Killing vectors in asymptotically flat space–times: II. Asymptotically translational Killing vectors and the rigid positive energy theorem in higher dimensions

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Abstract

We show that the borderline cases in the proof of the positive energy theorem for initial data sets, on spin manifolds, in dimensions $n \geq 3$, are only possible for initial data arising from embeddings in Minkowski space-time.

1 Introduction

Witten's proof of the positive energy theorem [10] shows that, under appropriate conditions, the time-component of the energy-momentum vector p is non-negative. For various reasons it is of interest to understand precisely the borderline cases, with a vanishing, or perhaps light-like, p. In the context of initial data sets this has been done in detail in an accompanying paper [5] in space-dimension three. It is the purpose of this note to generalise the results proved there to all spin initial data manifolds of dimension $n \geq 3$.

The argument presented in [5] proceeds as follows: in the borderline cases, Witten's proof provides one or more covariantly constant "KIDs" (by definition, those are the initial data counterparts of space-time Killing vectors). A careful

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study of such KIDs shows that their existence implies the vanishing of mass, and then flatness of space-time along the initial data. One then concludes by showing that the Killing development of the initial data set is flat.

Not unexpectedly, all those arguments can be extended to higher dimensions, after adjustment of the rates of decay of the fields. The only part of the proof where essential work is needed is the algebra proving existence of KIDs. This is based on [8], and presented in Section 3. On the other hand, the analysis of the KIDs is essentially identical to that in [5], so we will (mainly) only present the statements of the results needed for the positive energy theorem here.

The notation and conventions of [5] are used throughout. We assume that the space-dimension n is larger than or equal to three.

Our main results can be summarised as follows:

THEOREM 1.1 *Let $(\mathscr{M}, g_{\mu\nu})$ be an $(n+1)$–dimensional space–time, $n \geq 3$, with a Killing vector field which is asymptotically null along an (appropriately regular, see Section 2 below) asymptotically flat spacelike hypersurface $\mathscr{S}$. Then the ADM energy–momentum vector of $\mathscr{S}$ vanishes.*

The precise hypotheses needed for Theorem 1.1 are the conditions on the asymptotic behavior of (g, K) in (2.18)-(2.19) below, together with the matter decay conditions (2.20) and (2.22). Theorem 1.1 is a special case of Theorem 2.5 below.

THEOREM 1.2 ("Timelike "future–pointing" energy–momentum theorem") *Under natural regularity and matter–energy conditions (see the conditions of Theorem 3.2 below), the ADM energy–momentum vector p^μ of a <u>spin</u> initial-data manifold $\mathscr{S}$ satisfies*

$$p^0 > \sqrt{\sum_{i=1}^{n} (p^i)^2} \; ,$$

unless $(\mathscr{S}, g_{ij}, K_{ij})$ are initial data for Minkowski space–time.

Theorem 1.1 is a loose rephrasing of Theorem 3.2 below.

There are well known counterparts of this with trapped boundaries, which are of no concern to us here because they always lead to a strict inequality.

It would be natural to extend the result to cover the Bondi mass, both in three and higher dimensions. The starting point of the calculations of the proof of Theorem 3.2 is the existence of a parallel spinor, the existence of which follows from the analysis in [7] when the Bondi mass is null in space-dimension three. The calculations that follow apply without modifications, yielding a parallel isotropic KID. One expects that this is incompatible with a non-vanishing Trautman-Bondi mass, but a complete analysis of this has not been carried out so far.

2 KIDs in n-dimensional asymptotically flat initial data sets, $n \geq 3$

We have the following string of propositions, which are the building stones of the proof of Theorem 2.5 below:

PROPOSITION 2.1 *Let $R > 0$ and let (g_{ij}, K_{ij}) be initial data on $\mathscr{S}_R \equiv \mathbb{R}^n \backslash B(R)$ satisfying*

$$g_{ij} - \delta_{ij} = O_k(r^{-\alpha}), \qquad K_{ij} = O_{k-1}(r^{-1-\alpha}), \tag{2.1}$$

with some $k > 1$ and some $\alpha > 0$. Let N be a C^2 scalar field and Y^i a C^2 vector field on $\mathscr{S}_R$ such that

$$2NK_{ij} + \mathcal{L}_Y g_{ij} = 0 . \tag{2.2}$$

Define ρ, J^i and τ_{ij} by the equations

$$2\rho = {}^nR + (K^i{}_i)^2 - K^{ij}K_{ij} , \tag{2.3}$$

$$J^i = D_j(K^{ij} - K^k{}_k g^{ij}) , \tag{2.4}$$

$$\tau_{ij} - \tfrac{1}{2}g^{k\ell}\tau_{k\ell}g_{ij} = {}^nR_{ij} + K^k{}_k K_{ij} - 2K_{ik}K^k{}_j$$
$$- N^{-1}(\mathcal{L}_Y K_{ij} + D_i D_j N) - \tfrac{\rho}{2} g_{ij} , \tag{2.5}$$

and assume that ρ and τ_{ij} satisfy

$$\rho = O_{k-2}(r^{-2-\alpha}) , \qquad \tau_{ij} = O_{k-2}(r^{-2-\alpha}) . \tag{2.6}$$

