The index form of a warped product

Paul E. Ehrlich[a] and Seon-Bu Kim[b*]

[a]Department of Mathematics, The University of Florida, Gainesville, Florida 32611,
E-mail: ehrlich@math.ufl.edu

[b]Department of Mathematics, Chonnam National University, Kwangju, 500-757 Korea,
E-mail: sbk@chonnam.ac.kr

We construct the index form along timelike geodesics on a Lorentzian warped product manifold and apply this index form to generalized Robertson-Walker(GRW) space-times.

1. Preliminaries

Since many space-times in relativity may be represented by Lorentzian warped product manifolds([1], [2], [3], [4], [5], [6], [7], [9]), a study of the timelike index form for this class of metrics is relevant.

Because $Hess(f^2)(v,w) = 2fHess(f)(v,w) + 2 < \nabla f, v >< \nabla f, w >$ and calculations shows that the index form contains Hessian terms, we employ the formalism $\bar{g} = g_B \oplus f g_F$, $f : B \to (0, +\infty)$ for the warped product as originally treated in [3] and [1], rather than the currently more common formalism $\bar{g} = g_B \oplus f^2 g_F$ which has been more popular since the publication of O'Neill's monograph[8]. Thus let (B, g_B) be a Lorentzian manifold (or interval $((a,b), -dt^2)$) and let (F, g_F) be a Riemannian manifold. For $M = B \times F$, let $\pi : M \to B$ and $\sigma : M \to F$ denote the projection maps. Then given any curve $\gamma : I \to M$, we may set $\gamma_B = \pi \circ \gamma$, $\gamma_F = \sigma \circ \gamma$ and so obtain

$$\gamma(t) = (\gamma_B(t), \gamma_F(t)). \tag{1}$$

Correspondingly, since $T_{(b,f)}M = T_bB \oplus T_fF$, we may decompose $V \in \chi(\gamma)$ as

$$V = (V_B, V_F). \tag{2}$$

If $\alpha : I \times (-\epsilon, \epsilon) \to M$ is a variation of $\gamma(t)$, then setting $\alpha_B = \pi \circ \alpha$, $\alpha_F = \sigma \circ \alpha$: $I \times (-\epsilon, \epsilon) \to M$, we obtain

$$\alpha(t,s) = (\alpha_B(t,s), \alpha_F(t,s)) \tag{3}$$

so that a variation α of γ gives rise to a variation α_B of γ_B and α_F of γ_F. Conversely, given such variations of γ_B and γ_F, equation (3) defines a corresponding variation α of γ.

In the language of O'Neill[8], but with the convention on the warping function as in [1], for $f : B \to (0, +\infty)$ smooth, (B, g_B) Lorentzian (or $((a,b), -dt^2)$) and (F, g_F)

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Riemannian, the warped product $(M, \bar{g}) = (B \times_f F, \bar{g})$ is the product manifold $M = B \times F$ furnished with the Lorentz metric

$$\bar{g} = \pi^*(g_B) + (f \circ \pi)\sigma^*(g_F) \tag{4}$$

or

$$\bar{g}(x, x) = g_B(\pi_* x, \pi_* x) + f(p) g_F(\sigma_* x, \sigma_* x) \tag{5}$$

for $x \in T_{(p,q)}M$. As is common notation nowadays, we identify $X \in \chi(B)$ with $(X, 0) = \bar{X}$ in TM and $V \in \chi(F)$ with $(0, V) = \bar{V}$ in TM in the sequel. Also, we may write (4) more succinctly as in [1] as

$$\bar{g} = g_B \oplus f g_F. \tag{6}$$

With convention (6) employed, the basic Proposition 35 in [8](p.206) translates as

Proposition 1.1. For $X, Y \in \mathfrak{L}(B)$, $V, W \in \mathfrak{L}(F)$

(1) $D_X Y \in \mathfrak{L}(B)$ is the lift of $D_X Y$ on B

(2) $D_X V = D_V X = \dfrac{X(f)}{2f} V$

(3) $\mathrm{nor}(D_V W) = -\dfrac{1}{2} g_F(V, W) \nabla f$

(4) $\tan(D_V W) \in \mathfrak{L}(F)$ is the lift of $\nabla_V W$ on F

Applying the techniques of Proposition 38 of [8](p.208) and Proposition 1.1, we obtain the following formula for the acceleration vector γ'' of a smooth curve $\gamma(t)$ in $(M, \bar{g})$:

$$\begin{aligned} \gamma''(t) &= \nabla_{\frac{\partial}{\partial t}} \gamma' \\ &= \left(\nabla_{\frac{\partial}{\partial t}} \gamma'_B - \frac{1}{2} g_F(\gamma'_F(t), \gamma'_F(t)) \nabla f, \nabla_{\frac{\partial}{\partial t}} \gamma'_F + \frac{(f \circ \gamma_B)'(t)}{(f \circ \gamma_B)(t)} \gamma'_F(t) \right). \end{aligned} \tag{7}$$

Or if one prefers the sum notation,

$$\gamma''(t) = \nabla_{\frac{\partial}{\partial t}} \gamma'_B - \frac{1}{2} g_F(\gamma'_F(t), \gamma'_F(t)) \nabla f + \nabla_{\frac{\partial}{\partial t}} \gamma'_F + \frac{(f \circ \gamma_B)'(t)}{(f \circ \gamma_B)(t)} \gamma'_F(t) \tag{8}$$

Hence, the geodesic equations corresponding to Proposition 38 in [8](p.208) ensures $\gamma(t)$ is a smooth geodesic in $(M, \bar{g})$ iff

$$\begin{cases} \nabla_{\frac{\partial}{\partial t}} \gamma'_B = \dfrac{1}{2} g_F(\gamma'_F(t), \gamma'_F(t)) \nabla f \\ \nabla_{\frac{\partial}{\partial t}} \gamma'_F = -\dfrac{(f \circ \gamma_B)'(t)}{(f \circ \gamma_B)(t)} \gamma'_F(t) \end{cases} \tag{9}$$

and so the well known result follows that $\gamma_F(t)$ is always a pregeodesic in (F, g_F) if $\gamma(t)$ is a geodesic in $(M, \bar{g})$.

