The classification of static electro–vacuum space–times containing an asymptotically flat spacelike hypersurface with compact interior

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Abstract

We show that static electro–vacuum black hole space–times containing an asymptotically flat spacelike hypersurface with compact interior and with both degenerate and non–degenerate components of the event horizon do not exist. This is done by a careful study of the near-horizon geometry of degenerate horizons, which allows us to eliminate the last restriction of the static electro-vacuum no-hair theory.

1 Introduction

A classical question in general relativity, first raised and partially answered by Israel [9], is that of classification of regular static black hole solutions of the Einstein–Maxwell equations. The most complete results existing in the literature so far are due to Simon [16], Masood–ul–Alam [12], Heusler [7, 8] and one of us (PTC) [2] (compare Ruback [15]) leading, roughly speaking, to the following:

Suppose that

$$\forall\, i,j \qquad Q_iQ_j \geq 0\ , \tag{1.1}$$

where Q_i is the charge of the i–th connected *degenerate* component of the black hole. Then the black hole is either a Reissner-Norsdström black hole, or a Majumdar–Papapetrou black hole.

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The above results settled the classification question in the connected case. The general case, however, remained open. The aim of this work is to remove the sign conditions (1.1), finishing the problem. More precisely, we prove:

THEOREM 1.1 *Let* (M, g, F) *be a static solution of the Einstein–Maxwell equations with defining Killing vector* X. *Suppose that* M *contains a connected and simply connected space-like hypersurface* Σ, *the closure* $\bar{\Sigma}$ *of which is the union of an asymptotically flat end and of a compact interior, such that:*

1. *The Killing vector field* X *is timelike on* Σ.
2. *The topological boundary* $\partial\Sigma \equiv \overline{\Sigma} \backslash \Sigma$ *of* Σ *is a nonempty, two-dimensional, topological manifold, with* $g_{\mu\nu}X^\mu X^\nu = 0$ *on* $\partial\Sigma$.

Then, after performing a duality rotation of the electromagnetic field if necessary:

(i) If $\partial\Sigma$ *is* connected, *then* Σ *is diffeomorphic to* $\mathbb{R}^3$ *minus a ball. Moreover there exists a neighborhood of* Σ *in* M *which is isometrically diffeomorphic to an open subset of the (extreme or non–extreme) Reissner–Nordström space–time.*

(ii) If $\partial\Sigma$ *is* not *connected, then* Σ *is diffeomorphic to* $\mathbb{R}^3$ *minus a finite union of disjoint balls. Moreover the space–time contains* only *degenerate horizons, and there exists a neighborhood of* Σ *in* M *which is isometrically diffeomorphic to an open subset of the standard Majumdar–Papapetrou space–time.*

The property that the set $\{g_{\mu\nu}X^\mu X^\nu = 0\}$ is a topological manifold, as well as simple connectedness of Σ, will hold when appropriate further global hypotheses on M are made. In fact, Corollary 1.2 and Theorem 1.3 of [2], as well as the associated remarks, are valid now without the sign restriction (1.1), and will not be repeated here.

The definitions and conventions used here coincide with those of the papers [1, 2], except for Section 2 where a different signature is used.

The idea of the proof is to show that degenerate components of the horizon are only possible in standard Majumdar–Papapetrou space–times, as follows: we start by showing that the space-metric of a static degenerate horizon is spherical, with vanishing rotation one-form, and with constant "second-order surface gravity". This leads to very precise information on the geometry of the orbit-space metric near the horizon. (This part of our work is inspired by the calculations in [14].) Let φ be the electric potential normalised so that φ tends to zero at infinity. One then uses two conformal transformations of Masood-ul-Alam to prove that this geometry is possible with $\varphi = \pm 1$ on a component of the horizon if and only if the metric is a Majumdar–Papapetrou metric. This, together with [3] (compare [6]), reduces the problem to one where $|\varphi|$ is strictly bounded away from one, which has already been shown to lead to the Reissner-Nordström geometry in [2].

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2 The near-horizon geometry of static electrovacuum degenerate Killing horizons

In this section, we establish some results on the form of the metric of a static electrovac space-time near a degenerate Killing horizon. There is a range of formalisms available, and we shall exploit the Newman-Penrose spin-coefficient formalism[1] as reviewed in [13] or [17]. To agree with the equations as they appear in these references, we shall in this section take the space-time signature to be $(+ - --)$. As in [4] we introduce Gaussian null coordinates near a component $\mathcal{N}$ of the event horizon, but with the signature changed so that the metric is

$$g = r\phi du^2 - 2dudr - 2rh_a dx^a du - h_{ab}dx^a dx^b. \tag{2.1}$$

The Killing vector X is ∂_u with norm

$$g(X, X) = r\phi$$

and $\mathcal{N}$ is located at $r = 0$. The surface gravity is $\kappa = -\partial_r(r\phi)$ at $r = 0$, and degeneracy of $\mathcal{N}$ means that κ vanishes. It follows that

$$\phi = rA(r, x^a)$$

for some A. We shall show the following:

(i) The co-vector field h_a defined on the spheres $(r = r_0,\ u = u_0)$ vanishes to order r.