Then there exists numbers $\Lambda_{\mu\nu} = \Lambda_{[\mu\nu]}$ such that we have, for r large,

$$D_i Y_j - \Lambda_{ij} = O_{k-1}(r^{-\alpha}) , \qquad Y^i - \Lambda_{ij}x^j = \begin{cases} O(r^{1-\alpha}), & \alpha \neq 1; \\ O(\ln r), & \alpha = 1, \end{cases} \tag{2.7}$$

$$D_i N - \Lambda_{i0} = O_{k-1}(r^{-\alpha}) , \qquad N - \Lambda_{i0}x^i = \begin{cases} O(r^{1-\alpha}), & \alpha \neq 1; \\ O(\ln r), & \alpha = 1. \end{cases} \tag{2.8}$$

If $\Lambda_{\mu\nu} = 0$, then there exist numbers A^μ such that we have

$$Y^i - A^i = O_k(r^{-\alpha}), \qquad N - A^0 = O_k(r^{-\alpha}) . \tag{2.9}$$

If $\Lambda_{\mu\nu} = A^\mu = 0$, then $Y^i \equiv N \equiv 0$.

PROOF: See Section 2 and Appendix C of [5]. □

PROPOSITION 2.2 *Let $R > 0$ and let (g_{ij}, K_{ij}) be initial data on $\mathscr{S}_R$ satisfying*

$$g_{ij} - \delta_{ij} = O_2(r^{-\alpha}), \qquad K_{ij} = O_1(r^{-1-\alpha}), \qquad \alpha > (n-2)/2, \tag{2.10}$$

$$J^i = O(r^{-n-\epsilon}), \qquad \rho = O(r^{-n-\epsilon}), \qquad \epsilon > 0 . \tag{2.11}$$

Let N be a C^1 scalar field and Y^i a C^1 vector field on $\mathscr{S}_R$ such that

$$N - A^0 = O_1(r^{-\alpha}), \qquad Y^i \rightarrow_{r \to \infty} A^i , \tag{2.12}$$

for some set of constants $(A^\mu) \not\equiv 0$, satisfying

$$2NK_{ij} + \mathcal{L}_Y g_{ij} = O_1(r^{-(n-1)-\epsilon}). \tag{2.13}$$

Let p^μ be the ADM energy–momentum of $\mathscr{S}_R$. Then:

1. If $A^0 = 0$, then $p^0 = 0$.

2. If $A^0 \neq 0$, then p^μ is proportional to A^μ.

PROOF: See the proof of Proposition 3.1 in [5]. □

PROPOSITION 2.3 *Under the hypotheses of Proposition 2.2, suppose further that N is C^2 and that*

$$\tau_{ij} = O(r^{-n-\epsilon}) . \tag{2.14}$$

If

$$(A^0)^2 < \sum_i A^i A^i , \tag{2.15}$$

then p^μ vanishes.

PROOF: See the proof of Proposition 3.2 in [5]. □

PROPOSITION 2.4 *Under the hypotheses of Proposition 2.2, assume moreover that N is C^2, that (2.14) holds and that*

$$NK_{ij} + D_i Y_j = O_1(r^{-(n-1)-\epsilon}) , \tag{2.16}$$
$$K_{ij} Y^j + D_i N = O_1(r^{-(n-1)-\epsilon}) , \tag{2.17}$$
$$A^\mu A_\mu \neq 0.$$

Then the ADM energy–momentum p^μ vanishes.

PROOF: See the proof of Proposition 3.3 in [5]. Note that the proof in [5] uses the equality of the Komar and the ADM masses for translational, asymptotically timelike Killing vectors, while Proposition 2.3 shows that one only needs to consider timelike A^μ's to complete the proof. The equality of those masses, which is well known in space-dimension three [3], can also be established in higher dimensions by an asymptotic analysis of the stationary Einstein equations when the sources decay sufficiently fast. □

The notation used in the next theorem is explained in Appendix A:

THEOREM 2.5 *Let $R > 0$ and let (g_{ij}, K_{ij}) be initial data on $\mathscr{S}_R = \mathbb{R}^n \backslash B(R)$ satisfying*

$$g_{ij} - \delta_{ij} = O_{3+\lambda}(r^{-\alpha}), \qquad K_{ij} = O_{2+\lambda}(r^{-1-\alpha}), \tag{2.18}$$

$$\alpha > \begin{cases} 1/2, & n = 3; \\ n - 3, & n \geq 4, \end{cases} \qquad \epsilon > 0, \qquad 0 < \lambda < 1. \tag{2.19}$$

$$J^i = O_{1+\lambda}(r^{-n-\epsilon}), \qquad \rho = O_{1+\lambda}(r^{-n-\epsilon}) . \tag{2.20}$$

