We first check that $\mathcal{N} \cong \mathcal{M}$.

Lemma 2.5 *Let $\mathcal{N}'$ be the Fraïssé limit of the class κ' of finite structures in the language L' described above, and suppose κ has FEP_2. Then the reduct $\mathcal{N} := \mathcal{N}'|_L$ is isomorphic to $\mathcal{M}$.*

Proof Let $\mathcal{A}$ be a finite substructure of $\mathcal{N}$. We want to show that for any finite L-structure $\mathcal{B}$ such that $\mathcal{A} \leq \mathcal{B}$, $\mathcal{B}$ embeds into $\mathcal{N}$ over $\mathcal{A}$.

Since $\mathcal{N}' = (\mathcal{N}, f_1, f_2, d)$ is the Fraïssé limit of κ', it is a union of a chain of members of κ'. Hence there is a finite L'-structure $\mathcal{C}' := (\mathcal{C}, f_1, f_2, d) \leq (\mathcal{N}, f_1, f_2, d)$, such that $\mathcal{C}' \in \kappa'$ and $\mathcal{A} \leq \mathcal{C}'|_L$. Let $\mathcal{D}$ be an L-amalgam of $\mathcal{C}$ and $\mathcal{B}$ over $\mathcal{A}$ (we use here that $\mathcal{M}$ is homogeneous, so that κ has the amalgamation property). Let $\mathcal{D}'$ be the expansion of $\mathcal{D}$ to L', where d, f_1, f_2 are interpreted as in $\mathcal{C}'$ (so the f_i are partial).

By FEP_2 there is a finite L'-structure $\mathcal{E}' = (\mathcal{E}, f_1', f_2', d) \in \kappa'$ such that:
1. $\mathcal{D}' \leq \mathcal{E}'$;
2. $f_i \subseteq f_i'$ for $i = 1, 2$;
3. $\mathrm{fix}(f_i') = \{d\}$ for $i = 1, 2$.

By the universality and homogeneity of $\mathcal{N}'$ with respect to structures in κ', $\mathcal{E}'$ embeds in $\mathcal{N}'$ over $\mathcal{C}'$. It follows that $\mathcal{B}$ embeds in $\mathcal{N}$ over $\mathcal{A}$, as required. □

By the last lemma, we may write $\mathcal{M}' = (\mathcal{M}, f_1, f_2, d)$, in place of $\mathcal{N}'$. Clearly $f_i \in \mathrm{Aut}(\mathcal{M})$ and $\mathrm{fix}(f_i) = \{d\}$, for each i.

We now prove that the automorphisms f_1, f_2 of $\mathcal{M}$ are a generic pair.

Proposition 2.6 *Let $\mathcal{M}$ be an ω-categorical, transitive and homogeneous structure in the relational language $L = \{R_1, \ldots, R_n\}$. Adopt the notation $d, f_1, f_2, L', \kappa, \kappa'$ of the discussion above, and suppose that κ' has the free amalgamation property and that κ has FEP_2. Let $(\mathcal{M}, f_1, f_2, d)$ be the Fraïssé limit of κ', so $d = \mathrm{fix}(f_1) = \mathrm{fix}(f_2)$, let $G = \mathrm{Aut}(\mathcal{M})$, and put $\mathcal{D} = (f_1, f_2)^{G_d}$. Then $\mathcal{D}$ is comeagre in $X_d \times X_d$.*

Proof We play the Banach-Mazur game of $\mathcal{D}$. Let

$$P := \{f : \mathcal{M} \to \mathcal{M} : f \text{ is a finite partial isomorphism with } \mathrm{fix}(f) = d\}.$$

Then P is partially ordered by inclusion. Now let $P^2 = P \times P$. The game is played as follows: players I and II choose an increasing sequence of elements of P^2

$$(p_{1,0},\ p_{2,0}), (p_{1,1},\ p_{2,1}), (p_{1,2},\ p_{2,2}), \ldots$$

so that $p_{1,i} \subseteq p_{1,i+1}$ and $p_{2,i} \subseteq p_{2,i+1}$ for all i. Player I starts the game and chooses $(p_{1,i}, p_{2,i})$ for i even, player II chooses at odd stages. Player II wins if and only if $(p_1, p_2) := (\bigcup_{i\in\omega} p_{1,i}, \bigcup_{i\in\omega} p_{2,i}) \in \mathcal{D}$. Player II has a winning strategy iff $\mathcal{D}$ is comeagre in $X_d \times X_d$. Player II can always play so that at stage i, for $i > 1$ and even,