(ii) The metric h_{ab} on the spheres $\mathcal{S} = (r = 0,\ u = u_0)$ has constant Gauss curvature K.

(iii) On $\mathcal{N}$, $A = K > 0$, so that $Ah_{ab}|_{r=0}$ is the unit round metric on S^2.

(iv) In the purely electric case, the electrostatic potential φ satisfies $\partial_r\varphi = \pm\sqrt{A}$ at $\mathcal{N}$.

From [4] we know that staticity implies that, at $\mathcal{N}$, h_a is a gradient, say $h_a dx^a|_{r=0} = d\lambda$. Now in the metric (2.1), choose the coordinates x^a so that they are isothermal on $\mathcal{N}$ and then introduce $\zeta = x^1 + ix^2$. Choosing m to be proportional to $d\bar{\zeta}$ at $r = 0$, the metric becomes

$$g = r^2 Adu^2 - 2dudr - 2r(hd\zeta + \overline{h}d\overline{\zeta})du - 2m\overline{m}\ , \tag{2.2}$$

[1] Alternatively, one could introduce the near-horizon geometry as in [14] (compare [4]), and use the discussion of Kundt's class of metrics in [10]. We are grateful to H. Reall for this observation.

where

$$
\begin{aligned}
m &= -\mathring{Z} d\bar{\zeta} + O(r) \,, \\
h &= \frac{\partial \lambda}{\partial \zeta} + O(r) \,,
\end{aligned}
$$

in terms of functions λ (real) and $\mathring{Z}$ (complex) of ζ and $\overline{\zeta}$.

We shall investigate the metric (2.2) in the spin-coefficient formalism. We introduce the null tetrad $(l^\mu, n^\mu, m^\mu, \overline{m}^\mu)$ by

$$
\begin{aligned}
l^\mu \partial_\mu &= D = \partial_u + \tfrac{r^2 A}{2} \partial_r \,, \\
n^\mu \partial_\mu &= \Delta = -\partial_r \,, \\
m^\mu \partial_\mu &= \delta = \tfrac{1}{Z} \partial_\zeta + \tfrac{r}{Y} \partial_{\overline{\zeta}} - \left(\tfrac{rh}{\overline{Z}} + \tfrac{r^2 \overline{h}}{\overline{Y}} \right) \partial_r \,,
\end{aligned}
$$

where $Z = \mathring{Z} + O(r)$. Here and elsewhere, a circle over a quantity indicates the value at $r = 0$.

We follow the numbering of [13] in the following. The spin-coefficients are calculated following (A.2) with the result[2]

$$
\begin{aligned}
\alpha &= -\frac{1}{2\mathring{Z}\overline{\mathring{Z}}} \frac{\partial \overline{\mathring{Z}}}{\partial \overline{\zeta}} - \frac{1}{4\mathring{Z}} \overline{h} + O(r) \,, \\
\beta &= \frac{1}{2\mathring{Z}\overline{\mathring{Z}}} \frac{\partial \mathring{Z}}{\partial \zeta} - \frac{1}{4\overline{\mathring{Z}}} h + O(r) \,, \\
\gamma &= -\frac{1}{4} \frac{\partial}{\partial r} \log \left(\frac{Z}{\overline{Z}} \right) + O(r) \,, \\
\epsilon &= \frac{1}{2} r \mathring{A} + O(r^2) \,, \\
\mu &= -\frac{1}{2} \frac{\partial}{\partial r} \log(\mathring{Z}\overline{\mathring{Z}}) + O(r) \,, \\
\tau &= \frac{1}{2\overline{\mathring{Z}}} h + O(r) \,,
\end{aligned}
$$

together with $\pi = -\overline{\tau}, \nu = 0, \lambda = 1 + O(r), \rho = O(r^2), \kappa = O(r^2)$, and $\sigma = O(r^2)$.

From these, we calculate the curvature components from (A.3), setting the scalar curvature to zero. For the Weyl spinor, we find $\Psi_0 = O(r^2), \Psi_1 = O(r), \Psi_3 = O(1)$, and $\Psi_4 = O(1)$ together with two expressions for Ψ_2:

$$
\begin{aligned}
\Psi_2 &= \frac{1}{2\mathring{Z}\overline{\mathring{Z}}} \left(\partial_\zeta \partial_{\overline{\zeta}} \lambda - \frac{1}{2} \partial_\zeta \lambda \partial_{\overline{\zeta}} \lambda \right) + O(r) \,, && (2.3) \\
&= \frac{1}{4} \left(\mathring{A} - K + \frac{1}{2\mathring{Z}\overline{\mathring{Z}}} \partial_\zeta \lambda \partial_{\overline{\zeta}} \lambda \right) + O(r) && (2.4)
\end{aligned}
$$

where $K = -\frac{1}{\mathring{Z}\overline{\mathring{Z}}} \partial_\zeta \partial_{\overline{\zeta}} \left(\log(\mathring{Z}\overline{\mathring{Z}}) \right)$, which is the Gauss curvature of $\mathcal{S}$.