Let N be a scalar field and Y^i a vector field on $\mathscr{S}_R$ such that

$$N \to_{r \to \infty} A^0, \quad Y^i \to_{r \to \infty} A^i, \qquad A^\mu A_\mu = 0 ,$$

for some constants $A^\mu \not\equiv 0$. Suppose further that

$$2NK_{ij} + \mathscr{L}_Y g_{ij} = O_{3+\lambda}(r^{-(n-1)-\epsilon}) , \tag{2.21}$$
$$\tau_{ij} = O_{1+\lambda}(r^{-n-\epsilon}) , \tag{2.22}$$

Then the ADM energy–momentum of $\mathscr{S}_R$ vanishes.

PROOF: See the proof of Theorem 3.4 in [5]. We note that in our context [5, Equation (3.40)] reads

$$g_{nA} = \begin{cases} C_{AB}(x^n)\partial_B \ln \rho + O_{(1)}(\rho^{-1-\epsilon}\ln\rho), & n = 3; \\ C_{AB}(x^n)\partial_B \frac{1}{\rho^{n-3}} + O_{(1)}(\rho^{-(n-2)-\epsilon}), & n \geq 4. \end{cases} \qquad (2.23)$$

Similarly instead of [5, Equation (3.47)] we have

$$\frac{\partial g_{AB}}{\partial x^n} = \begin{cases} D_{ABCD}\partial_C\partial_D \ln \rho + O_{(1)}(\rho^{-2-\epsilon}\ln\rho), & n = 3; \\ D_{ABCD}\partial_C\partial_D \frac{1}{\rho^{n-3}} + O_{(1)}(\rho^{-(n-1)-\epsilon}), & n \geq 4. \end{cases} \qquad (2.24)$$

Finally, there are misprints in the definitions of the quantities Ω and Ω' in the proof there; the correct definitions, in all dimensions, are[1]

$$\Omega = \lim_{\rho\to\infty} \sum_C \int_{S^{n-2}(\rho,x^n)} (x^A \partial_C g_{nA} - g_{nC}) dS_C \ .$$

$$\Omega' = \lim_{\rho\to\infty} \int_{S^{n-2}(\rho,x^n)} \left((n-1)(x^A x^B \partial_C \partial_n g_{AB} - 2x^B \partial_n g_{CB}) - x^A x^A \partial_C \partial_n g_{BB} + 2x_C \partial_n g_{AB} \right) dS_C \ ,$$

where summation over every repeated occurrence of indices is implicitly understood, regardless of their positions. Here $\rho^2 = (x^1)^2 + \ldots + (x^{n-1})^2$, while $S^{n-2}(\rho,a)$ is a sphere (or circle, when $n = 3$) of radius ρ centred at $x^1 = \ldots = x^{n-1} = 0$ lying in the plane $x^n = a$. Finally the dS_C's are the usual surface forms $dS_C = \partial_C \rfloor (dx^1 \wedge \cdots \wedge dx^{n-1})$, and $\rfloor$ denotes contraction. □

3 The rigid positive energy theorem

The following strengthens somewhat Theorem 4.1 of [5] in the case $n = 3$, and generalises that theorem to higher dimensions; the calculations here are closely related to those in [8]:

THEOREM 3.1 ((Rigid) positive energy theorem) *Consider a data set* $(\mathscr{S}, g_{ij}, K_{ij})$, *with* $(\mathscr{S}, g_{ij})$ *a complete Riemannian spin manifold of dimension* $n \geq 3$, *and with* $g_{ij} \in C^2$, $K_{ij} \in C^1$. *Suppose that* $\mathscr{S}$ *contains an asymptotically flat end* $\mathscr{S}_R$ *diffeomorphic to* $\mathbb{R}^n \setminus B(R)$ *for some* $R > 0$, *with* $B(R)$ — *a coordinate ball of radius* R, *where the fields* (g, K) *satisfy*

$$|g_{ij} - \delta_{ij}| + |r\partial_k g_{ij}| + |rK_{ij}| \leq Cr^{-\alpha} \ , \qquad (3.1)$$

for some constants $C > 0$ *and* $\alpha > \min(1/2, n-3)$, *with* $r = \sqrt{\sum_{i=1}^n (x^i)^2}$. *Suppose moreover that the quantities* ρ *and* J

$$2\rho := {}^3R + (K^k{}_k)^2 - K^{ij}K_{ij} \ , \qquad (3.2)$$

$$J^k := D_l(K^{kl} - K^k{}_k g^{kl}) \ , \qquad (3.3)$$

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[1]We take this opportunity to point out that equation (2.20) of [5] (which is equation (2.27) of the gr-qc version of that paper) should be replaced by $\rho = O_{k-2}(r^{-2-\alpha})$, $\tau_{ij} = O_{k-2}(r^{-2-\alpha})$. Furthermore, Equations (2.15) and (3.34) of [5] are mutually incompatible; the correct one is (2.15).

