Invariants, Equivariants and Characters in Symmetric Bifurcation Theory

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December 8, 2005

Abstract

In the analysis of stability in bifurcation problems it is often assumed that the (appropriate reduced) equations are in normal form. In the presence of symmetry, the truncated normal form is an equivariant polynomial map. Therefore, the determination of invariants and equivariants of the group of symmetries of the problem is an important step. In general, these are hard problems of invariant theory, and in most cases, they are tractable only through symbolic computer programs. Nevertheless, it is desirable to obtain some of the information about invariants and equivariants without actually computing them, for example, the number of linearly independent homogeneous invariants or equivariants of a certain degree. Generating functions for these dimensions are generally known as "Molien functions".

In this work we obtain formulas for the number of linearly independent homogeneous invariants or equivariants for Hopf bifurcation in terms of characters and we show that they are effectively computable in several concrete examples. This information allows to draw some predictions about the structure of the bifurcations. For example, by comparing the number of equivariants with the number of invariants of one higher degree, it can be checked immediately whether the dynamics is variational (gradient-like).

Mathematics Subject Classification: 37G40, 37C80, 13A50

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1 Introduction

Symmetry appears naturally in several important physical models, and in many cases the collection of all the symmetries of the problem forms a compact Lie group. Moreover, there is a fully symmetric solution that loses stability as a parameter is varied, and this loss of stability is due to the crossing of eigenvalues through the imaginary axis. When the eigenvalues are zero a steady-state bifurcation is expected to happen – that is, a bifurcation from the group-invariant equilibrium to equilibria with less symmetry. When the eigenvalues are imaginary, the bifurcation expected is a Hopf bifurcation to periodic solutions. A Liapunov-Schmidt or center-manifold reduction reduces the bifurcation problem to equations on the sum of the generalized eigenspaces of these eigenvalues. See for example Golubitsky and Schaeffer [10] and Carr [2].

In a generic parametrized family of equations for a given symmetry group, there are restrictions on the multiplicity of the eigenvalues passing through the imaginary axis for some critical value of the parameter. For steady-state bifurcation, the group of symmetries leaves the kernel V of the linearization at the group-invariant solution invariant. Moreover, generically the action of that symmetry group on V is absolutely irreducible. See [12, Proposition XIII 3.2]. For Hopf bifurcation, the group of symmetries leaves the imaginary eigenspace of the linearization at the group-invariant solution for the critical value invariant. Generically, the action of the symmetry group is simple – the sum of two absolutely irreducible representations, or irreducible but not absolutely irreducible. See [12, Proposition XVI 1.4].

In this paper we focus on the two situations: symmetric steady-state bifurcation problems posed on absolutely irreducible spaces, and symmetric Hopf bifurcation problems posed on simple spaces.

Because of the multiplicity of the eigenvalues, the linearized problem is highly degenerate – there is no preferred direction within the eigenspace. This degeneracy is partially resolved by the nonlinear terms, which are constrained by the symmetry; terms which respect the symmetry are said to be equivariant. In a particular problem, one can specify the action of the group on the eigenspace and construct equivariant polynomials of a given degree.

Our aim in this paper is to obtain formulas for the number of possible equivariant terms, using only the character (trace) of the irreducible representations. This has a number of advantages over working with the matrices of the representation. The characters of a representation are unique, but the matrices themselves are not. Secondly, the characters of the irreducible representations of many finite groups are tabulated, and are much easier to work with than the matrices. Formulas for the number of equivariants are useful, because in a specific problem they can be used to confirm that all possible equivariants have been found. Furthermore, by comparing the number of equivariants with the number of invariants of one higher degree, it can be checked immediately whether the dynamics is variational (gradient-like).

In Section 2 below we introduce the properties of representations and characters. Our main new results are formulas for the numbers of invariants and equivariants for Hopf bifurcation in Section 4; we also show how to generalize the Molien formula to the case of Hopf bifurcation in Section 6. Finally we present results of the application of our formulas, obtaining new results for the numbers of invariants and equivariants for finite groups and for the symmetry group of the sphere.