[2] Note that the term $(\alpha - \beta)\pi$ in (A.3g) is misprinted, and should read $(\alpha - \bar{\beta})\pi$.

For the Ricci spinor, we find $\Phi_{00} = O(r^2), \Phi_{01} = O(r)$ and the remaining components are $O(1)$. In particular we have

$$\Phi_{11} = \frac{1}{4}\left(\mathring{A} + K + \frac{1}{2\mathring{Z}\overline{\mathring{Z}}}\partial_\zeta\lambda\partial_{\bar{\zeta}}\lambda\right) + O(r) \; , \tag{2.5}$$

$$\Phi_{02} = \frac{1}{2\overline{\mathring{Z}}^2}\left(\partial_\zeta\partial_\zeta\lambda - \frac{\partial_\zeta(\mathring{Z}\overline{\mathring{Z}})}{(\mathring{Z}\overline{\mathring{Z}})}\partial_\zeta\lambda - \frac{1}{2}(\partial_\zeta\lambda)^2\right) + O(r) \; . \tag{2.6}$$

Since we are concerned with electrovac solutions, the Ricci spinor $\Phi_{ABA'B'}$ is obtained from the Maxwell spinor ϕ_{AB} according to

$$\Phi_{ABA'B'} = k\phi_{AB}\overline{\phi}_{A'B'} \; ,$$

where $k = \frac{2G}{c^4}$. (We shall often assume $G = c = 1$.) In particular this means that

$$\Phi_{00} = k\phi_0\overline{\phi}_0; \; \Phi_{02} = k\phi_0\overline{\phi}_2; \; \Phi_{11} = k\phi_1\overline{\phi}_1 \; . \tag{2.7}$$

We saw above that $\Phi_{00} = O(r^2)$ so that, from the first equation in (2.7), we deduce that $\phi_0 = O(r)$ and since $\phi_2 = O(1)$ we must have $\Phi_{02} = O(r)$. By (2.6) this is

$$\partial_\zeta\partial_\zeta\lambda - \frac{\partial_\zeta(\mathring{Z}\overline{\mathring{Z}})}{(\mathring{Z}\overline{\mathring{Z}})}\partial_\zeta\lambda - \frac{1}{2}(\partial_\zeta\lambda)^2 = 0. \tag{2.8}$$

A second equation on λ follows from (2.3), (2.4) and (2.5) as

$$\frac{1}{2\mathring{Z}\overline{\mathring{Z}}}\left(\partial_\zeta\partial_{\bar{\zeta}}\lambda - \frac{1}{2}\partial_\zeta\lambda\partial_{\bar{\zeta}}\lambda\right) = k\phi_1\overline{\phi}_1 - \frac{1}{2}K. \tag{2.9}$$

The component ϕ_1 of the Maxwell field is constrained by the Maxwell equations, specifically by (A.5b) of [13] which here becomes

$$\overline{\delta}\phi_1 + 2\pi\phi_1 = O(r) \; ,$$

or

$$\partial_{\bar{\zeta}}\phi_1 - \phi_1\partial_{\bar{\zeta}}\lambda = O(r) \; . \tag{2.10}$$

This integrates at once to give $\phi_1 = \chi e^\lambda + O(r)$ where χ is holomorphic in ζ on $\mathcal{S}$. It is also bounded (since it is the contraction of the self-dual part of the Maxwell field with the volume form of $\mathcal{S}$), and so it must be constant (the value of this constant is proportional to the charge of the black hole). We use this in (2.9), and then we can write (2.8) and (2.9) jointly as a tensor equation on $\mathcal{S}$ as

$$\nabla_a\nabla_b\lambda - \frac{1}{2}\nabla_a\lambda\nabla_b\lambda = -\mathring{R}_{ab} + 2k|\chi|^2e^{2\lambda}\mathring{h}_{ab} \tag{2.11}$$

where, as before, a circle over a quantity indicates the value at $r = 0$. Note that this equation has appeared in the literature before, as equation (50) of [11] (also our (2.10) is equivalent to their (47)). We shall deduce from (2.11) that necessarily λ is constant and $\mathring{h}_{ab}$ is the metric of a round sphere.