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satisfy

$$\sqrt{g_{ij}J^i J^j} \le \rho \le C(1+r)^{-n-\epsilon}, \qquad \epsilon > 0. \tag{3.4}$$

Then the ADM energy–momentum (m, p^i) of any of the asymptotic ends of $\mathscr{S}$ satisfies

$$m \ge \sqrt{p_i p^i} \,. \tag{3.5}$$

If $m = 0$, then $\rho \equiv J^i \equiv 0$, and there exists an isometric embedding i of $\mathscr{S}$ into Minkowski space–time $(\mathbb{R}^{n+1}, \eta_{\mu\nu})$ such that K_{ij} represents the extrinsic curvature tensor of $i(\mathscr{S})$ in $(M, \eta_{\mu\nu})$. Moreover $i(\mathscr{S})$ is an asymptotically flat Cauchy surface in $(\mathbb{R}^{n+1}, \eta_{\mu\nu})$.

Theorem 3.2 has been formulated under differentiability requirements which are stronger than necessary, compare [2, 9]. Unfortunately our proof that ADM energy–momentum cannot be null requires even more differentiability and asymptotic decay conditions:

THEOREM 3.2 *Under the hypotheses of Theorem 3.1, suppose moreover that*

$$g_{ij} - \delta_{ij} = O_{3+\lambda}(r^{-\alpha}), \qquad K_{ij} = O_{2+\lambda}(r^{-1-\alpha}), \tag{3.6}$$
$$\rho = O_{1+\lambda}(r^{-n-\epsilon}), \tag{3.7}$$

with some $0 < \lambda < 1$. Then the ADM energy–momentum cannot be null.

PROOFS OF THEOREMS 3.1 AND 3.2: We use a Witten-type argument, as follows. Let $(\mathfrak{S}, \langle\cdot,\cdot\rangle)$ be any Riemannian bundle of real spinors over (M, g) with scalar product $\langle\cdot,\cdot\rangle$, such that Clifford multiplication (which we denote by $X\cdot$) is anti-symmetric. We suppose that there exists a bundle isomorphism $\gamma_0 : \mathfrak{S} \to \mathfrak{S}$ with the following properties:

$$\gamma_0^2 = 1 \,, \tag{3.8a}$$
$$\forall X \in TM \quad \gamma_0 X \cdot = -X \cdot \gamma_0 \,, \tag{3.8b}$$
$${}^t\gamma_0 = \gamma_0 \,, \tag{3.8c}$$
$$D\gamma_0 = \gamma_0 D \,, \tag{3.8d}$$

where ${}^t\gamma_0$ denotes the transpose of γ_0 with respect to $\langle\cdot,\cdot\rangle$, and D is the usual Riemannian spinorial connection associated with the metric g.

(Such a map always exists if $\mathfrak{S}$ is obtained by pulling-back a space-time spinor bundle, using an externally oriented isometric embedding of (M, g) in a Lorentzian space-time. Then the Clifford product $n\cdot$, where n is the field of Lorentzian unit normals to the image of M, has the required properties. If, however, such a map does not exist, we proceed as follows: let $\mathfrak{S}' = \mathfrak{S} \oplus \mathfrak{S}$ be the direct sum of two copies of $\mathfrak{S}$, equipped with the direct sum metric $\langle\cdot,\cdot\rangle_\oplus$:

$$\langle(\psi_1, \psi_2), (\varphi_1, \varphi_2)\rangle_\oplus := \langle\psi_1, \varphi_1\rangle + \langle\psi_2, \varphi_2\rangle \,. \tag{3.9}$$

We set, for $X \in TM$,

$$\gamma_0(\psi_1, \psi_2) := (\psi_2, \psi_1) \,, \tag{3.10a}$$

$$X \cdot (\psi_1, \psi_2) := (X \cdot \psi_1, -X \cdot \psi_2) , \qquad (3.10b)$$

$$D_X(\psi_1, \psi_2) := (D_X\psi_1, D_X\psi_2) . \qquad (3.10c)$$

One readily verifies that (3.10b) defines a representation of the Clifford algebra on $\mathfrak{S}'$, and that (3.8) holds.)

