The Cauchy problem for Schrödinger flows into Kähler manifolds

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November 25, 2005

Abstract

We prove local well-posedness of the Schrödinger flow from $\mathbb{R}^n$ into a compact Kähler manifold N with initial data in $H^{s+1}(\mathbb{R}^n, N)$ for $s \geq \left[\frac{n}{2}\right] + 4$.

1 Introduction

We consider maps

$$u : \mathbb{R}^n \to N$$

where N is a k-dimensional compact Kähler manifold with complex structure J and Kähler form ω (so that ω is a nondegenerate, skew-symmetric two-form). Thus N is a complex manifold and J is an endomorphism of the tangent bundle whose square, at each point, is minus the identity. N has a Riemannian metric g defined by

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot).$$

The condition that N is Kähler is equivalent to assuming that $\nabla J = 0$ where ∇ is the Levi-Civita covariant derivative with respect to g. The energy of a map u is defined by

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} |du|^2 dx$$

*Partially supported by the NSF under Grant DMS-9988711
†Partially supported by the NSF under Grant DMS-0305048
‡Partially supported by the NSF under Grant DMS-0330731
§Partially supported by the NSF under Grant DMS-0244834 and an ADVANCE TSP grant at the University of Washington

where the energy density $|du|^2$ is simply the trace with respect to the Euclidean metric of the pullback of the metric g under u, $|du|^2 = \operatorname{Tr} u^*(g)$. In local coordinates we have

$$|du|^2(x) = \sum_{\alpha=1}^{n} g_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\alpha}.$$

(We use the Einstein summation convention and sum over repeated indices.)

The L^2-gradient of $E(u)$ is given by minus the tension of the map, $-\tau(u)$, $\tau(u)$ is a vector field on N which can be expressed in local coordinates as $\tau(u) = (\tau(u)^1, \ldots, \tau(u)^k)$ with

$$\tau(u)^i = \Delta u^i + \sum_{\alpha=1}^{n} \Gamma^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\alpha} \qquad \text{for} \qquad i = 1, \ldots, k \tag{1.1}$$

where $\Gamma^i_{jk}(u)$ are the Christoffel symbols of the metric g at $u(x)$. Critical points of the energy are Harmonic maps and are characterized by the equation $\tau(u) = 0$. The foundational result on the existence of harmonic maps is due to Eells and Sampson [10] and is achieved by studying the harmonic map flow.

$$\frac{\partial u}{\partial t} = \tau(u)$$

which is simply the gradient flow for the energy functional on the space of maps. Eells and Sampson proved the existence of harmonic maps as stationary points of this flow when the domain is a compact manifold and the target is a compact manifold of non-positive curvature. In our setting, the symplectic structure on N induces a symplectic structure on the space of maps. Let $X_s = H^s(\mathbb{R}^n, N)$ be the Sobolev space of maps between $\mathbb{R}^n$ and N as defined below. For $s \geq \frac{n}{2} + 1$, X_s is a Banach manifold with a symplectic structure Ω induced form that of (N, ω) as follows. The tangent space to X_s at a map u is identified with sections of the pull-back tangent bundle over $\mathbb{R}^n$. We let $\Gamma(V)$ denote the space of sections of the bundle V, for example $du \in \Gamma(T^s(\mathbb{R}^n) \otimes u^{-1}(TN))$. For $\sigma, \mu \in \Gamma(u^{-1}(TN)) = T_u X_s$ we define

$$\Omega(\sigma, \mu) = \int_{\mathbb{R}^n} \omega(\sigma, \mu) dx.$$

In this setting we are interested in the Hamiltonian flow for the energy functional $E(\cdot)$ on (X_s, Ω). This is the Schrödinger flow which takes the form

$$\frac{\partial u}{\partial t} = J(u)\tau(u). \tag{1.2}$$

This natural geometric motivation for the flow (1.2) was elucidated in [8].

A key aspect of our approach to understanding the flow (1.2) is to isometrically embed N in some Euclidean space $\mathbb{R}^p$ and study "ambient" flows of maps from $\mathbb{R}^n$ to $\mathbb{R}^p$ which are related to (1.2). This is also central to the Eells-Sampson treatment of the harmonic map flow. Toward this end we use the Nash embedding theorem to assume that we have an isometric embedding

$$\omega : (N, g) \to (\mathbb{R}^p, \delta).$$

Using this we can now define $H^s(\mathbb{R}^n; N)$, the L^2-based Sobolev spaces of maps from $\mathbb{R}^n$ to N as follows. Note that since the domain is noncompact some care must be taken even when $s = 0$.

Definition 1.1 *For $s \geq 0$ let*

$$H^s(\mathbb{R}^n; N) = \{u : \mathbb{R}^n \to \mathbb{R}^p \;:\; u(x) \in N \text{ a.e. and } \exists\, y_u \in N \text{ such that } v - w(y_u) \in H^s(\mathbb{R}^n; \mathbb{R}^p) \text{ where } v = w \circ u\}. \tag{1.3}$$

With this definition in mind we can state our main result.

Theorem 1.1 *Given $\beta \geq 0$, the initial value problem*

$$\begin{cases} \frac{\partial u}{\partial t} &= J(u)\tau(u) + \beta\tau(u) \\ u(0) &= u_0 \end{cases} \tag{1.4}$$

for the generalized Schrödinger flow has a solution whenever the initial data $u_0 \in H^{s+1}(\mathbb{R}^n, N)$ for $s \geq \left[\frac{n}{2}\right]+4$. Moreover if $\beta = 0$ (1.4) is locally well posed in $H^{s+1}(\mathbb{R}^n, N)$ for $s \geq \left[\frac{n}{2}\right]+4$.

The question of the local and global well-posedness of equation (1.4) with data in Sobolev spaces has been previously studied by many authors (see [8, 9, 7, 30, 29, 35, 6, 27, 28, 36, 37, 25, 26, 14, 17, 15]). A common feature in all existence results for smooth solutions of Schrödinger maps is that they are obtained by using the energy method. This method consists in finding an appropriate regularizing equation which approximates the Schrödinger flow, and for which smooth solutions exist. One then proves that the regularizing equations satisfy a priori bounds in certain Sobolev norms, independent of the approximation, and that they converge to a solution of the original equation. The differences in the distinct results and proofs lie in the type of regularization used.

Ding and Wang [9] established a similar result to Theorem 1.1 for $s \geq \left[\frac{n}{2}\right] + 3$. Their work proceeds by direct study of equation (1.4) with $\beta > 0$, with a passage to the limit for $\beta = 0$. Thus the regularizing equation they use is obtained by adding the second order dissipative term $\beta\tau(u)$. In this paper we analyze equation (1.4) by adding a fourth order dissipative term (note that we allow the case $\beta = 0$ from the start). This term arises naturally in the geometric setting as the first variation of the L^2-norm of the tension. We believe that our regularization of (1.4) by a fourth order equation, which is geometric in nature, is of intrinsic and independent interest. Recently, H. McGahagan [25, 26] in her doctoral dissertation also proved a version of Theorem 1.1. Her work proceeds by a different regularization, this time hyperbolic, implemented by adding a term of the form $-\epsilon\frac{\partial^2 u}{\partial t^2}$ which transforms the equation into one whose solutions are wave maps.

Equations of the type (1.4), but with N being Euclidean space are generally known as derivative Schrödinger equations and have been the object of extensive study recently (see [20, 12, 5, 11, 21, 22, 18, 19]. The results in these investigations however do not apply directly to (1.4) for two reasons. The first one is the constraint imposed by the target being the manifold N. The second one is that in these works one needs to have data u_0 in weighted Sobolev space, a condition that we would like to avoid in the study of (1.4).

It turns out that for special choices of the target N, the equations (1.4) are related to various theories in mechanics and physics. They are examples of gauge theories which are abelian in the case of Riemann surfaces (Kähler manifolds of dimension 1 such as the 2-sphere S^2 or hyperbolic 2-space H^2). In the case of the 2-sphere S^2, Schrödinger maps arise naturally from the Landau-Lifschitz equations (a $U(1)$-gauge theory) governing the static as well as dynamical properties of magnetization [24, 32]. More precisely, for maps $s : \mathbb{R} \times \mathbb{R}^n \to S^2 \hookrightarrow \mathbb{R}^3$, equation (1.4) takes the form

$$\partial_t s = s \times \Delta s, \qquad |s| = 1 \tag{1.5}$$

which is the Landau-Lifschitz equation at zero dissipation, when only the exchange field is retained [23, 32]. When $n = 2$ this equation is also known as the two-dimensional classical continuous isotropic Heisenberg spin model (2d-CCIHS); i.e. the long wave length limit of the isotropic Heisenberg ferromagnet ([23, 32, 35]). It also occurs as a continuous limit of a cubic lattice of classical spins evolving in the magnetic field created by their close neighbours [35]. The paper [35] contains, in fact, for the cases $n = 1, 2$, $N = S^2$ the first local well-posedness results for equation (1.4) or (1.5) that we are aware of. In [6], Chang-Shatah-Uhlenbeck showed that, when $n = 1$, (1.5) is globally (in time) well-posed for data in the energy space $H^1(\mathbb{R}^1; S^2)$. When $n = 2$, for either radially symmetric or S^1-equivarient maps, they show that small energy implies global existence. In [27, 28], the authors show that, when $n = 2$, the problem is locally well-posed in the space $H^{2+\varepsilon}(\mathbb{R}^2; S^2)$, while the existence was extended to the space $H^{3/2+\varepsilon}(\mathbb{R}^2; S^2)$ in [14] and [17], and the uniqueness to the space $H^{7/4+\epsilon}(\mathbb{R}^2; S^2)$ in [17].

Remark Our proof of Theorem 1.1 actually only shows that the mapping $u_0 \mapsto u \in C([0, T], H^{s'+1}(\mathbb{R}^n, N))$, with $s' < s$, is continuous. However, one can show, by means of the standard Bond-Smith regularization procedure ([4, 13, 16]) that the statement in Theorem 1.1 also holds.

ACKNOWLEDGEMENTS: A significant portion of this work was carried out in Spring, 2002 in Cambridge, MA. At that time DP was visiting Massachusetts Institute of Technology, GS was visiting Harvard University and TT was visiting Harvard University and the Radcliffe Institute for Advanced Study. This final version was completed in Autumn, 2005 while DP and TT were visiting the Newton Institute of Mathematical Sciences in Cambridge, U.K. and TT was also visiting the University College London in London. The authors would like to thank all of these institutions for their hospitality.

2 A fourth order parabolic regularization

The method we employ in order to establish short-time existence to (1.4) is in part inspired by the work of Ding and Wang [8]. We seek to approximate equation (1.4) by a family (parametrized by $0 < \varepsilon < 1$) of parabolic equations. We establish short time existence for these systems and use energy methods to show that the time of existence is independent of ε and obtain ε independent bounds which allow us to pass to the limit as $\varepsilon \to 0$ and thus obtain a solution to (1.4). The regularization we use differs substantially from that of Ding and Wang because we wish to view the right hand side of (1.4) as a lower order term (in the regularization) so that we can use Duhamel's principle and a contraction mapping argument to establish and study the existence of our derived parabolic system.

The energy method we employ ultimately depends on establishing ε independent L^2-estimates for the tension, $\tau(u)$ and its derivatives. This suggests that we regularize (1.2) by ε times the gradient flow for the functional

$$G(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\tau(u)|^2 dx.$$

2.1 Geometric Preliminaries

We perform many of our computations in the appropriate pull-back tensor bundles over $\mathbb{R}^n$. We begin by recalling alternative formulations of the tension $\tau(u)$ in this setting (see [10]). First note that du is a closed 1-form with values in $u^{-1}(TN)$. The tension is simply minus the divergence of the differential of u

$$\tau(u) = -\delta du \in \Gamma(u^{-1}(TN))$$

where δ denotes the divergence operator with respect to the metric g. In particular, this shows that a map u is harmonic if and only if its differential is a harmonic 1-form. Let ∇ denote the covariant derivative on $T^*(\mathbb{R}^n) \otimes u^{-1}(TN)$ defined with respect to the Levi-Civita connection of the Euclidean metric on $\mathbb{R}^n$ (i.e. the ordinary directional derivative) and the Riemannian metric g on N. For $\alpha = 1, \ldots, n$ we let $\nabla_\alpha u \in \Gamma(u^{-1}(TN))$ be the vector field given by

$$\nabla_\alpha u = \partial_\alpha u = \frac{\partial u^i}{\partial x^\alpha}\frac{\partial}{\partial u^i} \tag{2.1}$$

where $(u^1, \ldots, u^k)$ are coordinates about $u(x) \in N$. In particular

$$du = \frac{\partial u^i}{\partial x^\alpha} dx^\alpha \otimes \frac{\partial}{\partial u^i} = (\nabla_\alpha u)^i dx^\alpha \otimes \frac{\partial}{\partial u^i}.$$

The second fundamental form of the map u is defined to be the covariant derivative of du, $\nabla du \in \Gamma((T^2\mathbb{R}^n) \otimes u^{-1}(TN))$. In local coordinates we have for $i = 1, \ldots, k$ and $\alpha, \beta \in 1, \ldots n$,

$$\begin{aligned} (\nabla du)^i_{\alpha\beta} &= \nabla_\alpha \nabla_\beta u^i \\ &= \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} + \Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta}. \end{aligned} \tag{2.2}$$

Note that here the subscript α actually denotes covariant differentiation with respect to the vector field $\nabla_\alpha u$ as defined in (2.1) and we have $\nabla_\alpha \nabla_\beta u = \nabla_\beta \nabla_\alpha u$. The tension is simply the trace of ∇du with respect to the Euclidean metric, $\delta = \delta_{\alpha\beta}$

$$\begin{aligned} \tau(u)^i &= \nabla_\alpha \nabla_\alpha u^i \\ &= \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\alpha} + \Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\alpha} \end{aligned} \tag{2.3}$$

from which we recover (1.1).

2.2 The gradient flow for $G(u)$

For a given vector field $\xi \in \Gamma(u^{-1}(TN))$, we construct a variation of $u : \mathbb{R}^n \to N$ with initial velocity ξ as follows. Define the map

$$U : \mathbb{R}^n \times \mathbb{R} \to N$$

by setting

$$U(x,s) = \exp_{u(x)} s\xi(x)$$

where $\exp_{u(x)} : T_{u(x)}N \to N$ denotes the exponential map. Set $u_s(x) = U(x,s)$ and now let ∇ denote the natural covariant derivative on $T^*(\mathbb{R}^n \times \mathbb{R}) \otimes U^{-1}(TN)$. Then

$$\begin{aligned}
\frac{d}{ds}G(u_s)\Big|_{s=0} &= \frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^n} |\tau(u_s)|^2 dx\Big|_{s=0} \\
&= \frac{1}{2}\int_{\mathbb{R}^n} \frac{\partial}{\partial s}\langle \tau(u_s), \tau(u_s)\rangle dx\Big|_{s=0} \\
&= \int_{\mathbb{R}^n} \langle \nabla_s \tau(u_s), \tau(u_s)\rangle dx\Big|_{s=0}
\end{aligned}$$

where the inner products are taken with respect to g and we have used the metric compatibility of ∇. Let $R = R(\cdot,\cdot)\cdot$ denote the Riemann curvature endomorphism of ∇. Using (2.3) and the definition of R we see that

$$\begin{aligned}
\nabla_s \tau(u_s) &= \nabla_s \nabla_\alpha \nabla_\alpha u_s \\
&= \nabla_\alpha \nabla_s \nabla_\alpha u_s - R(\nabla_\alpha u_s, \nabla_s u_s)\nabla_\alpha u_s \\
&= \nabla_\alpha \nabla_\alpha \nabla_s us - R(\nabla_\alpha u_s, \nabla_s u_s)\nabla_\alpha u_s.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{ds}G(u_s)\Big|_{s=0} &= \int_{\mathbb{R}^n} \langle \nabla_\alpha \nabla_\alpha \nabla_s us - R(\nabla_\alpha u_s, \nabla_s us)\nabla_\alpha u_s, \tau(u_s)\rangle dx\Big|_{s=0} \\
&= \int_{\mathbb{R}^n} \langle \nabla_\alpha \nabla_\alpha \xi, \tau(u)\rangle dx - \int_{\mathbb{R}^n} \langle R(\nabla_\alpha u, \xi)\nabla_\alpha u, \tau(u)\rangle dx.
\end{aligned}$$

By the symmetries of the curvature we have

$$\int_{\mathbb{R}^n} \langle R(\nabla_\alpha u, \xi)\nabla_\alpha u, \tau(u)\rangle dx = \int_{\mathbb{R}^n} \langle R(\nabla_\alpha u, \tau(u))\nabla_\alpha u, \xi\rangle dx$$

and provided that $\tau(u)$ and $\nabla_\alpha \tau(u)$, for $\alpha = 1, \ldots, n$, are in L^2 (and likewise for v) we may integrate by parts to obtain

$$\frac{d}{ds}G(u_s)\Big|_{s=0} = \int_{\mathbb{R}^n} \langle \xi, \nabla_\alpha \nabla_\alpha \tau(u)\rangle dx - \int_{\mathbb{R}^n} \langle R(\nabla_\alpha u, \tau(u))\nabla_\alpha u, \xi\rangle dx. \tag{2.4}$$

Proposition 2.1 *The Euler-Lagrange equation for $G(\cdot)$ acting on $H^{s+1}(\mathbb{R}^n, N)$, for $s \geq 3$ is*

$$F(u) \equiv \nabla_\alpha \nabla_\alpha \tau(u) - R(\nabla_\alpha u, \tau(u))\nabla_\alpha u = 0. \tag{2.5}$$

The parabolic regularization of (1.4) which we now proceed to study is

$$\left\{ \begin{array}{rcl} \frac{\partial u}{\partial t} & = & -\varepsilon F(u) + J(u)\tau(u) + \beta\tau(u) \\ u(0) & = & u_0 \end{array} \right. \tag{2.6}$$

2.3 The ambient flow equations

Rather than attempting to study the parabolic equations (2.6) directly we will focus on the induced "ambient flow equations" for $v = w \circ u$ where $w : (N, g) \to \mathbb{R}^p$ is a fixed isometric embedding. We fix a $\delta > 0$, chosen sufficiently small so that on the δ-tubular neighborhood $w(N)_\delta \subset \mathbb{R}^p$, the nearest point projection map

$$\Pi : w(N)_\delta \to w(N)$$

is a smooth map (cf. [33] §2.12.3). For a point $Q \in w(N)_\delta$ set

$$\rho(Q) = Q - \Pi(Q) \in \mathbb{R}^p$$

so that $|\rho(Q)| = \operatorname{dist}(Q, w(N))$, and viewing ρ and π as maps from $w(N)_\delta$ into itself we have

$$\Pi + \rho = \mathrm{Id}|_{w(N)_\delta} . \tag{2.7}$$

Note that then the differentials of the maps satisfy

$$d\Pi + d\rho = \mathrm{Id} \tag{2.8}$$

as a linear map from $\mathbb{R}^p$ to itself. For any map $v : \mathbb{R}^n \to w(N)_\delta$ we set

$$T(v) = \Delta v - \Pi_{ab}(v) v_\alpha^a v_\alpha^b$$

where $\Pi_{ab}(v)$, $1 \le a, b \le p$ are the components of the Hessian of Π at $v(\cdot)$. At a point $y \in N$ the Hessian of Π is minus the second fundamental form of N at y. So if $v = w \circ u$, with $u : \mathbb{R} \to N$, then $T(v)$ is simply the tangential component of the Laplacian of v which corresponds to the tension of the map u, i.e.

$$dw(\tau(u)) = (\Delta v)^T = d\Pi(\Delta v) = T(v).$$

Therefore, in direct analogy with the functional $G(\cdot)$, we now consider

$$\begin{aligned} \mathcal{G}(v) &= \frac{1}{2} \int_{\mathbb{R}^n} |T(v)|^2 dx \\ &= \frac{1}{2} \int_{\mathbb{R}^n} |\Delta v - \Pi_{ab}(v) v_\alpha^a v_\alpha^b|^2 dx \end{aligned}$$

Our point here (and hence the seemingly odd notation) is that we wish to consider $T(v)$ for arbitrary maps into $w(N)_\delta$ whose image does not necessarily lie on N.

Definition 2.1 *For $v : \mathbb{R}^n \to w(N)_\delta$, let $\mathcal{F}(v)$ denote the vector field on $\mathbb{R}^n$ whose components are given by*

$$(\mathcal{F}(v))^c = (\Delta T(v))^c - \sum_{\alpha,\beta=1}^{n} \left(T(v)^e \Pi^e_{abc}(v) v^a_\alpha v^b_\beta - (T(v)^e \Pi^e_{ac}(v) v^a_\alpha)_\beta - (T(v)^e \Pi^e_{cb}(v) v^b_\beta)_\alpha \right).$$

Note that the subscripts here refer to coordinate differentiation in $\mathbb{R}^n$ (Greek indices) or $\mathbb{R}^p$ (Roman indices).