2 Background

In this section we review some important concepts concerning the representation theory of compact Lie groups. For details, see for example Fulton *et al.* [8] and James *et al.* [16].

2.1 Representations

Let G be a compact Lie group *acting* linearly on a finite-dimensional real or complex vector space V. Thus this action corresponds to a *representation* T of the group G on the vector space V through a linear homomorphism from G to the group $\mathrm{GL}(V)$ of invertible linear transformations on V. The space V itself can be regarded as a G-module.

A subspace W of V is *invariant* under G if $gW \subseteq W$ for all $g \in G$; in this case, we say that W is a *G-submodule* of the G-module V. The action is said to be *reducible* if V possesses a proper invariant subspace. Otherwise it is said to be *irreducible*. A representation T of G is *absolutely irreducible* if the only linear maps on V commuting with G are the scalar multiples of the identity.

Two representations T_1 and T_2 of a group G are *equivalent* if there exists a nonsingular linear transformation S such that

$$T_1(g) = ST_2(g)S^{-1}$$

for all $g \in G$.

2.2 Haar Measure and Haar Integral

Since G is a compact group there exists an *invariant measure* μ_G on G such that

$$\int_G f(hg)\,\mathrm{d}\mu(g) \;=\; \int_G f(gh)\,\mathrm{d}\mu(g) \;=\; \int_G f(g)\,\mathrm{d}\mu(g) \;=\; \int_G f(g^{-1})\,\mathrm{d}\mu(g) \tag{2.1}$$

for any continuous function f on G and for any $h \in G$. We assume that the measure is normalised so that $\int_G \mathrm{d}\mu(g) = 1$ and $\int_G f(g)\mathrm{d}\mu(g)$ is called the *normalised Haar integral* of f. See Hochschild [14, page 9] for the proof and existence of the Haar integral. For finite groups the Haar integral reduces to the "averaging over the group" formula

$$\int_G f(g)\,\mathrm{d}\mu(g) \;=\; \frac{1}{|G|}\sum_{g\in G} f(g)\,.$$

Using the Haar integral it is possible to construct a G-invariant inner product on V, that is

$$(T(g)u,\, T(g)v) \;=\; (u,v)\,,$$

for all $u, v \in V$ and $g \in G$. See for example [12, Proposition XII 1.3]. Moreover, we can choose an orthogonal basis with respect to such a G-invariant inner product where if M_g denotes the matrix representing $T(g)$ at this basis, then M_g is unitary for all $g \in G$, that is,

$$M_{g^{-1}} \;=\; M_g^* \;=\; \overline{M}_g^{\,t}\,,$$

Here, M_g^t denotes the transpose matrix of M_g. In particular, if V is real then M_g is orthogonal for all $g \in G$.

2.3 Real and Complex Representations

There are two ways to understand the relation between real and complex representations, depending on which class of representations we choose as the most fundamental one. In all texts on representation theory, the complex representations are considered as most fundamental – mainly to take advantage of the algebraic completeness of the complex field – and then the real representations are defined as a special class of those representations. On the other hand, in bifurcation theory, all the representations are real with the complex ones arising due to extra structure, as for example, in Hopf bifurcation where the circle action induces a complex structure on V. In this paper we assume the second point of view.

