

Inextendibility of expanding cosmological models with symmetry

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Abstract

A new criterion for inextendibility of expanding cosmological models with symmetry is presented. It is applied to derive a number of new results and to simplify the proofs of existing ones. In particular it shows that the solutions of the Einstein-Vlasov system with T^2 symmetry, including the vacuum solutions, are inextendible in the future. The technique introduced adds a qualitatively new element to the available tool-kit for studying strong cosmic censorship.

1 Introduction

A central open question in mathematical relativity is that of strong cosmic censorship [10, 7]. It is convenient to formulate this in terms of the maximal Cauchy development [5]. Intuitively, this is the largest globally hyperbolic spacetime satisfying the Einstein equations which evolves from prescribed initial data on a Cauchy surface. Implicit in this is the assumption that

the Einstein equations are coupled to some suitable equations describing the motion of matter and that the matter model chosen does not suffer from unphysical singularities. (See [12], section 4 for a discussion of the latter point.) The strong cosmic censorship hypothesis says that the maximal Cauchy development should be inextendible. In other words it should not be possible to embed the maximal Cauchy development non-trivially in any other regular spacetime, at least for generic initial data.

There are some ambiguities in this formulation as just stated. When we say an extension do we mean only that the geometry can be extended, or that this should also hold for the matter fields? Should the extended metric and matter fields satisfy the Einstein equations and the equations of motion coming from the given matter model? To say that the extension is regular means that it satisfies certain differentiability properties. What degree of differentiability is to be chosen? This is not the place to give a general discussion of these questions - in the following they will only be answered insofar as this is directly relevant to the main discussion.

When discussing strong cosmic censorship for solutions which are suitable for being used as cosmological models there are two typical regimes which come up. These are the approach to a singularity and an epoch of unending expansion. This Letter is chiefly concerned with the second of these regimes although some remarks will also be made on the first. The approach to the singularity will be considered in detail in a related paper [8].

How can inextendibility be shown? A standard approach [7] is to show that either causal geodesics are complete or that, if they are incomplete, some curvature invariant blows up along any incomplete direction. Usually this requires detailed information about the asymptotic behaviour of solutions of the Einstein-matter equations in the given asymptotic regime and this is hard, even under strong symmetry assumptions on the solutions considered. This paper presents a method which in many cases is much simpler. It applies to spacetimes with Killing vectors.

2 Extensions of Killing vectors

Let X^α be a Killing vector field on a Lorentzian manifold $(M, g_{\alpha\beta})$. It satisfies the well-known relation

$$\nabla_\alpha \nabla_\beta X_\gamma = R_{\alpha\beta\gamma\delta} X^\delta. \tag{1}$$

Next suppose that the original spacetime can be embedded isometrically in another spacetime $(\tilde{M}, \tilde{g}_{\alpha\beta})$. Let H be the boundary of the image of M in \tilde{M} . Suppose further that if $p \in H$ there is a local coordinate system $\{x^1, x^2, x^3, x^4\}$ such that p is at the origin, the part of H covered by these coordinates coincides with the set defined by the equation $x^4 = f(x^1, x^2, x^3)$ for some continuous function f and the part of M covered coincides with the set defined by $x^4 < f(x^1, x^2, x^3)$. We now think of the equation (1) as referring to components with respect to these local coordinates.

It will be assumed that the metric $\tilde{g}_{\alpha\beta}$ is C^2 , so that the components of its curvature are continuous. Define $Y_{\alpha\beta} = \partial_\alpha X_\beta$. Then (1) implies, by writing out the covariant derivatives explicitly in terms of Christoffel symbols, the relation:

$$\partial_4 Y_{\beta\gamma} = A_{\beta\gamma}^{\sigma\tau} Y_{\sigma\tau} + B_{\beta\gamma}^\delta X_\delta \quad (2)$$

for some continuous coefficients $A_{\beta\gamma}^{\sigma\tau}$ and B^δ . Moreover $\partial_4 X_\alpha = Y_{4\alpha}$. Hence along the coordinate lines of x^4 the collection $(X_\alpha, Y_{\alpha\beta})$ satisfies a linear system of ordinary differential equations with coefficients which are continuous up to and including H . It follows that the components X_α and their first order partial derivatives are bounded as H is approached. Hence these components are uniformly continuous and as a consequence they extend continuously to H within the given coordinate system. To sum up, under the given assumptions, Killing vectors of $g_{\alpha\beta}$ have continuous local extensions to H .

3 Symmetric spacetimes

Consider first the case of a Lorentzian metric with T^2 -symmetry. In this case it is assumed that there are two commuting spacelike Killing vectors without fixed points. Call them X and Y . It is possible to choose local coordinates so that they take the form $\partial/\partial x$ and $\partial/\partial y$. Then the components of the metric in these coordinates are independent of x and y . It is common to define a quantity R as a constant times the square root of the determinant of the metric induced on an orbit of the local group action generated by the Killing vectors. This can also be defined in a coordinate-invariant way as the square root of the determinant of the matrix of inner products of the Killing vectors X and Y . With the latter definition it is clear from the discussion of the last section that when the metric has a suitable C^2 extension the function R extends continuously to the boundary H of the extension.