Given an initial data set (M, g, K), a vector field X, and a spinor field ξ we set

$$K(X) := K_i{}^j X^i e_j \cdot , \qquad (3.11)$$

$$\nabla_X \xi := D_X \xi + \frac{1}{2} K(X) \gamma_0 \xi . \qquad (3.12)$$

Here e_i is a local orthonormal basis of TM; it is straightforward to check that (3.11) does not depend upon the choice of this basis. (To make things clear, (3.12) defines ∇ in terms of the Riemannian spin connection D. If the spin bundle arises from a space-time bundle, then ∇ coincides with the canonical space-time spinorial derivative, when restricted to space directions.)

We will need an explicit expression for the curvature of ∇:

PROPOSITION 3.3 *For every $X, Y \in \Gamma(T\mathscr{S})$ we have*

$$R_{X,Y} = {}^n R_{X,Y} + \frac{1}{2} \mathrm{d}^D K(X, Y) \gamma_0 - \frac{1}{4} \Big(K(X) K(Y) - K(Y) K(X) \Big) , \quad (3.13)$$

where R is the curvature of ∇, ${}^n R$ is that of D, and

$$\mathrm{d}^D K(e_i, e_j) = (K^k{}_{j;i} - K^k{}_{i;j}) e_k \cdot .$$

PROOF: We have

$$
\begin{aligned}
\nabla_X \nabla_Y \psi &= \Big(D_X + \frac{1}{2} K(X) \gamma_0 \Big) \Big(D_Y \psi + \frac{1}{2} K(Y) \gamma_0 \psi \Big) \\
&= D_X D_Y \psi + \frac{1}{2} K(X) \gamma_0 D_Y \psi \\
&\quad + \frac{1}{2} \Big((D_X K)(Y) \gamma_0 \psi + K(D_X Y) \gamma_0 \psi + K(Y) \gamma_0 D_X \psi \Big) \\
&\quad + \frac{1}{4} K(X) \gamma_0 K(Y) \gamma_0 \psi \\
&= D_X D_Y \psi + \frac{1}{2} \Big(K(X) \gamma_0 D_Y \psi + K(Y) \gamma_0 D_X \psi \Big) \\
&\quad + \frac{1}{2} \Big((D_X K)(Y) \gamma_0 \psi + K(D_X Y) \gamma_0 \psi \Big) \\
&\quad - \frac{1}{4} K(X) K(Y) \psi ,
\end{aligned}
$$

so that

$$
\begin{aligned}
R_{X,Y} \psi &= \nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X,Y]} \psi \\
&= D_X D_Y \psi - D_Y D_X \psi - D_{[X,Y]} \psi - \frac{1}{2} K([X,Y]) \gamma_0 \psi \\
&\quad + \frac{1}{2} \Big(((D_X K)(Y) - (D_Y K)(X)) \gamma_0 \psi + K(D_X Y - D_Y X) \gamma_0 \psi \Big) \\
&\quad - \frac{1}{4} \Big(K(X) K(Y) - K(Y) K(X) \Big) \psi ,
\end{aligned}
$$

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and the vanishing of the torsion of the Levi-Civita connection gives the result.
$\square$

We now run the usual Witten argument (see, *e.g.*, [2]) using the connection ∇ and the associated Dirac operator $\mathscr{D} = e^i \cdot \nabla_i$. Under the current conditions the ADM energy–momentum of $\mathscr{S}$ is finite and well defined [1, 6], and the Witten boundary integral reproduces the ADM energy–momentum. The arguments in [2] show that, again under the current conditions, for every spinor field $\mathring{\psi}$, with constant entries in the natural spin frame in the asymptotic region, one can find a solution ψ to the Witten equation which asymptotes to $\mathring{\psi}$. Witten's identity subsequently implies that

$$\langle \mathring{\psi}, p \cdot \mathring{\psi} \rangle \geq 0 \ , \tag{3.14}$$

where

$$p \cdot := m\gamma_0 + p^i e_i \cdot \ ,$$

and $p = (m, p^i)$ is the ADM momentum. This gives (3.5).

The equality case, which is of main interest here, is only possible if p is lightlike or vanishes. In either case one obtains a spinor field $\psi \in \Gamma(\mathfrak{S})$ which is asymptotic to $\mathring{\psi}$, and satisfies

$$\nabla \psi = 0 \ , \tag{3.15}$$

$$\langle \psi, \mathscr{R}\psi \rangle = 0 \ . \tag{3.16}$$

Here

$$\mathscr{R} := \frac{1}{2}\left(\rho + J^i e_i \cdot \gamma_0 \right)$$

is the (non-negative) spinorial endomorphism which appears in the identity:

$$\mathscr{D}^*\mathscr{D} = \nabla^*\nabla + \mathscr{R} \ .$$

The idea of the calculations that follow is to show, roughly speaking, that the space-time is a pp-wave space-time, perhaps with matter decaying at inifinity, with a null Killing vector, which by the results in the previous section is only possible if we are in Minkowski space-time. We start with an analysis of the curvature tensor.