Let V be a real vector space. The *complexification* of V is the complex vector space $V^{\mathbf{C}}$ given by the following tensor product over $\mathbf{R}$,

$$V^{\mathbf{C}} \;=\; V \otimes_{\mathbf{R}} \mathbf{C}.$$

The map $v \mapsto v \otimes 1$ allows us to identify V canonically with a subset of $V^{\mathbf{C}}$. If $\{e_i\}$ is a basis of V over $\mathbf{R}$, then $\{e_i \otimes 1\}$ (which we often write simply as $\{e_i\}$) is a basis of $V^{\mathbf{C}}$ over $\mathbf{C}$. If W is a complex vector space, we can restrict the definition of scalar multiplication to scalars in $\mathbf{R}$, thereby obtaining a vector space over $\mathbf{R}$. This vector space we denote by $W^{\mathbf{R}}$. The operations $(\cdot)^{\mathbf{C}}$ and $(\cdot)^{\mathbf{R}}$ are not inverse to each other: $(V^{\mathbf{C}})^{\mathbf{R}}$ has twice the real dimension of V, and $(W^{\mathbf{R}})^{\mathbf{C}}$ has twice the complex dimension of W. More precisely,

$$(V^{\mathbf{C}})^{\mathbf{R}} \;=\; V \oplus iV \tag{2.2}$$

as real vector spaces, where V means $V \otimes 1$ in $V \otimes_{\mathbf{R}} \mathbf{C}$ and i refers to the real linear mapping "multiplication-by-i".

When a complex vector space W and a real vector space V are related by

$$W^{\mathbf{R}} \;=\; V \oplus iV$$

we say that V is a *real form* of the complex vector space W. The formula (2.2) says that any real space is a real form of its complexification.

It is convenient – especially when V or W is a G-module – to express these structures by certain maps on V and W.

A *real structure* or *conjugation* on a complex vector space W is an involutive anti-linear map $\sigma : W \to W$, that is,

$$\sigma(\alpha v) \;=\; \overline{\alpha} v \;, \qquad \sigma^2 \;=\; \mathrm{id} \;.$$

The real subspace of W defined by

$$V \;=\; \mathrm{Fix}(\sigma) \;=\; \{w \in W^{\mathbf{R}} \;:\; \sigma(w) = w\} \tag{2.3}$$

is a real form of W. Obviously, real structures in W are in bijective correspondence with real forms $V \subset W^{\mathbf{R}}$. For example, let $V^{\mathbf{C}}$ be the complexification of a real vector space V. The map $\sigma : (V^{\mathbf{C}})^{\mathbf{R}} \to (V^{\mathbf{C}})^{\mathbf{R}}$ given by $\sigma(u + iv) = u - iv$, $u, v \in V$ is called the *canonical conjugation* of the complex vector space $V^{\mathbf{C}}$ and the real form V of $(V^{\mathbf{C}})^{\mathbf{R}}$ is given by (2.3).

A *complex structure* on a real vector space V is a linear map $J : V \to V$ satisfying $J^2 = -1$, which is simply the "multiplication-by-i" map. A complex vector space W can be regarded as the real vector space $W^{\mathbf{R}}$ endowed with the complex structure $w \mapsto iw$, $w \in W^{\mathbf{R}}$. Conversely, if a complex structure J in a real vector space V is given, then we may regard (V, J) as a complex vector space with the realification V with multiplication by scalar given by $(a + bi)v = av + bJv$, $a, b \in \mathbf{R}$, $v \in V$.

For any complex vector space W, let us denote by $\overline{W}$ the complex vector space which coincides with W as an additive group, but is endowed with the following multiplication by complex scalars: $c \cdot w = \bar{c}w$, $c \in \mathbf{C}$, $w \in \overline{W}$. In other words, if J is the given complex structure in W, then $\overline{W} = (W^{\mathbf{R}}, -J)$. The vector space $\overline{W}$ is called the *complex conjugate* of W.

Now suppose there is a group G acting on the real vector space V. Then the action of $g \in G$ on V can be extended to an action on $V^{\mathbf{C}}$ by

$$g(v \otimes z) \;=\; gv \otimes z$$

for all $z \in \mathbf{C}$ and $v \in V$. This is equivalent to having $gJ = Jg$ for all $g \in G$, where J is the complex structure on V and we say that J is *G-invariant* in that case. On the other hand, if W is a complex G-module with a G-invariant real structure σ, that is, $g\sigma = \sigma g$ for all $g \in G$ then the real form $V = \mathrm{Fix}(\sigma)$ of W is a real representation of G.