For $v = w \circ u$, we wish to consider compactly supported tangential variations of $\mathcal{G}(v)$. Such variations correspond to (compactly supported) vector fields ϕ on $w(N)_\delta$ which satisfy $d\rho(\phi) = 0$.

Proposition 2.2 *For all $\phi \in \Gamma(Tw(N)_\delta)$ with compact support such that $d\rho(\phi) = 0$ we have*

$$\frac{d}{ds}\mathcal{G}(v + s\phi)\Big|_{s=0} = \int_{\mathbb{R}^n} \langle \mathcal{F}(v), \phi \rangle dx.$$

Therefore if

$$\frac{d}{ds}\mathcal{G}(v + s\phi)\Big|_{s=0} = 0$$

for all such ϕ then v satisfies

$$\mathcal{F}(v) = d\rho(\mathcal{F}(v)).$$

Definition 2.2 *If $v = w \circ u$, then the ambient form of the Schrödinger vector field $J(u)\tau(u)$, is given by the vector field f_v with*

$$f_v = dw_{|_{w^{-1}\Pi(v(x))}}[J(w^{-1}\Pi(v))(dw)^{-1}_{|_{\Pi(v(x))}}(d\Pi_{|_{(v(x))}}(\Delta v))]. \tag{2.9}$$

Note that f_v is defined for maps $v : \mathbb{R}^n \to w(N)_\delta$ whose image does not necessarily lie on N.

The regularized ambient equations are given by

$$\begin{cases} \frac{\partial v}{\partial t} &= -\varepsilon(\mathcal{F}(v) - d\rho(\mathcal{F}(v))) + f_v + \beta dw_{|_{w^{-1}\Pi(v(x))}}(Tv) \\ v(0) &= v_0 \end{cases} \tag{2.10}$$

The basic relationship between the regularized geometric flows (2.6) and the regularized ambient flows (2.10) is provided by the following two Lemmas (cf. §7 of [10]).

Lemma 2.3 *Fix $\varepsilon \in [0,1]$. Given $u_0 \in H^{s+1}(\mathbb{R}^n, N)$ with $s \geq 3$, $w : N \to \mathbb{R}^p$ an isometric embedding, and $T_\varepsilon > 0$, a flow $u : \mathbb{R}^n \times [0, T_\varepsilon] \to N$ satisfies (2.6) if and only if the flow $v = w \circ u : \mathbb{R}^n \times [0, T_\varepsilon] \to \mathbb{R}^p$ satisfies (2.10) with $v_0 = w \circ u_0$.*

Proof. First note that since w is an isometry we have

$$|T(v)|^2 = |\tau(u)|^2 \tag{2.11}$$

and therefore $\mathcal{G}(v) = G(u)$. Given $\xi \in \Gamma(u^{-1}(TN))$ a smooth compactly supported vector field set $\phi = dw(\xi) \in \Gamma(u^{-1}(T\mathbb{R}^p))$. As before we consider the variation of u given by $u_s(x) = \exp_{u(x)} s\xi$. We then have

$$w \circ u_s = v + s\phi + \mathcal{O}(s^2)$$

so that

$$G(u_s) = \mathcal{G}(v + s\phi) + \mathcal{O}(s^2).$$

Therefore

$$\int_{\mathbb{R}^n} \langle F(u), \xi \rangle dx = \int_{\mathbb{R}^n} \langle \mathcal{F}(v), \phi \rangle dx.$$

Observe that

$$\int_{\mathbb{R}^n} \left\langle \frac{\partial u}{\partial t}, \xi \right\rangle dx = \int_{\mathbb{R}^n} \left\langle dw\left(\frac{\partial u}{\partial t}\right), dw(\xi) \right\rangle dx = \int_{\mathbb{R}^n} \left\langle \frac{\partial v}{\partial t}, \phi \right\rangle dx. \tag{2.12}$$

Since $d\rho(\phi) = 0$ we also have

$$-\varepsilon \int_{\mathbb{R}^n} \langle F(u), \xi \rangle dx = -\varepsilon \int_{\mathbb{R}^n} \langle \mathcal{F}(v), \phi \rangle dx = -\varepsilon \int_{\mathbb{R}^n} \langle \mathcal{F}(v) - d\rho(\mathcal{F}(v)), \phi \rangle dx. \tag{2.13}$$

Note that

$$\begin{aligned} \int_{\mathbb{R}^n} \langle J(u)\tau(u), \xi \rangle dx &= \int_{\mathbb{R}^n} \langle dw(J(u)\tau(u)), dw(\xi) \rangle dx \\ &= \int_{\mathbb{R}^n} \langle dw[J(w^{-1}(\Pi(v)))(dw)^{-1}(T(v))], dw(\xi) \rangle dx \\ &= \int_{\mathbb{R}^n} \langle f_v, \phi \rangle dx \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \tau(u), \xi \rangle dx &= \int_{\mathbb{R}^n} \langle dw(\tau(u)), dw(\xi) \rangle dx \\ &= \int_{\mathbb{R}^n} \langle dw_{w^{-1}\Pi(v(x))}(Tv), dw(\xi) \rangle dx \\ &= \int_{\mathbb{R}^n} \langle dw_{w^{-1}\Pi(v(x))}(Tv), \phi \rangle dx \end{aligned}$$

where here, since $v = w \circ u$, we have $\Pi(v) = v$. This together with (2.12) and (2.13) implies that the flows correspond as claimed. □

We end this section by exhibiting in a more practical form the structure of the parabolic operator appearing in the regularized ambient flow equations (2.10).

Definition 2.3 *For $v : \mathbb{R}^n \to \mathbb{R}^p$, and $j \in \mathbb{N}$ we let $\partial^j v$ denote an arbitrary $j^t h$-order partial derivative of v*

$$\partial^j v = \frac{\partial^j v}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_r}} \qquad \textit{with} \qquad \alpha_1 + \cdots \alpha_r = j$$

and let

$$\partial^{j_1} v * \cdots * \partial^{j_l} v$$

denote terms which are a sum of products of terms of the form $\partial^{j_1} v, \ldots, \partial^{j_l} v$.

Proposition 2.4 *Let* $v : \mathbb{R}^n \to w(N)_\delta \subset \mathbb{R}^p$, *then*

$$-\varepsilon(\mathcal{F}(v) - d\rho(\mathcal{F}(v))) + f_v + \beta dw\Big|_{w^{-1}\Pi(v(x))}(Tv)$$

$$= -\varepsilon\Delta^2 v - \varepsilon \sum_{l=2}^{4} \sum_{j_1+\cdots+j_l=4} A_{(j_1\cdots j_l)}(v)\partial^{j_1} v * \cdots * \partial^{j_l} v + B_0(v)\partial^2 v + B_1(v)\partial v * \partial v$$

where each $j_s \geq 1$ *and each of* $A_{(j_1\cdots j_l)}(v)$, $B_0(v)$ *and* $B_1(v)$ *are bounded smooth functions of* v.

Lemma 2.5 *Fix* $\varepsilon > 0$. *Let* $v : \mathbb{R}^n \times [t_0, t_1] \to w(N)_\delta$ *satisfy*

$$\left\{ \begin{array}{rcl} \frac{\partial v}{\partial t} &=& -\varepsilon(\mathcal{F}(v) - d\rho(\mathcal{F}(v))) + f_v + \beta dw|_{w^{-1}\Pi(v(x))}(Tv) \\ v(x, t_0) &=& v_0(x) \in w(N), \end{array} \right. \tag{2.14}$$

where $v_0 \in H^{s+1}(\mathbb{R}^n, w(N))$ *with* $s \geq [\frac{n}{2}] + 4$. *Then* $v(x,t) \in w(N)$ *for all* $x \in \mathbb{R}^n$ *and all* $t \in [t_0, t_1]$.

Note that in this case Lemma 2.3 $u(x,t) = w^{-1} \circ v(x,t)$ satisfies

$$\left\{ \begin{array}{rcl} \frac{\partial u}{\partial t} &=& -\varepsilon F(u) + J(u)\tau(u) + \beta\tau(u) \\ u(x, t_0) &=& u_0(x) = w^{-1} \circ v_0(x). \end{array} \right.$$

Proof. First note that

$$\mathcal{F}(v) = dp(\mathcal{F}(v)) - d\Pi\mathcal{F}(v) \in T\omega(N).$$

Moreover $d\Pi\big|_{v(x)}(\Delta v) \in T_{\Pi v(x)}\omega(N)$, $J(\omega^{-1}(\Pi v(x))d\omega^{-1}\big|_{\Pi v(x)}(d\Pi\big|_{v(x)}\Delta(v) \in T_{\omega^{-1}(\Pi v(x))}N$, and $f_v \in T_{\Pi v(x)}\omega(N)$. Thus $\partial_t v \in T_{\Pi v(x)}\omega(N)$. Furthermore $\partial_t(\Pi \circ v) \in T_{\Pi v(x)}\omega(N)$ thus $\partial_t(p \circ v) = \partial_t v - \partial_t(\Pi \circ v) \in T_{\Pi v(x)}\omega(N)$. Therefore

$$\frac{1}{2}\frac{d}{dt}\langle \rho \circ v, \rho \circ v\rangle = \langle \rho \circ v, \partial_t(\rho \circ v)\rangle = 0$$

since $\rho \circ v$ is orthogonal to $T_{\Pi v(x)}\omega(N)$, which implies that $|\rho \circ v|^2$ is constant on $[t_0, t_1]$. Since $v \circ (x) \in \omega(N)$, $|\rho \circ v_0| = 0$ which implies that $|\rho \circ v(x,t)| = 0$ $\forall\, x \in \mathbb{R}^n$ and $\forall\, t \in [t_0, t_1])$, i.e. $v(x,t) \in \omega(N)$. $\square$

3 The Duhamel solution to the ambient flow equations

In this section we introduce a fixed point method that solves the initial value problem (2.10) in the Sobolev space $H^{s+1}(\mathbb{R}^n, \mathbb{R}^p)$, for $s \geq \frac{n}{2} + 4$. To simplify the notation, using Proposition 2.4, we rewrite (2.10) as

$$\left\{ \begin{array}{rcl} \frac{\partial v}{\partial t} & = & -\varepsilon \Delta^2 v + N(v) \\ v(x,0) & = & v_0, \end{array} \right. \tag{3.1}$$

where

$$N(v) = -\varepsilon \sum_{l=2}^{4} \sum_{j_1+\cdots+j_l=4} A_{(j_1,\ldots,j_l)}(v) \partial^{j_1} v * \cdots * \partial^{j_l} v + B_0(v) \partial^2 v + B_1(v) \partial v * \partial v. \tag{3.2}$$

We now state the well-posedness theorem for (3.1). For any fixed v_0, define the spaces

$$L^2_\delta = \{ v : \mathbb{R}^n \to \mathbb{R}^p / \|v - v_0\|_{L^2} < \delta \}.$$

and

$$L^{2,\infty}_\delta = \{ v : \mathbb{R}^n \to \mathbb{R}^p / \|v - v_0\|_{L^2}, \|v - v_0\|_{L^\infty} < \delta \}.$$

We then have the following theorem:

Theorem 3.1 *Assume $\delta > 0$, $\varepsilon > 0$, and $\gamma \in \mathbb{R}^p$ are fixed. Then for any $(v_0 - \gamma) \in H^{s+1}(\mathbb{R}^n, \mathbb{R}^p)$, $s > n/2 + 4$ there exist $T_\varepsilon = T(\varepsilon, \delta, \|\partial v_0\|_{H^s}, \|v_0 - \gamma\|_{L^2})$ and a unique solution $v = v_\varepsilon$ for (3.1) such that $v \in C([0, T_\varepsilon], H^{s+1} \cap L^{2,\infty}_\delta)$.*

To prove the theorem we rewrite (3.1) as an integral equation using the Duhamel principle:

$$v(x,t) = S_\varepsilon(t)(v_0 - \gamma) + \int_0^t S_\varepsilon(t - t') N(v)(x, t') dt' + \gamma, \tag{3.3}$$

where for $f \in H^{s+1}(\mathbb{R}^n, \mathbb{R}^p)$

$$S_\varepsilon(t) f(x) = \int_{\mathbb{R}^n} e^{(i\langle x, \xi \rangle - \varepsilon |\xi|^4 t)} \widehat{f}(\xi) d\xi \tag{3.4}$$

is the solution of the linear and homogeneous initial value problem associated to (3.1). The main idea is to consider the operator

$$Lv(x,t) = S_\varepsilon(t)(v_0 - \gamma) + \int_0^t S_\varepsilon(t - t') N(v)(x, t') dt' + \gamma \tag{3.5}$$

and prove that for a certain T_ε the operator L is a contraction from a ball of $C([0, T_\varepsilon], H^s \cap L^2_\delta)$ into itself.

To estimate L we need to study the smoothing properties of the linear solution $S_\varepsilon(t) v_0$. Because the order of derivatives that appears in $N(v)$ is 3, in order to be able to estimate the nonlinear part of L in H^{s+1}, we should prove that the operator $S_\varepsilon(t)$ provides a smoothing effect also of order 3. We have in fact the following lemma:

Lemma 3.2 *Define the operator D^s, $s \in \mathbb{R}$ as the multiplier operator such that $\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}$. Then for any $t > 0$ and $i = 1, 2, 3$,*

$$\|S_\varepsilon(t) f\|_{L^2} \lesssim \|f\|_{L^2}, \tag{3.6}$$

$$\|D^s S_\varepsilon(t) f\|_{L^2} \lesssim t^{-\frac{i}{4}} \varepsilon^{-\frac{i}{4}} \|D^{s-i} f\|_{L^2}. \tag{3.7}$$

Proof. The proof follows from Plancherel theorem and the two estimates

$$\begin{aligned} \left| e^{-\varepsilon|\xi|^4 t} \right| &\lesssim 1 \\ |\xi|^s \left| e^{-\varepsilon|\xi|^4 t} \right| &\lesssim |\xi|^{s-i} t^{-\frac{i}{4}} \varepsilon^{-\frac{i}{4}}. \end{aligned}$$

□

The next lemma shows how for small intervals of time the evolution $S_\varepsilon(t)(v_0 - \gamma)$ stays close to $v_0 - \gamma$.

Lemma 3.3 *Let $\sigma \in (0, 1)$ and assume $f \in H^{4\sigma} \cap \dot{H}^s$, and $s > \frac{n}{2} + 4\sigma$. Then*

$$\|S_\varepsilon(t) f - f\|_{L^\infty} \leq \varepsilon^\sigma t^\sigma [\|f\|_{\dot{H}^s} + \|f\|_{\dot{H}^{4\sigma}}], \text{ and } \|S_\varepsilon(t) f - f\|_{L^2} \leq \varepsilon^\sigma t^\sigma \|f\|_{\dot{H}^{4\sigma}}. \tag{3.8}$$

Proof. By the mean value theorem

$$\left| e^{-\varepsilon|\xi|^4 t} - 1 \right| \lesssim |\xi|^4 t \varepsilon,$$

which combined with the trivial bound

$$\left| e^{-\varepsilon|\xi|^4 t} - 1 \right| \leq 2$$

gives, for any $\sigma \in [0, 1]$

$$\left| e^{-\varepsilon|\xi|^4 t} - 1 \right| \lesssim (|\xi|^4 t \varepsilon)^\sigma. \tag{3.9}$$

We now write

$$\begin{aligned} |(S_\varepsilon(t) - 1) f(x)| &\leq \left| \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} [e^{-\varepsilon|\xi|^4 t} - 1] \widehat{f}(\xi) d\xi \right| \qquad (3.10) \\ &\leq (t\varepsilon)^\sigma \int_{\mathbb{R}^n} |\widehat{f}|(\xi) |\xi|^{4\sigma} \\ &\leq (t\varepsilon)^\sigma [\int_{|\xi| \leq 1} |\widehat{f}|(\xi) |\xi|^{4\sigma} + \int_{|\xi| \geq 1} |\widehat{f}|(\xi) |\xi|^s \frac{1}{|\xi|^{s-4\sigma}}, \qquad (3.11) \end{aligned}$$

and Cauchy-Schwarz concludes the argument. Note also that

$$\begin{aligned}\|(S_\varepsilon(t)-1)f\|_{L^2} &\leq \|(e^{-|\xi|^4\varepsilon t}-1)\widehat{f}\|_{L^2} \\ &\lesssim (t\varepsilon)^\sigma \left(\int (|\xi|^{4\sigma})^2|\widehat{f}|^2\right)^{\frac{1}{2}} \\ &\lesssim (t\varepsilon)^\sigma \|f\|_{\dot{H}^{4\sigma}} \lesssim (t\varepsilon)^\sigma\|\partial f\|_{H^{4\sigma-1}}.\end{aligned} \tag{3.12}$$

Here the space $\dot{H}^s$ denotes the homogeneous Sobolev space defined as the set of all functions f such that $D^s f \in L^2$.

□

We are now ready to prove Theorem 3.1.

Proof. For $T_\varepsilon, r > 0$ and $s > n/2+4$ consider the ball

$$B_r = \{\partial v \in H^s : \|\partial(v-v_0)\|_{L^\infty_{T_\varepsilon},H^s} \leq r\} \cap L^{2,\infty}_\delta.$$

We want to prove that for the appropriate T_ε and f, the operator L maps B_r to itself and is a contraction. We start with the estimate of the linear part of L. By (3.6) we have

$$\|\partial(S_\varepsilon(t)(v_0-\gamma)-(v_0-\gamma))\|_{H^s} \lesssim \|(1+D^s)S_\varepsilon(t)\partial v_0\|_{L^2} + \|\partial v_0\|_{H^s} \lesssim 2\|\partial v_0\|_{H^s}. \tag{3.13}$$

To estimate the nonlinear term we use (3.6) (3.7), and interpolation :

$$\begin{aligned}&\left\|\partial\left(\int_0^t S_\varepsilon(t-t')N(v)(x,t')dt'\right)\right\|_{H^s} \\ &= \left\|\int_0^t S_\varepsilon(t-t')\partial N(v)(x,t')dt'\right\|_{L^2} + \left\|\int_0^t D^s S_\varepsilon(t-t')\partial N(v)(x,t')dt'\right\|_{L^2} \\ &\lesssim \int_0^t \|\partial N(v)\|_{L^2_x}(t')\,dt' + \int_0^t (t')^{-3/4}\varepsilon^{-3/4}\|D^{s-3}\partial N(v)\|_{L^2_x}(t')\,dt' \\ &\lesssim \int_0^t (1+t^{-3/4}\varepsilon^{-3/4})\|\partial v\|^m_{H^s_x}(t')\,dt'.\end{aligned} \tag{3.14}$$

Note that to control $\partial N(v)$ and $D^{s-3}\partial N(v)$ in the previous inequality we are never in the position of estimating v in L^2. By (3.5), (3.7), (3.14) and (3.13), we obtain the estimate

$$\|\partial(Lv-v_0)\|_{H^s_x}(t) \leq C_0\|\partial v_0\|_{H^s} + C_1\int_0^t (1+t^{-3/4}\varepsilon^{-3/4})\|\partial v\|^m_{H^s_x}(t')\,dt'. \tag{3.15}$$

Thus

$$\|\partial(Lv-v_0)\|_{L^\infty_{T_\varepsilon}H^s_x} \leq C_0\|\partial v_0\|_{H^s} + C_1\varepsilon^{-3/4}T_\varepsilon^{1/4}\|\partial v\|^m_{L^\infty_{T_\varepsilon}H^s_x}. \tag{3.16}$$

We still need to check that Lv is continuous in time and that $Lv \in L^{\infty,2}_\delta$. The continuity follows directly from the continuity of the operator $S_\varepsilon(t)$. To prove the L^∞ and L^2 estimates one uses (3.8) with $\sigma = 1/4$ applied to $f = v_0-\gamma$, the Sobolev inequality and estimates similar to the ones used to obtain (3.15). One gets

$$\|Lv-v_0\|_{L^\infty_{T_\varepsilon}L^2_x} + \|Lv-v_0\|_{L^\infty_{T_\varepsilon}L^\infty_x} \leq C_1\varepsilon^{1/4}T_\varepsilon^{1/4}\|\partial v_0\|_{H^s} + C_1\varepsilon^{-\frac{3}{4}}T_\varepsilon^{1/4}\|\partial v\|^m_{L^\infty_{T_\varepsilon}H^s_x}. \tag{3.17}$$

We now take $r = 3C_0\|v_0\|_{H^s}$ and

$$T_\varepsilon \le \min(\delta^4 C_1^{-4}\varepsilon^3 6^{-4m} C_0^{-4m}\|v_0\|_{H^s}^{-m+1}, \delta^{\frac{1}{\sigma}} 2^{-\frac{1}{\sigma}} C_1^{-\frac{1}{\sigma}}\varepsilon^{-1}\|v_0\|_{H^s}^{-\frac{1}{\sigma}}) \tag{3.18}$$

so that (3.15) and (3.17) guarantees that L maps B_r into itself. Note that (3.14) yields for $v, w \in B_r$

$$\left\| \int_0^t S_\varepsilon(t-t')\partial[N(v) - N(w)](x,t')\,dt' \right\|_{H^s} \lesssim \varepsilon^{-3/4} T_\varepsilon^{1/4} \|\partial[N(v) - N(w)]\|_{L^\infty_{T_\varepsilon} H^{s-3}_x}. \tag{3.19}$$

Therefore

$$\begin{aligned} &\|\partial(Lv - Lw)\|_{L^\infty_{T_\varepsilon} H^s_x} \\ &\quad\lesssim \varepsilon^{-3/4} T_\varepsilon^{1/4} \|\partial[N(v) - N(w)]\|_{L^\infty_{T_\varepsilon} H^{s-3}_x} \\ &\quad\lesssim \varepsilon^{-3/4} T_\varepsilon^{1/4} C(\delta)(\|\partial v\|^p_{L^\infty_{T_\varepsilon} H^s_x} + \|\partial w\|^p_{L^\infty_{T_\varepsilon} H^s_x})\|\partial(v - w)\|_{L^\infty_{T_\varepsilon} H^s_x} \\ &\quad\lesssim \varepsilon^{-3/4} T_\varepsilon^{1/4} C(\delta, \|\partial v_0\|_{H^s_x})\|\partial(v - w)\|_{L^\infty_{T_\varepsilon} H^s_x}. \end{aligned} \tag{3.20}$$

Similarly one shows that

$$\|Lv - Lw\|_{L^\infty_{T_\varepsilon} H^s_x} \lesssim \varepsilon^{-3/4} T_\varepsilon^{1/4} C(\delta, \|\partial v_0\|_{H^s_x})\|\partial(v - w)\|_{L^\infty_{T_\varepsilon} H^s_x}.$$

By shrinking T_ε further by an absolute constant if necessary, from (3.20) and (3.21) we obtain

$$\|Lv - Lw\|_{L^\infty_{T_\varepsilon} H^{s+1}_x} \le \frac{1}{2}\|v - w\|_{L^\infty_{T_\varepsilon} H^{s+1}_x}. \tag{3.21}$$

The contraction mapping theorem ensures that there exists a unique function $v = v_\varepsilon$ in $L^2_\delta \cap \{\partial v \in H^s : \|\partial(v - v_0)\|_{L^\infty_{T_\varepsilon}, H^s} \le r\}$ which solves the integral equation (3.3) in the time interval $[0, T_\varepsilon]$ defined in (3.18). Moreover $v \in B_r$ by our choice of T_ϵ. The uniqueness in the whole space $H^{s+1}_x \cap L^{2,\infty}_\delta$ follows by similar and by now classical arguments. □

4 Analytic preliminaries

In this section we state and present the detailed proof of an interpolation inequality for Sobolev sections on vector bundles which appears in [9] (see Theorem 2.1). This inequality was first proved for functions on $\mathbb{R}^n$ by Gagliardo and Nirenberg, and for functions on Riemannian manifolds by Aubin [1]. The justification for presenting a complete proof is that this estimate plays a crucial role in the energy estimates and therefore in the proof of the results this paper. The precise dependence of the constants involved in this inequality is vital to our argument and we feel compelled to emphasize it.