As ψ is ∇-parallel we have $R_{XY}\psi = 0$, and from Proposition 3.3 one finds, for all $X, Y \in T\mathscr{S}$,

$$\langle {}^n R_{X,Y}\psi, \psi \rangle - \frac{1}{2}\langle \mathrm{d}^D K(X,Y)\cdot\gamma_0\psi, \psi \rangle - \frac{1}{4}\langle (K(X)K(Y) - K(Y)K(X))\psi, \psi \rangle = 0.$$

Both the first and third term vanish since the spinorial curvature can be written as

$$^n R_{XY}\psi = -\frac{1}{2}\sum_{i<j} \ {}^n R(X, Y, e_i, e_j) e_i \cdot e_j \cdot \psi \ ,$$

and since the Clifford product of two distinct elements of an ON basis is anti-symmetric. (We use the conventions

$$^n R(e_i, e_j)e_k = D_{e_i}D_{e_j}e_k - D_{e_j}D_{e_i}e_k - D_{[e_i,e_j]}e_k = {}^n R^s{}_{kij}e_s = {}^n R(e_m, e_k, e_i, e_j)g^{sm}e_s \ ,$$

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$$^{n}R_{ij} = {}^{n}R^{k}{}_{ikj} \ ,$$

where $^{n}R_{ij}$ is the Ricci tensor of g.) Thus we obtain

$$\langle \mathrm{d}^{D}K(X,Y)\gamma_{0}\psi , \psi \rangle = 0 \ . \tag{3.17}$$

Let us denote by N the function

$$N = \langle \psi , \psi \rangle \ , \tag{3.18}$$

and by Y the real 1-form defined as

$$Y(X) = -\langle \gamma_{0} X \cdot \psi , \psi \rangle \ . \tag{3.19}$$

Using this notation, (3.17) can be rewritten as

$$K_{ki;j}Y^{k} = K_{kj;i}Y^{k} \ . \tag{3.20}$$

We continue with the following calculation:

$$
\begin{aligned}
\sum_{k=1}^{n} e_{k} \cdot R_{e_{s},e_{k}} &= \sum_{k=1}^{n} e_{k} \cdot \Big({}^{n}R_{e_{s},e_{k}} - \frac{1}{4}\Big(K(e_{s})K(e_{k}) - K(e_{k})K(e_{s}) \Big) \\
&\quad + \frac{1}{2}\mathrm{d}^{D}K(e_{s},e_{k})\gamma_{0} \Big) \\
&= -\frac{1}{4}\Big({}^{n}R_{s}{}^{kij} + K_{s}{}^{i}K^{kj} - K^{ki}K_{s}{}^{j} \Big) e_{k} \cdot e_{i} \cdot e_{j} \cdot \\
&\quad + \frac{1}{2}(K^{mk}{}_{;s} - K^{m}{}_{s}{}^{;k}) e_{k} \cdot e_{m} \cdot \gamma_{0} \ . \tag{3.21}
\end{aligned}
$$

In order to analyse the curvature terms in the before-last line of (3.21), recall the convenient identity[2]

$$e_{k} \cdot e_{i} \cdot e_{j} \cdot = e_{[k} \cdot e_{i} \cdot e_{j]} \cdot - g_{ki}e_{j} \cdot + g_{ij}e_{k} \cdot - g_{kj}e_{i} \cdot \ . \tag{3.22}$$

(Square brackets around indices denote anti-symmetrisation, and round brackets denote symmetrisation.) The Bianchi identity $^{n}R_{s}{}^{[kij]} = 0$ immmediately implies

$$^{n}R_{s}{}^{kij} e_{k} \cdot e_{i} \cdot e_{j} \cdot = 2 \ {}^{n}R_{s}{}^{i} e_{i} \cdot \ .$$

Next, the undifferentiated extrinsic curvature terms in next-to-last line of (3.21) can be manipulated as

$$
\begin{aligned}
K_{s}{}^{i}K^{kj} &\underbrace{e_{k} \cdot e_{i} \cdot}_{-2g_{ki}-e_{i}\cdot e_{k}\cdot} e_{j} \cdot - K_{s}{}^{j} \underbrace{K^{ki}e_{k} \cdot e_{i} \cdot}_{-K^{ki}g_{ki}} e_{j} \cdot \\
&= -2K_{s}{}^{i}K^{kj}g_{ki}e_{j} \cdot - K_{s}{}^{i}e_{i} \cdot \underbrace{K^{kj}e_{k} \cdot e_{j} \cdot}_{-K^{kj}g_{kj}} + K_{s}{}^{j}K^{ki}g_{ki}e_{j} \cdot \\
&= 2\Big(-K^{kj}K_{sk} + K^{k}{}_{k}K_{s}{}^{j} \Big) e_{j} \cdot \ ,
\end{aligned}
$$