Employing again the proof idea of Proposition 38 of [8] and Proposition 1.1 above, the formula for the covariant derivative of a smooth vector field $V = (V_B, V_F)$ along the smooth curve $\gamma(t)$ may be obtained:

$$\begin{aligned} V'(t) &= \nabla_{\frac{\partial}{\partial t}} V \\ &= \left(V_B'(t) - \frac{1}{2} g_F(\gamma_F', V_F) \nabla f \right) \\ &\quad + \left(V_F'(t) + \frac{\gamma_B'(f)}{2f} V_F(t) + \frac{g_B(\nabla f, V_B)}{2f} \gamma_F'(t) \right) \end{aligned} \tag{10}$$

This may be recast in the direct sum language as

$$\begin{aligned} V'(t) &= (V_B'(t), V_F'(t)) \\ &\quad + \frac{1}{2} \left(-g_F(\gamma_F', V_F) \nabla f,\ \frac{\gamma_B'(f)}{f} V_F(t) + \frac{V_B(f)}{f} \gamma_F'(t) \right) \end{aligned} \tag{11}$$

which manifests how far the warped product covariant derivative formula is from the simple product case $f = 1$ for which $V'(t) = (V_B'(t), V_F'(t))$.

With formula (10) in hand, a recent result of Raposo and del Riego[10] on lifts of parallel vector fields in semi-Riemannian warped products may be given in the present context.

Proposition 1.2. (a) Let V_B be a smooth vector field along the smooth curve $\gamma_B : I \to (B, g_B)$ and for fixed $q \in F$, let $\gamma(t) = (\gamma_B(t), q)$ and $V(t) = (V_B(t), 0_q)$ along γ. Then V is parallel along $\gamma : I \to (M, \bar{g})$ if and only if V_B is parallel along $\gamma_B : I \to (B, g_B)$.

(b) Let V_F be a smooth vector field along the smooth curve $\gamma_F : I \to (F, g_F)$ and for fixed $b \in B$, let $\gamma(t) = (b, \gamma_F(t))$ and $V(t) = (0_b, V_F(t))$ along γ. Then V is parallel along $\gamma : I \to (M, \bar{g})$ if and only if V_F is parallel along $\gamma_F : I \to (F, g_F)$ and $g_F(V_F(t), \gamma_F'(t)) = 0$ for all $t \in I$ for which $\nabla f(t) \neq 0$.

Proof. (a) Given $V_F = 0_q$ and $\gamma_F'(t) = 0$, formula (10) reduces to $V'(t) = V_B'(t)$.

(b) Given $V_B = 0_b$ and $\gamma_B'(t) = 0$, formula (11) reduces to $V'(t) = (-\frac{1}{2} g_F(\gamma_F'(t), V_F(t)) \nabla f, V_F'(t)))$. ∎

2. Construction of the Index Form

With some effort, assuming that $\gamma(t)$ is a timelike geodesic, the curvature formula $\bar{g}(R(V, \gamma')\gamma', V)$ of the index form may be calculated. However, given the rather complicated formula for this term as well as the somewhat complicated formula (11) for the covariant differentiation, the direct approach of calculating these terms directly in the general index form formula

$$I(X, Y) = -\int_{t=a}^{b} [< X', Y' > - < R(X, \gamma')\gamma', Y >] \, dt$$

seems less helpful than deriving formula from a variational approach. Thus we shall not present the curvature formula for the warped product here.

Now let $\gamma : [a, b] \to (M, \bar{g})$ be a unit timelike curve. Further, let $\alpha : I \times (-\epsilon, \epsilon) \to (M, \bar{g})$ be a variation of $\gamma(t)$. Then from general results, for a compact interval $I = [a, b]$, the curves $\alpha(0, s)$ will be timelike if ϵ is taken to be sufficiently small. As above, decompose

$$\alpha_s(t) := \alpha(t, s) = (\alpha_B(t, s), \alpha_F(t, s)) \tag{12}$$

and define corresponding variation vector fields

$$\begin{aligned} &W = \alpha_* \frac{\partial}{\partial s} = \frac{\partial \alpha}{\partial s},\ W_B = \alpha_{B_*} \frac{\partial}{\partial s} = \frac{\partial \alpha_B}{\partial s}, \\ &W_F = \alpha_{F_*} \frac{\partial}{\partial s} = \frac{\partial \alpha_F}{\partial s}, \end{aligned} \tag{13}$$

and

$$V(t) = W(t, 0),\ V_B(t) = W_B(t, 0),\ V_F(t) = W_F(t, 0). \tag{14}$$

Also,

$$\alpha_s' = \left(\alpha_{B_*} \frac{\partial}{\partial t}, \alpha_{F_*} \frac{\partial}{\partial t} \right) = \left(\frac{\partial \alpha_B}{\partial t}, \frac{\partial \alpha_F}{\partial t} \right) \tag{15}$$