Let $\Pi : E \to \mathbb{R}^n$ be a Riemannian vector bundle over $\mathbb{R}^n$. We have the bundle $\Lambda^P T^*\mathbb{R}^n \otimes E \to \mathbb{R}^n$ over $\mathbb{R}^n$ which is a tensor product of the bundle E and the induced P-form bundle over $\mathbb{R}^n$, with $p = 1, 2, \ldots, n$. We define $T(\Lambda^P T^*\mathbb{R}^n \otimes E)$ as the set of all smooth sections of $\Lambda^P T^*\mathbb{R}^n \otimes E \to \mathbb{R}^n$. There exists an induced metric on $\Lambda^P T^*\mathbb{R}^n \otimes E \to \mathbb{R}^n$ from the metric on $T^*\mathbb{R}^n$ and E such that for any $s_1, s_2 \in \Gamma(\Lambda^P T^*\mathbb{R}^n \otimes E)$

$$\langle s_1, s_2\rangle = \sum_{i_1\leq\cdots\leq i_p} \langle s_1(e_{i_1},\dots e_{i_p}), s_2(e_1,\dots e_{i_p})\rangle \tag{4.1}$$

where $\{e_i\}$ is an orthonormal local frame for $T\mathbb{R}^n$. We define the inner product on $\Gamma(\Lambda^P T^*\mathbb{R}^n \otimes E)$ as follows

$$(s_1, s_2) = \int_{\mathbb{R}^n} \langle s_1, s_2\rangle(x)dx. \tag{4.2}$$

The Sobolev space $L^2(\mathbb{R}^n, \Lambda^P T^*\mathbb{R}^n\otimes E)$ is the completion of $\Gamma(\Lambda^P T^*\mathbb{R}^n\otimes E)$ with respect to the above inner product. To define the bundle-valued Sobolev space $H^{k,r}(\mathbb{R}^n, \Lambda^P T^*\mathbb{R}^n\otimes E)$ consider ∇ the covariant derivative induced by the metric on E, then take the completion of smooth sections of E in the norm

$$\|s\|_{H^{k,r}} = \|s\|_{k,r} = \left(\sum_{i=0}^{k} \int_{\mathbb{R}^n} |\nabla^i s|^r dx\right)^{\frac{1}{r}} \tag{4.3}$$

where

$$|\nabla^i s|^2 = \langle \underbrace{\nabla\cdots\nabla}_{i-\text{times}} s, \underbrace{\nabla\cdots\nabla}_{i-\text{times}} s\rangle. \tag{4.4}$$

If $r = 2$, $H^{k,r} = H^k$.

Proposition 4.1 *Let $s \in C_c^\infty(E)$ where E is a finite dimensional C^∞ vector bundle over $\mathbb{R}^n$. Then given $q, r \in [1,\infty]$ and integers $0 \leq j \leq k$ we have that*

$$\|\nabla^j s\|_{L^p} \leq C\|\nabla^k s\|_{L^q}^a \|s\|_{L^r}^{1-a} \tag{4.5}$$

with $p \in [2,\infty)$, $a \in \left(\frac{j}{k}, 1\right]$ and satisfying

$$\frac{1}{p} = \frac{j}{n} + \frac{1}{r} + a\left(\frac{1}{q} - \frac{1}{r} - \frac{k}{n}\right). \tag{4.6}$$

If $r = n/k - 1 \neq 1$ then (4.5) does not hold for $a = 1$. The constant C that appears in (4.5) only depends on n, k, j, q, r and a.

Proof. If f is a real valued smooth function with compact support on E then Theorem 3.70 in [2] ensures that (4.5) holds.

Case 1: Let $j = 0$ and $k = 1$. Then for $f = |s|$ we have by (4.5) that

$$\|s\|_{L^p} \leq C\|\nabla|s|\|_{L^q}^a \|s\|_{L^r}^{1-a}. \tag{4.7}$$

Kato's inequality ensures that $|\nabla|s|| \leq |\nabla s|$ which using (4.7) yields

$$\|s\|_{L^p} \leq C\|\nabla s\|_{L^q}^a \|s\|_{L^r}^{1-a}, \tag{4.8}$$

which proves (4.5) for $j = 0$ and $k = 1$. In general if $f = |\nabla^j s|$ Kato's inequality ensures that $|\nabla|\nabla^j s|| \leq |\nabla^{j+1} s|$ which yields using (4.8)

$$\begin{aligned} \|\nabla^j s\|_{L^p} &\leq C\|\nabla|\nabla^j s|\,\|_{L^q}^a \|\nabla^j s\|_{L^r}^{1-a} \\ &\leq C\|\nabla^{j+1} s\|_{L^q}^a \|\nabla^j s\|_{L^r}^{1-a} \end{aligned} \tag{4.9}$$

where $a \in (0, 1)$ and

$$\frac{1}{p} = \frac{1}{r} + a\left(\frac{1}{q} - \frac{1}{r} - \frac{1}{n}\right). \tag{4.10}$$

Note that so far the condition $p \geq 2$ has not played a role.

Case 2: Let $j = 1$, $k = 2$ and $\frac{1}{2} \leq a \leq 1$. If $a = 1$ (4.9) yields

$$\|\nabla s\|_{L^p} \leq C\|\nabla^2 s\|_{L^q} \tag{4.11}$$

with

$$\frac{1}{p} = \frac{1}{q} - \frac{1}{n}. \tag{4.12}$$

If $a = \frac{1}{2}$, assume $p \geq 2$ then

$$\begin{aligned} \operatorname{div}\langle|\nabla s|^{p-2}\nabla s, s\rangle = |\nabla s|^p + |\nabla s|^{p-2}\langle\nabla_\alpha\nabla_\alpha s, s\rangle + \\ +(p-2)|\nabla s|^{p-4}\langle\nabla_\beta s, \nabla_\alpha\nabla_\beta s\rangle\langle\nabla_\alpha s, s\rangle. \end{aligned} \tag{4.13}$$

Since

$$\int_{\mathbb{R}^n} \operatorname{div}\langle|\nabla s|^{p-2}\nabla s, s\rangle = 0 \tag{4.14}$$

then (4.13) gives

$$\int_{\mathbb{R}^n} |\nabla s|^p \leq (n+p-2)\int_{\mathbb{R}^n} |\nabla s|^{p-2}|\nabla^2 s|\,|s|. \tag{4.15}$$

Given our choice of $j = 1$, $k = 2$ and $a = \frac{1}{2}$ we have $\frac{1}{q} + \frac{1}{r} = \frac{2}{p}$, i.e. $\frac{1}{q} + \frac{1}{r} + \frac{p-2}{p} = 1$.
Thus Hölder's inequality yields

$$\|\nabla s\|_{L^p}^p \leq (n+p-2)\|\nabla^2 s\|_{L^q}\|s\|_{L^r}\|\nabla s\|_{L^p}^{p-2}, \tag{4.16}$$

thus

$$\|\nabla s\|_{L^p} \leq \sqrt{n+p-2}\,\|\nabla^2 s\|_{L^q}^{\frac{1}{2}}\|s\|_{L^r}^{\frac{1}{2}} \tag{4.17}$$

with

$$\frac{1}{p} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{n}\right). \tag{4.18}$$

For $a \in \left(\frac{1}{2}, 1\right)$ we consider two cases: $q < n$, and $q \geq n$. For $q < n$ using the convexity of $\log \|f\|_{L^p}^p$ as a function of p we have

$$\|\nabla s\|_{L^p} \leq \|\nabla s\|_{L^t}^{\alpha} \|\nabla s\|_{L^\sigma}^{1-\alpha} \text{ with } \alpha = \frac{p^{-1} - \sigma^{-1}}{t^{-1} - \sigma^{-1}} \in (0,1) \tag{4.19}$$

where $t < p < \sigma$ are such that

$$\frac{2}{t} = \frac{1}{q} + \frac{1}{r} \text{ and } \frac{1}{\sigma} = \frac{1}{q} - \frac{1}{n}. \tag{4.20}$$

Using (4.17) and (4.17) we have that

$$\|\nabla s\|_{L^\sigma} \leq C\|\nabla^2 s\|_{L^q} \tag{4.21}$$

and

$$\|\nabla s\|_{L^t} \leq C\|\nabla^2 s\|_{L^q}^{\frac{1}{2}} \|s\|_{L^r}^{\frac{1}{2}}. \tag{4.22}$$

Combining (4.19), (4.21) and (4.22) we obtain

$$\|\nabla s\|_{L^p} \leq C\|\nabla^2 s\|_{L^q}^{1-\frac{\alpha}{2}} \|s\|_{L^r}^{\frac{\alpha}{2}} \tag{4.23}$$

where

$$\frac{1}{p} = \frac{1}{n} + \frac{1}{r} + \left(1 - \frac{\alpha}{2}\right)\left(\frac{1}{q} - \frac{1}{r} - \frac{2}{n}\right), \tag{4.24}$$

which proves the case $a \in \left(\frac{1}{2}, 1\right)$ and $q < n$.

For $q \geq n$ $t > 0$ and $b \in (0,1)$ such that

$$\frac{1}{p} = \frac{1}{t} + b\left(\frac{1}{q} - \frac{1}{t} - \frac{1}{n}\right) \tag{4.25}$$

we have by (4.9)

$$\|\nabla s\|_{L^p} \leq C\|\nabla^2 s\|_{L^q}^{b} \|\nabla s\|_{L^t}^{1-b}. \tag{4.26}$$

Choosing $t > 0$ so that

$$\frac{2}{t} = \frac{1}{q} + \frac{1}{r} \tag{4.27}$$

we have by (4.17)

$$\|\nabla s\|_{L^t} \leq C\|\nabla^2 s\|_{L^q}^{\frac{1}{2}} \|s\|_{L^r}^{\frac{1}{2}}. \tag{4.28}$$

Combining (4.26) and (4.28) we obtain

$$\|\nabla s\|_{L^p} \leq C\|\nabla^2 s\|_{L^q}^{\frac{b+1}{2}} \|s\|_{L^r}^{\frac{1-b}{2}} \tag{4.29}$$

with

$$\frac{1}{p}=\frac{1}{n}+\frac{1}{r}+\left(\frac{b+1}{2}\right)\left(\frac{1}{q}-\frac{1}{r}-\frac{2}{n}\right) \tag{4.30}$$

by (4.25) and (4.27). This concludes the proof of Case 2.

Case 3: Let $j=0$ and $k=2$. From (4.8) we have

$$\|s\|_{L^p}\leq C\|\nabla s\|_{L^{q_1}}^{a_1}\|s\|_{L^r}^{1-a_1} \tag{4.31}$$

with $a_1\in(0,1)$ and

$$\frac{1}{p}=\frac{1}{r}+a_1\left(\frac{1}{q_1}-\frac{1}{r}-\frac{1}{n}\right). \tag{4.32}$$

Choosing q_1 so that

$$\frac{1}{q_1}=\frac{1}{r}+\frac{1}{n}+a_2\left(\frac{1}{q}-\frac{1}{r}-\frac{2}{n}\right) \tag{4.33}$$

then $a_2\in\left(\frac{1}{2},1\right)$ and

$$\|\nabla s\|_{L^{q_1}}\leq C\|\nabla^2 s\|_{L^q}^{a_2}\|s\|_{L^r}^{1-a_2}. \tag{4.34}$$

Combining (4.31) and (4.34) we have that

$$\|s\|_{L^p}\leq C\|\nabla^2 s\|_{L^q}^{a_1a_2}\|s\|_{L^r}^{1-a_1a_2} \tag{4.35}$$

with

$$\frac{1}{p}=\frac{1}{r}+a_1a_2\left(\frac{1}{q}-\frac{1}{r}-\frac{2}{n}\right) \tag{4.36}$$

from (4.32) and (4.33).

Case 4: We now proceed by induction on k. Assume that for $k\geq 2$ and $j<k$ we have proved (4.5). Let $j<k<k+1$. By (4.9) we have

$$\|\nabla^k s\|_{L^{q_1}}\leq C\|\nabla^{k+1}s\|_{L^{q_2}}^{a_2}\|\nabla^k s\|_{L^{r_2}}^{1-a_2} \tag{4.37}$$

with

$$\frac{1}{q_1}=\frac{1}{r_2}+a_2\left(\frac{1}{q_2}-\frac{1}{r_2}-\frac{1}{n}\right). \tag{4.38}$$

By the induction hypothesis, applied to $\nabla^{k-1}s$, we also have

$$\|\nabla^k s\|_{L^{r_2}}\leq C\|\nabla^{k+1}s\|_{L^{q_3}}^{a_3}\|\nabla^{k-1}s\|_{L^{r_3}}^{1-a_3} \tag{4.39}$$

with

$$\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{n}+a_3\left(\frac{1}{q_3}-\frac{1}{r_3}-\frac{2}{n}\right) \tag{4.40}$$

and

$$\|\nabla^{k-1}s\|_{L^{r_0}} \le C\|\nabla^k s\|_{L^{q_4}}^{a_4}\|s\|_{L^{r_4}}^{1-a_4} \tag{4.41}$$

with

$$\frac{1}{r_3} = \frac{1}{r_4} + \frac{k-1}{n} + a_4\left(\frac{1}{q_4} - \frac{1}{r_4} - \frac{k}{n}\right). \tag{4.42}$$

Letting $q_4 = r_2$, $q_3 = q$, $r_4 = r$, $r_2 = p$ we obtain

$$\|\nabla^k s\|_{L^p} \le C\|\nabla^{k+1}s\|_{L^q}^{a}\|s\|_{L^r}^{1-a} \text{ with } a = \frac{a_3}{1-a_4+a_3a_4} \in \left[\frac{k}{k+1}, 1\right] \tag{4.43}$$

and

$$\frac{1}{p} = \frac{1}{r} + \frac{k}{n} + a\left(\frac{1}{q} - \frac{1}{r} + \frac{k+1}{n}\right). \tag{4.44}$$

By hypothesis for $j < k$ and using (4.43) we have

$$\begin{aligned} \|\nabla^j s\|_{L^p} &\le C\|\nabla^k s\|_{L^{q_1}}^{a_1}\|s\|_{L^r}^{1-a_1} \\ &\le C\|\nabla^{k+1}s\|_{L^q}^{a_0a_1}\|s\|_{L^r}^{1-a_0a_1} \end{aligned} \tag{4.45}$$

with $a_0a_1 \in \left[\frac{j}{k+1}, 1\right]$ and

$$\frac{1}{p} = \frac{1}{r} + \frac{j}{n} + a_1a_0\left(\frac{1}{q} - \frac{1}{r} - \frac{k+1}{n}\right) \tag{4.46}$$

which finishes the proof of the proposition. □

Corollary 4.2 *Let $u \in C^\infty(\mathbb{R}^n, N)$ be constant outside a compact set. Then for $k \ge 1$, $q, r \in [1,\infty)$ and $0 \le j \le k-1$ we have*

$$\|\nabla^{j+1}u\|_{L^p} \le C\|\nabla^k u\|_{L^q}^{a}\|\nabla u\|_{L^r}^{1-a} \tag{4.47}$$

with

$$\frac{1}{p} = \frac{j}{n} + \frac{1}{r} + a\left(\frac{1}{q} + \frac{1}{r} - \frac{k-1}{n}\right). \tag{4.48}$$

If $r = \frac{n}{k-1-j} \neq 1$ then (4.47) does not hold for $a = 1$. The constant C that appears in (4.48) only depends on n, k, j, q, r and a.

Proof. Apply (4.5) to $s = \nabla u$ a section of the bundle $u^*(TN) \otimes T^*\mathbb{R}^n$. Since ∇u is not necessarily compactly supported a standard approximation argument might be needed to complete the proof. □

In the second part of this section we establish the equivalence of the Sobolev norms defined in either the intrinsic, geometric setting or in the ambient, Euclidean setting. These results hold when we are above the range in which these spaces have suitable multiplication properties. Since we are working with the gradients of the maps we must consider the H^s spaces with $s > \frac{n}{2} + 1$.

We begin by assuming that we have chosen coordinate systems on (N, g) so that the eigenvalues of g are bounded above and below by a fixed constant $C > 1$, i.e. we assume that

$$C^{-1}|\xi|^2 \leq g_{ij}\xi_i\xi_j \leq C|\xi|^2 \qquad \text{for all} \qquad \xi \in \mathbb{R}^k.$$

We denote these coordinates by either $(y^1, \ldots, y^k)$ or $(u^1, \ldots, u^k)$. As before $(x^1, \ldots, x^n)$ denotes Euclidean coordinates on $\mathbb{R}^n$.

For $v : \mathbb{R}^n \to \mathbb{R}^p$ we let

$$\partial_\alpha v = \frac{\partial v^a}{\partial x^\alpha} e_a$$

where $\{e_1, \ldots, e_p\}$ is an orthonormal basis for $\mathbb{R}^p$. Recall that if $X \in \Gamma(u^{-1}(TN))$ then

$$(\nabla_\alpha X)^j = \frac{\partial X^j}{\partial x^\alpha} + \Gamma^j_{ik} X^i \frac{\partial u^k}{\partial x^\alpha}$$

and $\nabla_\alpha u = \partial_\alpha u \in \Gamma(u^{-1}(TN))$ denotes the vector field along u defined in (4). We use the following notation for higher order derivatives.