Finally, a G-module V is *G-simple* if either: $V = U \oplus U$ where U is absolutely irreducible for G, or V is irreducible but not absolutely irreducible for G. In any case, there exists a G-invariant complex structure J on V and in the first case we have $(V, J) = U \otimes \mathbf{C}$ and $(V, J)^{\mathbf{R}} = U \oplus iU$.

2.4 Symmetric Tensor Power Representation

Let T_1 and T_2 be representations of a group G on the vector spaces V and W respectively. The *tensor product representation* $T_1 \otimes T_2$ on $V \otimes W$ of G is defined by

$$(T_1(g) \otimes T_2(g))\,(v \otimes w) \;=\; T_1(g)(v) \otimes T_2(g)(w)$$

on elements of the type $v \otimes w$ and extending to the full space $V \otimes W$ by linearity. Using this rule, we can define the *n-th tensor power representation* $T^{\otimes n}$ on the n-th tensor product space $V^{\otimes n}$.

The symmetric group $\mathbf{S}_n$ acts on the n-th tensor product space $V^{\otimes n}$ by permuting the factors. This action commutes with the action of G on $V^{\otimes n}$ and therefore the *n-th symmetric tensor power*

$$S^n V \;=\; \{x \in V^{\otimes n} : \sigma x = x \text{ for all } \sigma \in \mathbf{S}_n\}$$

is a submodule of $V^{\otimes n}$.

In the next proposition we collect some properties of the n-th symmetric tensor power.

Proposition 2.1 *Let V and W be finite dimensional vector spaces over a field. Then there are canonical isomorphisms:*

$$\mathrm{Hom}(S^n V, V) \;\cong\; L^n_s(V, V)\,, \tag{2.4}$$

$$S^n(V \oplus W) \;\cong\; \bigoplus_{i=0}^{n} S^i V \otimes S^{k-i} W\,. \tag{2.5}$$

Here $L_s^n(V,V)$ denotes the vector space of V-valued symmetric n-multi-linear maps on $V \times \cdots \times V$ and $S^0(V)$ the ground field.

Proof: For (i) see [13, page 621] and for (ii) see [8, page 473]. □

Let V^* be the *dual space* of V and denote the natural bilinear pairing between V and V^* by $\langle \cdot\,,\cdot \rangle$. The *dual representation* of G on V^* is defined by $T^* :\; G \to \mathrm{GL}(V^*)$ where $[T^*(g)\psi](v) = \psi(T(g^{-1})v)$, or equivalently, $\langle T(g)v, T^*(g)\psi \rangle = \langle v, \psi \rangle$ for all $g \in G$, $v \in V$ and $\psi \in V^*$. Of particular importance for representation theory is the isomorphism

$$V^* \otimes W \;\cong\; \mathrm{Hom}(V, W) \tag{2.6}$$

which maps $v^* \otimes w$ to the homomorphism $u \mapsto v^*(u)w$. If V is a complex representation of a compact group G then one can choose a G-invariant hermitian inner product $(\cdot\,,\cdot)$ on V which induces an equivalence of representations

$$\overline{V} \;\cong\; V^*\,. \tag{2.7}$$

Finally, combining the isomorphisms (2.4) and (2.6) we have

$$L_s^n(V,V) \;\cong\; \mathrm{Hom}(S^n V, V) \;\cong\; (S^n V)^* \otimes V\,. \tag{2.8}$$

2.5 Characters

Recall that two elements $g_1, g_2 \in G$ are *conjugate* if there is an element $h \in G$ such that $g_1 = h g_2 h^{-1}$. Note that conjugacy is an equivalence relation on G and so partitions G into separate classes, called *conjugacy classes*. A function $f : G \to \mathbf{C}$ is called a *class function* if it is constant on the conjugacy classes. The *character* of a representation T of a group G is the trace

$$\chi_T(g) \;=\; \mathrm{tr}\, T(g) \qquad \text{for all} \quad g \in G\,.$$

Note that characters are constant on conjugacy classes. In fact, the characters of the irreducible representations form a basis for the vector space of class functions; therefore two representations are equivalent if and only if they have the same character. The character of a one-dimensional representation is said to be a *linear character*.