Since the curves $\alpha_s(t)$ are timelike, if

$$\begin{aligned} h(t, s) &= -g_B \left(\frac{\partial \alpha_B}{\partial t}, \frac{\partial \alpha_B}{\partial t} \right) - (f \circ \alpha_B)(t, s) g_F \left(\frac{\partial \alpha_F}{\partial t}, \frac{\partial \alpha_F}{\partial t} \right) \\ &= -\bar{g} \left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \end{aligned} \tag{16}$$

and

$$\begin{aligned} F(t, s) = g_B &\left(\nabla_{\frac{\partial}{\partial s}} \frac{\partial \alpha_B}{\partial t}, \frac{\partial \alpha_B}{\partial t} \right) + \frac{1}{2} \frac{\partial}{\partial s} (f \circ \alpha_B)(t, s) g_F \left(\frac{\partial \alpha_F}{\partial t}, \frac{\partial \alpha_F}{\partial t} \right) \\ &+ (f \circ \alpha_B)(t, s) g_F \left(\nabla_{\frac{\partial}{\partial s}} \frac{\partial \alpha_F}{\partial t}, \frac{\partial \alpha_F}{\partial t} \right) \end{aligned} \tag{17}$$

then

$$h(t, 0) = 1, \tag{18}$$

$$\frac{\partial h}{\partial s} = -2F(t, s), \tag{19}$$

and

$$L(\alpha_s) = \int_{t=a}^{b} \sqrt{-\bar{g}\left(\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\right)} dt \tag{20}$$
$$= \int_{t=a}^{b} (h(t,s))^{1/2} dt.$$

Thus

$$L'(\alpha_s) = -\int_{t=a}^{b} (h(t,s))^{1/2} F(t,s) dt \tag{21}$$

and one obtains the first variation formula (for γ' smooth, unit timelike)

$$\begin{aligned} L'(0) = & -\bar{g}(V,\gamma') + \int_{t=a}^{b} \left[g_B\left(\nabla_{\frac{\partial}{\partial t}}\gamma_B' - \frac{1}{2} g_F(\gamma_F',\gamma_F')\nabla f, V_B\right)\right. \\ & \left. + (f\circ\gamma_B)(t) g_F\left(\nabla_{\frac{\partial}{\partial t}}\gamma_F' + \frac{(f\circ\gamma_B)'(t)}{(f\circ\gamma_B)(t)}\gamma_F'(t), V_F\right)\right] dt \\ = & -\bar{g}(V,\gamma') + \int_{t=a}^{b} \bar{g}(\gamma'',V) dt \end{aligned} \tag{22}$$

consistent with equations (8) and (9) above.
Still assuming only that $\gamma(t)$ is a unit timelike curve,

$$L''(\alpha_s) = -\int_{t=a}^{b} \frac{\partial}{\partial s}[(h(t,s))^{-1/2} F(t,s)] dt$$
$$= -\int_{t=a}^{b} \left[(h(t,s))^{-3/2}(F(t,s))^2 + (h(t,s))^{-1/2}\frac{\partial F}{\partial s}\right] dt.$$

Thus recalling (18),

$$L''(0) = -\int_{t=a}^{b} \left[(F(t,0))^2 + \frac{\partial F}{\partial s}(t,0)\right] dt. \tag{23}$$

Lemma 2.1. Assuming that $\gamma(t)$ is a unit timelike geodesic in $(M,\bar{g})$, then

$$F(t,0) = \frac{d}{dt}[\bar{g}(V,\gamma')] = \bar{g}(V',\gamma').$$

Proof. Since $\alpha_*\left[\frac{\partial}{\partial s},\frac{\partial}{\partial t}\right]=0$, etc., one has

$$\begin{aligned}
F(t,0) &= g_B\left(\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha_B}{\partial t},\frac{\partial\alpha_B}{\partial t}\right)\Big|_{s=0}+\frac{1}{2}\frac{\partial}{\partial s}(f\circ\alpha_B)(t,0)g_F\left(\frac{\partial\alpha_F}{\partial t},\frac{\partial\alpha_F}{\partial t}\right)\Big|_{s=0}\\
&\quad+(f\circ\alpha_B)(t,0)g_F\left(\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha_F}{\partial t},\frac{\partial\alpha_F}{\partial t}\right)\Big|_{s=0}\\
&= g_B(V_B',\gamma_B')+\frac{1}{2}g_B(\nabla f,V_B)g_F(\gamma_F',\gamma_F')+(f\circ\gamma_B)(t)g_F(V_F',\gamma_F')\\
&= \frac{d}{dt}[g_B(V_B,\gamma_B')]-g_B(V_B,\gamma_B'')+\frac{1}{2}g_B(\nabla f,V_B)g_F(\gamma_F',\gamma_F')\\
&\quad+(f\circ\gamma_B)(t)g_F(V_F',\gamma_F').
\end{aligned}$$

Substituting for γ_B'' from formula (9), the second and third terms cancel so that

$$\begin{aligned}
F(t,0) &= \frac{d}{dt}[g_B(V_B,\gamma_B')]+(f\circ\gamma_B)(t)g_F(V_F',\gamma_F')\\
&= \frac{d}{dt}[g_B(V_B,\gamma_B')]+(f\circ\gamma_B)(t)\left[\frac{d}{dt}(g_F(V_F,\gamma_F'))-g_F(V_F,\gamma_F'')\right].
\end{aligned}$$