1. he can choose to put any particular $x \in \mathcal{M}$ into the domain and range of $p_{1,i}, p_{2,i}$;
2. $(p_{1,i}, p_{2,i}) \in P^2$ and $\mathrm{dom}(p_{1,i}) = \mathrm{ran}(p_{1,i}) = \mathrm{dom}(p_{2,i}) = \mathrm{ran}(p_{2,i})$;

Player II will also ensure

3. $(\mathcal{M}, p_1, p_2, d)$ is *weakly homogeneous*, that is: if $(\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d)$, $(\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d)$ are finite L-structures, $(\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d) \leq (\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d)$, and $\alpha : (\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d) \to (\mathcal{M}, p_1, p_2, d)$ is an embedding, there is an embedding $\tilde{\alpha} : (\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d) \to (\mathcal{M}, p_1, p_2, d)$ extending α.

At stage $i+1$, i even, player II is given a finite structure $(\Delta_i, p_{1,i}, p_{2,i}, d)$, where the $p_{j,i}$ are finite partial isomorphisms of Δ_i. Here, $\Delta_i = \mathrm{dom}(p_{1,i}) \cup \mathrm{dom}(p_{2,i}) \cup \mathrm{ran}(p_{1,i}) \cup \mathrm{ran}(p_{2,i})$. For points 1. and 2., for any $x \in \mathcal{M}$, II can consider $\Delta^*_{i+1} := \Delta_i \cup \{x\}$ and use FEP_2 to obtain extensions Δ_{i+1} of Δ^*_{i+1}, and $p_{1,i+1}, p_{2,i+1} \in \mathrm{Aut}(\Delta_{i+1})$ of $p_{1,i}, p_{2,i}$, each fixing only d. By homogeneity of $\mathcal{M}$, Δ_{i+1} can be chosen to be a substructure of $\mathcal{M}$ containing Δ_i.

In order for 3. to hold, a typical task for II is the following: for $(\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d) \leq (\Delta_i, p_{1,i}, p_{2,i}, d)$ and $(\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d) \geq (\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d)$, II has to ensure that $(\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d)$ embeds in $(\Delta_{i+1}, p_{1,i+1}, p_{2,i+1}, d)$ over $(\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d)$. First, Δ_i is a structure in κ containing $d \in \mathcal{M}$, and admits partial isomorphisms $p_{1,i}, p_{2,i}$ each with fixed point d. Thus, by FEP_2, there is $(\Delta^*_i, p^*_{1,i}, p^*_{2,i}, d) \in \kappa'$ with $(\Delta_i, p_{1,i}, p_{2,i}, d) \leq (\Delta^*_i, p^*_{1,i}, p^*_{2,i}, d)$. Now, using free amalgamation in κ', there is $(\Delta^*_{i+1}, p^*_{1,i+1}, p^*_{i+2}, d) \in \kappa'$, the free amalgam over $(\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d)$ of $(\Delta^*_{i+1}, p^*_{1,i+1}, p^*_{i+2}, d) \in \kappa'$ and $(\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d)$ (replacing $\mathcal{B}$ by a copy $\mathcal{B}'$ with $\mathcal{B}' \cap \Delta^*_i = \mathcal{A}$, if necessary). Finally, since $(\mathcal{M}, d)$ is homogeneous, there is an embedding g of Δ^*_{i+1} into $\mathcal{M}$ over Δ_i. Let $\Delta_{i+1} := g(\Delta^*_{i+1})$, and put $p_{j,i+1} := g \circ p^*_{j,i+1} \circ g^{-1}$, for $j = 1, 2$.

It follows that Player II can play so that $(\mathcal{M}, p_1, p_2, d)$ has property (EMB). By (1) and (2) it is the union of a countable chain of members of κ', so $(\mathcal{M}, p_1, p_2, d) \cong \mathcal{M}'$. Thus, there is $h \in G_d$ with $(p_1^h, p_2^h) = (f_1, f_2)$, as required. $\square$