First, introduce $\psi = e^{-\lambda/2}$ so that (2.11) becomes

$$\nabla_a \nabla_b \psi = \frac{\psi}{2} \mathring{R}_{ab} - k|\chi|^2 \psi^{-3} \mathring{h}_{ab} , \tag{2.12}$$

and from the trace of this find

$$\Delta\psi = \frac{\psi}{2} \mathring{R} - 2k|\chi|^2 \psi^{-3} . \tag{2.13}$$

Take ∇^b on (2.12) and use (2.13) to find

$$\nabla_a \left(\psi^3 \mathring{R} - 12k|\chi|^2 \psi^{-1} \right) = 0 ,$$

so that

$$\mathring{R} = \frac{c_1}{\psi^3} + \frac{12k}{\psi^4} |\chi|^2 \tag{2.14}$$

for some constant c_1. Insert this into (2.13), and then into (2.12), to obtain

$$\Delta\psi = \frac{c_1 \psi + 8k|\chi|^2}{2\psi^3} , \tag{2.15}$$

$$\nabla_a \nabla_b \psi = \frac{c_1 \psi + 8k|\chi|^2}{4\psi^3} \mathring{h}_{ab} . \tag{2.16}$$

The possibility that $c_1 \geq 0$ or $\chi = 0$ leads to $\Delta\psi$ of constant sign, which is possible on a compact manifold only if $c_1 = \chi = 0$ and ψ is constant, leading to $\mathring{R}_{ab} = 0$. But there are no such metrics on S^2, hence

$$c_1 < 0 \quad \text{and} \quad \chi \neq 0 . \tag{2.17}$$

Multiply (2.12) by $\nabla_b \psi$ and integrate using (2.14) to find

$$|\nabla\psi|^2 = c_2 - \frac{c_1}{2\psi} - \frac{2k}{\psi^2} |\chi|^2 \tag{2.18}$$

for some constant c_2.

In order to analyse the critical set of ψ, consider any geodesic $\gamma(s)$, $s \in I$, with unit tangent $\dot{\gamma}$. From (2.16) we find

$$\frac{d^2\psi}{ds^2} = \frac{c_1 \psi + 8k|\chi|^2}{4\psi^3} .$$

Suppose that p is a critical point of ψ and that the right-hand-side vanishes at p. Then $\psi(\gamma(s)) = \psi(p)$ is a solution satisfying the right initial data at p, and uniqueness of solutions of ODEs shows then that ψ is constant on all geodesics through p. It easily follows from (2.12) and (2.15) that both ψ and the metric are analytic in an appropriate chart, so this situation arises if and only if ψ is constant. Supposing it is not, we conclude that the Hessian of ψ is strictly definite at critical points, and the Laplacian does not vanish there. Consequently, the critical set of ψ is either a finite collection of points, or the whole sphere. In the former case, each critical point is a strict local extremum. In both cases the set

$$\Omega := \{\nabla\psi \neq 0\} \tag{2.19}$$

is connected (possibly empty).

We next show that c_2 is negative. Let

$$a := \inf\psi \ , \quad b := \sup\psi \ ,$$

then $0 < a \le b < \infty$ since ψ is positive and smooth. Suppose that $a \neq b$. At any p such that $\psi(p) = b$ we have $\nabla\psi(p) = 0$ and $\Delta\psi(p) < 0$.[3] The latter together with (2.15) gives $c_1 b < -8k|\chi|^2$, while the former reads, in view of (2.18),

$$c_1 b = 2c_2 b^2 - 4k|\chi|^2 \ ,$$

leading to

$$c_2 b^2 < -2k|\chi|^2 \ ,$$

implying $c_2 < 0$.

Our next target is to derive (2.21) below. Let p_- be a minimum of ψ and let γ be any geodesic starting at p_-, with tangent $\dot\gamma$ of unit length. Then $\psi\circ\gamma$ is a solution of the Cauchy problem

$$\frac{d^2\psi}{ds^2} = \frac{c_1\psi + 8k|\chi|^2}{4\psi^3} \ , \quad \psi(0) = a \ , \quad \frac{d\psi}{ds}(0) = 0 \ ,$$

which shows that ψ depends only upon the geodesic distance from p_-, and not on the direction of the geodesic. Thus, the level sets of ψ coincide with the geodesic spheres centred at p_-, within the injectivity radius of p_-. A similar conclusion holds at any maximum of ψ.

On Ω, as defined in (2.19), we may use ψ locally as a coordinate, leading to the following form of the metric

$$\frac{d\psi^2}{F^2(\psi)} + H^2(\psi,\phi)d\phi^2 \ ,$$

where ϕ is a local coordinate on the level sets of ψ, and

$$F^2(\psi) = |\nabla\psi|^2 = c_2 - \frac{c_1}{2\psi} - \frac{2k}{\psi^2}|\chi|^2. \tag{2.20}$$

Equation (2.13) implies $H^2 = F^2(\psi)G(\phi)$ and we may redefine ϕ to make $G = 1$, leading to the local form

$$\frac{d\psi^2}{F^2(\psi)} + F^2(\psi)d\phi^2 \tag{2.21}$$

Within the radius of injectivity of p_- and away from p_- we have $d\psi/F = d\rho$, where ρ is the distance function from p_-. Normalising ϕ to run from zero to 2π by a redefinition $\phi \to \lambda\phi$, elementary regularity requires that

$$F^2(\psi) = \lambda^2\rho^2 + o(\rho^2) \tag{2.22}$$

[3]The inequality has to be strict, otherwise ψ is constant, either by the geodesic argument just given, or by the fact that $\Delta\psi$ would again have constant sign.

for small ρ.