Definition 4.1 *Let $\alpha = (\alpha_1, \ldots, \alpha_{l+1})$ denote a multi-index of length $l + 1$ ($|\alpha| = l + 1$) with each $a_s \in \{1, \ldots, n\}$. We let $\nabla^{l+1}u \in \Gamma(u^{-1}(TN))$ denote any covariant derivative of U of order $l + 1$ e.g.*

$$\nabla^{l+1}u = \sum_{\alpha=(\alpha_1,\ldots,\alpha_{l+1})} \nabla_{\alpha_1} \cdots \nabla_{\alpha_{l+1}} u$$

Similarly

$$\partial^{l+1}v = \sum_{\alpha=(\alpha_1,\ldots,\alpha_{l+1})} \frac{\partial^{l+1} v^a}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_{l+1}}} e_a$$

and

$$\partial^{l+1}u = \sum_{\alpha=(\alpha_1,\ldots,\alpha_{l+1})} \frac{\partial^{l+1} u^a}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_{l+1}}} \frac{\partial}{\partial y^a}.$$

Remark 1 *Note that our use of the multi-index notation differs from the usual one.*

Recall that for

$$\begin{aligned} v &: \mathbb{R}^n \to \mathbb{R}^p \\ u &: \mathbb{R}^n \to N \end{aligned}$$

the Sobolev norms of ∂v and ∇u for $k \in \mathbb{N}$ are defined by

$$\begin{aligned} \|\partial v\|_{H^k} &= \sum_{l=0}^{k} \|\partial^{l+1} v\|_{L^2(\mathbb{R}^n)} \\ \|\nabla u\|_{H^k} &= \sum_{l=0}^{k} \|\nabla^{l+1} u\|_{L^2(\mathbb{R}^n)} \\ &= \sum_{l=0}^{k} \left(\int_{\mathbb{R}^n} g_{ij} (\nabla^{l+1} u)^i (\nabla^{l+1} u)^j \right)^{\frac{1}{2}} \end{aligned}$$

where here

$$\|\nabla^{l+1} u\|_{L^2(\mathbb{R}^n)} = \sum_{|\alpha|=l+1} \|\nabla_{\alpha_1} \cdots \nabla_{\alpha_{l+1}} u\|_{L^2(\mathbb{R}^n)}$$

and the sum is taken over all distinct multi-indices of length $l+1$ (see (4.5)). The L^2 norm of each of these is computed with respect to the metric g as indicated. We use the obvious analogous definition for $\|\partial^{l+1} v\|_{L^2(\mathbb{R}^n)}$.

Note that by definition $u \in H^k(\mathbb{R}^n, N)$ if $\exists\, y_u \in N$ such that for $v = w \circ u$

$$\|v - w(y_u)\|_{L^2} + \|\partial v\|_{H^{k-1}} < \infty.$$

Our immediate goal is to show that for $k > \frac{n}{2} + 1$ if $v = w \circ u$ then

$$\|\partial v\|_{H^k} < \infty \quad \text{if and only if} \quad \|\nabla u\|_{H^k} < \infty.$$

Lemma 4.3 *For each $k \geq 0$ we have*

$$\nabla^{k+1} u = \partial^{k+1} u + \sum_{l=2}^{k+1} \sum_{j_1+\cdots+j_l=k+1} G_{(j_1,\ldots,j_l)}(u) \partial^{j_1} u * \cdots * \partial^{j_l} u \tag{4.49}$$

$$\partial^{k+1} u = \nabla^{k+1} u + \sum_{l=2}^{k+1} \sum_{j_1+\cdots+j_l=k+1} E_{(j_1,\ldots,j_l)}(u) \nabla^{j_1} u * \cdots * \nabla^{j_l} u \tag{4.50}$$

$$\partial^{k+1} v = \frac{\partial w^a}{\partial y^j} \partial^{k+1} u^j e_a + \sum_{l=2}^{k+1} \sum_{j_1+\cdots+j_l=k+1} F_{(j_1,\ldots,j_l)}(u) \partial^{j_1} u * \cdots * \partial^{j_l} u \tag{4.51}$$

where each subscript $j_s \geq 1$ and

$$\begin{aligned} G_{(j_1,\ldots,j_l)}(u) &= G^j_{(j_1,\ldots,j_l)}(u) \frac{\partial}{\partial y^j} \\ E_{(j_1,\ldots,j_l)}(u) &= E^j_{(j_1,\ldots,j_l)}(u) \frac{\partial}{\partial y^j} \\ F_{(j_1,\ldots,j_l)}(u) &= F^a_{(j_1,\ldots,j_l)}(u) e_a \end{aligned}$$

and each G, E and F are smooth, bounded functions of u.

The notation $a_{j_1} * \cdots * a_{j_l}$ corresponds to a product of the $a_{j)k}$'s.

Remark 2 *Throughout this section whenever expressions similar to the right hand side of (4.49), (4.50) or (4.51) occur, a key point is to note that all the subscripts $j_s \geq 1$, for $s \in \{1, \ldots, l\}$. This is always to be understood even if it is not explicitly stated.*

Proof. We establish each of these by induction, beginning with (4.49). Note that for $k = 0$, $\nabla u = \partial u$. For $k = 1$

$$\begin{aligned}
\nabla_{\alpha_2}\nabla_{\alpha_1} u &= \nabla_{\alpha_2}\left(\frac{\partial u^j}{\partial x^{\alpha_1}}\frac{\partial}{\partial y^j}\right) \\
&= \frac{\partial^2 u^j}{\partial x^{\alpha_2}\partial x^{\alpha_1}}\frac{\partial}{\partial y^j} + \Gamma^j_{ik}\frac{\partial u^i}{\partial x^{\alpha_1}}\frac{\partial u^k}{\partial x^{\alpha_2}}\frac{\partial}{\partial y^j} \quad .
\end{aligned}$$

Assume now that (4.49) holds for some $k \geq 1$. Then

$$\begin{aligned}
\nabla^{k+2} u &= \nabla_{\alpha_{k+2}}\nabla^{k+1} u \\
&= \nabla_{\alpha_{k+2}}(\nabla_{\alpha_{k+1}}\cdots\nabla_{\alpha_1} u) \\
&= \nabla_{\alpha_{k+2}}\left(\frac{\partial^{k+1} u^j}{\partial x^{\alpha_{k+1}}\cdots\partial x^{\alpha_1}}\frac{\partial}{\partial y^j}\right) \\
&\quad +\nabla_{\alpha_{k+2}}\left(\sum_{l=2}^{k+1}\sum_{j_1+\cdots+j_l=k+1} G^j_{(j_1,\ldots,j_l)}(u)\partial^{j_1}u * \cdots * \partial^{j_l}u\frac{\partial}{\partial y^j}\right) \\
&= \partial^{k+2}u + \Gamma^j_{il}\frac{\partial^{k+1}u^i}{\partial x^{\alpha_{k+1}}}\cdots\partial x^{\alpha_1}\frac{\partial u^l}{\partial x^{\alpha_{k+2}}}\frac{\partial}{\partial y^j} \\
&\quad +\sum_{l=2}^{k+1}\sum_{j_1+\cdots+j_l=k+1}\left(G^j_{(j_1,\ldots,j_l)}(u)\right)'\partial^{j_1}u * \cdots * \partial^{j_l}u\frac{\partial}{\partial y^j} \\
&\quad +\sum_{l=2}^{k+2}\sum_{j_1+\cdots+j_l=k+2} G^i_{(j_1,\ldots,j_l)}(u)\partial^{j_1}u * \cdots * \partial^{j_l}u\frac{\partial}{\partial y^j} \\
&\quad +\sum_{l=2}^{k+2}\sum_{j_1+\cdots+j_l=k_2} G^i_{(j_1,\ldots,j_l)}(u)\partial^{j_1}u * \cdots * \partial^{j_l}u\Gamma^j_{il}\frac{\partial u^l}{\partial x^{\alpha_{k+2}}}\frac{\partial}{\partial y^j}.
\end{aligned}$$

Therefore

$$\nabla^{k+2}u = \partial^{k+2}u + \sum_{l=2}^{k+2}\sum_{j_1+\cdots+j_l=k+2} G_{(j_1,\ldots,j_l)}(u)\partial^{j_1}u * \cdots * \partial^{j_l}u$$

which completes the proof of (4.49). The proof of (4.50) proceeds in a similar fashion and is left to the reader.

To prove (4.51) we recall that $v = w \circ u$ and thus

$$\frac{\partial v^a}{\partial x^{\alpha_1}} = \frac{\partial w^a}{\partial y^j}\frac{\partial u^j}{\partial x^{\alpha_1}} \tag{4.52}$$

(which is the case $k = 0$). When $k = 1$ we differentiate this to obtain

$$\frac{\partial^2 v^a}{\partial x^{\alpha_2} \partial x^{\alpha_1}} = \frac{\partial w^a}{\partial y^j} \frac{\partial^2 u^j}{\partial x^{\alpha_2} \partial x^{\alpha_1}} + \frac{\partial^2 w^a}{\partial y^i \partial y^j} \frac{\partial u^i}{\partial x^{\alpha_2}} \frac{\partial u^j}{\partial x^{\alpha_1}}.$$

Assume now that (4.51) holds for some $k \geq 1$. Then

$$\begin{aligned} \partial^{k+2} v &= \partial_{\alpha_{k+2}} (\partial^{k+1} v) \\ &= \frac{\partial w^a}{\partial y^j} \partial^{k+2} u^j e_a + \frac{\partial^2 w^a}{\partial y^i \partial y^j} \partial^{k+1} u^j \frac{\partial u^i}{\partial x^{\alpha_{k+2}}} e_a \\ &\quad + \sum_{l=2}^{k+2} \sum_{j_1 + \cdots + j_l = k+2} F_{(j_1, \ldots, j_l)}(u) \partial^{j_1} u * \cdots * \partial^{j_l} u. \end{aligned}$$

This implies (4.51) and completes the proof of the Lemma. □

Combining (4.50) and (4.51) in Lemma 4.3 we obtain the following.

Lemma 4.4 *For $v = w \circ u$ and $k \geq 0$ we have*

$$\partial^{k+1} v^a = \frac{\partial w^a}{\partial y^j} (\nabla^{k+1} u)^j + \sum_{l=2}^{k+1} \sum_{j_1 + \cdots + j_l = k+1} H^a_{(j_1, \ldots, j_l)}(u) \nabla^{j_1} u * \cdots * \nabla^{j_l} u \tag{4.53}$$

where, as before, each subscript $j_s \geq 1$ and each H^a is a smooth, bounded function of u.

We now proceed to bound the pointwise norms in terms of each other.

Lemma 4.5 *For $v = w \circ u$ and $k \geq 0$ there is a constant $C > 1$ depending only on n and k such that*

$$|\partial^{k+1} v|^2 \leq C |\nabla^{k+1} u|^2 + C \sum_{l=2}^{k+1} \sum_{j_1 + \cdots + j_l = k+1} |\nabla^{j_1} u|^2 \cdots |\nabla^{j_l} u|^2 \tag{4.54}$$

and

$$|\nabla^{k+1} u|^2 \leq C |\partial^{k+1} v|^2 + C \sum_{l=2}^{k+1} \sum_{j_1 + \cdots + j_l = k+1} |\partial^{j_1} v|^2 \cdots |\partial^{j_l} v|^2 \tag{4.55}$$

Proof. Using (4.53) we have

$$\begin{aligned} \sum_{a=1}^{p} |\partial^{k+1} v^a|^2 &= \sum_{a=1}^{p} \sum_{|\alpha| = k+1} |\partial_{\alpha_{k+1}} \cdots \partial_{\alpha_1} v^a|^2 \\ &= \sum_{a=1}^{p} \frac{\partial w^a}{\partial y^j} (\nabla^{k+1} u)^j \frac{\partial w^a}{\partial y^i} (\nabla^{k+1} u)^i \\ &\quad + \left| \sum_{l=2}^{k+1} \sum_{j_1 + \cdots + j_l = k+1} H^a_{(j_1, \ldots, j_l)}(u) \nabla^{j_1} u * \cdots * \nabla^{j_l} u \right|^2 \\ &\quad + 2 \frac{\partial w^a}{\partial y^j} (\nabla^{k+1} u)^j \sum_{l=2}^{k+1} \sum_{j_1 + \cdots + j_l = k+1} H^a_{(j_1, \ldots, j_l)}(u) \nabla^{j_1} u * \cdots * \nabla^{j_l} u. \end{aligned}$$

Since $w : N \to \mathbb{R}^p$ is an isometric embedding we note that

$$g_{ij} = \sum_{a=1}^{p} \frac{\partial w^a}{\partial y^i} \frac{\partial w^a}{\partial y^j}. \tag{4.56}$$

Therefore, using the fact that for any $l \geq 1$

$$C^{-1} \sum_{i=1}^{k} |(\nabla^l u)^i|^2 \leq |\nabla^l u|^2 = g_{ij}(\nabla^l u)^i (\nabla^l u)^j \leq C \sum_{i=1}^{k} |(\nabla^l u)^i|^2$$

we have

$$\sum_{a=1}^{p} |\partial^{k+1} v^a|^2 \leq 2|\nabla^{k+1} u|^2 + 2C \sum_{l=2}^{k+1} \sum_{j_1+\cdots+j_l=k+1} |\nabla^{j_1} u|^2 \cdots |\nabla^{j_l} u|^2$$

which establishes (4.54). To prove (4.55) we proceed by induction. For $k = 0$ we have

$$\nabla_a u = \frac{\partial u^i}{\partial x^\alpha} \frac{\partial}{\partial y^i}.$$

Using (4.52) and (4.56) this implies

$$|\nabla_\alpha u|^2 = g_{ij} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\alpha} = \frac{\partial v^a}{\partial x^\alpha} \frac{\partial va}{\partial x^\alpha}.$$

Therefore

$$|\nabla u|^2 = |\partial v|^2. \tag{4.57}$$

Note that for $k = 1$, by (4.53) we have

$$\frac{\partial w^a}{\partial y^j} (\nabla^2 u)^j = \partial^2 v^a - H^a(u) \nabla u * \nabla u.$$

So that

$$|\nabla^2 u|^2 = 2|\partial^2 v|^2 + C|\nabla u|^4$$

or

$$|\nabla^2 u|^2 = 2|\partial^2 v|^2 |\partial v|^4. \tag{4.58}$$

Assume now that (4.55) holds for any $k \geq 1$. Again using (4.53) we then have

$$\begin{aligned} |\nabla^{k+2} u|^2 &\leq 2|\partial^{k+2} v|^2 + C \sum_{l=2}^{k+2} \sum_{j_1+\cdots+j_l=k+2} |\nabla^{j_1} u|^2 \cdots |\nabla^{j_l} u|^2 \\ &\leq 2|\partial^{k+2} v|^2 + C \sum_{l=2}^{k+2} \sum_{j_1+\cdots+j_l=k+2} |\partial^{j_1} v|^2 \cdots |\partial^{j_l} v|^2 \end{aligned}$$

which completes the proof of Lemma 4.5. □

Lemma 4.6 *Assume that $k > \frac{n}{2}+1$. There exists a constant $C = C(N,k,n)$ such that for $u \in C^\infty(\mathbb{R}^n, N)$ constant outside a compact set of $\mathbb{R}^n$ if $v = w \circ u$ then*

$$\|\nabla^{k+1}u\|_{L^2} \leq C\sum_{l=1}^{k}\|\partial v\|^l_{H^k} \tag{4.59}$$

$$\|\partial^{k+1}v\|_{L^2} \leq C\sum_{l=1}^{k}\|\nabla u\|^l_{H^k}. \tag{4.60}$$

Proof. By (4.55) we have

$$\|\nabla^{k+1}u\|_{L^2} \leq C\|\partial^{k+1}v\|_{L^2} + C\sum_{l=2}^{k+1}\sum_{j_1+\cdots+j_l=k+1}\left(\int_{\mathbb{R}^n}|\partial^{j_1}_v|^2\cdots|\partial^{j_l}v|^2\right)^{\frac{1}{2}}. \tag{4.61}$$

Let $2 \leq p_i \leq \infty$, $i = 1,\dots l$ be such that

$$\frac{1}{p_1}+\cdots+\frac{1}{p_l} = \frac{1}{2}. \tag{4.62}$$

Then by Hölder's inequality

$$\||\partial^{j_1}v|\cdots|\partial^{j_l}v|\|_{L^2} \leq c\|\partial^{j_1}v\|_{L^{p_1}}\cdots\|\partial^{j_l}v\|_{L^{p_l}}. \tag{4.63}$$

Since $k \geq \frac{n}{2}+1$ then

$$\frac{j_i-1}{k} < a_i = \frac{j_i-1}{k} + \frac{n}{2k^2}\left(k - j_i + \frac{1}{l}\right) < 1 \tag{4.64}$$

and

$$\frac{1}{2} \geq \frac{1}{p_i} = \frac{j_i-1}{n} + \frac{1}{2} - \frac{ka_i}{n} > 0 \quad . \tag{4.65}$$

Note that to ensure that $a_i < 1$ in (4.64) we either need $n \leq 3$ or $\left(\frac{n}{2k}-1\right)\left(k-j+\frac{1}{e}\right) < 1-\frac{1}{e}$. Since $2 \leq l \leq k+1$ and $1 \leq j \leq k$ the previous inequality holds provided $\left(\frac{n}{2k}-1\right)\left(k-\frac{1}{2}\right) < \frac{1}{2}$ which requires $k > \frac{n+\sqrt{n(n-4)}}{4}$. Thus to accommodate all values of n simultaneously, it is enough to choose $k \geq \frac{n}{2}+1$ and $k \in \mathbb{N}$. Thus (4.5) in Proposition 4.1 yields

$$\begin{aligned} |\partial^{j_i}v\|_{L^{p_i}} &\leq C\|\partial^{k+1}v\|^{a_i}_{L^2}\|\partial v\|^{1-a_i}_{L^2} \\ &\leq C\|\partial v\|_{H^k}. \end{aligned} \tag{4.66}$$

Therefore combining (4.61), (4.63) and (4.66) we have

$$\|\nabla^{k+1}u\|_{L^2} \leq C\sum_{\rho=1}^{k}\|\partial v\|^\rho_{H^k}. \tag{4.67}$$

□

A similar argument to the one above where Proposition 4.1 is now applied to ∇u rather than ∂v yields (4.60).

Lemma 4.7 *There exists a constant $C = C(N,n)$ such that if $u \in C^\infty(\mathbb{R}^n, N)$ constant outside a compact set of $\mathbb{R}^n$ and $v = w \circ u$ then for $1 \le k \le \frac{n}{2} + 1$*

$$\|\nabla^{k+1} u\|_{L^2} \le C \sum_{l=1}^{k} \|\partial v\|^l_{H^{[\frac{n}{2}]+2}} \tag{4.68}$$

$$\|\partial^{k+1} v\|_{L^2} \le C \sum_{l=1}^{k} \|\nabla u\|^l_{H^{[\frac{n}{2}]+2}}. \tag{4.69}$$

Proof. The proof is very similar to that of Lemma 4.6, where the a_i's and p_i's in the interpolation are taken as follows

$$\frac{j_i - 1}{s_0} < a_i = \frac{j_i - 1}{s_0} + \frac{n}{2ks_0}(k - j_i + l^{-1}) < 1, \tag{4.70}$$

where $s_0 = \left[\frac{n}{2}\right] + 2$, and

$$\frac{1}{2} \ge \frac{1}{p_i} = \frac{j_i - 1}{n} + \frac{1}{n} - \frac{s_0}{n} a_i > 0. \tag{4.71}$$

□

Remark Proposition 4.1 holds $s \in C^m_c(E)$ where E is a finite dimensional C^m vector bundle over $\mathbb{R}^n$ provided $k < m$. Similarly Lemma 4.6 holds for $u \in C^m(\mathbb{R}^n, N)$ and u constant outside a compact set of $\mathbb{R}^n$, provided once again that $m > k$. A simple approximation theorem ensures that Lemma 4.6 holds for $u \in C^m(\mathbb{R}^n, N) \cap H^k(\mathbb{R}^n, N)$ with $m > k$.