Now applying formula (9) for γ_F'' we obtain

$$\begin{aligned}
F(t,0) &= \frac{d}{dt}[g_B(V_B,\gamma_B')]+(f\circ\gamma_B)(t)\frac{d}{dt}[g_F(V_F,\gamma_F')]\\
&\quad+(f\circ\gamma_B)'(t)g_F(V_F,\gamma_F')\\
&= \frac{d}{dt}[g_B(V_B,\gamma_B')+(f\circ\gamma_B)(t)g_F(V_F,\gamma_F')]\\
&= \frac{d}{dt}[\bar{g}(V,\gamma')]=\bar{g}(V',\gamma')+\bar{g}(V,\gamma'')\\
&= \bar{g}(V',\gamma')
\end{aligned}$$

since $\gamma''=0$ is assumed. ∎

Thus $L''(0)=-\int_{t=a}^{b}\left[(\bar{g}(V',\gamma'))^2+\frac{\partial F}{\partial s}(t,0)\right]dt.$
It remains to calculate $\frac{\partial F}{\partial s}(t,0)$. Since

$$\begin{aligned}
F(t,s) &= g_B\left(\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha_B}{\partial t},\frac{\partial\alpha_B}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial s}(f\circ\alpha_B)(t,s)g_F\left(\frac{\partial\alpha_F}{\partial t},\frac{\partial\alpha_F}{\partial t}\right)\\
&\quad+(f\circ\alpha_B)(t,s)g_F\left(\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha_F}{\partial t},\frac{\partial\alpha_F}{\partial t}\right),
\end{aligned}$$

commuting the differentiation, we have

$$\begin{aligned}
\frac{\partial F}{\partial s} &= g_B\left(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial \alpha_B}{\partial s}, \frac{\partial \alpha_B}{\partial t}\right) + g_B\left(\nabla_{\frac{\partial}{\partial t}}\frac{\partial \alpha_B}{\partial s}, \nabla_{\frac{\partial}{\partial s}}\frac{\partial \alpha_B}{\partial t}\right) \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial s^2}(f\circ\alpha_B)(t,s) g_F\left(\frac{\partial \alpha_F}{\partial t}, \frac{\partial \alpha_F}{\partial t}\right) \\
&\quad + \frac{\partial}{\partial s}(f\circ\alpha_B)(t,s) g_F\left(\nabla_{\frac{\partial}{\partial s}}\frac{\partial \alpha_F}{\partial t}, \frac{\partial \alpha_F}{\partial t}\right) \\
&\quad + \frac{\partial}{\partial s}(f\circ\alpha_B)(t,s) g_F\left(\nabla_{\frac{\partial}{\partial t}}\frac{\partial \alpha_F}{\partial s}, \frac{\partial \alpha_F}{\partial t}\right) \\
&\quad + (f\circ\alpha_B)(t,s) g_F\left(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial \alpha_F}{\partial s}, \frac{\partial \alpha_F}{\partial t}\right) \\
&\quad + (f\circ\alpha_B)(t,s) g_F\left(\nabla_{\frac{\partial}{\partial s}}\frac{\partial \alpha_F}{\partial t}, \nabla_{\frac{\partial}{\partial s}}\frac{\partial \alpha_F}{\partial t}\right).
\end{aligned}$$

Hence, commuting more derivatives,

$$\begin{aligned}
\frac{\partial F}{\partial s}\Big|_{s=0} &= g_B\left(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial \alpha_B}{\partial s}, \frac{\partial \alpha_B}{\partial t}\right)\Big|_{s=0} + g_B(V_B', V_B') \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial s^2}(f\circ\alpha_B)(t,0) g_F(\gamma_F'(t), \gamma_F'(t)) + 2\frac{\partial}{\partial s}(f\circ\alpha_B)(t,0) g_F(V_F', \gamma_F') \\
&\quad + (f\circ\gamma_B)(t) g_F\left(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial \alpha_F}{\partial s}, \frac{\partial \alpha_F}{\partial t}\right)\Big|_{s=0} + (f\circ\gamma_B)(t) g_F(V_F', V_F') \\
&= g_B(V_B', V_B') - g_B(R(V_B, \gamma_B')\gamma_B', V_B) \\
&\quad + g_B\left(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial \alpha_B}{\partial s}, \frac{\partial \alpha_B}{\partial t}\right)\Big|_{s=0} + (f\circ\gamma_B)(t) g_F(V_F', V_F') \\
&\quad - (f\circ\gamma_B)(t) g_F(R(V_F, \gamma_F')\gamma_F', V_F) \\
&\quad + (f\circ\gamma_B)(t) g_F\left(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial \alpha_F}{\partial s}, \frac{\partial \alpha_F}{\partial t}\right)\Big|_{s=0} \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial s^2}(f\circ\alpha_B)(t,0) g_F(\gamma_F', \gamma_F') + 2\frac{\partial}{\partial s}(f\circ\alpha_B)(t,0) g_F(V_F', \gamma_F').
\end{aligned}$$

Now we again bring the geodesic equation (9) to bear

$$\begin{aligned}
&g_B\left(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial \alpha_B}{\partial s}, \frac{\partial \alpha_B}{\partial t}\right)\Big|_{s=0} \\
&= \frac{d}{dt}\left[g_B\left(\nabla_{\frac{\partial}{\partial s}}W_B, \gamma_B'\right)\right] - g_B\left(\nabla_{\frac{\partial}{\partial s}}W_B, \nabla_{\frac{\partial}{\partial t}}\gamma_B'\right) \\
&= \frac{d}{dt}\left[g_B\left(\nabla_{\frac{\partial}{\partial s}}W_B, \gamma_B'\right)\right] - g_B\left(\nabla_{\frac{\partial}{\partial s}}W_B, \frac{1}{2}g_F(\gamma_F', \gamma_F')\nabla f\right) \\
&= \frac{d}{dt}\left[g_B\left(\nabla_{\frac{\partial}{\partial s}}W_B, \gamma_B'\right)\right] - \frac{1}{2}g_B\left(\nabla_{\frac{\partial}{\partial s}}W_B, \nabla f\right) g_F(\gamma_F', \gamma_F').
\end{aligned}$$