The integral curves of $\nabla\psi$ can be used to obtain a diffeomorphism between the level sets of ψ within Ω, which shows that (2.21) provides a global representation of the metric on Ω.

Let p_+ be any point in $\mathcal{S}$ such that $\psi(p_+) = b$, then $p_+ \in \overline{\Omega}$, and likewise the level sets of ψ near p_+ are geodesic spheres. This, and what has been said, implies that the set

$$\hat{\mathcal{S}} := \{p_-\} \cup \Omega \cup \{p_+\}$$

is both open and closed in $\mathcal{S}$, hence $\hat{\mathcal{S}} = \mathcal{S}$. Furthermore, within the radius of injectivity of p_+ and away from p_+, we have $d\psi/F = d\hat{\rho}$, where $\hat{\rho}$ is the distance function from p_+. Since ϕ has already been normalised to run from zero to 2π we obtain

$$F^2(\psi) = \lambda^2\hat{\rho}^2 + o(\hat{\rho}^2) \tag{2.23}$$

for small $\hat{\rho}$. Eliminating λ between (2.22) and (2.23), a standard calculation leads to

$$(F^2)'(a) = -(F^2)'(b) \ .$$

Equivalently,

$$c_1 ab(a^2 + b^2) = -8k|\chi|^2(b^3 + a^3) \ .$$

Eliminating c_2 from the equations $F(a) = F(b) = 0$ one finds

$$c_1 ab = -4k|\chi|^2(a + b) \ .$$

Substitute into the previous equation to obtain $a = b$, which is a contradiction.

We conclude that regularity of the metric of $\mathcal{S}$ requires that $|\nabla\psi| = 0$, so ψ and therefore λ are constant, which establishes (i). From (2.11), if λ is constant then the metric $\mathring{h}_{ab}$ is that of a round sphere, establishing (ii). Next, from (2.3), with λ constant, Ψ_2 is zero at $\mathcal{N}$ and then from (2.4), $\mathring{A} = K$, establishing (iii).

Finally, recall that the electrostatic potential φ is defined by the equation

$$d\varphi = i_X F \ ,$$

where F is the Maxwell two-form and i_X denotes contraction with X. Since $X = l + r^2 An/2$, in the purely electric case we have

$$\varphi_1 = \frac{1}{2}F_{ab}l^a n^b = \frac{1}{2}\frac{\partial\varphi}{\partial r} \ .$$

But (see (2.7) and (2.5))

$$\Phi_{11} = 2\varphi_1\bar{\varphi}_1 = \frac{1}{2}\mathring{A} + O(r) \ ,$$

so $\partial_r\varphi = \pm\sqrt{A}$ at $\mathcal{N}$, establishing (iv).

3 Proof of Theorem 1.1

As argued by Heusler [7] (compare [2, Lemma 3.2]), a duality rotation guarantees that the Maxwell field is purely electric. Following [1], we equip Σ with the orbit space metric[4] γ defined as

$$\gamma(Y,Z) = g(Y,Z) - \frac{g(X,Y)g(X,Z)}{g(X,X)} , \tag{3.1}$$

where X is the defining Killing vector, that is, the Killing vector which asymptotes $\partial/\partial t$ in the asymptotic regions, and satisfies the staticity condition.

As in [12], we consider the functions

$$\Omega_\pm = \frac{(1 \pm V)^2 - \varphi^2}{4} , \tag{3.2}$$

and the metrics

$$g_\pm := \Omega_\pm^2 \gamma . \tag{3.3}$$

(The interest in those metrics arises from the positivity of their scalar curvatures [12].) Proposition 3.4 of [2] shows that

$$0 \le |\varphi| \le 1 - V , \tag{3.4}$$

hence the functions $\Omega_\pm$ are non-negative, with the inequalities being strict in the interior unless the metric is locally a Majumdar–Papapetrou metric (compare [12]). From now on we suppose that this is *not* the case. The possibility that $|\varphi|$ is strictly bounded away from one leads to the Reissner-Nordström solutions [2], so we assume, for contradiction, that there exists a component of the horizon $\mathcal{N}$ on which $\varphi|_{r=0} =: \epsilon \in \{\pm 1\}$, then $\mathcal{N}$ is degenerate [2, Prop. 3.4]. By the results in Section 2 and by (3.4) we have