Lemma 2.7 *The set $f_1^{G_d}$ is comeagre in X_d.*

Proof Consider the projections $\mathcal{D}_1$, $\mathcal{D}_2$ of $\mathcal{D}$ to the first and second coordinates respectively. Clearly $\mathcal{D} \subseteq \mathcal{D}_1 \times \mathcal{D}_2$. Since $\mathcal{D}$ is comeagre in $X_d \times X_d$, $\mathcal{D}_1 \times \mathcal{D}_2$ also is. Via the Kuratowski-Ulam theorem (see e.g. Theorem 8.41 of [9]) , it is easy to see that $\mathcal{D}_1$ is comeagre in X_d. Note that $f_1^{G_d} = \mathcal{D}_1$. □

We can now prove our main result:

Proposition 2.8 *Let $g \in f_1^{G_d}$ and $\mathcal{D}_g := \{h \in X_d : (g, h) \in \mathcal{D}\}$. Then $\mathcal{D}_g$ is comeagre in X_d for all $g \in f_1^{G_d}$.*

Proof Since $\mathcal{D}$ has the Baire Property, by the Kuratowski-Ulam theorem, the set

$$\{h \in X_d : \mathcal{D}_h \text{ is comeagre in } X_d\}$$

is comeagre in X_d. Also, $f_1^{G_d}$ is comeagre in X_d, so

$$\{h \in X_d : \mathcal{D}_h \text{ is comeagre in } X_d\} \cap f_1^{G_d} \neq \emptyset.$$

Pick $g \in \{h \in X_d : \mathcal{D}_h \text{ is comeagre in } X_d\} \cap f_1^{G_d}$, so that $\mathcal{D}_g$ is comeagre in X_d. Note that G_d is transitive on $f_1^{G_d}$. Also, if $\mathcal{D}_g$ is comeagre in X_d and h is conjugate to g under G_d, then $\mathcal{D}_h$ is also comeagre in X_d. Therefore, $\mathcal{D}_g$ is comeagre in X_d for all $g \in f_1^{G_d}$. □

3 The interpretation

Let $\mathcal{M}$ be an ω-categorical, transitive and homogeneous structure in a relational language which satisfies the hypotheses of Proposition 2.6. Then the Fraïssé limit $(\mathcal{M}, f_1, f_2, d)$ constructed in 2.6 exists, $\text{fix}(f_1) = \text{fix}(f_2) = \{d\}$, and $(f_1, f_2) \in \text{Aut}(\mathcal{M})^2$ is a G_d-generic pair of automorphisms in $X_d \times X_d$. We give a weak $\forall\exists$ interpretation for $\mathcal{M}$ based on an equivalence relation defined in terms of our comeagre orbit on pairs $\mathcal{D} = (f_1, f_2)^{G_d}$, with the notation of Section 2.

Define $\mathcal{D}^G = \{(g_1, g_2)^g : (g_1, g_2) \in \mathcal{D},\ g \in G\}$, and let $\mathcal{D}_1^G$ be the projection of $\mathcal{D}^G$ to the first coordinate; so $\mathcal{D}_1^G$ is a conjugacy class of G. Since we assume G to be transitive, for each $a \in \mathcal{M}$ there is $g \in G$ such that $a^g = d$. The set $\mathcal{D}^G$ consists of certain pairs (h_1, h_2) such that $\text{fix}(h_1) = \text{fix}(h_2)$ is a singleton, and for each $a \in \mathcal{M}$ there is a pair in $\mathcal{D}^G$ fixing a. We shall define an equivalence relation on $\mathcal{D}_1^G$ which identifies automorphisms having the same fixed point.

Lemma 3.1 *Let E be the following equivalence relation on $\mathcal{D}_1^G$:*

$$g_1 E g_2 \iff \mathrm{fix}(g_1) = \mathrm{fix}(g_2).$$

Then for $g_1, g_2 \in \mathcal{D}_1^G$

$$g_1 E g_2 \iff \exists f \in G((g_1, f),\ (g_2, f) \in \mathcal{D}^G),$$

so E is $\exists$-definable with parameters in the language of groups.

Proof ($\Leftarrow$) is immediate. Indeed, if $(g_1, f), (g_2, f) \in \mathcal{D}^G$, then $\mathrm{fix}(g_1) = \mathrm{fix}(f) = \mathrm{fix}(g_2)$.