Thus

$$
\begin{aligned}
&g_B\left(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha_B}{\partial s},\frac{\partial\alpha_B}{\partial t}\right)\Big|_{s=0}+\frac{1}{2}\frac{\partial^2}{\partial s^2}(f\circ\alpha_B)(t,0)g_F(\gamma_F'(t),\gamma_F'(t))\\
&=\frac{d}{dt}\left[g_B\left(\nabla_{\frac{\partial}{\partial s}}W_B,\gamma_B'\right)\right]\\
&\quad+\frac{1}{2}\left[\frac{\partial^2}{\partial s^2}(f\circ\alpha_B)(t,0)-\left(\nabla_{\frac{\partial}{\partial s}}W_B\right)(f)\right]g_F(\gamma_F'(t),\gamma_F'(t))\\
&=\frac{d}{dt}\left[g_B\left(\nabla_{\frac{\partial}{\partial s}}W_B,\gamma_B'\right)\right]+\frac{1}{2}Hess(f)(V_B,V_B)g_F(\gamma_F'(t),\gamma_F'(t)),
\end{aligned}
$$

recalling $Hess(f)(X,Y) =< \nabla_X\nabla f, Y >= X(< \nabla f, Y >)- < \nabla f, \nabla_X Y >= X(Y(f)) - (\nabla_X Y)(f)$.

Applying also the geodesic equation (9) to the term $(f\circ\gamma_B)(t)g_F\left(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha_F}{\partial s},\frac{\partial\alpha_F}{\partial t}\right)$, we obtain

$$
\begin{aligned}
&(f\circ\gamma_B)(t)g_F\left(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha_F}{\partial s},\frac{\partial\alpha_F}{\partial t}\right)\Big|_{s=0}\\
&\qquad=(f\circ\gamma_B)(t)\frac{d}{dt}\left[g_F\left(\nabla_{\frac{\partial}{\partial s}}W_F,\gamma_F'\right)\right]-(f\circ\gamma_B)(t)g_F\left(\nabla_{\frac{\partial}{\partial s}}W_F,\nabla_{\frac{\partial}{\partial t}}\gamma_F'\right)\\
&\qquad=(f\circ\gamma_B)(t)\frac{d}{dt}\left[g_F\left(\nabla_{\frac{\partial}{\partial s}}W_F,\gamma_F'\right)\right]\\
&\qquad\quad-(f\circ\gamma_B)(t)g_F\left(\nabla_{\frac{\partial}{\partial s}}W_F,\frac{-(f\circ\gamma_B)'(t)}{(f\circ\gamma_B)(t)}\gamma_F'\right)\\
&\qquad=(f\circ\gamma_B)(t)\frac{d}{dt}\left[g_F\left(\nabla_{\frac{\partial}{\partial s}}W_F,\gamma_F'\right)\right]+(f\circ\gamma_B)'(t)g_F\left(\nabla_{\frac{\partial}{\partial s}}W_F,\gamma_F'\right)\\
&\qquad=\frac{d}{dt}\left[(f\circ\gamma_B)(t)g_F\left(\nabla_{\frac{\partial}{\partial s}}W_F,\gamma_F'\right)\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{\partial F}{\partial s}(t,0)&=g_B(V_B',V_B')-g_B(R(V_B,\gamma_B')\gamma_B',V_B)\\
&\quad+(f\circ\gamma_B)(t)[g_F(V_F',V_F')-g_F(R(V_F,\gamma_F')\gamma_F',V_F)]\\
&\quad+\frac{d}{dt}\left[g_B\left(\nabla_{\frac{\partial}{\partial s}}W_B,\gamma_B'\right)+(f\circ\gamma_B)(t)g_F\left(\nabla_{\frac{\partial}{\partial s}}W_F,\gamma_F'\right)\right]\\
&\quad+\frac{1}{2}Hess(f)(V_B(t),V_B(t))g_F(\gamma_F'(t),\gamma_F'(t))\\
&\quad+2\frac{\partial}{\partial s}(f\circ\alpha_B)|_{s=0}g_F(V_F'(t),\gamma_F'(t)).
\end{aligned}
$$

If desired, the last term may be rewritten as $2g_B(\nabla f,V_B)g_F(V_F'(t),\gamma_F'(t))$ since $\frac{\partial\alpha_B}{\partial s}(t,0)$ $= V_B(t)$ so that $\frac{\partial}{\partial s}(f\circ\alpha_B)|_{s=0}=V_B(t)(f)=g_B(\nabla f,V_B(t))$.

The second variation formula may now be stated.