$$\varphi = \epsilon(1 - \sqrt{\mathring{A}}r + O(r^2)) ,$$

and since $V = \sqrt{|g_{uu}|} = \sqrt{\mathring{A}}r + O(r^2)$ we obtain

$$\Omega_+ = \frac{1}{4}\Big((1+\sqrt{\mathring{A}}r + O(r^2))^2 - (1-\sqrt{\mathring{A}}r + O(r^2))^2\Big) = \sqrt{\mathring{A}}r + O(r^2) , \tag{3.5}$$

$$\Omega_- = \frac{1}{4}\Big((1-\sqrt{\mathring{A}}r + O(r^2))^2 - (1-\sqrt{\mathring{A}}r + O(r^2))^2\Big) = O(r^2) . \tag{3.6}$$

We can use the space-times coordinates r and x^a of Section 2 as coordinates on the orbit space near $\{r=0\}$. From (3.1) and Section 2 we obtain

$$\gamma_{rr} = g_{rr} - \frac{g_{ru}^2}{g_{uu}} = \frac{1}{\mathring{A}r^2 + O(r^3)} , \tag{3.7}$$

$$\gamma_{ra} = g_{ra} - \frac{g_{ru}g_{au}}{g_{uu}} = \frac{O(r^2)}{\mathring{A}r^2 + O(r^3)} , \tag{3.8}$$

$$\gamma_{ab} = g_{ab} - \frac{g_{au}g_{bu}}{g_{uu}} = \mathring{h}_{ab} + O(r) + \frac{O(r^4)}{\mathring{A}r^2 + O(r^3)} . \tag{3.9}$$

[4] In [1] the symbol h is used; to avoid a clash of notation with the previous section we use γ instead.

This leads to the following form of the metric g_+:

$$\begin{aligned} g_+ &= \frac{\Omega_+^2}{g_{uu}} \times g_{uu}\gamma = \frac{\Omega_+^2}{g_{uu}} \Big(dr^2 + O(r^2) dx^a dr + (g_{uu} g_{ab} + O(r^4)) dx^a dx^b \Big) \\ &= \Big(1 + O(r)\Big) \Big(dr^2 + O(r^2) dx^a dr + \Big(r^2 (1 + O(r)) \mathring{h}_{ab} + O(r^3) \Big) dx^a dx^b \Big) . \end{aligned} \tag{3.10}$$

We want to think of the coordinate r above as a radial coordinate near the origin of $\mathbb{R}^3$. First, since $\mathring{h}_{ab}$ is the unit round metric on S^2 we have

$$r^2 \mathring{h}_{ab} dx^a dx^b = \sum (dx^i)^2 - dr^2 ,$$

so this part of the metric combines with the leading part of the dr^2 term in (3.10) to give a smooth tensor field. Next, the form $rdr = \sum x^i dx^i$ is smooth with respect to the standard differentiable structure on $\mathbb{R}^3$, and vanishes at the origin as $O(|\vec{x}|)$, so that the term $O(r)dr^2$ gives a contribution which, in the coordinates x^i, vanishes at the origin as $O(|\vec{x}|)$, with bounded first derivatives, and second derivatives dominated by a multiple of r^{-1}. To understand the remaining terms, a seemingly straightforward approach is to use spherical coordinates on S^2. However, those coordinates are singular at the z-axis, which leads to problems when one wishes to capture the regularity of the resulting metric. An alternative way of handling this proceeds as follows:

Think of the sphere S^2 as a subset of $\mathbb{R}^3$ with global coordinates $\hat{x}^i$, and let β be any smooth one-form on S^2. Then β can be uniquely extended to a smooth one-form $\hat{\beta}_i(\hat{x}^j) d\hat{x}^i$ defined on $\mathbb{R}^3 \setminus \{0\}$ by requiring that

$$\hat{x}^i \partial_{\hat{x}^i} \hat{\beta}_j = 0 , \quad \hat{\beta}_j \hat{x}^j = 0 , \quad i^*_{S^2}(\hat{\beta}_i d\hat{x}^i) = \beta ,$$

where $i^*_{S^2}$ is the pull-back map. Similarly any two-covariant tensor field α on S^2 can be uniquely extended to a smooth tensor field $\hat{\alpha}_{k\ell}(\hat{x}^j) d\hat{x}^k d\hat{x}^\ell$ defined on $\mathbb{R}^3 \setminus \{0\}$ by requiring that

$$\hat{x}^i \partial_{\hat{x}^i} \hat{\alpha}_{k\ell} = 0 , \quad \hat{\alpha}_{k\ell} \hat{x}^k = \hat{\alpha}_{k\ell} \hat{x}^\ell = 0 , \quad i^*_{S^2}(\hat{\alpha}_{k\ell} d\hat{x}^k d\hat{x}^\ell) = \alpha .$$