($\Rightarrow$) Let $g_1, g_2 \in \mathcal{D}_1^G$ have the same fixed point e. Then, by transitivity of G, find a conjugating element $h \in G$ so that $\mathrm{fix}(g_1^h) = \mathrm{fix}(g_2^h) = d$. By 2.8, $\mathcal{D}_{g_1^h}$ and $\mathcal{D}_{g_2^h}$ are comeagre in X_d. Hence $\mathcal{D}_{g_1^h} \cap \mathcal{D}_{g_2^h} \neq \emptyset$. Choose $k \in \mathcal{D}_{g_1^h} \cap \mathcal{D}_{g_2^h}$, so that both $(g_1^h, k) \in \mathcal{D}$ and $(g_2^h, k) \in \mathcal{D}$. But then $(g_1, k^{h^{-1}}) \in \mathcal{D}^G$ and $(g_2, k^{h^{-1}}) \in \mathcal{D}^G$, so $k^{h^{-1}}$ is our required f.

It follows that E is $\exists$-definable in the language of groups via the following formula.

$$xEy \leftrightarrow \exists vwz\ (x, v)^w = (g_1, g_2) \wedge (y, v)^z = (g_1, g_2),$$

where g_2, g_2 are parameters with $(g_1, g_2) \in \mathcal{D}$. It is easy to check that the formula

$$xEy\ \wedge E \text{ is an equivalence relation on } \mathcal{D}_1^G$$

is an $\forall\exists$ equivalence formula. $\square$

The following theorem follows from the above discussion.

Theorem 3.2 *Let $\mathcal{M}$ be an ω-categorical, transitive and homogeneous structure in a relational language which satisfies the hypotheses of Proposition 2.6. Then $\mathcal{M}$ has a weak $\forall\exists$ interpretation.*

4 Extension lemmas

We state a range of extension lemmas which will yield FEP_2, so make the Banach-Mazur game described above work for various relational structures. The proofs are essentially due to Bernhard Herwig. The motivation in Herwig's work was to obtain a proof of the small index property for the structures treated, by producing an analogue of Hrushovski's extension lemma for graphs [8] used in [7]. Herwig's proofs cover the extension property for partial isomorphisms without any restriction on the cycle type of the isomorphisms involved. However, minimal modifications of his proofs yield the extension property for finite partial isomorphisms having a unique fixed point. We shall give a brief indication of the changes needed. More details can be found in the first author's PhD thesis [3].

Herwig's proofs are by induction on the maximal arity k of the relation symbols in the language S concerned, and, later, on the maximal size of certain forbidden configurations. In both cases the induction hypothesis is used by reducing k as follows: a k-ary relation symbol $R\bar{x}$ is replaced by $(k-1)$-ary symbols R_a, one for each element a of the smaller structure, to be interpreted in the obvious way ($R_a\bar{b} \iff Ra\bar{b}$).

Herwig produces separate extension lemmas for three different classes of structures:

1. the class of all finite structures in a given finite relational language S;
2. the class of finite K_m-free graphs, for $m \in \omega$;
3. the class of all finite irreflexive structures omitting certain configurations, described below.

Case 1. is needed in order to prove the base step in the induction arguments for 2. and 3. The class of K_m-free graphs in 2. is included in the class covered by 3. Nevertheless, it is treated separately as a paradigm of the more intricate case 3. All of Herwig's proofs can be adapted to yield the fixed point extension property FEP required in our argument. We indicate briefly the change needed in Herwig's argument in case 1, and state the result for the structures in 3. For our version of the arguments we need the following fact, which is easy to prove:

Fact 4.1 *Let X, Y be finite sets such that $|X| = |Y| \geq 2$. Then there is a fixed-point free bijection $\alpha : X \to Y$.*

The theorem we require in the case of general relational structures (i.e., for case (1) above) is the following:

Theorem 4.2 *Let S be a finite relational language, and let κ be the class of all finite S-structures. Then κ has FEP, the fixed point extension property for partial isomorphisms.*