The information about R just obtained will now be compared with the results of [2]. If $g_{\alpha\beta}$ is the metric defined by the maximal globally hyperbolic development of some Cauchy data then the boundary H of this spacetime in any extension is called the Cauchy horizon. The boundary is known to satisfy the conditions required in the previous section - the function f can even be taken to be Lipschitz continuous [9]. Thus R extends continuously to H . It follows from Theorem 2 of [2] that if the spacetime concerned is a solution of the Einstein-Vlasov system which is initially expanding then R tends to infinity along any future-directed causal curve. As a consequence, no future-directed causal curve can tend to a point of H and no future extension of the original spacetime is possible. Thus the part of cosmic censorship referring to the future is settled for these spacetimes. To say it another way, it has been proved that the future Cauchy horizon is empty. Note that this result in particular covers vacuum spacetimes with this symmetry. With the method of the present paper it follows immediately from [3]. This result will now be formulated as a theorem:

Theorem The maximal globally hyperbolic development of data for the Einstein-Vlasov equation with T^2 symmetry which are expanding cannot be extended in a C^2 way so as to allow an inextendible future-directed causal curve in the original spacetime to be continued.

The inextendibility result just derived was not previously known for general T^2 -symmetric vacuum spacetimes or for plane-symmetric solutions of the Einstein-Vlasov equations, a very special case. It was known for Gowdy spacetimes but the proof was based on a detailed determination of the asymptotics [13] which was very complicated. Of course the asymptotics is in itself of great interest but the point of this paper is to show that the question of inextendibility can be handled with much simpler methods.

Consider next the case of spherical or hyperbolic symmetry. In these symmetry types there are three Killing vectors but these do not all commute and they have fixed points. Hence the approach used for T^2 -symmetry must be modified. To do this we use the fact that there are three Killing vectors X , Y and Z on the standard sphere with the property that the sum of the squares of their lengths is constant. This can be seen by considering the standard embedding of the unit sphere in \mathbf{R}^3 for which these Killing vectors can be taken as $x\partial_y - y\partial_x$, $y\partial_z - z\partial_y$ and $z\partial_x - x\partial_z$. There are corresponding Killing vectors on the spacetime and locally each of them extends continuously to H . The sum of the squares of the lengths of X , Y and Z is proportional to R and so it can be concluded that R extends

continuously to H . It follows that a spherically symmetric spacetime which is a maximal globally hyperbolic development where R goes to infinity along any future-directed causal geodesic cannot be extended to the future. In fact spherically symmetric spacetimes tend to recollapse but in the presence of a positive cosmological constant this result can be applied. In the case where it is known that r tends to infinity geodesic completeness has also been proved [14]. Nevertheless it represents a simplified proof of inextendibility in that case and related applications are likely to come up in the future.

Spacetimes with hyperbolic symmetry can be handled in a similar way. This time we choose Killing vectors X , Y and Z on hyperbolic space with the property that the combination $|X|^2 + |Y|^2 - |Z|^2$ is constant, where the modulus denotes the length with respect to the given metric. To see that vectors of this kind exist, consider the standard embedding of hyperbolic space in three-dimensional Minkowski space with metric $dx^2 + dy^2 - dz^2$ as the hyperboloid $x^2 + y^2 - z^2 = -1$ and take the vectors $x\partial_z + z\partial_x$, $y\partial_z + z\partial_y$ and $x\partial_y - y\partial_x$. Now take the corresponding Killing vectors on spacetime and put them into the quadratic form introduced above. This quantity is proportional to R . Following the same procedure as above gives a criterion for inextendibility in this case too. Combining it with a result of [1] shows that solutions of the Einstein-Vlasov system with hyperbolic symmetry cannot be extended to the future. This was previously known only in a small data situation [11].

Using the results of Tegankong [15] it can be shown that maximal globally hyperbolic developments are inextendible in the future in the case of solutions of the Einstein-Vlasov-scalar field system with plane or hyperbolic symmetry. The methods used here can be extended to the case of spacetimes with only one spacelike Killing vector (spacetimes with $U(1)$ symmetry). In that case the analogue of R is the length of the Killing vector. An inextendibility result for vacuum spacetimes with small initial data follows from [6]. In fact in that case a lot more control on the asymptotics is available and geodesic completeness can be proved [4].

4 Concluding remarks

Strong cosmic censorship is a hard problem and for this reason it is common to study restricted problems for spacetimes with symmetry. In this paper a powerful method has been presented for proving inextendibility of cosmo-

logical spacetimes with symmetry in the future. No symmetry is assumed of possible extensions. Straightforward consequences include a number of new results on classes of spacetimes frequently studied in the literature together with major simplifications of proofs of known theorems.

It should be noted that the method presented here only involves the geometry of an extension. Properties of matter fields play no role. On the other hand the control of the quantity R which is necessary in order for the method to be applicable in any given example relies on a previous analysis of the coupled Einstein-matter equations.

The results which have been presented here shift the emphasis in proving cosmic censorship to the analysis of the initial singularity. For that case the procedure of this paper says that at any potential singularity on the boundary of the maximal Cauchy development is such that the Killing vectors extend continuously (for any of the symmetry types considered here), as does the function R . It is known in certain cases by other arguments that the extension of R is constant on the boundary. The limiting constant value may be positive or zero. The further insights which may be obtained in this way on the structure of singularities will be discussed elsewhere [8].

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