Let $B^*(\vec{0}, a) := B(\vec{0}, a) \setminus \{\vec{0}\}$ denote a punctured coordinate ball centred at $\vec{0}$, of radius a, in $\mathbb{R}^3$ and consider the map

$$\begin{aligned} \Psi : B^*(\vec{0}, a) &\to (0, \infty) \times \mathbb{R}^3 , \\ x^i &\mapsto \left(r = \sqrt{\sum (x^i)^2}, \hat{x}^i = \frac{x^i}{r} \right) . \end{aligned}$$

A term $\beta_a dx^a dr$ in the metric extends as above to a tensor field $\hat{\beta}_i d\hat{x}^i dr$ on $(0, \infty) \times \mathbb{R}^3$, and its pull-back by Ψ produces a term

$$\Psi^*(\hat{\beta}_i d\hat{x}^i) = \hat{\beta}_i d\left(\frac{x^i}{r}\right) = \hat{\beta}_i \frac{dx^i}{r} - \underbrace{\hat{\beta}_i \frac{x^i}{r^2}}_{=0} dr = \hat{\beta}_i \frac{dx^i}{r} .$$

This shows that the terms $O(r^2)dx^a dr$ in (3.10) give contributions, in the coordinates x^i, of the form

$$r \times \Big(\text{smooth function of } \frac{x^i}{r}\Big) dx^j dx^k \ .$$

A similar analysis of the remaining $dx^a dx^b$ terms shows that, in the coordinates x^i, the metric g_+ can be extended by continuity through the origin to a metric still denoted by the same symbol, of the form

$$g_+ = (\delta_{ij} + O(|\vec{x}|))dx^i dx^j \ , \tag{3.11}$$

with derivatives satisfying, for some constant C,

$$|\partial_j (g_+)_{k\ell}| \le C \ , \quad |\partial_i \partial_j (g_+)_{k\ell}| \le C|\vec{x}|^{-1} \ . \tag{3.12}$$

It will be convenient to rescale g_+ so that the scalar curvature vanishes:

PROPOSITION 3.1 *There exists $b > 0$ and a positive function $\psi \in C(B(\vec{0}, b)) \cap C^\infty(B^*(\vec{0}, b))$, bounded away from zero, such that the scalar curvature of $\psi^4 g_+$ vanishes on $B^*(\vec{0}, b)$.*

PROOF: We want to construct a solution ψ in $B^*(\vec{0}, b)$ of the equation

$$R(\psi^4 g_+)\psi^5 = -8\Delta_{g_+}\psi + R(g_+)\psi = 0 \ .$$

We look for ψ of the form $\psi = 1 + u$, where u vanishes on a coordinate sphere of radius b. The equation for u reads

$$-8\Delta_{g_+} u + R(g_+)u = -R(g_+) \ . \tag{3.13}$$

From (3.11)-(3.12) one finds that the scalar curvature $R(g_+)$ of the metric g_+ satisfies

$$R(g_+) \le \frac{C'}{|\vec{x}|} \tag{3.14}$$

for some constant C'. By scaling $\vec{x} \to b^{-1}\vec{x}$ we can assume $b = 1$, note that (3.14) becomes then

$$R(g_+) \le \frac{C' b}{|\vec{x}|} \ . \tag{3.15}$$

It follows that the right-hand-side of (3.13) is in $L^2(B(\vec{0}, 1))$. To solve (3.13) one can proceed as follows: Let $0 < \epsilon < 1$; we wish, first, to show the existence of a solution $u_\epsilon \in C^\infty(B(\vec{0}, 1) \backslash B(\vec{0}, \epsilon)$ of (3.13) that vanishes both on the coordinate sphere of radius one and on that of radius ϵ. This will follow from the standard theory if we can show that the solutions of the homogeneous equation, still denoted by u_ϵ are unique. For this, extend u_ϵ by zero to the interior ball of radius ϵ, and recall the Hardy inequality

$$\int_{B(\vec{0},1)} \frac{u^2}{r^2} \le C \int_{B(\vec{0},1)} |du|^2$$

(note that the standard version thereof uses the flat metric, but by uniform ellipticity both the measure and the norm of du can be taken with respect to the current metric with an appropriately modified constant C). We then have

$$\int_{B(\vec{0},1)} |R(g_+)|u^2 \leq C'b \int_{B(\vec{0},1)} \frac{u^2}{r} \leq CC'b \int_{B(\vec{0},1)} |du|^2 \ .$$

We can choose b small enough to obtain $CC'b \leq 1$, then

$$\begin{aligned} 0 &= \int_{B(\vec{0},1)\setminus B(\vec{0},\epsilon)} u_\epsilon(-8\Delta_{g_+}u_\epsilon + R(g_+)u_\epsilon) \\ &= \int_{B(\vec{0},1)\setminus B(\vec{0},\epsilon)} 8|du_\epsilon|^2 + R(g_+)u_\epsilon^2 \geq 7 \int_{B(\vec{0},1)\setminus B(\vec{0},\epsilon)} |du_\epsilon|^2 \ , \end{aligned}$$

giving uniqueness, as desired.