Proposition 2.2. Let $\gamma : [a,b] \to (M,\bar{g}) = (B \times_f F, g_B \oplus f g_F)$ be a unit timelike geodesic and let $\alpha : [a,b] \times (-\epsilon,\epsilon) \to (M,\bar{g})$be a smooth variation of $\gamma(t)$ with variation vector field $V = (V_B, V_F)$ and $W(t,s) = \alpha_* \frac{\partial}{\partial s}\Big|_{(t,s)}$. Then

$$\begin{aligned} L''(0) = -& \int_{t=a}^{b} \left[(\bar{g}(V',\gamma'))^2 + (g_B(V_B',V_B') - g_B(R(V_B,\gamma_B')\gamma_B',V_B)) \right. \\ &+ (f\circ\gamma_B)(t)\,(g_F(V_F',V_F') - g_F(R(V_F,\gamma_F')\gamma_F',V_F)) \\ &+ \frac{1}{2} Hess(f)(V_B(t),V_B(t)) g_F(\gamma_F'(t),\gamma_F'(t)) \\ &\left. + 2g_B(\nabla f, V_B(t)) g_F(V_F'(t),\gamma_F'(t)) \right] dt \\ -& \left[g_B\left(\nabla_{\frac{\partial}{\partial s}} W_B, \gamma_B'(t)\right) + f\circ\gamma_B(t) g_F\left(\nabla_{\frac{\partial}{\partial s}} W_F, \gamma_F'(t)\right)\right] \Big|_{t=a}^{b}. \end{aligned}$$

Remark 2.3. The second and fourth terms could be combined as

$$\bar{g}((V_B',V_F'),\gamma')$$

with similar expressions for the two curvature terms and the last evaluation term. But $\bar{g}((V_B',V_F'),\gamma')$ and the curvature term

$$\bar{g}((R(V_B,\gamma_B')\gamma_B', R(V_F,\gamma_F')\gamma_F'),(V_B,V_F))$$

are so far removed from $\bar{g}(V',\gamma')$ (cf. equation (11)) and the warped product curvature, that such a reworking of the formula in Proposition 2.2 does not seem warranted. Note that this result bears some resemblance (with $\epsilon = -1$, $c = 1$ and the opposite sign convention on the curvature tensor) to what O'Neill[8](p.266) terms "Synge's formula for the second variation."

As is well known, in studying the second variation and index form, it suffices to consider vector fields perpendicular to the given geodesic $\gamma(t)$. But since $\gamma(t)$ is a geodesic, differentiating $\bar{g}(V,\gamma') = 0$ implies $\bar{g}(V',\gamma') = 0$ as well, and hence in this setup, the first term in the formula for $L''(0)$ in Proposition 2.2 vanishes. As in [2], let $V^\perp(\gamma)$ denote the vector space of piecewise smooth vector fields V along γ with $\bar{g}(V,\gamma') = 0$ and let $V_0^\perp(\gamma) = \{V \in V^\perp(\gamma) \mid V(a) = V(b) = 0\}$. Then guided by the result of Proposition 2.2, the index form

$$I : V_0^\perp(\gamma) \times V_0^\perp(\gamma) \to \mathbb{R}$$

should be given by

$$\begin{aligned} I(V,V) = &-\int_{t=a}^{b} [g_B(V_B',V_B') - g_B(R(V_B,\gamma_B')\gamma_B',V_B)]dt \\ &-\int_{t=a}^{b} f\circ\gamma_B(t)[g_F(V_F',V_F') - g_F(R(V_F,\gamma_F')\gamma_F',V_F)]dt \\ &-\int_{t=a}^{b} [\frac{1}{2} Hess(f)(V_B(t),V_B(t)) g_F(\gamma_F'(t),\gamma_F'(t)) \\ &\qquad + 2g_B(\nabla f, V_B(t)) g_F(V_F'(t),\gamma_F'(t))]dt \end{aligned} \tag{24}$$

and $I(V,W)$ could be obtained from (24) by polarization.
All terms in this formula seem satisfactory except for the term containing

$$g_F(V_F'(t),\gamma_F'(t)),$$

where it would be desirable to get rid of the derivative of $V_F(t)$.

By employing formula (10) for $V'(t)$, this last term may at least be shifted from F to B which may perhaps best be utilized for generalized Robertson Walker space-times for which $\dim B = 1$. Given $0 = \bar{g}(V,\gamma')$ and $\gamma'' = 0$, differentiation yields $0 = \bar{g}(V',\gamma')$. Hence for$V \in V_0^{\perp}(\gamma)$, we obtain from formula (10):

$$\begin{aligned} &g_F(V_F'(t),\gamma_F'(t)) \\ &\quad = -\frac{1}{f\circ\gamma_B(t)}\left[g_B(V_B'(t),\gamma_B'(t)) + \frac{1}{2}g_B(V_B(t),\nabla f)g_F(\gamma_F'(t),\gamma_F'(t))\right]. \end{aligned} \tag{25}$$

Substituting (25) into (24) results in the following alternative formula for $I(V,V)$:

$$\begin{aligned} I(V,V) = &-\int_{t=a}^{b}[g_B(V_B',V_B') - g_B(R(V_B,\gamma_B')\gamma_B',V_B)]dt \\ &-\int_{t=a}^{b} f\circ\gamma_B(t)[g_F(V_F',V_F') - g_F(R(V_F,\gamma_F')\gamma_F',V_F)]dt \\ &-\int_{t=a}^{b}\left[\left\{\frac{1}{2}Hess(f)(V_B(t),V_B(t)) - \frac{1}{f\circ\gamma_B(t)}\left(g_B(V_B(t),\nabla f)\right)^2\right\}\right. \\ &\qquad \left. g_F(\gamma_F'(t),\gamma_F'(t)) - \frac{2}{f\circ\gamma_B(t)}g_B(\nabla f, V_B(t))g_B(V_B'(t),\gamma_B'(t))\right] dt. \end{aligned} \tag{26}$$

