On the black hole limit of rotating fluid bodies in equilibrium*

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Abstract

Recently, it was shown that the extreme Kerr black hole is the only candidate for a (Kerr) black hole limit of stationary and axisymmetric, uniformly rotating perfect fluid bodies with a zero temperature equation of state. In this paper, necessary and sufficient conditions for reaching the black hole limit are presented.

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* Fondly dedicated to Gernot Neugebauer on the occasion of his 65$^{\text{th}}$ birthday.

I. INTRODUCTION

It was shown in [1] that a continuous sequence of stationary and axisymmetric, uniformly rotating perfect fluid bodies with a "cold matter" equation of state and finite baryonic mass can reach a (Kerr) black hole limit only if the relation

$$M = 2\Omega J \tag{1}$$

between the gravitational mass M, the angular velocity Ω and the angular momentum J is satisfied in the limit.[1] This result implies the impossibility of black hole limits of non-rotating equilibrium configurations (cf. "Buchdahl's inequality"), and moreover, since Ω must become equal to the "angular velocity of the horizon"

$$\Omega^H = \frac{J}{2M^2\left[M + \sqrt{M^2 - (J/M)^2}\right]}\,, \tag{2}$$

the relation

$$J = \pm M^2, \tag{3}$$

characteristic of an *extreme* Kerr black hole, must hold in the limit. The possibility of such a limit was first demonstrated for infinitesimally thin disks – numerically by Bardeen and Wagoner [2] and analytically by Neugebauer and Meinel [3], see also [4]. Further numerical examples, for genuine fluid bodies, were provided by the "relativistic Dyson rings" [5] and their generalizations [6].

The aim of the present paper is to show that condition (1) is not only necessary, but also sufficient for reaching the black hole limit. An equivalent condition will turn out to be the statement that a zero angular momentum photon emitted from the fluid's surface suffers an infinite redshift.

II. BASIC RELATIONS

In the following, some basic relations for rotating fluids in equilibrium are provided, for more details see, for example, [2, 7–11]. The four-velocity of the fluid must point in

[1] We use units in which the speed of light as well as Newton's gravitational constant are equal to 1. Strictly speaking, Ω and J are the components of angular velocity and angular momentum with respect to the axis of symmetry and can have either sign, corresponding to the two possible directions of rotation.

the direction of a linear combination of the two commuting Killing vectors $\xi = \partial/\partial t$ and $\eta = \partial/\partial\phi$ corresponding to stationarity and axisymmetry:

$$u^i = e^{-V}(\xi^i + \Omega\,\eta^i), \quad \Omega = \text{constant}. \tag{4}$$

The Killing vector ξ is fixed by the normalization $\xi^i\xi_i \to -1$ at spatial infinity (we assume asymptotic flatness).[2] The orbits of the spacelike Killing vector η are closed and η is zero on the axis of symmetry. The constant of eq. (4), $\Omega = u^\phi/u^t$, is the angular velocity of the fluid body with respect to infinity. Using $u^i u_i = -1$, the factor $e^{-V} = u^t$ is given by

$$(\xi^i + \Omega\,\eta^i)(\xi_i + \Omega\,\eta_i) = -e^{2V}. \tag{5}$$

The energy-momentum tensor is

$$T_{ik} = (\epsilon + p)\,u_i u_k + p\,g_{ik}, \tag{6}$$

where the mass-energy density ϵ and the pressure p are related by a "cold" equation of state, $\epsilon = \epsilon(p)$, following from

$$p = p(\rho, T), \quad \epsilon = \epsilon(\rho, T) \tag{7}$$

for $T = 0$, where ρ is the baryonic mass-density and T the temperature. The specific enthalpy

$$h = \frac{\epsilon + p}{\rho} \tag{8}$$

can be calculated from $\epsilon(p)$ via the thermodynamic relation

$$dh = \frac{1}{\rho}\,dp \qquad (T = 0) \tag{9}$$

leading to

$$\frac{dh}{h} = \frac{dp}{\epsilon + p} \Rightarrow h(p) = h(0)\,\exp\left[\int_0^p \frac{dp'}{\epsilon(p') + p'}\right]. \tag{10}$$