---

[2] To prove (3.22), note first that the result is clearly true if all indices are distinct or equal; the final formula follows by inspection of the remaining possibilities.

which results in

$$\left({}^nR_s{}^{kij} + K_s{}^i K^{kj} - K^{ki}K_s{}^j\right)e_k \cdot e_i \cdot e_j \cdot$$
$$= 2\left({}^nR_s{}^i + K^k{}_k K_s{}^i - K^{ki}K_{sk}\right)e_i \cdot =: 2E_s{}^i e_i \cdot =: 2E(e_s) \ . \quad (3.23)$$

Using again that ψ is ∇–parallel we have $\sum_{k=1}^{n} e_k \cdot R_{e_s,e_k}\psi = 0$. Equations (3.21) and (3.23) show that

$$\left(E(e_s) - (K^{mk}{}_{;s} - K^m{}_s{}^{;k})e_k \cdot e_m \cdot \gamma_0\right)\psi = 0 \ .$$

Multiplying by $e_r \cdot$ and taking a scalar product with ψ we obtain

$$
\begin{aligned}
-NE_{rs} &= (K^{mk}{}_{;s} - K^m{}_s{}^{;k})\langle\psi, \underbrace{e_r \cdot e_k \cdot e_m \cdot}_{=e_{[r} \cdot e_k \cdot e_{m]} \cdot - g_{rk}e_m \cdot + g_{rm}e_k \cdot - g_{km}e_r \cdot} \gamma_0\psi\rangle \\
&= (K^{mk}{}_{;s} - K^m{}_s{}^{;k})\langle\psi, (-g_{rk}e_m \cdot + g_{rm}e_k \cdot - g_{km}e_r \cdot)\gamma_0\psi\rangle \\
&= (K^{mk}{}_{;s} - K^m{}_s{}^{;k})(-g_{rk}Y_m + g_{rm}Y_k - g_{km}Y_r) \\
&= -(K_{rs;k} - K_{ks;r})Y^k + J_s Y_r \ , \quad\quad (3.24)
\end{aligned}
$$

where we have used the fact that the products $e_r \cdot e_m$ and $e_r \cdot e_k \cdot e_m \cdot \gamma_0$ are anti-symmetric when all indices are distinct, and therefore give no contribution in (3.24). Hence

$$N\left({}^nR_{ij} + K^k{}_k K_{ij} - K_{ik}K^k{}_j\right) = (K_{ij;k} - K_{kj;i})Y^k - J_j Y_i \ . \quad (3.25)$$

Taking a trace implies

$$N\rho = -J^i Y_i \ . \quad (3.26)$$

Anti-symmetrising (3.25) in i and j and using (3.20) one finds

$$J_i = \sigma Y_i \quad (3.27)$$

for some function σ.

We wish, now, to show that the pair (N, Y^i) defined by (3.18)-(3.19) satisfies (2.2). It is convenient to choose an ON basis $\{e_i\}_{i=1}^n$ which satisfies $e_i = \partial_i$ and $D_{e_i}e_j = 0$ at the point under consideration, then

$$
\begin{aligned}
-D_i Y_j &= \partial_i\langle\gamma_0 e_j \cdot \psi, \psi\rangle = \langle\gamma_0 e_j \cdot D_i\psi, \psi\rangle + \langle\gamma_0 e_j \cdot \psi, D_i\psi\rangle \\
&= 2\langle\gamma_0 e_j \cdot \underbrace{D_i\psi}_{-\frac{1}{2}K_i{}^k e_k \cdot \gamma_0\psi}, \psi\rangle = -K_i{}^k\langle e_j \cdot e_k \cdot \psi, \psi\rangle \\
&= -K_i{}^k\underbrace{\langle e_{[j} \cdot e_{k]} \cdot \psi, \psi\rangle}_{0} - K_i{}^k\underbrace{\langle e_{(j} \cdot e_{k)} \cdot \psi, \psi\rangle}_{-g_{jk}} \\
&= NK_{ij} \ ,
\end{aligned}
$$

as desired.

Next,

$$
\begin{aligned}
D_i N &= \partial_i\langle\psi, \psi\rangle = 2\langle\psi, D_i\psi\rangle = -\langle\psi, K_i{}^k e_k \gamma_0\psi\rangle \\
&= -K_{ik}Y^k
\end{aligned}
$$