The pleasantest simplification occurs when $V_B = 0$ and (26) reduces to

$$I(V,V) = -\int_{t=a}^{b} f\circ\gamma_B(t)[g_F(V_F',V_F') - g_F(R(V_F,\gamma_F')\gamma_F',V_F)]dt. \tag{27}$$

3. The GRW Case

Now we specialize to the index form to the case that $(M,\bar{g}) = ((a,b)\times_f F, \bar{g} = -dt^2 \oplus f g_F)$ where (F,g_F) is an arbitrary Riemannian manifold and $(B,g_B) = (I,g_I)$ with $I = (a,b)$ and $g_I = -dt^2$. Such Lorentzian manifolds have been called "generalized Robertson-Walker space-time", especially when (F,g_F) is Riemannian complete. (In the original Robertson-Walker cosmological models, (F,g_F) was assumed to have constant curvature.) Let $\gamma : [\alpha,\beta] \to (M,\bar{g})$, $\gamma(t) = (\gamma_I(t),\gamma_F(t))$, denote a unit timelike geodesic segment.

Consider variation vector fields $V = (V_I,V_F)$ along γ with $\bar{g}(V,\gamma') = 0$. We begin with a physically important special case in which the timelike geodesic $\gamma(t)$ is of the form $\gamma_q(t) = (t,q)$ with $q \in F$ fixed. All such timelike geodesics are known to be globally maximal, hence free of conjugate points, and have been termed "stationary" in [9] and "galaxies" in [8, p.341]. Hence $\gamma_I(t) = t$ and $\gamma_F(t) = q$. If we write generally for $V = (V_I,V_F)$ the first term as

$$V_I = v(t)\frac{\partial}{\partial t}\circ\gamma_I(t)$$

then for the stationary geodesic $\gamma_q(t)$ the equation $0 = \bar{g}(V, \gamma_q')$ becomes

$$0 = -\upsilon(t) \cdot 1 + f(t) g_F(V_F, 0)$$

where $\upsilon(t) = 0$ and $V_I(t) = 0$. Hence we are exactly in the setting where (27) applies and we obtain

Proposition 3.1. Let $(M, \bar{g}) = ((a, b) \times_f F, \bar{g} = -dt^2 \oplus f g_F)$ be a generalized Robertson-Walker space-time and let $\gamma_q : [\alpha, \beta] \to (M, \bar{g})$ be a stationary maximal timelike geodesic $\gamma_q(t) = (t, q)$ for some fixed $q \in F$ and α, β with $a < \alpha < \beta < b$. Then $I : V_0^\perp(\gamma_q) \times V_0^\perp(\gamma_q) \to \mathbb{R}$ is given by

$$I(V, V) = -\int_{t=\alpha}^{\beta} f(t) g_F(V_F'(t), V_F'(t)) dt \tag{28}$$

and I is negative definite.

Proof. As above, $\gamma_F'(t) = 0$ and $V \in V_0^\perp(\gamma_q)$ implies $V_I = 0$. Hence formula (28) holds and thus the index form $I(V, V) \leq 0$. However as in the general theory (cf. [2, p.341], proof of Theorem 10.22), the negative semidefiniteness of the index form implies the negative definiteness by algebraic arguments. ■

Consider now a "non-stationary" unit timelike geodesic $\gamma = (\gamma_I, \gamma_F) : [\alpha, \beta] \to M$ and to avoid ambiguity, write $\gamma_I(t) = \tau(t)$ so we have the decompositions

$$\begin{cases} \gamma_I' = \tau'(t) \dfrac{\partial}{\partial t} \circ \gamma_I|_t \\[2ex] V_I = \upsilon(t) \dfrac{\partial}{\partial t} \circ \gamma_I|_t \end{cases} \tag{29}$$

where we let $\upsilon(t) = \upsilon_I(t) : [\alpha, \beta] \to \mathbb{R}$.
Regarding $\upsilon(t)$ as being prescribed, seek to find V_F along γ_F so that with V_I as in (29), we have $\bar{g}(V, \gamma') = 0$.

Since $\gamma_F(t)$ is a pregeodesic in (F, g_F) and $-1 = \bar{g}(\gamma', \gamma') = -(\tau'(t))^2 + f \circ \gamma_I(t) g_F(\gamma_F', \gamma_F')$, we may suppose that $\tau'(t) > 1$ and $g_F(\gamma_F', \gamma_F') > 0$. Let us try to find V_F of the form

$$V_F(t) = \phi(t) \gamma_F'(t) \tag{30}$$

where $\phi : [\alpha, \beta] \to \mathbb{R}$ is an unknown function.
Writing out $0 = \bar{g}(V, \gamma')$ with (30), we obtain

$$\upsilon(t) \tau'(t) = \phi(t) (f \circ \gamma_I)(t) g_F(\gamma_F', \gamma_F')$$

so we may solve for $\phi(t)$ as

$$\phi(t) := \frac{\upsilon(t) \tau'(t)}{(f \circ \gamma_I)(t) g_F(\gamma_F'(t), \gamma_F'(t))}$$

since the expression in the denominator is positive. Moreover, using $-1 = -(\tau'(t))^2 + f \circ \gamma_I(t) g_F(\gamma_F', \gamma_F')$, we also obtain

$$\phi(t) = \frac{\tau'(t)}{\tau'(t)^2 - 1} \upsilon(t).$$

Hence for nonstationary timelike geodesics, the assumption that $V_I = 0$ in considering the index form on $V_0^{\perp}(\gamma)$ is invalid.