Note that $h(0) = 1$ in most cases. For our purposes, however, it is sufficient to assume $0 < h(0) < \infty$. From $T^{ik}{}_{;k} = 0$ we obtain

$$h(p)\,e^V = h(0)\,e^{V_0} = \text{constant}, \tag{11}$$

[2] The spacetime signature is chosen to be $(-+++)$.

where V_0, the constant surface value (corresponding to $p = 0$) of the function V defined in (5), is related to the relative redshift z of zero angular momentum photons[3] emitted from the surface of the fluid and received at infinity:

$$z = e^{-V_0} - 1. \tag{12}$$

Equilibrium models, for a given equation of state, are fixed by two parameters, for example Ω and V_0. (When we discuss a "sequence" of solutions, what is meant is a curve in the two-dimensional parameter space.) The gravitational mass and the angular momentum can be calculated by

$$M = 2\int_\Sigma (T_{ik} - \frac{1}{2}T g_{ik}) n^i \xi^k d\mathcal{V}, \quad J = -\int_\Sigma T_{ik}\, n^i \eta^k d\mathcal{V}, \tag{13}$$

where Σ is a spacelike hypersurface ($t =$ constant) with the volume element $d\mathcal{V} = \sqrt{^{(3)}g}\, d^3x$ and the future pointing unit normal n^i, see for example [12]. The baryonic mass M_0 corresponding to the local conservation law $(\rho u^i)_{;i} = 0$ is

$$M_0 = -\int_\Sigma \rho\, u_i\, n^i d\mathcal{V}. \tag{14}$$

Note that nearby equilibrium configurations with the same equation of state are related by [2, 7]

$$\delta M = \Omega\, \delta J + \mu\, \delta M_0, \qquad \mu = h(0)\, e^{V_0}. \tag{15}$$

The parameter μ ("chemical potential") represents the specific injection energy of zero angular momentum baryons.

III. CONDITIONS FOR A BLACK HOLE LIMIT

A combination of (13) and (14) leads to the formula

$$M = 2\Omega J + \int \frac{\epsilon + 3p}{\rho}\, e^V dM_0, \tag{16}$$

cf. equation (II.28) in [2]. With (8) and (11) we get

$$M = 2\Omega J + h(0)\, e^{V_0} \int \frac{\epsilon + 3p}{\epsilon + p}\, dM_0. \tag{17}$$

[3] Zero angular momentum means $\eta_i p^i = 0$ (p^i: four-momentum of the photon).

Since $1 \leq (\epsilon + 3p)/(\epsilon + p) \leq 3$ (we assume ϵ and p to be non-negative), condition (1) is equivalent to[4]

$$V_0 \to -\infty \quad (z \to \infty). \tag{18}$$

We now want to show that this condition is not only necessary (as discussed in [1]), but also sufficient for approaching a black hole limit.

Because of (5) and (11), the surface of the fluid is characterized in general by

$$\chi^i \chi_i = -e^{2V_0}, \quad \chi^i \equiv \xi^i + \Omega \eta^i. \tag{19}$$

The Killing vector χ^i is tangential to the hypersurface $\mathcal{H}$ generated by the timelike world lines of the fluid elements of the surface of the body with four-velocity $u^i = e^{-V_0}\chi^i$, see (4). Each of the Killing vectors ξ^i and η^i must itself be tangential to $\mathcal{H}$ because of the symmetries of the spacetime. In the limit $V_0 \to -\infty$, we approach a situation in which χ^i becomes null on $\mathcal{H}$:

$$\chi^i \chi_i \to 0. \tag{20}$$

Moreover, with the reasonable assumption[5]

$$0 \leq -\xi^i u_i \leq 1, \tag{21}$$

we find that χ^i also becomes orthogonal to ξ^i (and thus to η^i) on $\mathcal{H}$ in the limit:

$$\chi^i \xi_i \to 0, \quad \chi^i \eta_i \to 0. \tag{22}$$

Together with the orthogonal transitivity[6] of the spacetime, χ^i therefore becomes orthogonal to three linearly independent tangent vectors at each point of $\mathcal{H}$, i.e. normal to $\mathcal{H}$. Because of (20), we thus approach a situation in which $\mathcal{H}$ is a null hypersurface and satisfies all defining conditions for a horizon of a stationary black hole with Ω being the angular velocity of the horizon, see [14]. Assuming that the Kerr family represents the only stationary black hole solutions surrounded by a vacuum (this is partially proved by the black hole uniqueness theorems (see [15, 16] and also [11]), we conclude with (1), that the metric of an extreme Kerr black hole (outside the horizon) results, whenever a sequence of fluid bodies admits a (parameter) limit $V_0 \to -\infty$.

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[4] We assume $0 < M_0 < \infty$.

[5] The condition $-\xi^i u_i \leq 1$ ensures that a particle resting on the surface of the fluid is (at least marginally) bound, i.e. cannot escape to infinity on a geodesic; $-\xi^i u_i \geq 0$ follows from $-\xi^i u_i = -(\chi^i - \Omega \eta^i) u_i = e^{V_0} + \Omega \eta^i u_i$, since $\eta^i u_i$ will always have the same sign as Ω ($\eta^i u_i = 0$ on the axis, of course).

[6] The conditions of the theorem by Kundt and Trümper [13] are satisfied.

IV. DISCUSSION

The special properties of the *extreme* Kerr metric with its degenerate horizon and the infinitely long "throat region" allow for the existence of a black hole limit independent of the fluid body's topology. Indeed, such a limit was found numerically for bodies of toroidal topology [5]. Strictly speaking, there is not yet a horizon in the limit. Instead, a separation of an "inner" and an "outer" world occurs. The "inner world" contains the fluid body and is not asymptotically flat, but approaches the "extreme Kerr throat geometry" [17] at spatial infinity. The "outer world" is given by the $r > M$ part of the extreme Kerr metric, where r is the radial Boyer-Lindquist coordinate. Note that the horizon as well as the "throat region" are characterized by $r = M$. Here, the whole "inner world" corresponds to $r = M$. It should be mentioned in this connection that the conditions (20) and (22) are also satisfied *inside* the fluid as $V_0 \to -\infty$, cf. (10) and (11). More details can be found in [2, 4, 5].

It is interesting to note that a similar separation of spacetimes has been observed for some limiting solutions of the static, spherically symmetric Einstein-Yang-Mills-Higgs equations [18, 19] leading to the extreme Reissner-Nordström metric in the "outer world".

As discussed in [20], the slightest dynamical perturbation will lead to a genuine black hole. Therefore, it is tempting to continue a sequence of fluid bodies beyond the black hole limit as a sequence of Kerr black holes and to discuss the transition from the "normal matter state" to the "black hole state" as a (one-way) phase transition [21]. From the exterior point of view, this transition is continuous, i.e. all gravitational multipole moments change continuously. It is interesting to compare the mass formula (17) as well as the differential relation (15) with the black hole formulas [22, 23]

$$M = 2\Omega^H J + \frac{\kappa}{4\pi} A, \tag{23}$$

$$\delta M = \Omega^H \delta J + \frac{\kappa}{8\pi} \delta A, \tag{24}$$

where κ denotes the "surface gravity" and A the area of the horizon. In the two-dimensional parameter space, the transition line is characterized by $M = 2\Omega J = 2\Omega^H J$. On the "fluid side" of this line, the parameter $\mu = h(0)e^{V_0}$ of the chemical potential vanishes ($V_0 \to -\infty$), whereas κ (related to the temperature in black hole thermodynamics) vanishes on the "black hole side" of the transition line ($\kappa = 0$ for extreme Kerr black holes). The quantities M_0 (baryonic mass of the fluid body) and A (related to the black hole entropy) are defined in the corresponding regions of the parameter space only. This is consistent with $\mu \equiv 0$ in the black hole region and $T \equiv 0$ in the fluid region, i.e. μ and T are continuous across the transition line. (An alternative interpretation was given in [24].)

The parametric transitions from fluid bodies to black holes discussed here may be used as a starting-point for dynamical collapse investigations far from the spherically symmetric case.

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