Corollary 4.8 *Assume that $k \ge \frac{n}{2} + 4$. There exists a constant $C = C(N,k,n)$ such that for $u \in C^{k+1}(\mathbb{R}^n, N) \cap H^k(\mathbb{R}^n, N)$*

$$\|\nabla u\|_{L^\infty} \le C \sum_{j=1}^{[\frac{n}{2}]+2} \|\nabla u\|^j_{H^{[\frac{n}{2}]+2}} \tag{4.72}$$

$$\|\nabla^2 u\|_{L^\infty} \le C \sum_{l=1}^{2[\frac{n}{2}]+4} \|\nabla u\|^l_{H^{[\frac{n}{2}]+3}} \tag{4.73}$$

$$\|\nabla^3 u\|_{L^\infty} \le C \sum_{l=1}^{3[\frac{n}{2}]+12} \|\nabla u\|^l_{H^{[\frac{n}{2}]+4}}. \tag{4.74}$$

Proof. Recall that $\|\nabla u\| = |\partial v|$ if $v = w \circ u$, and by Sobolev embedding theorem

$$\|\partial v\|_{L^\infty} \le c \|\partial v\|_{H^{[\frac{n}{2}]+2}} \tag{4.75}$$

$$\|\partial^2 v\|_{L^\infty} \le c \|\partial v\|_{H^{[\frac{n}{2}]+3}} \tag{4.76}$$

and

$$\|\partial^3 v\|_{L^\infty} \le c\|\partial v\|_{H^{[\frac{n}{2}]+4}} \tag{4.77}$$

Therefore combining (4.60), (4.69) and (4.75) we have

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\le C\sum_{j=0}^{[\frac{n}{2}]+2}\|\partial^{j+1}v\|_{L^2} \\ &\le C\left(\|\nabla u\|_{L^2} + \sum_{j=1}^{[\frac{n}{2}]+2}\|\nabla u\|^j_{H^{[\frac{n}{2}]+2}}\right) \\ &\le C\sum_{j=1}^{[\frac{n}{2}]+2}\|\nabla u\|^j_{H^{[\frac{n}{2}]+2}}. \end{aligned} \tag{4.78}$$

Note that (4.35) ensures that

$$|\nabla^2 u| \le C|\partial^2 v| + c|\partial v|^2. \tag{4.79}$$

Combining (4.75), (4.76), (4.60) and (4.69) we have

$$\begin{aligned} \|\nabla^2 u\|_{L^\infty} &\le C\|\partial^2 v\|_{L^\infty} + C\|\partial v\|^2_{L^\infty} \\ &\le C\|\partial v\|_{H^{[\frac{n}{2}]+3}} + C\|\partial v\|^2_{H^{[\frac{n}{2}]+2}} \\ &\le C\sum_{j=0}^{[\frac{n}{2}]+3}\|\partial^{j+1}v\|_{L^2} + C\sum_{j=0}^{[\frac{n}{2}]+2}\|\partial^{j+1}v\|^2_{L^2} \\ &\le C\sum_{l=1}^{[\frac{n}{2}]+3}\|\nabla u\|^l_{H^{[\frac{n}{2}]+2}} \end{aligned} \tag{4.80}$$

Note that (4.55) also ensures that

$$|\nabla^3 u| \le C(|\partial^3 v| + |\partial^2 v|\,|\partial v| + |\partial v|^3). \tag{4.81}$$

Combining (4.75), (4.76), (4.77), (4.60) and (4.69) we have

$$\|\nabla^3 u\|_{L^\infty} \le C\sum_{j=1}^{3([\frac{n}{4}]+4)}\|\nabla u\|^j_{H^{[\frac{n}{2}]+4}}. \tag{4.82}$$

□

5 ε-independent energy estimates

Theorem 3.1 ensures that the initial value problem

$$\begin{cases} \frac{\partial v}{\partial t} &= -\varepsilon\Delta^2 v + N(v) \\ v(0) &= v_0 \end{cases}$$

has a unique solution $v_\varepsilon \in C([0,T_\varepsilon], H^{s+1} \cap L^{2,\infty}_\delta)$ provided $v_0 \in H^{s+1}(\mathbb{R}^n, \mathbb{R}^p)$ for $s > [\frac{n}{2}]$. To prove that (1.4) has a solution we need to show that (2.10) has a solution for $\varepsilon = 0$. To do this we need to show that each v_ε extends to a solution in $C([0,T], H^{s+1} \cap L^{2,\infty}_\delta)$ where $T > 0$ is independent of ε. This is accomplished by proving ε-independent energy estimates for the function v_ε. It turns out that thanks to the geometric nature of this flow, if one assumes enough regularity (i.e. $s > [\frac{n}{2}]+4$), it is easier to prove ε-independent energy estimates for the corresponding u_ε. Lemma 4.6 and Lemma 4.7 then allows us to translate these into estimates for v_ε.

Let $u_\varepsilon = u \in C([0,T_\varepsilon], H^{s+1}(\mathbb{R}^n, N))$ with s large enough[1] be a solution of

$$\begin{cases} \partial_t u &= -\varepsilon\Delta\tau(u) + \varepsilon R(\nabla u, \tau(u))\nabla u + J(u)\tau(u) + \beta\tau(u) \\ u(0) &= u_0, \end{cases} \tag{5.1}$$

where $\varepsilon \in (0,1]$, $\beta > 0$, $\Delta = \sum_{\alpha=1}^n \nabla_\alpha\nabla_\alpha$. Our goal is to understand how $\|\nabla u\|_{H^k}(t)$ varies with time.

Let $l \in \mathbb{N}$. We denote by $\boldsymbol{\alpha}$ the multi-index of length l $\boldsymbol{\alpha} = (\alpha_1 \cdots \alpha_l)$, and $\nabla_{\boldsymbol{\alpha}} u = \nabla_{\alpha_1 \cdots} \nabla_{\alpha_l} u$. The following lemma and corollaries establish some computational identities which are very useful.

Lemma 5.1 *Let $u \in C^1([0,T], H^s(\mathbb{R}^n, N))$, $s \in \mathbb{N}$, $s > \frac{n}{2} + 2$. Let $X \in TN$ for $1 \le l \le s$ and $|\boldsymbol{\alpha}| = l$. We have*

$$\nabla_{\alpha_0}\nabla_{\boldsymbol{\alpha}} u = \nabla_{\boldsymbol{\alpha}}\nabla_{\alpha_0} u + \sum_{j=0}^{l-2} \nabla_{\alpha_1} \cdots \nabla_{\alpha_j}[R(\nabla_{\alpha_0} u, \nabla_{\alpha_{j+1}} u)\nabla_{\alpha_{j+2}} \cdots \nabla_{\alpha_l} u] \tag{5.2}$$

$$\nabla_t\nabla_{\boldsymbol{\alpha}} u = \nabla_{\boldsymbol{\alpha}}\nabla_t u + \sum_{j=0}^{l-2} \nabla_{\alpha_1} \cdots \nabla_{\alpha_j}[R(\nabla_t u, \nabla_{\alpha_{j+1}} u)\nabla_{\alpha_{j+2}} \cdots \nabla_{\alpha_l} u] \tag{5.3}$$

$$\nabla_{\alpha_0}\nabla_{\boldsymbol{\alpha}} X = \nabla_{\boldsymbol{\alpha}}\nabla_{\alpha_0} X + \sum_{j=0}^{l-1} \nabla_{\alpha_1} \cdots \nabla_{\alpha_j}[R(\nabla_{\alpha_0} u, \nabla_{\alpha_{j+1}} u)\nabla_{\alpha_{j+2}} \cdots \nabla_{\alpha_l} X]. \tag{5.4}$$

Proof. The proof is done by induction on the length of the multi-index $\boldsymbol{\alpha}$, i.e., on l. We prove (5.4) and leave (5.2) and (5.3) to the reader, as the two proofs are very similar. If $l = 1$

$$\nabla_{\alpha_0}\nabla_{\alpha_1} X = \nabla_{\alpha_1}\nabla_{\alpha_0} X + R(\nabla_{\alpha_0} u, \nabla_{\alpha_1} u)X. \tag{5.5}$$

[1] We will see later that $s > [\frac{n}{2}] + 4$ will be enough. In this paper we do not attempt to obtain the lowest possible exponent s.

Suppose (5.4) holds for $l \geq 1$ and consider

$$
\begin{aligned}
(5.6) \qquad & \nabla_{\alpha_0}\nabla_{\alpha_1}\cdots\nabla_{\alpha_{l+1}}X \\
&= \nabla_{\alpha_1}[\nabla_{\alpha_0}\nabla_{\alpha_2}\cdots\nabla_{\alpha_{l+1}}X] + R(\nabla_{\alpha_0}u, \nabla_{\alpha_1}u)\nabla_{\alpha_2}\cdots\nabla_{\alpha_{l+1}}X \\
&= \nabla_{\alpha_1}\cdots\nabla_{\alpha+1}\nabla_{\alpha_0}X + R(\nabla_{\alpha_0}u, \nabla_{\alpha_1}u)\nabla_{\alpha_2}\cdots\nabla_{\alpha_{l+1}}X \\
&\quad +\nabla_{\alpha_1}[\sum_{j=1}^{l}\nabla_{\alpha_2}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{l+1}X] \\
&= \nabla_{\alpha_1}\cdots\nabla_{\alpha_{l+1}}\nabla_{\alpha_0}X \\
&\quad +\sum_{j=0}^{l}\nabla_{\alpha_1}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_{l+1}}X]
\end{aligned}
$$

□

Corollary 5.2 *Let $u \in C^1([0,T], H^s(\mathbb{R}^n, N))$ $s \in \mathbb{N}$, $s > \frac{n}{2}+2$, then for $1 \leq l \leq s$, $|\boldsymbol{\alpha}| = l$ we have*

$$
\begin{aligned}
(5.7) \qquad \Delta\nabla_{\boldsymbol{\alpha}}u &= \nabla_{\boldsymbol{\alpha}}\tau(u) + \sum_{j=0}^{l-1}\nabla_{\alpha_1}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}\nabla_{\alpha_0}u] \\
&\quad + \sum_{j=1}^{l-2}\nabla_{\alpha_0}\nabla_{\alpha_1}\nabla_{\alpha_j}[R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}u] \\
(5.8) \qquad \nabla_t\nabla_{\boldsymbol{\alpha}}\nabla_{\alpha_0}u &= \nabla_{\boldsymbol{\alpha}}\nabla_{\alpha_0}\nabla_t u \\
&\quad \sum_{j=0}^{l-1}\nabla_{\alpha_1}\cdots\nabla_{\alpha_j}[R(\nabla_t u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}\nabla_{\alpha_0}u] \\
(5.9) \qquad \nabla_{\beta_0}\nabla_{\boldsymbol{\alpha}}\nabla_{\alpha_0}u &= \nabla_{\boldsymbol{\alpha}}\nabla_{\alpha_0}\nabla_{\beta_0}u \\
&\quad \sum_{j=0}^{l-1}\nabla_{\alpha_1}\cdots\nabla_{\alpha_j}[R(\nabla_{\beta_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}\nabla_{\alpha_0}u]
\end{aligned}
$$

Proof. The proof of (5.7) is an application of (5.2) and (5.4). To prove (5.8) and (5.9) apply (5.4) to $\nabla_{\alpha_0}u = X$ and note that ∇_t and ∇_{β_0} behave the same way. Moreover recall that $\nabla_{\alpha_0}\nabla_t u = \nabla_t\nabla_{\alpha_0}u$. □

Corollary 5.3 *Let $u \in C^1([0,T], H^s(\mathbb{R}^n, N))$, $s \in \mathbb{N}$ $s > \frac{n}{2}+2$. Let $X \in TN$ then for $l \geq 1$ and $|\boldsymbol{\alpha}| = l$ we have*

$$
\begin{aligned}
(5.10) \quad \Delta\nabla_{\boldsymbol{\alpha}}X &= \nabla_{\boldsymbol{\alpha}}\Delta X + \sum_{j=0}^{l-1}\nabla_{\alpha_1}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}\nabla_{\alpha_0}X] \\
&\quad + \sum_{j=0}^{l-1}\nabla_{\alpha_0}\nabla_{\alpha_1}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\nabla_{\alpha_l}X].
\end{aligned}
$$

Proof. To prove (5.10) we apply (5.4) twice, first to x then $\nabla_{\alpha_0} X$.

$$
\begin{aligned}
(5.11) \Delta \nabla_{\boldsymbol{\alpha}} X &= \nabla_{\alpha_0}\nabla_{\alpha_0}\nabla_{\boldsymbol{\alpha}} X \\
&= \nabla_{\alpha_0}[\nabla_{\boldsymbol{\alpha}}\nabla_{\alpha_0} X + \nabla_{\alpha_0}[\sum_{j=0}^{l-1} \nabla_{\alpha_1}\cdots\nabla_{\alpha_j}(R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l} X] \\
&= \nabla_{\boldsymbol{\alpha}}\nabla_{\alpha_0}\nabla_{\alpha_0} X + \sum_{j=0}^{l-1} \nabla_{\alpha_1}\cdots\nabla_{\alpha_j}(R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\nabla_{\alpha_l}\nabla_{\alpha_0} X) \\
&\quad \sum_{j=0}^{l-1} \nabla_{\alpha_0}\nabla_{\alpha_1}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l} X].
\end{aligned}
$$

□

Remark Note that in particular (5.10) applied to $X = \tau(u)$ yields

$$
\begin{aligned}
(5.12) \quad \Delta \nabla_{\boldsymbol{\alpha}} \tau(u) &= \nabla_{\boldsymbol{\alpha}} \Delta \tau(u) \\
&\quad + \sum_{j=0}^{l-1} \nabla_{\alpha_1}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}\nabla_{\alpha_0}\tau(u)] \\
&\quad + \sum_{j=0}^{l-1} \nabla_{\alpha_0}\nabla_{\alpha_1}\nabla_{\alpha_j}[R(\nabla_{\alpha_0}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}(u)]
\end{aligned}
$$

Lemma 5.4 *Let $u \in C^1([0,T], H^s(\mathbb{R}^n, N))$ with $s \in \mathbb{N}$ and $s > \left[\frac{n}{2}\right] + 4$ be a solution of (5.1). Then for $\left[\frac{n}{2}\right] + 4 \le l \le s$ and $l \in \mathbb{N}$ we have*

$$
(5.13) \qquad \frac{d}{dt}\|\nabla^l u\|_{L^2}^2 \le C\|\nabla u\|_{H^{l-1}}^2 (1 + \|\nabla u\|_{H^{l-1}}^{3n+2l+14})
$$

Proof. We first compute the evolution

$$
\begin{aligned}
(5.14) \quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u|^2 dx &= \sum_{\alpha_0=1}^{n} \int \langle \nabla_t \nabla_{\alpha_0} u, \nabla_{\alpha_0} u\rangle \\
&= \int \nabla_\alpha(\langle \nabla_t u, \nabla_{\alpha_0} u\rangle) - \int \langle \nabla_t u, \tau(u)\rangle - \int \langle \nabla_t u, \tau(u)\rangle \\
&= \varepsilon \int \langle \Delta\tau(u), \tau(u)\rangle - \varepsilon \int \langle R(\nabla u, \tau(u))\nabla u, \tau(u)\rangle \\
&\quad - \int \langle J(u)\tau(u), \tau(u)\rangle - \beta \int |\tau(u)|^2 \\
&= -\varepsilon \int |\nabla \tau(u)|^2 - \beta \int |\tau(u)|^2 \\
&\quad -\varepsilon \int \langle R(\nabla u, \tau(u))\nabla u, \tau(u)\rangle,
\end{aligned}
$$

where we have used the fact that for $f \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ $\int_{\mathbb{R}^n} \operatorname{div} f = 0$ as well as integration by parts. Note that using integration by parts and Cauchy-Schwarz we have

$$
\begin{aligned}
(5.15) \qquad \left| \int \langle R(\nabla u, \tau(u)) \nabla u, \tau(u) > \right| &\leq C \|\nabla u\|_{L^\infty}^2 \int |\tau(u)|^2 \\
&\leq C \|\nabla u\|_{L^\infty}^2 \|\nabla u\|_{L^2} \|\nabla \tau(u)\|_{L^2} \\
&\leq \frac{1}{2} \|\nabla \tau(u)\|_{L^2}^2 + C \|\nabla u\|_{L^\infty}^4 \|\nabla u\|_{L^2}^2.
\end{aligned}
$$

Combining (5.14) and (5.15) we have

$$
(5.16) \qquad \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 \leq C \|\nabla u\|_{L^\infty}^4 \|\nabla u\|_{L^2}^2.
$$

For $1 \leq l \leq s$ applying (5.3) we have

$$
\begin{aligned}
(5.17) \frac{1}{2} \frac{d}{dt} \|\nabla^l u\|_{L^2}^2 &= \sum_{|\boldsymbol{\alpha}|=l} \int \langle \nabla_t \nabla_{\boldsymbol{\alpha}} u, \nabla_{\boldsymbol{\alpha}} u \rangle \\
&= \sum_{|\boldsymbol{\alpha}|=l} \int \langle \nabla_{\boldsymbol{\alpha}} \nabla_t u; \nabla_{\boldsymbol{\alpha}} u \rangle \\
&+ \sum_{|\boldsymbol{\alpha}|=l} \sum_{j=0}^{l-2} \int \langle \nabla_{\alpha_1} \cdots \nabla_{\alpha_j} [R(\nabla_t u, \nabla_{\alpha_{j+1}} u) \nabla_{\alpha_{j+2}} \cdots \nabla_{\alpha_l} u]; \nabla_{\boldsymbol{\alpha}} u \rangle.
\end{aligned}
$$

Consider each term separately

$$
\begin{aligned}
(5.18) \qquad \int \langle \nabla_{\boldsymbol{\alpha}} \nabla_t u, \nabla_{\boldsymbol{\alpha}} u \rangle &= -\varepsilon \int \langle \nabla_{\boldsymbol{\alpha}} \Delta \tau(u), \nabla_{\boldsymbol{\alpha}} u \rangle \\
&+ \varepsilon \int \langle \nabla_{\boldsymbol{\alpha}} (R(\nabla u, \tau(u)) \nabla u; \nabla_{\boldsymbol{\alpha}} u \rangle \\
&+ \int \langle \nabla_{\boldsymbol{\alpha}} J(u) \tau(u), \nabla_{\boldsymbol{\alpha}} u \rangle + \beta \int \langle \nabla_{\boldsymbol{\alpha}} \tau(u), \nabla_{\boldsymbol{\alpha}} u \rangle.
\end{aligned}
$$

Using (5.12) and (5.7) and integrating by parts we have that

$$
\begin{aligned}
(5.19)\quad & \int \langle \nabla_{\boldsymbol{\alpha}} \Delta\tau(u), \nabla_{\boldsymbol{\alpha}} u\rangle \\
= \quad & \int \langle \nabla_{\boldsymbol{\alpha}} \tau(u), \Delta \nabla_{\boldsymbol{\alpha}} u\rangle \\
& - \sum_{j=0}^{l-1} \int \langle \nabla_{\alpha_1} \cdots \nabla_{\alpha_j} [R(\nabla_{\alpha_0} u, \nabla_{\alpha_{j+1}} u) \nabla_{\alpha_{j+2}} \cdots \nabla_{\alpha_l} \nabla_{\alpha_0} \tau(u)], \nabla_{\boldsymbol{\alpha}} u\rangle \\
& - \sum_{j=0}^{l-1} \int \langle \nabla_{\alpha_0} \nabla_{\alpha_1} \cdots \nabla_{\alpha_j} [R(\nabla_{\alpha_0} u, \nabla \alpha_{j+1} u) \nabla_{\alpha_{j+2}} \cdots \nabla_{\alpha_l} \tau(u)]; \nabla_{\boldsymbol{\alpha}} u\rangle \\
= \quad & \int |\nabla_{\boldsymbol{\alpha}} \tau(u)|^2 \\
& + \sum_{j=0}^{l-1} \int \langle \nabla_{\alpha_1} \cdots \nabla_{\alpha_j} [R(\nabla_{\alpha_0} u, \nabla_{\alpha_{j+1}} u) \nabla_{\alpha_{j+2}} \cdots \nabla_{\alpha_l} \nabla_{\alpha_0} u] \nabla_{\boldsymbol{\alpha}} \tau(u)\rangle \\
& - \sum_{j=0}^{l-1} \int \langle \nabla_{\alpha_1} \cdots \nabla_{\alpha_j} [R(\nabla_{\alpha_0) u, \nabla_{j+1}} u) \nabla_{\alpha_{j+2}} \cdots \nabla_{\alpha_l} \nabla_{\alpha_0} \tau(u)]; \nabla_{\boldsymbol{\alpha}} u\rangle \\
& - \sum_{j=0}^{l-2} \int \langle \nabla_{\alpha_0} \nabla_{\alpha_1} \cdots \nabla_{\alpha_j} [R(\nabla_{\alpha_0} u, \nabla_{\alpha_{j+1}} u) \nabla_{\alpha_{j+2}} \cdots \nabla_{\alpha_l} \tau(u)]; \nabla_{\boldsymbol{\alpha}} u\rangle \\
& + \sum_{j=1}^{l-2} \langle \nabla_{\alpha_0} \nabla_{\alpha_1} \cdots \nabla_{\alpha_j} [R(\nabla_{\alpha_0} u, \nabla_{\alpha_{j+1}} u) \nabla_{\alpha_{j+2}} \cdots \nabla_{\alpha_l} u], \nabla_{\alpha} \tau(u)\rangle
\end{aligned}
$$

(5.19) yields

$$
\begin{aligned}
(5.20)\quad -\varepsilon \sum_{|\boldsymbol{\alpha}|=l} \langle \nabla_{\boldsymbol{\alpha}} \Delta\tau(u), \nabla_{\boldsymbol{\alpha}} u\rangle \quad \le \quad & -\varepsilon \int |\nabla^l \tau(u)|^2 \\
& + C\varepsilon \sum_{m=3}^{l+2} \sum_{\substack{j_1+\cdots+j_m=l+2 \\ j_s \ge 1}} \int |\nabla^l \tau(u)|\, |\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \\
& + C\varepsilon \sum_{m=3}^{l+2} \sum_{\substack{j_1+\cdots+j_m=l+2 \\ j_s \ge 1}} \int |\nabla^{j_1} \tau(u)|\, |\nabla^{j_2} u| \cdots |\nabla^{j_m} u|\, |\nabla^l u|.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(5.21)\qquad & \sum_{|\boldsymbol{\alpha}|=l} \int \langle \nabla_{\boldsymbol{\alpha}} [R(\nabla u, \tau(u)) \nabla u], \nabla_{\boldsymbol{\alpha}} u\rangle \le \\
& \le C \sum_{m=3}^{l+2} \sum_{\substack{j_1+\cdots+j_m=l+2 \\ j_s \ge 1 \text{ if } s \ge 2}} \int |\nabla^l u|\, |\nabla^{j_1} \tau(u)|\, |\nabla^{j_2} u| \cdots |\nabla^{j_m} u|
\end{aligned}
$$