Before specializing index formula (26) for the GRW case, we note how Proposition 12.22 in [8, p.353] translates with our convention on the power of f in the formula for the warped product metric $\bar{g}$.

Lemma 3.2. The smooth curve $\gamma(t) = (\tau(t), \gamma_F(t))$ is a geodesic in $(M, \bar{g})$ if and only if

$$\tau''(t) + \frac{1}{2} g_F(\gamma_F'(t), \gamma_F'(t)) f'(\tau(t)) = 0 \tag{31}$$

$$\gamma_F''(t) + \frac{f'(\tau(t))}{f(\tau(t))} \tau'(t) \gamma_F'(t) = 0. \tag{32}$$

In this GRW setting, Remark 7.39 of [8, p.208] translates as the product

$$(f(\tau(t)))^2 g_F(\gamma_F'(t), \gamma_F'(t)) = C_\gamma \tag{33}$$

is a constant, depending on the geodesic γ in $(M, \bar{g})$. Formula (33) then offers an elementary proof of the following basic result.

Corollary 3.3. If the warped function $f : (a, b) \to (0, +\infty)$ is nonconstant on $[\alpha, \beta]$ and γ_F is a nontrivial geodesic in (F, g_F), (i.e., $\gamma_F'(t) \neq 0$), then $\sigma(t) = (t, \gamma_F(t))$ is not a geodesic in warped product $(M, \bar{g})$.

Proof. First, $g_F(\gamma_F'(t), \gamma_F'(t)) = d > 0$. Hence, if $\sigma(t)$ were a geodesic in $(M, \bar{g})$, then $f(t) = \sqrt{\frac{C_\gamma}{d}}$, in contradiction. ∎

Returning to the arbitrary nonstationary unit timelike geodesic $\gamma : [\alpha, \beta] \to (M, \bar{g})$, $\gamma(t) = (\tau(t), \gamma_F(t))$ and $V = (V_I, V_F) \in V_0^{\perp}(\gamma)$, formula (26) above translates as

$$\begin{aligned} I(V, V) = &- \int_{t=\alpha}^{\beta} [g_I(V_I', V_I') - g_I(R(V_I, \gamma_I')\gamma_I', V_I)] dt \\ &- \int_{t=\alpha}^{\beta} f(\tau(t)) [g_F(V_F', V_F') - g_F(R(V_F, \gamma_F')\gamma_F', V_F)] dt \\ &- \int_{t=\alpha}^{\beta} \left[\left\{ \frac{1}{2} Hess(f)(V_I(t), V_I(t)) - \frac{1}{f(\tau(t))} (g_I(V_I(t), \nabla f))^2 \right\} \right. \\ &\qquad \left. g_F(\gamma_F'(t), \gamma_F'(t)) - \frac{2}{f(\tau(t))} g_I(\nabla f, V_I(t)) g_I(V_I'(t), \gamma_I'(t)) \right] dt. \end{aligned}$$

Since $\frac{d}{dt}$ is a parallel vector field in $(I, g_I) = ((a,b), -dt^2)$ and $V_I(t) = \upsilon(t)\frac{d}{dt}\Big|_{\tau(t)}$, also $V_I'(t) = \upsilon'(t)\frac{d}{dt}\Big|_{\tau(t)}$ and $g_I(V_I', V_I') = -(\upsilon'(t))^2$. Further, $R(V_I, \gamma_I')\gamma_I' = 0$ since $\dim I = 1$. Thus the first integral term in the above formula for $I(V,V)$ reduces to $\int_{t=\alpha}^{\beta} (\upsilon'(t))^2 dt$. With $g_I = -dt^2$, also $\nabla f(u) = -f'(u)\frac{d}{dt}\Big|_u$. Hence, $g_I(V_I'(t), \gamma_I'(t)) = -\upsilon'(t)\tau'(t)$, $g_I(V_I(t), \nabla f) = f'(\tau(t))\upsilon(t)$ and $Hess(f)(V_I(t), V_I(t)) = (\upsilon(t))^2 Hess(f)\left(\frac{d}{dt}\Big|_{\tau(t)}, \frac{d}{dt}\Big|_{\tau(t)}\right)$ $= f''(\tau(t))(\upsilon(t))^2$.
Combining these calculations, we obtain

Theorem 3.4. Let $\gamma : [\alpha, \beta] \to (M, \bar{g})$, $\gamma(t) = (\tau(t), \gamma_F(t))$ be a unit timelike geodesic segment in $(M, \bar{g}) = ((a,b) \times_f F, \bar{g} = -dt^2 \oplus f g_F)$. Let $V = (V_I, V_F) \in V_0^{\perp}(\gamma)$ and denote $V_I(t) = \upsilon(t)\frac{d}{dt}\Big|_{\tau(t)}$. Then

$$\begin{aligned} I(V,V) = &- \int_{t=\alpha}^{\beta} f(\tau(t))[g_F(V_F', V_F') - g_F(R(V_F, \gamma_F')\gamma_F', V_F)]dt \\ &+ \int_{t=\alpha}^{\beta} \left[\frac{(f'(\tau(t)))^2}{f(\tau(t))} - \frac{1}{2}f''(\tau(t))\right](\upsilon(t))^2 g_F(\gamma_F'(t), \gamma_F'(t))dt \\ &+ \int_{t=\alpha}^{\beta} \left[(\upsilon'(t))^2 - 2\frac{f'(\tau(t))}{f(\tau(t))}\tau'(t)\upsilon'(t)\upsilon(t)\right] dt. \end{aligned}$$

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