Herwig's proof involves what he calls partial *permorphisms*, rather than isomorphisms. For the purposes of our explanation, in what follows a permorphism can be thought of as an isomorphism. Let $\mathcal{A} \in \kappa$ be given, let $d \in \mathcal{A}$, and let $p_1, \dots p_n$ be partial permorphisms of $\mathcal{A}$ such that $\text{fix}(p_i) = \{d\}$ for each $i = 1, \dots, n$. We want to find $\mathcal{B} \in \kappa$ and total permorphisms $f_1, \dots, f_n$ of $\mathcal{B}$ such that $f_i \supseteq p_i$ and $\text{fix}(f_i) = \{d\}$. The first part of Herwig's argument consists in embedding $\mathcal{A}$ in a finite S-structure $\mathcal{C}$ by adding enough realisations of positive atomic types over $\mathcal{A}$ so that each such type has the same number of realisations in $\mathcal{C}$. An inclusion/exclusion argument shows that if ϕ is a positive atomic type over $\mathrm{dom}(p_i)$, then the translate ϕ^{p_i} by p_i (which will be a type over $\mathrm{ran}(p_i)$) has exactly the same number of realisations in $\mathcal{C}$ as ϕ. Then each p_i can be extended to a map $h_i : \mathcal{C} \to \mathcal{C}$ which maps the realisations of ϕ bijectively to the realisations of ϕ^{p_i}. The h_i are not total permorphisms of $\mathcal{C}$, but they essentially determine the required extensions f_i of the p_i. We can ensure that $\mathrm{fix}(h_i) = \{d\}$ as follows: for each positive atomic type ϕ over $\mathcal{A}$ we arrange that ϕ has at least two realisations in $\mathcal{C} \setminus \mathcal{A}$. Then, in virtue of 4.1, $h_i \setminus p_i$ can be chosen to be fixed point-free for each $i = 1, \ldots, n$.

The rest of the proof goes through exactly as in [6], with one further check needed, namely, that the extensions f_i of the p_i that one obtains eventually have a single fixed point. But this follows easily from the fact that the h_i have a single fixed point.

Herwig's other proofs can be modified in a similar way to yield FEP for the classes of structures he describes in [6]. His methods cover, for each of the following homogeneous structures $\mathcal{M}$, the class of structures isomorphic to a finite substructure of $\mathcal{M}$.

- the universal homogeneous k-hypergraph;
- the universal homogeneous K_m-free graph, for any $m \geq 3$;
- each Henson digraph;
- the arity k analogues of triangle free graphs, namely, for any fixed k, the homogeneous k-hypergraph which is universal subject to not admitting a $(k+1)$-set all of whose k-subsets are hyperedges.

Henson digraphs and K_m-free graphs are also handled by Rubin, as they are in fact 'simple' in the sense of Rubin ([10], §3).

We refer the reader to [6] for the definitions involved in the statement of the following theorem, a slight adaptation of the main theorem of [6]:

Theorem 4.3 *Let S be a finite relational language, $\mathfrak{F}$ a set of finite S structures which are irreflexive and packed, $\mathfrak{L}$ a set of irreflexive link structures. Then $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ has FEP, the fixed point extension property for finite partial isomorphisms.*

The discussion in Sections 1 and 2 yields the following as a corollary:

Theorem 4.4 *Let $\mathfrak{L}$ be a set of link structures and $\mathfrak{F}$ be a set of finite irreflexive packed structures in a finite relational language. Let $\mathcal{M}$ be the Fraïssé limit of the class $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$, and assume that $\mathcal{M}$ is transitive. Then $\mathcal{M}$ has a weak $\forall\exists$ interpretation.*

References

[1] G. Ahlbrandt, M. Ziegler. Quasi-finitely axiomatisable totally categorical theories. *Stability in model theory (Trento, 1984)*, Ann. Pure Appl. Logic 30:63–82, 1986.

[2] S. Barbina. Reconstruction of classical geometries from their automorphism groups. Provisionally accepted for publication in the *Journal of the London Mathematical Society* .

[3] S. Barbina. *Automorphism groups of omega-categorical structures.* PhD thesis. University of Leeds, 2004.

[4] D. Evans. Examples of $\aleph_0$-categorical structures, in *Automorphisms of first order structures* (ed. R. Kaye and D. Macpherson, Cambridge University Press, 1994) 33-72.

[5] C.W. Henson. Countable homogeneous relational structures and $\aleph_0$-categorical theories, J. Symb. Logic 37:494–500, 1972.

[6] B. Herwig. Extending partial isomorphisms for the small index property of many ω-categorical structures. *Israel Journal of Mathematics* 107:93–123, 1998.

[7] W. A. Hodges, I. M. Hodkinson, D. Lascar and S. Shelah. The small index property for ω-stable ω-categorical structures and for the random graph. *Journal of the London Mathematical Society*, 48(2):204–218, 1993.

[8] E. Hrushovski. Extending partial automorphisms. *Combinatorica*, 12(4):411–416, 1992.

[9] A. Kechris. *Classical descriptive set theory*, Springer, New York, 1994.

[10] M. Rubin. On the reconstruction of $\aleph_0$-categorical structures from their autmorphism groups, *Proceedings of the London Mathematical Society* 69(3):225-249, 1992.