We now look at the third term in (5.18) and recall that $\nabla J = 0$ and $\langle JX, X\rangle = 0$ for $X \in TN$. Integrating by parts and applying (5.7) we obtain for $j = (\alpha_2 \cdots \alpha_e)$

$$(5.22) \qquad \int \langle \nabla_{\boldsymbol{\alpha}} J(u)\tau(u), \nabla_{\boldsymbol{\alpha}} u\rangle$$

$$\begin{aligned}
&= -\int \langle \nabla_{\boldsymbol{\gamma}} J(u)\tau(u), \nabla_{\alpha_1}\nabla_{\boldsymbol{\alpha}} u\rangle \\
&= -\langle \nabla_{\boldsymbol{\gamma}} J(u)\tau(u), \Delta\nabla_{\boldsymbol{\gamma}} u\rangle \\
&= -\int \langle J(u)\nabla_{\boldsymbol{\gamma}}\tau(u); \nabla_{\boldsymbol{\gamma}}\tau(u)\rangle \\
&\quad \sum_{j=1}^{l-1}\int \langle J(u)\nabla_{\boldsymbol{\gamma}}\tau(u), \nabla_{\alpha_2}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_1}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}\nabla_{\alpha_1}u]\rangle \\
&\quad +\sum_{j=2}^{l-2}\int \langle J(u)\nabla_{\boldsymbol{\gamma}}\tau(u), \nabla_{\alpha_1}\nabla_{\alpha_2}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_1}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}u]\rangle \\
&= -\sum_{j=1}^{l-1}\int \langle J(u)\nabla_{\alpha_3}\cdots\nabla_{\alpha_l}\tau(u), \\
&\qquad \nabla_{\alpha_2}\nabla_{\alpha_2}\nabla_{\alpha_3}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_1}u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}\nabla_{\alpha_1}u]\rangle \\
&\quad -\sum_{j=2}^{l-2}\int \langle J(u)\nabla_{\alpha_3}\cdots\nabla_{\alpha_l}\tau(u), \nabla_{\alpha_2}\nabla_{\alpha_1}\nabla_{\alpha_2}\cdots\nabla_{\alpha_j}[R(\nabla_{\alpha_1}u, \nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}u]\rangle.
\end{aligned}$$

Thus (5.22) yields

$$(5.23) \qquad \sum_{|\boldsymbol{\alpha}|=l}\int \langle \nabla_{\boldsymbol{\alpha}} J(u)\tau(u), \nabla_{\boldsymbol{\alpha}} u\rangle \le C\sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ j_s\ge 1}}\int |\nabla^{l-2}\tau(u)|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u|.$$

A very similar computation yields
(5.24)

$$\int \langle \nabla_{\boldsymbol{\alpha}}\tau(u), \nabla_{\boldsymbol{\alpha}} u\rangle \le -\int |\nabla_{\boldsymbol{\gamma}}\tau(u)|^2 + C\sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ j_s\ge 1}}\int |\nabla^{l-2}\tau(u)|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u|.$$

Combining (5.18), (5.20), (5.21), (5.23) and (5.24) we obtain

$$
\begin{aligned}
(5.25)\qquad & \sum_{|\boldsymbol{\alpha}|=l}\int \langle \nabla_{\boldsymbol{\alpha}}\nabla_t u, \nabla_{\boldsymbol{\alpha}} u\rangle \\
\leq\ & -\varepsilon\int |\nabla^l \tau(u)|^2 + C\varepsilon \sum_{m=3}^{l+2}\sum_{j_1+\cdots+j_m=l+2}\int |\nabla^l\tau(u)|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \\
& + C\varepsilon \sum_{m=3}^{l+2}\sum_{j_1+\cdots+j_m=l+2}\int |\nabla^{j_1}\tau(u)|\,|\nabla^{j_2}u|\cdots|\nabla^{j_m}u|\,|\nabla^l u| \\
& + C\varepsilon \sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ j_s\geq 1 \text{ if } s\geq 2}}\int |\nabla^l u|\,|\nabla^{j_1}\tau(u)|\,|\nabla^{j_2}u|\cdots|\nabla^{j_m}u| \\
& -\beta\int |\nabla^{l-1}\tau(u)|^2 + C\sum_{m=3}^{l+2}\sum_{j_1+\cdots+j_m=l+2}\int |\nabla^{l-2}\tau(u)|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u|.
\end{aligned}
$$

We now look at the second term in (5.17). Using equation (5.1) we obtain

$$
\begin{aligned}
(5.26)\qquad & \sum_{|\boldsymbol{\alpha}|=l}\sum_{j=0}^{l-2}\int \langle \nabla_{\alpha_1}\cdots\nabla_{\alpha_j}[R(\nabla_t u, \nabla_{\alpha_{j+1}}u)\nabla_{\alpha_{j+2}}\cdots\nabla_{\alpha_l}u], \nabla_{\boldsymbol{\alpha}} u\rangle \\
\leq\ & C\sum_{m=3}^{l+1}\sum_{\substack{j_1+\cdots+j_m=l\\ j_s\geq 1 \text{ if } s\geq 2}}\int |\nabla^l u|\,|\nabla^{j_1}\nabla_t u|\,|\nabla^{j_2}u|\cdots|\nabla^{j_l}u| \\
\leq\ & C\varepsilon\sum_{m=3}^{l+1}\sum_{\substack{j_1+\cdots+j_m=l\\ j_s\geq 1 \text{ if } s\geq 2}}\int |\nabla^l u|\,|\nabla^{j_1}\Delta\tau(u)|\,|\nabla^{j_2}u|\cdots|\nabla^{j_m}u| \\
& + C\varepsilon\sum_{m=6}^{l+3}\sum_{\substack{j_1+\cdots+j_m=l+2\\ j_s\geq 1 \text{ if } s\geq 2}}\int |\nabla^l u|\,|\nabla^{j_1}\tau(u)|\,|\nabla^{j_2}u|\cdots|\nabla^{j_m}u| \\
& + C\varepsilon\sum_{m=3}^{l}\sum_{\substack{j_1+\cdots+j_m=l\\ j_s\geq 1 \text{ if } s\geq 2}}\int |\nabla^l u|\,|\nabla^{j_1}\tau(u)|\,|\nabla^{j_2}u|\cdots|\nabla^{j_m}u|.
\end{aligned}
$$

Combining (5.17), (5.25) and (5.26) we obtain

$$
\begin{aligned}
(5.27)\qquad & \frac{1}{2}\frac{d}{dt}\|\nabla^l u\|^2_{L^2} \\
\le\ & -\varepsilon\int |\nabla^l\tau(u)|^2 \\
& +C\varepsilon\int |\nabla^l\tau(u)|\,|\nabla^l u|\,|\nabla u|^2 + C\varepsilon\int |\nabla^{l-1}\tau(u)|\,|\nabla^l u|\ \left(|\nabla u|^3+|\nabla u|\,|\nabla^2 u|\right) \\
& +C\varepsilon\sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ 1\le j_s\le l-1}}\int |\nabla^l\tau(u)|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \\
& C\varepsilon\sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ j_s\ge 1\ \ \text{if } s\ge 2}}\int |\nabla^l u|\,|\nabla^{j_1}\tau(u)|\cdots|\nabla^{j_m}u| \\
& +C\varepsilon\sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ j_s\ge 1}}\int |\nabla^l u|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \\
& +C\varepsilon\sum_{m=5}^{l+3}\sum_{\substack{j_1+\cdots+j_m=l+4\\ j_s\ge 1}}\int |\nabla^l u|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \\
& +C\sum_{m=3}^{l+2}\sum_{j_1+\cdots+j_m=l+2}\int |\nabla^l u|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| - \beta\int |\nabla^{l-1}u|^2 \\
\le\ & -\varepsilon\int |\nabla^l\tau(u)|^2 + C\varepsilon\int |\nabla^l\tau(u)|\,|\nabla^l u|\,|\nabla u|^2 \\
& +C\varepsilon\int |\nabla^{l-1}\tau(u)|\,|\nabla^l u|\ \left(|\nabla u|^3+|\nabla^2 u|\,|\nabla u|\right) + C\varepsilon\int |\nabla^l u|\,|\tau(u)|\,|\nabla^{l+1}u|\,|\nabla u| \\
& +C\varepsilon\int |\nabla^l u|^2\left(|\tau(u)|\,|\nabla u|^2+|\tau(u)|\,|\nabla^2 u|\right) + C\varepsilon\int |\nabla^l u|^2\left(|\nabla u|^4+|\nabla\tau(u)|\,|\nabla u|\right) \\
& +C\varepsilon\sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ 1\le j_s\le l-1}}\int |\nabla^l\tau(u)|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \\
& +C\varepsilon\sum_{m=5}^{l+3}\sum_{\substack{j_1+\cdots+j_m=l+4\\ 1\le j_s\le l-1}}\int |\nabla^l u|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \\
& +C\sum_{m=3}^{l+2}\sum_{j_1+\cdots+j_m=l+2}\int |\nabla^l u|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \\
& +C\varepsilon\sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ 1\le j_s\le l-1,\, s\ge 2\\ j_1\le l-2}}\int_l |\nabla^l u|\,|\nabla^{j_1}\tau(u)|\cdots|\nabla^{j_m}u|.
\end{aligned}
$$

We now look at each term of (5.27) separately. Apply Cauchy-Schwarz we have

$$
\begin{aligned}
(5.28)\qquad C\varepsilon\int|\nabla^l\tau(u)|\,|\nabla^l u|\,|\nabla u|^2 &\leq C\varepsilon\|\nabla u\|_{L^\infty}^2\Big(\int|\nabla^l\tau(u)|^2\Big)^{\frac12}\Big(\int|\nabla^l u|^2\Big)^{\frac12}\\
&\leq \frac{\varepsilon}{64}\int|\nabla^l\tau(u)|^2 + C\|\nabla u\|_{L^\infty}^4\int|\nabla^l u|^2.
\end{aligned}
$$

Using Cauchy-Schwarz and integration by parts we have

(5.29)

$$
\begin{aligned}
C\varepsilon\int|\nabla^{l-1}\tau(u)|\,|\nabla^l u|\,|\nabla u|^3 &\leq C\varepsilon\|\nabla u\|_{L^\infty}^3\Big(\int|\nabla^{l-1}\tau(u)|^2\Big)^{\frac12}\Big(\int|\nabla^l u|^2\Big)^{\frac12}\\
&\leq \frac{\varepsilon}{64}\int|\nabla^{l-1}\tau(u)|^2 + C\|\nabla u\|_{L^\infty}^6\int|\nabla^l u|^2\\
&\leq \frac{\varepsilon}{64}\int|\nabla^l\tau(u)|\,|\nabla^{l-2}\tau(u)| + C\|\nabla u\|_{L^\infty}^6\int|\nabla^l u|^2\\
&\leq \frac{\varepsilon}{64}\Big(\int|\nabla^l\tau(u)|^2\Big)^{\frac12}\Big(\int|\nabla^l u|^2\Big)^{\frac12} + C\|\nabla u\|_{L^\infty}^6\int|\nabla^l u|^2\\
&\leq \frac{\varepsilon}{64}\int|\nabla^l\tau(u)|^2 + C(1+\|\nabla u\|_{L^\infty}^6)\int|\nabla^l u|^2.
\end{aligned}
$$

$$
\begin{aligned}
(5.30)\ & C\varepsilon\int|\nabla^{l-1}\tau(u)|\,|\nabla^l u|\,|\nabla^2 u|\,|\nabla u|\\
&\leq C\varepsilon\|\nabla u\|_{L^\infty}\|\nabla^2 u\|_{L^\infty}\left(\int|\nabla^{l-1}\tau(u)|^2\right)^{\frac12}\left(\int|\nabla^l u|^2\right)^{\frac12}\\
&\leq \frac{\varepsilon}{64}\int|\nabla^l\tau(u)|^2 + C\left(1+\|\nabla u\|_{L^\infty}^6+|\nabla^2 u\|_{L^\infty}^3\right)\left(\int|\nabla^l u|^2\right)^{\frac12}.
\end{aligned}
$$

Using Cauchy-Schwarz integration by parts and (5.17) we have

$$
\begin{aligned}
(5.31)\qquad & C\varepsilon\int |\nabla^l u|\,|\tau(u)|\,|\nabla^{l+1}u|\,|\nabla u| \\
\leq\ & C\varepsilon\|\tau(u)\|_{L^\infty}\|\nabla u\|_{L^\infty}(\int|\nabla^l u|^2)^{\frac{1}{2}}(\int|\nabla^{l+1}u|^2)^{\frac{1}{2}} \\
\leq\ & \frac{\varepsilon}{64}\int|\nabla^{l+1}u|^2 + C\|\tau(u)\|^2_{L^\infty}\|\nabla u\|^2_{L^\infty}\int|\nabla^l u|^2 \\
\leq\ & -\frac{\varepsilon}{64}\int\langle\Delta\nabla^l u,\nabla^l u\rangle + C\|\tau(u)\|^2_{L^\infty}\|\nabla u\|^2_{L^\infty}\int|\nabla^l u|^2 \\
\leq\ & \frac{\varepsilon}{64}\int|\nabla^l\tau(u)|\,|\nabla^l u| + C\|\tau(u)\|^2_{L^\infty}\|\nabla u\|^2_{L^\infty}\int|\nabla^l u|^2 \\
& +C\varepsilon\int|\nabla^l u|\sum_{m=3}^{l+2}\sum_{j_1+\cdots|j_m=l+2}|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \\
\leq\ & \frac{\varepsilon}{64}\int|\nabla^l\tau(u)|^2 + C(\|\tau(u)\|^2_{L^\infty}\|\nabla u\|^2_{L^\infty}+1)\int|\nabla^l u|^2 \\
& +C\varepsilon\sum_{m=3}^{l+2}\sum_{j_1+\cdots+j_m=l+2}\int|\nabla^l u|\,|\nabla^{j-1}u|\cdots|\nabla^{j_m}u|.
\end{aligned}
$$

Combining (5.27), (5.28), (5.29), (5.30) and (5.31) and using the fact that $ab \leq \frac{a^p}{p}+\frac{b^q}{q}$ if $\frac{1}{p}+\frac{1}{q}=1$ we have

$$
\begin{aligned}
(5.32)\qquad \frac{1}{2}\frac{d}{dt}\|\nabla^l u\|^2_{L^2}\ \leq\ & \frac{-3\varepsilon}{4}\int|\nabla^l\tau(u)|^2 \\
& +C(1+\|\nabla u\|^6_{L^\infty}+\|\nabla^2 u\|^3_{L^\infty})\int|\nabla^l u|^2 \\
& +C\varepsilon\sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ 1\leq j_s\leq l-1}}\int|\nabla^l\tau(u)|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \\
& +C\varepsilon\sum_{m=5}^{l+1}\sum_{\substack{j_1+\cdots+j_m=l+4\\ 1\leq j_s\leq l-1}}\int|\nabla^l u|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \\
& +C\varepsilon\sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ l-1\geq j_s\geq 1\ \text{if}\ s\geq 2\\ j_1\leq l-2}}\int|\nabla^l u|\,|\nabla^{j_1}\tau(u)|\cdots|\nabla^{j_m}u| \\
& +C\sum_{m=3}^{l+2}\sum_{\substack{j_1+\cdots+j_m=l+2\\ 1\leq j_s\leq l-1}}\int|\nabla^l u|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u|.
\end{aligned}
$$

To finish the estimate we need to use the interpolation result that appears in Proposition 4.1. Consider $3\leq m\leq l+2$, $1\leq j_s\leq l-1$ and $j_1+\cdots+j_m=l+2$ then by Cauchy-Schwarz we have

$$\int |\nabla^l \tau(u)|\,|\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \le \left(\int |\nabla^l \tau(u)|^2\right)^{\frac{1}{2}} \left(\int |\nabla^{j_1} u|^2 \cdots |\nabla^{j_m} u|^2\right)^{\frac{1}{2}}. \tag{5.33}$$

Let $p_i \in [2,\infty]$ for $i = 1, \dots, m$ be such that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{2}, \tag{5.34}$$

by Hölder's inequality

$$\left(\int |\nabla^{j_1} u|^2 \cdots |\nabla^{j_m} u|^2\right)^{\frac{1}{2}} \le \|\nabla^{j_1} u\|_{L^{p_1}} \cdots \|\nabla^{j_m} u\|_{L^{p_m}}. \tag{5.35}$$

Since $l > \left[\frac{n}{2}\right] + 1$ for

$$\frac{j_i - 1}{l-1} < a_i = \frac{j_i - 1}{l-1} + \frac{n}{2(l-1)^2}\left(l - 1 - j_i + \frac{3}{m}\right) < 1 \tag{5.36}$$

and when $m > 3$ or $m = 3$ and $j_i \ge 2$

$$\frac{1}{2} \ge \frac{1}{p_i} = \frac{j_i - 1}{n} + \frac{1}{2} - \frac{l-1}{n} a_i > 0. \tag{5.37}$$

Thus (4.5) yields

$$\|\nabla^{j_i} u\|_{L^{p_i}} \le C \|\nabla^l u\|_{L^2}^{a_i} \|\nabla u\|_{L^2}^{1-a_i} \le C \|\nabla u\|_{H^{l-1}}. \tag{5.38}$$

Combining (5.33), (5.35) and (5.38) we have in the case $m > 3$ that

$$\int |\nabla^l \tau(u)|\,|\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \le c \left(\int |\nabla^l \tau(u)|^2\right)^{\frac{1}{2}} \|\nabla u\|_{H^{l-1}}^m. \tag{5.39}$$

In the case when $m = 3$, $j_1 \ge j_2 \ge j_3$, and $j_3 = 1$ we have $j_1 + j_2 = l + 1$ and

$$\left(\int |\nabla^{j_1} u|^2 |\nabla^{j_2} u|^2 |\nabla u|^2\right)^{\frac{1}{2}} \le \|\nabla u\|_{L^\infty} \left(\int |\nabla^{j_1} u|^2 |\nabla^{j_2} u|^2\right)^{\frac{1}{2}}. \tag{5.40}$$

If $j_2 = 1$ then (5.33) becomes

$$\int |\nabla^{l} \tau(u)|\,|\nabla^l u|\,|\nabla u|^2 \le c \|\nabla u\|_{L^\infty}^2 \left(\int |\nabla^l \tau(u)|^2\right)^{\frac{1}{2}} \left(\int |\nabla^l u|^2\right)^{\frac{1}{2}}. \tag{5.41}$$

If $j_2 > 1$ then for $i = 1, 2$ since $l > \left[\frac{n}{2}\right] + 3$ if

$$\frac{j_1 - 1}{l-1} \le a_1 = \frac{j_1 - 1}{l-1} + \frac{n}{2(l-1)^2}(l - j_i) < 1 \tag{5.42}$$

and

$$\frac{1}{2} \geq \frac{1}{p_i} = \frac{j_i - 1}{n} + \frac{1}{2} - \frac{l-1}{n} a_i > 0. \tag{5.43}$$

Hölder's inequality and Proposition 4.1 yield

$$\begin{aligned} \left(\int |\nabla^{j_1} u|^2 |\nabla^{j_2} u|^2 \right)^{\frac{1}{2}} &\leq \|\nabla^{j_1} u\|_{L^{P_1}} \|\nabla^{j_2} u\|_{L^{P_2}} \\ &\leq C \|\nabla^l u\|_{L^2}^{a_1} \|\nabla u\|_{L^2}^{1-a_1} \|\nabla^l u\|_{L^2}^{a_2} \|\nabla u\|_{L^2}^{1-a_2} \\ &\leq C \|\nabla \mathrm{l} u\|_{H^{l-1}}^2. \end{aligned} \tag{5.44}$$

Thus in this case (5.33) becomes combining (5.40) and (5.44)

$$\int |\nabla^l \tau(u)| \, |\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \leq c \|\nabla u\|_{L^\infty} \|\nabla u\|_{H^{l-1}}^2 \left(\int |\nabla^l \tau(u)|^2 \right)^{\frac{1}{2}}. \tag{5.45}$$

Combining (5.39), (5.41) and (5.45) we can estimate the third term on the right hand side of (5.32)

$$\begin{aligned} &\sum_{m=3}^{l+2} \sum_{\substack{j_1 + \cdots + j_m = l+2 \\ 1 \leq j_s \leq l-1}} \int |\nabla^l \tau(u)| \, |\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \\ &\qquad \leq C \left(\int |\nabla^l \tau(u)|^2 \right)^{\frac{1}{2}} \left(1 + \|\nabla u\|_{L^\infty}^2\right) \sum_{m=1}^{l+2} \|\nabla u\|_{H^{l-1}}^m. \end{aligned} \tag{5.46}$$