In the case of the non-homogeneous equation the last calculation further gives

$$\begin{aligned} 7\int_{B(\vec{0},1)} |du_\epsilon|^2 &\leq \int_{B(\vec{0},1)} u_\epsilon(-8\Delta_{g_+}u_\epsilon + R(g_+)u_\epsilon) = -\int_{B(\vec{0},1)} R(g_+)u_\epsilon \\ &\leq \frac{1}{2\epsilon}\int_{B(\vec{0},1)} (R(g_+)r)^2 + \frac{\epsilon}{2}\int_{B(\vec{0},1)} \frac{u_\epsilon^2}{r^2} \\ &\leq \frac{1}{2\epsilon}\int_{B(\vec{0},1)} (R(g_+)r)^2 + \frac{C\epsilon}{2}\int_{B(\vec{0},1)} |du_\epsilon^2| \ , \end{aligned}$$

where we have used $2xy \leq \epsilon^{-1}x^2 + \epsilon y^2$. Choosing $\epsilon = 2C^{-1}$, we can carry the last term to the left-hand-side, which shows that the sequence u_ϵ is bounded in H^1. Standard arguments imply the existence of a function u solving (3.13) on $B^*(\vec{0},1)$. We can use Sobolev's inequality and [18, Corollary 1.1, p. 29] to conclude that u is a weak solution of (3.13) on $B(\vec{0},1)$. By the last calculation one has $\Delta_{g_+}u \in L^2$ (in a weak sense), and since the metric g_+ is Lipschitz continuous we can invoke elliptic theory [5, Theorem 8.8] to conclude that $u \in H^2 \subset C^0$. Since all the norms involved can be made arbitrarily small when b is small enough, we will have $\|u\|_{L^\infty} \leq 1/2$ for appropriate b, hence $\psi \geq 1/2$. ■

We turn our attention now to the metric g_-; the sign of its scalar curvature will provide the desired contradiction. In the coordinates $\vec{x}$ introduced above, on $B^*(\vec{0},b)$ we have

$$g_- = \Omega_-^2\gamma = \left(\frac{\Omega_-^2}{\Omega_+^2}\right)g_+ = \underbrace{\left(\frac{\Omega_-^2}{\Omega_+^2}\psi^{-4}\right)}_{=:\hat{\psi}^4}\underbrace{\psi^4 g_+}_{=:\hat{g}} \ .$$

with $\hat{\psi}$ nonnegative, and vanishing precisely at the origin. The transformation law of the scalar curvature under conformal transformation gives, on $B^*(\vec{0},b)$,

$$-8\Delta_{\hat{g}}\hat{\psi} + \underbrace{R(\hat{g})}_{=0}\psi = R(g_-)\hat{\psi}^5 \geq 0 \ .$$

Let H be the $\Delta_{\hat{g}}$–harmonic function in $B(\vec{0}, b)$ which equals $\hat{\psi}$ on the boundary of the ball. The function H can be constructed by minimising

$$\int_{B(\vec{0},b)} |dH|^2_{\hat{g}} d\mu_{\hat{g}}$$

in the class of functions satisfying the boundary data, hence $H \in H^1(B(\vec{0}, b))$. By the maximum principle [5, Theorem 8.1] (note that uniform ellipticity of the metric, guaranteed by (3.11), and the H^1 character of H, suffice for this) H is bounded away from zero.

Applying the maximum principle on the set $B(\vec{0}, b) - B(\vec{0}, e)$ for $e < b$ tells us that $\hat{\psi}$ is greater than or equal to the harmonic function which equals $\hat{\psi}$ on the outer boundary $S_b(\vec{0})$ and zero on the inner boundary $S_e(\vec{0})$.[5] In the limit as e goes to zero, this harmonic function converges, uniformly on compact subsets, to a bounded function $\hat{H}$ which solves the Laplace equation on $B^*(\vec{0}, b)$. By Serrin's removable singularity theorem [18, Theorem 1.19, p. 30] (note again that (3.11), suffices for this) it holds that $\hat{H} = H|_{B^*(\vec{0},b)}$. Thus, $\hat{\psi} \geq H(x)$ in $B^*(\vec{0}, b)$. Since $H(x)$ is strictly positive in $B(\vec{0}, b)$, and $\hat{\psi}(z)$ is continuous at $\vec{0}$, by the comparison principle [5, Theorem 8.1] we obtain $\hat{\psi}(\vec{0}) > 0$. This contradicts the fact that Ω_-/Ω_+ tends to zero as $\vec{x}$ approaches the origin. Hence $\varphi(\vec{0}) = \pm 1$ is only possible for Majumdar–Papapetrou solutions, and the remarks at the end of the Introduction complete the proof.

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