(compare (2.17)). For further use we note that $d(N^2 - |Y|^2) = 0$, and as $N^2 - |Y|^2 \to_{r\to\infty} 0$ (since equality is attained in (3.14)) we conclude that

$$N^2 = |Y|^2 .$$

Further differentiation yields

$$D_i D_j N = N(K \circ K)_{ij} - D_i K_{jk} Y^k .$$

Inserting this into (2.5) and using the relations above leads to our key formula

$$N^2 \tau_{ij} = \rho Y_i Y_j . \tag{3.28}$$

Note that $N \to_{r\to\infty} 0$ implies $Y \to_{r\to\infty} 0$. The last part of Proposition 2.1 gives then $N \equiv 0$, hence $\psi = 0$, contradicting the fact that we have a non-trivial solution of the Witten equation. Thus N approaches a non-zero constant at infinity by (2.9), and our hypothesis on the decay of ρ provides decay of τ_{ij}. We can therefore apply Proposition 2.4 and Theorem 2.5 to conclude that the ADM momentum vanishes. But then for any $\mathring{\psi}$ there exists an associated ∇-parallel ψ. Let $\mathring{\psi}_a$, $a = 1, \ldots, m$, form a basis and let ψ_a be the parallel spinor that asymptotes to $\mathring{\psi}_a$. Now,

$$\nabla \langle \psi_a, \psi_b \rangle = 0 ,$$

which implies that the ψ_a's form a basis of $\mathfrak{S}_p$ at every $p \in \mathscr{S}$. It follows that $R_{XY}\psi_a = 0$ for a collection of spinors forming a basis at each point, hence

$$R_{XY} = 0 . \tag{3.29}$$

Choose $\mathring{\psi}$ so that $N \to 1$ and $Y \to 0$. (If no such $\mathring{\psi}$ exists, we pass to $\mathfrak{S}'$ with the structures defined by (3.9)-(3.10), choose any $\mathring{\chi}$ with norm one-half, then $\mathring{\psi} = (\mathring{\chi}, \mathring{\chi})$ will have the desired property.) Let $\widetilde{\mathscr{S}}$ be the universal covering space of $\mathscr{S}$ with corresponding data $\left(\widetilde{\mathscr{S}}, \widetilde{g}_{ij}, \widetilde{K}_{ij}, \widetilde{N}, \widetilde{Y}^j \right)$, and consider the *Killing development* thereof: by definition, this is $\widetilde{\mathscr{M}} = \mathbb{R} \times \widetilde{\mathscr{S}}$ endowed with the metric

$$\widetilde{g}_{\mu\nu} = -\widetilde{N}^2 \mathrm{d}u^2 + \widetilde{g}_{ij} \left(\mathrm{d}x_i + \widetilde{Y}^i \mathrm{d}u \right) \left(\mathrm{d}x_j + \widetilde{Y}^j \mathrm{d}u \right) ,$$

where $\widetilde{N}(u, x) = \widetilde{N}(x), \quad \widetilde{g}_{ij}(u, x) = \widetilde{g}_{ij}(x), \quad \widetilde{Y}^j(u, x) = \widetilde{Y}^j(x)$. Similarly let $(\mathscr{M}, g_{\mu\nu})$ be the Killing development of $\left(\mathscr{S}, g_{ij}, K_{ij}, N, Y^j \right)$. It should be clear that $(\widetilde{\mathscr{M}}, \widetilde{g}_{\mu\nu})$ is the universal pseudo-Riemannian covering of $(\mathscr{M}, g_{\mu\nu})$.

Equations (3.26)-(3.29) and the Codazzi-Mainardi embedding equations (compare (3.13)) show then that both $(\widetilde{\mathscr{M}}, \widetilde{g}_{\mu\nu})$ and $(\mathscr{M}, g_{\mu\nu})$ are flat. The remaining arguments of the proof of [5, Theorem 4.1] apply to show that $(\widetilde{\mathscr{M}}, \widetilde{g}_{\mu\nu}) = (\mathscr{M}, g_{\mu\nu}) = (\mathbb{R}^{n+1}, \eta_{\mu\nu})$, as desired. $\qquad\square$

A   Weighted Hölder spaces

Consider a function f defined on $\mathscr{S}_R \equiv \mathbb{R}^n \setminus B(R)$, where $B(R)$ is a closed ball of radius $R > 0$. We shall write $f = O_k(r^\beta)$ if there exists a constant C such that we have

$$0 \le i \le k \qquad |\partial^i f| \le Cr^{\beta - i}.$$

For $\sigma \in (0,1)$ we shall write $f = O_{k+\sigma}(r^\beta)$ if $f = O_k(r^\beta)$ and if there exists a constant C such that we have

$$|y - x| \le r(x)/2 \quad \Rightarrow \quad |\partial^k f(x) - \partial^k f(y)| \le C|x - y|^\sigma r^{\beta - k - \sigma}.$$

Let us note that $f = O_{k+1}(r^\beta)$ implies $f = O_{k+\sigma}(r^\beta)$ for all $\sigma \in (0,1)$, so that the reader unfamiliar with Hölder type spaces might wish to simply replace, in the hypotheses of our theorems, the $k + \sigma$ by $k + 1$ wherever convenient.

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