To estimate the fourth term in (5.32) consider $5 \leq m \leq l+3$, $1 \leq j_s \leq l-1$ and $j_1 + \cdots + j_m = l+4$ then by Cauchy-Schwarz we have

$$\int |\nabla^l u| \, |\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \leq \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \left(\int |\nabla^{j_1} u|^2 \cdots |\nabla^{j_m} u|^2 \right)^{\frac{1}{2}}. \tag{5.47}$$

Let $2 \leq p_i \leq \infty$ for $i = 1, \ldots, m$ be such that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{2} \tag{5.48}$$

by Hölder's inequality (5.47) becomes

$$\int |\nabla^l u| \, |\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \leq \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla^{j_1} u\|_{L^{P_1}} \cdots \|\nabla^{j_m} u\|_{L^{P_m}} \tag{5.49}$$

since $l > \left[\frac{n}{2}\right] + 1$ for

$$\frac{j_i - 1}{l-1} \leq a_i = \frac{j_i - 1}{l-1} + \frac{n}{2(l-1)^2} \left(l - 1 - j_i + \frac{5}{m} \right) < 1 \tag{5.50}$$

and when $m > 5$ or $m = 5$ and $j_i \geq 2$

$$\frac{1}{2} \geq \frac{1}{p_i} = \frac{j_i - 1}{n} + \frac{1}{2} - \frac{l-1}{n} a_i > 0. \tag{5.51}$$

Thus (4.5) yields

$$\int |\nabla^l u| |\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \leq C \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla u\|^m_{H^{l-1}}. \tag{5.52}$$

If $m = 5$, $j_1 \geq j_2 \geq \cdots \geq j_5 \geq 1$, and $j_5 = 1$ then $j_1 + j_2 + j_3 + j_4 = l + 3$, by Cauchy-Schwarz and Hölder's inequality

$$\begin{aligned} \int |\nabla^l u| \, |\nabla^{j_1} u| \cdots |\nabla^{j_4} u| \, |\nabla u| \\ \leq \quad & \|\nabla u\|_{L^\infty} \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \left(\int |\nabla^{j_1} u|^2 \cdots |\nabla^{j_4} u|^2 \right)^{\frac{1}{2}} \\ \leq \quad & \|\nabla u\|_{L^\infty} \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla^{j_1} u\|_{L^{p_1}} \cdots \|\nabla^{j_4} u\|_{L^{p_4}} \end{aligned} \tag{5.53}$$

with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = 1$. For

$$\frac{j_i - 1}{l - 1} < a_i = \frac{j_i - 1}{l - 1} + \frac{n}{2(l-1)^2}(l - j_i) < 1 \tag{5.54}$$

if $j_4 > 1$ we have

$$\frac{1}{2} \geq \frac{1}{p_i} = \frac{j_i - 1}{n} + \frac{1}{2} - \frac{l-1}{n} a_i > 0 \tag{5.55}$$

and (5.53) becomes by Proposition 4.1

$$\int |\nabla^l u| \, |\nabla^{j_1} u| \cdots |\nabla^{j_4} u| \, |\nabla u| \leq C \|\nabla u\|_{L^\infty} \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla u\|^4_{H^{l-1}}. \tag{5.56}$$

If $j_4 = 1$ and $j_3 > 1$ a similar argument yields

$$\int |\nabla^l u| \, |\nabla^{j_1} u| \, |\nabla^{j_3} u| \, |\nabla u|^2 \leq C \|\nabla u\|^2_{L^\infty} \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla u\|^3_{H^{l-1}}. \tag{5.57}$$

If $j_3 = 1$ then $j_1 + j_2 = l + 1$ since $j_1 \leq l - 1$ then $j_2 > 1$ and we have

$$\int |\nabla^l u| \, |\nabla^{j_1} u| \, |\nabla^{j_2} u| \, |\nabla u|^3 \leq C \|\nabla u\|^3_{L^\infty} \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla u\|^2_{H^{l-1}}. \tag{5.58}$$

Combining (5.52), (5.54), (5.57) and (5.58) we can estimate the fourth term on the right hand side of (5.32) as follows

$$\sum_{m=5}^{l+3} \sum_{\substack{j_1+\cdots+j_m=l+4 \\ 1\le j_s\le l-1}} \int |\nabla^l u|\,|\nabla^{j_1}u|\cdots|\nabla^{j_m}u| \tag{5.59}$$
$$\le C\left(\int |\nabla^l u|^2\right)^{\frac{1}{2}} (1+\|\nabla u\|^3_{L^\infty}) \sum_{m=2}^{l+3} \|\nabla u\|^m_{H^{l-1}}$$
$$\le C\left(1+\|\nabla u\|^3_{L^\infty}\right) \sum_{m=3}^{l+4} \|\nabla u\|^m_{H^{l-1}}.$$

To estimate the fifth term in (5.32) consider $3 \le m \le l+2$, $j_1+\cdots+j_m = l+2$, $j_1 \le l-2$, $1 \le j_s \le l-1$ if $s \ge 2$. Cauchy-Schwarz and Hölder's inequality ensure that for $\frac{1}{p_1}+\cdots+\frac{1}{p_m} = \frac{1}{2}$

$$\int |\nabla^l u|\,|\nabla^{j_1}\tau(u)|\,|\nabla^{j_2}u|\cdots|\nabla^{j_m}u| \le \left(\int |\nabla^l u|^2\right)^{\frac{1}{2}} \|\nabla^{j_1}\tau(u)\|_{L^{P_1}}\cdots\|\nabla^{j_m}u\|_{L^{P_m}}. \tag{5.60}$$

since $l > \left[\frac{n}{2}\right]+3 > 1$ for $i \ge 2$

$$\frac{j_i-1}{l-1} < a_i = \frac{j_i-1}{l-1} + \frac{n}{2(l-1)^2}\left(l-1-j_i+\frac{3}{m}\right) < 1 \tag{5.61}$$

and

$$\frac{j_1}{l-1} < a_1 = \frac{j_1}{l-1} + \frac{n}{2(l-1)^2}\left(l-1-j_1+\frac{3}{m}\right) < 1 \tag{5.62}$$

when $m > 3$ or $m = 3$ and $j_i \ge 2$ for $i \ge 2$

$$\frac{1}{2} \ge \frac{1}{p_i} = \frac{j_i-1}{n} + \frac{1}{2} - \frac{l-1}{n}a_i > 0 \tag{5.63}$$

and $m > 3$ or $m = 3$ and $j_1 \ge 2$

$$\frac{1}{2} \ge \frac{1}{p_1} = \frac{j_1}{n} + \frac{1}{2} - \frac{l-1}{n}a_1 > 0. \tag{5.64}$$

In these cases (5.60) can be estimated by (4.5) as follows

$$\int |\nabla^l u|\,|\nabla^{j_1}\tau(u)|\cdots|\nabla^{j_m}u| \le \left(\int |\nabla^l u|^2\right)^{\frac{1}{2}} \|\nabla^{l-1}\tau(u)\|^{a_1}_{L^2}\|\tau(u)\|^{1-a_1}_{L^2}\|\nabla u\|^{m-1}_{H^{l-1}}. \tag{5.65}$$

If $m = 3$ and $j_1 \le 1$ then $j_2 \ge 2$ and $j_3 \ge 2$. Cauchy-Schwarz and Hölder's inequality yield for $\frac{1}{p_2}+\frac{1}{p_3} = \frac{1}{2}$

$$\int |\nabla^l u|\,|\tau(u)|\,|\nabla^{j_2}u|\,|\nabla^{j_3}u| \le \|\tau(u)\|_{L^\infty}\left(\int |\nabla^l u|^2\right)^{\frac{1}{2}} \|\nabla^{j_2}u\|_{L^{p_2}}\|\nabla^{j_3}u\|_{L^{p_3}}. \tag{5.66}$$

and

$$\frac{1}{2} \geq \frac{1}{p_i} = \frac{j_i - 1}{n} + \frac{1}{2} - \frac{l-1}{n} a_i > 0, \tag{5.67}$$

Proposition 4.1 ensures that

$$\int |\nabla^l u| \, |\tau(u)| \, |\nabla^{j_2} u| \, |\nabla^{j_3} u| \leq C \|\tau(u)\|_{L^\infty} \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla u\|^2_{H^{l-1}}. \tag{5.68}$$

Similarly

$$\int |\nabla^l u| \, |\nabla \tau(u)| \, |\nabla^{j_2} u| \, |\nabla^{j_3} u| \leq C \|\nabla \tau(u)\|_{L^\infty} \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla u\|^2_{H^{l-1}}. \tag{5.69}$$

If $m = 3$, $j_1 \geq 2$ and $j_2 = 1$, $j_3 > 1$ we have by Cauchy-Schwarz and Hölder's inequality for $\frac{1}{p_1} + \frac{1}{p_3} = \frac{1}{2}$

$$\int |\nabla^l u| \, |\nabla^{j_1} \tau(u)| \, |\nabla u| \, |\nabla^{j_3} u| \leq C \|\nabla u\|_{L^\infty} \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla^{j_1} \tau(u)\|_{L^{p_1}} \|\nabla^{j_3} u\|_{L^{p_3}}. \tag{5.70}$$

For

$$a_1 = \frac{j_1}{l-1} + \frac{n}{2(l-1)^2}(l - j_1) < 1 \text{ and } \frac{1}{2} \geq \frac{1}{p_1} = \frac{j_1}{n} + \frac{1}{2} - \frac{l-1}{n} a_1 > 0 \tag{5.71}$$

and

$$a_3 = \frac{j_3 - 1}{l-1} + \frac{n}{2(l-1)^2}(l - j_3) < 1 \text{ and } \frac{1}{2} \geq \frac{1}{p_3} = \frac{j_3 - 1}{n} + \frac{1}{2} - \frac{l-1}{n} a_3 > 0. \tag{5.72}$$

Proposition 4.1 ensures

$$\begin{aligned} &\int |\nabla^l u| \, |\nabla^{j_1} \tau(u)| \, |\nabla u| \, |\nabla^{j_3} u| \\ &\leq \; C \|\nabla u\|_{L^\infty} \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla^{l-1} \tau(u)\|^{a_1}_{L^2} \|\tau(u)\|^{1-a_1}_{L^2} \|\nabla u\|_{H^{l-1}}. \end{aligned} \tag{5.73}$$

In the case $j_1 = l$, $j_2 = j_3 = 1$ see (5.28).

Combining (5.65), (5.68), (5.67) and (5.73) we estimate the 5$^{\text{th}}$ term of (5.32) as follows

$$\begin{aligned} &\sum_{m=3}^{l+2} \sum_{\substack{j_1 + \cdots + j_m = l+2 \\ 1 \leq j_s \leq l-1 \text{ if } s \geq 2 \\ j_1 \leq l-2}} \int |\nabla^l u| \, |\nabla^{j_1} \tau(u)| \cdots |\nabla^{j_m} u| \\ &\leq \; C \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} (1 + \|\nabla u\|_{L^\infty}) \sum_{m=1}^{l+1} \|\nabla u\|^m_{H^{l-1}} \|\nabla^{l-1} \tau(u)\|^{a_1}_{L^2} \|\tau(u)\|^{1-a_1}_{L^2} \\ &\quad + C(\|\tau(u)\|_{L^\infty} + \|\nabla \tau(u)\|_{L^\infty}) \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla u\|^2_{H^{l-1}}. \end{aligned} \tag{5.74}$$

Finally we look at the last term of (5.32). Let $3 \leq m \leq l+2$, $j_1 + \cdots + j_m = l+2$. Applying the same argument as the one used to obtain (5.45) we conclude that

(5.75)
$$\sum_{m=3}^{l+2} \sum_{j_1+\cdots+j_l=l+2} \int |\nabla^l u| \, |\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \leq C(1 + \|\nabla u\|_{L^\infty}^2) \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \sum_{m=1}^{l+2} \|\nabla u\|_{H^{l-1}}^m .$$

Combining (5.32), (5.46), (5.59), (5.74) and (5.75), using (4.78), (4.80), (4.82) and the fact that $l > \left[\frac{n}{2}\right] + 1$ as well as $\varepsilon \in (0,1)$ and the fact that for $a \in (0,1) \leq r^a s^{1-a} \leq ar + (1-a)s$ we obtain

(5.76)
$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla^l u\|_{L^2}^2 \leq\; & -\frac{3\varepsilon}{4} \int |\nabla^l \tau(u)|^2 + C(1 + \|\nabla u\|_{L^\infty}^6 + \|\tau(\nabla^2 u)\|_{L^\infty}^3) \|\nabla^l u\|_{L^2}^2 \\
& + C\varepsilon \left(\int |\nabla^l \tau(u)|^2 \right)^{\frac{1}{2}} (1 + \|\nabla u\|_{L^\infty}^2) \sum_{m=1}^{l+2} \|\nabla u\|_{H^{l-1}}^m \\
& + C\varepsilon (1 + \|\nabla u\|_{L^\infty}^3) \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \sum_{m=2}^{l+3} \|\nabla u\|_{H^{l-1}}^m \\
& + C\varepsilon \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} (1 + \|\nabla u\|_{L^\infty}) \|\nabla^{l-1} \tau(u)\|_{L^2}^{a_1} \|\tau(u)\|_{L^2}^{1-a_1} \sum_{m=1}^{l+1} \|\nabla u\|_{H^{l-1}}^m \\
& + C(\|\tau(u)\|_{L^\infty} + \|\nabla \tau(u)\|_{L^\infty}) \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla u\|_{H^{l-1}}^2 \\
& + C(1 + \|\nabla u\|_{L^\infty}^2) \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \sum_{m=1}^{l+2} \|\nabla u\|_{H^{l-1}}^m \\
\leq\; & -\frac{3\varepsilon}{4} \int |\nabla^l \tau(u)|^2 + C \left(\|\nabla u\|_{H^{[\frac{n}{2}]+4}}^{6[\frac{n}{2}]+12} + 1 \right) \sum_{m=2}^{l+4} \|\nabla u\|_{H^{l-1}}^m \\
& + C\varepsilon \|\nabla^l u\|_{L^2} \left(1 + \|\nabla u\|_{H^{[\frac{n}{2}]+4}}^{[\frac{n}{2}]+2} \right) \sum_{m=1}^{l+2} \|\nabla u\|_{H^{l-1}}^m \|\nabla^{l-1} \tau(u)\|_{L^2} \\
\leq\; & -\frac{3\varepsilon}{4} \int |\nabla^l \tau(u)|^2 + \frac{\varepsilon}{64} \|\nabla^{l-1} \tau(u)\|_{L^2}^2 \\
& + C \left(1 + \|\nabla u\|_{H^{[\frac{n}{2}]+4}}^{3n+12} \right) \sum_{m=2}^{2l+4} \|\nabla u\|_{H^{l-1}}^m .
\end{aligned}$$

Using the same trick as in (5.29) we obtain from (5.76) for $l > \left[\frac{n}{2}\right] + 1$

(5.77)
$$\begin{aligned}
\frac{d}{dt} \|\nabla^l u\|_{L^2}^2 \leq\; & -\frac{\varepsilon}{2} \int |\nabla^l \tau(u)|^2 + C \left(1 + \|\nabla u\|_{H^{[\frac{n}{2}]+4}}^{3n+12} \right) \sum_{m=2}^{2l+4} \|\nabla u\|_{H^{L^2}}^m \\
\leq\; & C \left(1 + \|\nabla u\|_{H^{[\frac{n}{2}]+4}}^{3n+12} \right) \|\nabla u\|_{H^{l-1}}^2 \left(1 + \|\nabla\|_{H^{l-1}}^{2l+2} \right)
\end{aligned}$$

Thus for $l > \left[\frac{n}{2}\right] + 4$, (5.77) concludes the proof of Lemma 5.4. $\square$

Note that from the proof of Lemma 5.4 we also have the following statement.

Proposition 5.5 *Let $u \in C([0,T], H^{s+1}(\mathbb{R}^n, N))$ with $s \in \mathbb{N}$ and $s \leq \left[\frac{n}{2}\right]+4$ be a solution of (5.2) then for $s \geq l > \left[\frac{n}{2}\right] + 1$ and $l \in \mathbb{N}$ we have*

$$\frac{d}{dt}\|\nabla^l u\|_{L^2}^2 \leq C\left(1 + \|\nabla u\|_{H^{[\frac{n}{2}]+4}}^{3n+12}\right)\|\nabla^2 u\|_{H^{l-1}}\left(1 + \|\nabla u\|_{H^{l-1}}^{2l+2}\right). \tag{5.78}$$

Since our ultimate goal is to estimate $\frac{d}{dt}\|\nabla u\|_{H^{l-1}}^2$ for $l \geq 1$, we still need to analyze $\frac{d}{dt}\|\nabla^l u\|_{L^2}^2$ for $1 \leq l \leq \left[\frac{n}{2}\right] + 1$.

Lemma 5.6 *Let $u \in C([0,T], H^{[\frac{n}{2}]+4}(\mathbb{R}^n, N))$ be a solution of (5.2). Let $1 \leq l \leq \left[\frac{n}{2}\right] + 1$ then if $s_0 = \left[\frac{n}{2}\right] + 2$ we have*

$$\frac{d}{dt}\|\nabla^l u\|_{L^2}^2 \leq c\|\nabla u\|_{H^{s_0}}^2\left(1 + \|\nabla u\|_{H^{s_0+2}}^{M_l}\right). \tag{5.79}$$

where $M_l = 3n + 2l + 12$.

Proof. Note that (5.16) yields

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 \leq C\|\nabla u\|_{H^{[\frac{n}{2}]+1}}^4\left(1 + \|\nabla u\|_{H^{[\frac{n}{2}]+2}}^{2n+8}\right). \tag{5.80}$$

Note that for $l \geq 2$ computation (5.32) remains valid. In fact we only used $l > \left[\frac{n}{2}\right]+1$ when we started to interpolate as in Proposition 4.1. Let $s_0 = \left[\frac{n}{2}\right] + 2$. Consider $3 \leq m \leq l+2$ $1 \leq j_s \leq l-1$ and $j_1 + \cdots + j_m = l+2$ then by Cauchy-Schwarz, Hölder's inequality applied with $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{2}$ where

$$\frac{1}{p_i} = \frac{j_i - 1}{n} + \frac{1}{2} - \frac{s_0}{n}a_i \tag{5.81}$$

and

$$\frac{j_1 - 1}{s_0} \leq a_1 = \frac{j_i - 1}{s_0} + \frac{n}{2(l-1)s_0}\left(l - 1 - j_i + \frac{3}{m}\right) < 1 \tag{5.82}$$

and (4.5) in the case $m > 3$ or $m = 3$ and $j_i \geq 2$ we obtain as in (5.39)

$$\int |\nabla^l \tau(u)|^2 |\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \leq C\left(\int |\nabla^l \tau(u)|^2\right)^{\frac{1}{2}} \|\nabla u\|_{H^{s_0}}^m. \tag{5.83}$$

In the case $m = 3$ we proceed as in the proof of (5.45) and obtain

$$\int |\nabla^l \tau(u)|^2 |\nabla^{j_1} u| \cdots |\nabla^{j_m} u| \leq C\|\nabla u\|_{L^\infty}\left(\int |\nabla^l \tau(u)|^2\right)^{\frac{1}{2}} \|\nabla u\|_{H^{s_0}}^2. \tag{5.84}$$

Thus for $2 \le l \le \left[\frac{n}{2}\right] + 1$ (5.46) becomes

$$(5.85) \quad \sum_{m=3}^{l+2} \sum_{\substack{j_1+\cdots+j_m=l+2 \\ 1\le j_s \le l-1}} \int |\nabla^l \tau(u)|\, |\nabla^{j_1} u| \cdots |\nabla^{j_m} u|$$

$$\le \; C \left(\int |\nabla^l \tau(u)|^2 \right)^{\frac{1}{2}} \left(1 + \|\nabla u\|_{L^\infty}^2\right) \sum_{m=1}^{l+2} \|\nabla u\|_{H^{s_0}}^m .$$

The same type of argument as the one used to prove (5.59), (5.74) and (5.75) yields

$$(5.86) \; \sum_{m=5}^{l+3} \sum_{\substack{j_1+\cdots+j_m=l+4 \\ 1\le j_s \le l-1}} \int |\nabla^l u|\, |\nabla^{j_1} u| \cdots |\nabla^{j_m} u|$$

$$\le \; C \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} (1 + \|\nabla u\|_{l^\infty}^3 + \|\nabla \tau(u)\|_{L^\infty}) \|\nabla u\|_{H^{s_0}}^m$$

$$(5.87) \; \sum_{m=3}^{l+2} \sum_{\substack{j_1+\cdots+j_m=l+2 \\ 1\le j_s \le l-1 \quad s\ge 2 \\ j_1 \le l-2}} \int |\nabla^l u|\, |\nabla^{j_1} \tau(u)| \cdots |\nabla^{j_m} u|$$

$$\le \; C \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} (1 + \|\nabla u\|_{L^\infty}) \sum_{m=1}^{l+1} \|\nabla u\|_{H^{s_0}}^m \|\nabla^{s_0} \tau(u)\|_{L^2}^{a_1} \|\tau(u)\|_{L^2}^{1-a_1}$$

$$+C(\|\tau(u)\|_{L^\infty} + \|\nabla \tau(u)\|_{L^\infty}) \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla u\|_{H^{s_0}}^2$$

$$\le \; C \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} (1 + \|\nabla u\|_{L^\infty}) \sum_{m=2}^{l+2} \|\nabla u\|_{H^{s_0+1}}^m$$

$$+C(\|\tau(u)\|)_{L^\infty} + \|\nabla \tau(u)\|_{L^\infty} \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \|\nabla u\|_{H^{s_0}}$$

$$(5.88) \; \sum_{m=3}^{l+2} \sum_{j_1+\cdots+j_l=l+2} \int |\nabla^l u|^2 |\nabla^{j_1} u| \cdots |\nabla^{j_m} u|$$

$$\le \; C(1 + \|\nabla u\|_{L^\infty}^2) \left(\int |\nabla^l u|^2 \right)^{\frac{1}{2}} \sum_{m-1}^{l+2} \|\nabla u\|_{H^{s_0}}^m .$$

Combining (5.32), (5.85), (5.86), (5.87) and (5.88); using (4.75), (4.76) and (4.77) we have for $\varepsilon \in (0,1)$, $l \leq \left[\frac{n}{2}\right]+1$, $s_0 = \left[\frac{n}{2}\right]+2$

$$
\begin{aligned}
(5.89)\quad \frac{1}{2}\frac{d}{dt}\|\nabla^l u\|_{L^2}^2 \quad &\leq \quad -\frac{\varepsilon}{2}\int |\nabla^l \tau(u)|^2 + C\left(1+\|\nabla u\|_{L^\infty}^6 + \|\tau(u)\|_{L^\infty}^3\right)\int |\nabla^l u|^2 \\
&\quad + C\left(1+\|\nabla u\|_{L^\infty}^4\right)\sum_{m=2}^{2l+4}\|\nabla u\|_{H^{s_0}}^m \\
&\quad + C\left(1+\|\nabla u\|_{L^\infty}+\|\tau(u)\|_{L^\infty}+\|\nabla\tau(u)\|_{L^\infty}\right)\sum_{m=2}^{l+3}\|\nabla u\|_{H^{s_0+1}}^m \\
&\leq \quad C\|\nabla u\|_{H^{s_0}}^2\left(1+\|\nabla u\|_{H^{s_0+2}}^{3n+2l+12}\right)
\end{aligned}
$$

□

Corollary 5.7 *Let $u \in C([0,T], H^{s+1}(\mathbb{R}^n, N))$ with $s \in \mathbb{N}$ and $s \geq \left[\frac{n}{2}\right]+4$ be a solution of (5.2) then for $\left[\frac{n}{2}\right]+4 \leq l-1 \leq s$ and $l \in \mathbb{N}$ we have*

$$
\frac{d}{dt}\|\nabla u\|_{H^{l-1}}^2 \leq C_0\|\nabla u\|_{H^{l-1}}^2\left(1+\|\nabla u\|_{H^{l-1}}^{3n+21+14}\right). \tag{5.90}
$$

Proof. Combine (5.2), (5.77), (5.80) and (5.89). □

Corollary 5.8 (Uniform energy estimate) *Let $u_\varepsilon(t) \in H^{s+1}(\mathbb{R}^n, N)$, with $s \in \mathbb{N}$ and $s \geq \left[\frac{n}{2}\right]+4$ be a solution of (5.2). There exists $T_0 = T_0(\|\nabla u_0\|_{H^s})$ such that for $0 \leq t \leq T_0$*

$$
\|\nabla u_\varepsilon(t)\|_{H^s} \leq 3\|\nabla u_0\|_{H^s}. \tag{5.91}
$$

Proof. Let $E(t) = \|\nabla u\|_{H^{l-1}}^2(t)$. Then (5.90) implies $\left[\frac{n}{2}\right]+4 \leq l-1 \leq s$

$$
\frac{d}{dt}E \leq C_0 E(1+E^{2n+l+7}), \tag{5.92}
$$

which leads to, after integrating from 0 to t to

$$
\log\frac{E(t)}{E(0)} - \frac{1}{2n+l+7}\log\frac{E(t)^{2n+l+7}}{E(0)} - \frac{1}{2n+l+7}\log\frac{E(t)^{2n+l+7}+1}{E(0)^{2n+l+7}+1} \leq C_0 t \tag{5.93}
$$

which implies

$$
\frac{E(t)^{2n+l+7}}{1+E(t)^{2n+l+7}} \leq C_0 t e^{(2n+l+7)}\frac{E(0)^{2n+l+7}}{1+E(0)^{2n+l+y}}. \tag{5.94}
$$

We now consider two cases: either $E(0)^{2n+l+7} < \frac{1}{2}$ or $E(0)^{2n+l+7} \geq \frac{1}{2}$. If $E(0)^{2n+l+7} < \frac{1}{2}$ then for $0 < t \leq T_0 = \frac{1}{C_0}\log\left[\frac{1}{E(0)}\left(\frac{3}{4}\right)^{\frac{1}{2}+l+7}\right]$ we have

$$
\begin{aligned}
(5.95)\qquad E(t)^{2n+l+7} \quad &\leq \quad 4e^{C_0 t(2n+l+7)}\frac{E(0)^{2n+l+7}}{1+E(0)^{2n+l+7}} \\
E(t) \quad &\leq \quad 4^{\frac{1}{2n+l+7}}e^{C_0 t}E(0)
\end{aligned}
$$

In particular for $0 < t \leq \frac{1}{C_0} \log\left(\frac{3}{2}\right)^{\frac{1}{2n+l+7}}$ (5.95) ensures that

$$E(t) \leq 6E(0) \tag{5.96}$$

In the case when $E(0)^{2n+l+7} \geq \frac{1}{2}$ we note that (5.92) implies

$$\frac{d}{dt}(1+E) \leq C_0(1+E)^{2n+l+8}, \tag{5.97}$$

which by integration between 0 and t leads to

$$\frac{1}{(1+E(t))^{2n+l+7}} \geq -(2n+l+7)C_0 t + \frac{1}{(1+E(0))^{2n+l+7}}. \tag{5.98}$$

If $0 < t \leq T_0 = \frac{1}{2C_0 3^{2n+l+7} E(0)^{2n+l+7}}$, (5.98) implies

$$1 + E(t) \leq 2^{\frac{1}{2n+l+7}}(E(0)+1), \tag{5.99}$$

since $E(0)^{2n+l+7} \geq \frac{1}{2}$ (5.99) yields

$$E(t) \leq 6E(0). \tag{5.100}$$

Thus we have showed combining (5.96) and (5.100) that for $s \geq \left[\frac{n}{2}\right] + 4$

$$0 < t \leq T_0 = \min\left\{\frac{1}{C_0(2n+7+l)}\log\frac{3}{2}, \frac{1}{2C_0 3^{2n+l+7}\max\left\{\|\nabla u_0\|_{H^{l-1}}^{4n+21+14}, \frac{1}{2}\right\}}\right\} \tag{5.101}$$

$$\|\nabla u_\varepsilon(t)\|_{H^{l-1}} \leq 3\|\nabla u_0\|_{H^{l-1}}. \tag{5.102}$$

□

Lemma 5.9 *Let $u_\varepsilon(t) \in H^{s+1}(\mathbb{R}^n, N)$ with $s \in \mathbb{N}$ and $s \geq \left[\frac{n}{2}\right] + 4$ be a solution of (5.1). Let $v = v_\varepsilon = w \circ u_\varepsilon$. For $T_0 = T_0(\|\nabla u_0\|_{H^s})$ as in (5.101) we have*

$$\sup_{0<t\leq T_0} \|v(t) - v_0\|_{L^2} \leq C\|\nabla u_0\|_{H^{[\frac{n}{2}]+4}}\left(1 + \|\nabla u_0\|_{H^{[\frac{n}{2}]+4}}^{3[\frac{n}{2}]+6}\right) T_0. \tag{5.103}$$

Proof. Our goal is to study how $\|v(t) - v_0\|_{L^2}$ evolves. Using (3.1) and (3.2)) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |v-v_0|^2 = \int \langle \partial_t v, v - v_0\rangle &\leq \|v-v_0\|_{L^2}\left(\int (\partial_t v)^2\right)^{\frac{1}{2}} \\ &\leq C\left(\|\Delta^2 v\|_{L^2} + \|\partial^2 v\|_{L^2} + \|\partial v\|_{L^2}\|\partial v\|_{L^\infty}\right. \\ &\quad + \|\partial^2 v\|_{L^2}\|\partial^2 v\|_{L^\infty} + \|\partial v\|_{L^2}\|\partial v\|_{L^\infty}^3 \\ &\quad \left.+ \|\partial^2 v\|_{L^2}\|\partial v\|_{L^\infty}^2\right)\|v-v_0\|_{L^2}.\end{aligned} \tag{5.104}$$

Recall that

$$|\partial^2 v| \leq |\nabla^2 u| + C|\nabla u|^2 \text{ and } |\partial v| = |\nabla u|. \tag{5.105}$$

Moreover by (4.54) we have

$$(5.106)\qquad \begin{aligned} |\partial^4 v| &\le C|\nabla^4 u| + C\sum_{l=2}^{4}\sum_{j_1+\cdots+j_l=4} |\nabla_u^{j_1}|\cdots|\nabla^{j_p} u| \\ &\le C|\nabla^4 u| + C|\nabla^2 u|^2 + C|\nabla u|^4 + C|\nabla^3 u|\,|\nabla u|. \end{aligned}$$

Using (4.72), (4.73) and (5.106), (5.103) yields

$$(5.107)\qquad \begin{aligned} \frac{d}{dt}\int |v-v_0|^2 &\le C\left\{\|\nabla^4 u\|_{L^2} + \|\nabla^2 u\|_{L^\infty}\|\nabla^2 u\|_{L^2}\right. \\ &\qquad \|\nabla u\|_{L^\infty}^3\|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^3 u\|_{L^2} + \|\nabla^3 u\|_{L^2} \\ &\qquad \left. + \|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^\infty}^2\right\}\|v-v_0\|_{L^2} \\ &\le C\|\nabla u\|_{H^{[\frac{n}{2}]+3}}\left(1+\|\nabla u\|_{H^{[\frac{n}{2}]+3}}^{3[\frac{n}{2}]+6}\right)\|v-v_0\|_{L^2}. \end{aligned}$$

For $t\in[0,T_0]$ as in (5.100), (5.101)) combined with (5.106) yields

$$(5.108)\qquad \frac{d}{dt}\|v-v_0\|_{L^2}^2 \le C\|\nabla u_0\|_{H^{[\frac{n}{2}]+4}}\left(1+\|\nabla u_0\|_{H^{[\frac{n}{2}]+4}}^{3[\frac{n}{2}]+6}\right)\|v-v_0\|_{L^2}.$$

Integrating from 0 to T_0 (as defined in (5.100)) we deduce from (5.108) that

$$(5.109)\qquad \|v(t)-v_0\|_{L^2} \le CT_0\|\nabla u_0\|_{H^{[\frac{n}{2}]+4}}\left(1+\|\nabla u_0\|_{H^{[\frac{n}{2}]+4}}^{3[\frac{n}{2}]+6}\right).$$

□

Theorem 5.10 *Let $s\ge\left[\frac{n}{2}\right]+4$. Given $u_0\in H^{s+1}(\mathbb{R}^n,N)$ there exists $T_0=T_0(\|\nabla u_0\|_{H^s},N)>0$ and a solution $u_\varepsilon\in C([0,T_0],H^{s+1}(\mathbb{R}^n,N))$ of (5.2). Furthermore*

$$(5.110)\qquad \sup_{0\le t\le T_0}\|\nabla u_\varepsilon(t)\|_{H^s} \le 3\|\nabla u_0\|_{H^s}.$$

Proof. Lemma 2.3, Lemma 4.6, Theorem 3.1 and Lemma 4.7 imply that there exist $T_\varepsilon = T(\varepsilon,\|\nabla u_0\|_{H^s},\|v_0-\gamma\|_{L^2},N)$ for some $\gamma\in\mathbb{R}^p$ and $u_\varepsilon\in C([0,T_\varepsilon],H^{s+1}(\mathbb{R}^n,N))$ a solution of (5.2). Either $T_\varepsilon\ge T_0$ as defined in (5.101) and we are done or $T_\varepsilon<T_0$. Using the fact that $\|v(T_\varepsilon)-v_0\|_{L^2}\le CT_0\|\nabla u_0\|_{H^{[\frac{n}{2}]+4}}\left(1+\|\nabla u_0\|_{H^{[\frac{n}{2}]+4}}^{n+2}\right)$ the same argument as above ensures that there exists $T'_\varepsilon = T(\varepsilon,\|\nabla u_0\|_{H^s})$ and $u_\varepsilon\in C([T_\varepsilon,T_\varepsilon+T'_\varepsilon],H^s(\mathbb{R}^n,N))$ a solution of (5.2). The uniqueness statement in Theorem 3.1 ensures that we can extend $u_\varepsilon\in C([0,T_\varepsilon+T'_\varepsilon],H^s(\mathbb{R}^n,N))$ to be a solution of (5.2).

After a finite number of steps (namely l where $T_\varepsilon+lT'_\varepsilon\le T_0<T_\varepsilon+(l+1)T'_\varepsilon$) we manage to extend $\forall\varepsilon\in(0,1)$, u_ε to be a solution of (5.2) in $C([0,T_0],H^s(\mathbb{R}^n,N))$. Note that (5.110) is simply a restatement of (5.102). □

Proof of Theorem 1.1. For $s\ge\left[\frac{n}{2}\right]+4$, let $u_\varepsilon\in C([0,T_0],H^{s+1}(\mathbb{R}^n,N))$ be a solution of (5.1). Choosing a sequence $\varepsilon_i\to 0$ we conclude, by means of Theorem 5.10, Lemma 4.7 and Lemma 2.3 that there exist functions $u \in C([0,T_0], H^{s+1}(\mathbb{R}^n, N))$ and $v \in C([0,T_0], H^{s+1}(\mathbb{R}^n, \mathbb{R}^m))$ with $v = \omega \circ u$ satisfying the initial value problems (3.1) and (2.10) with $\varepsilon = 0$ and $v_0 = \omega \circ u_0$.

To prove the well-posedness of the Schrödinger flow (i.e. when $\beta = 0$ in (3.1)) we refer to work of Ding and Wang [8] and McGahagan [26]. By adapting the argument of Ding and Wang [8] one can show that if a solution, $u \in C([0,T_0], H^{s+1}(\mathbb{R}^n, N))$ with $s \geq \left[\frac{n}{2}\right] + 4$, to the initial value problem (1.4) (with $\beta = 0$) exists then it is unique. This argument makes explicit use of the fact that the target is compact and isometrically embedded in Euclidean space. We present here part of an argument that appears in the proof of Theorem 4.1 in [26]. These inequalities yield uniqueness and continuous dependence on the initial data. Let $u_1, u_2 \in C([0,T_0], H^{s+1}(\mathbb{R}^n, N))$ be solutions of (1.4) (with $\beta = 0$) with initial data $u_1^0, u_2^0 \in H^{s+1}(\mathbb{R}^n, N)$ with $s \geq \left[\frac{n}{2}\right] + 4$. Following the notation in [26] let $V = \nabla u_1$ and $W = \nabla u_2$. Let $\widetilde{V}(x)$ represent the parallel transport of V to the point $u_2(x)$ along the unique geodesic joining the points. McGahagan proves (see end of the proof of Theorem 4.1 in [26]) that whenever $\|u_1^0 - u_2^0\|_{H^{[\frac{n}{2}]+4}}$ is small enough (depending only on the geometry of N) then

$$\frac{d}{dt}\left(\|W - \widetilde{V}\|_{L^2}^2 + \|u_1 - u_2\|_{L^2}^2\right) \leq C\left(\|W - \widetilde{V}\|_{L^2}^2 + \|u_1 - u_2\|_{L^2}^2\right), \tag{5.111}$$

where C depends on the $H^{[\frac{n}{2}]+4}$ norms of u_1 and u_2. In the case that $u_1^0 = u_2^0$ McGahagan concludes (using Gronwall's) that $\|W - \widetilde{V}\|_{L^2}^2 = \|u_1 - u_2\|_{L^2}^2 = 0$, and that therefore $u_1 = u_2$ a.e.. Since the unique solution is constructed as a limit of solutions of equation (5.2) letting $\varepsilon \to 0$, the estimate in Theorem 5.10 yields that

$$\sup_{0 \leq t \leq T_0} \|\nabla u(t)\|_{H^s} \leq 3\|\nabla u_0\|_{H^s}. \tag{5.112}$$

To prove the continuous dependence on the initial data note that, in general, (5.111) yields

$$\|W - \widetilde{V}\|_{L^2}^2(t) + \|u_1 - u_2\|_{L^2}^2(t) \leq e^{Ct}\left(\|W^0 - \widetilde{V}^0\|_{L^2}^2 + \|u_1^0 - u_2^0\|_{L^2}^2\right), \tag{5.113}$$

where $W^0 = \nabla u_2^0$ and $\widetilde{V}^0(x)$ is the parallel transport of $V^0 = \nabla u_1^0$ to $u_2^0(x)$. Since

$$\|W - \widetilde{V}\|_{L^2}^2(t) \lesssim \|\partial u_1 - \partial u_2\|_{L^2}^2(t) + \|u_1 - u_2\|_{L^2}^2(t), \tag{5.114}$$

and

$$\|\partial u_1 - \partial u_2\|_{L^2}^2(t) \lesssim \|W - \widetilde{V}\|_{L^2}^2(t) + \|u_1 - u_2\|_{L^2}^2(t), \tag{5.115}$$

(5.113) yields

$$\|\partial u_1 - \partial u_2\|_{L^2}^2(t) + \|u_1 - u_2\|_{L^2}^2(t) \lesssim e^{Ct}\left(\|\partial u_1^0 - \partial u_2^0\|_{L^2}^2 + \|u_1^0 - u_2^0\|_{L^2}^2\right). \tag{5.116}$$

Note that (5.116) ensures that $C([0,T_0], H^{s+1}(\mathbb{R}^n, N))$ solutions to (1.4) (with $\beta = 0$) with $s \geq \left[\frac{n}{2}\right] + 4$ depend continuously in H^1 on the initial data. To show continuous dependence in $H^{s'}$ for $s' < s$ we need to use a classic interpolation inequality in $\mathbb{R}^n$. If $v_i = \omega \circ u_i$ for $i = 1, 2$, where ω denotes the embedding of N into $\mathbb{R}^p$ then combining (4.52) and (5.116) we have

$$\|\partial v_1 - \partial v_2\|_{L^2}^2(t) + \|v_1 - v_2\|_{L^2}^2(t) \lesssim e^{Ct}\Big(\|\partial v_1^0 - \partial v_2^0\|_{L^2}^2 + \|v_1^0 - v_2^0\|_{L^2}^2\Big). \tag{5.117}$$

Interpolation, Lemma 4.6, Lemma 4.7, (5.112) and (5.117) yield for $s' < s$

$$\begin{aligned} \|\partial v_1 - \partial v_2\|_{H^{s'}}(t) &\lesssim \|\partial v_1 - \partial v_2\|_{H^s}^{\frac{s'}{s}}(t)\|\partial v_1 - \partial v_2\|_{L^2}^{1-\frac{s'}{s}}(t) \\ &\lesssim \Big(\|\partial v_1\|_{H^s}^{\frac{s'}{s}}(t) + \|\partial v_2\|_{H^s}^{\frac{s'}{s}}(t)\Big)\|\partial v_1 - \partial v_2\|_{L^2}^{1-\frac{s'}{s}}(t) \\ &\lesssim \Big(\|u_1^0\|_{H^s}^m + \|u_2^0\|_{H^s}^m\Big)\|\partial v_1 - \partial v_2\|_{L^2}^{1-\frac{s'}{s}}(t) \\ &\lesssim \Big(\|u_1^0\|_{H^s}^m + \|u_2^0\|_{H^s}^m\Big)e^{Ct}\Big(\|\partial v_1^0 - \partial v_2^0\|_{L^2}^2 + \|v_1^0 - v_2^0\|_{L^2}^2\Big) \end{aligned} \tag{5.118}$$

Inequalities (5.117) and (5.118) prove that if $u_1, u_2 \in C([0, T_0], H^{s+1}(\mathbb{R}^n, N))$ to (1.4) (with $\beta = 0$) and $\|u_1^0 - u_2^0\|_{H^{[\frac{n}{2}]+4}}$ is small enough then the functions $v_1 = \omega \circ u_1$ and $v_2 = \omega \circ u_2$, which are solutions to the ambient equation, depend continuously in the $H^{s'+1}(\mathbb{R}^n, \mathbb{R}^p)$-norm on the initial data for $s' < s$. As mentioned in the introduction by means of the standard Bond-Smith regularization procedure ([4, 13, 16]) one can prove that the dependence on the initial data is continuous in $H^{s+1}(\mathbb{R}^n, \mathbb{R}^p)$. It is in this sense that we express the well-posedness of (1.4) for $\beta = 0$. This concludes the proof Theorem 1.1. □

References

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