Since all representations of a compact Lie group are equivalent to unitary representations (by choosing an invariant inner product) we have $\mathrm{tr}(M_{g^{-1}}) = \mathrm{tr}\left(\overline{M}_g\right)$. Then an inner product can be defined on characters:

$$\langle \chi_1, \chi_2 \rangle \;=\; \int_G \chi_1(g)\overline{\chi_2(g)}\,\mathrm{d}\mu_G(g) \;=\; \int_G \chi_1(g)\chi_2(g^{-1})\,\mathrm{d}\mu_G(g)\,,$$

or for finite groups,

$$\langle \chi_1, \chi_2 \rangle \;=\; \frac{1}{|G|}\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)} \;=\; \frac{1}{|G|}\sum_{g\in G}\chi_1(g)\chi_2(g^{-1})\,.$$

With respect to this inner product, the characters of irreducible inequivalent representations are orthonormal and the relation

$$\langle \chi_1, \chi_2 \rangle \;=\; \langle \chi_2, \chi_1 \rangle \tag{2.9}$$

holds for any two characters χ_1, χ_2. See [16, Proposition 14.5].

Let V and W be two (real or complex) G-modules, with characters χ_V and χ_W, respectively. Then

$$\chi_{V\oplus W} \;=\; \chi_V + \chi_W, \qquad \chi_{V\otimes W} \;=\; \chi_V\chi_W$$

$$\chi_{V^*} \;=\; \overline{\chi}_V \;=\; \overline{\chi_V}\,, \qquad \chi_{S^2V}(g) \;=\; \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right)$$

For the proof of these facts see for example Fulton *et al.* [8, Proposition 2.1]. More generally, given a representation V with character χ we shall denote the character of the n-th symmetric tensor power S^nV by $\chi_{(n)}$.

Usually the irreducible characters are defined for the complex representations and then the characters of the irreducible real representations are computed from the complex ones. In order to do this one should be able to decide when a complex representation has a real form, that is, it is a complexification of an absolutely irreducible real representation. A necessary condition is that the complex character χ must be real valued. However this is not sufficient. The sufficient condition is supplied by the *Frobenius-Schur indicator*:

$$\iota_\chi \;=\; \int_G \chi(g^2)\mathrm{d}\mu(g)$$

for an irreducible complex character χ according to the following:

Theorem 2.2 *For each irreducible complex character χ of G, we have*

$$\iota_\chi = \begin{cases} 0, & \text{if } \chi \text{ is not real valued,} \\ 1, & \text{if } \chi \text{ can be realized over } \mathbf{R}, \\ -1, & \text{if } \chi \text{ is real but cannot be realized over } \mathbf{R}. \end{cases}$$

Proof: See for example [16, p. 274]. □

Using this theorem one can list the real irreducible characters from the complex irreducible characters:

(i) The complex characters χ_V with indicator 1 are exactly the real absolutely irreducible characters, that is, $V = U^{\mathbf{C}}$ where U is absolutely irreducible and $\chi_V(g) = \chi_U(g)$ for all $g \in G$.

(ii) For each non real valued character χ_V, its conjugate $\overline{\chi_V}$ is also an irreducible complex character and
$$\chi_V^{\mathbf{R}} = \chi_V + \overline{\chi_V} \tag{2.10}$$
is the real irreducible character of $V^{\mathbf{R}}$.

(iii) For each complex character χ_V with indicator -1, the character $\chi_V^{\mathbf{R}} = 2\chi_V$ is the real irreducible character of $V^{\mathbf{R}}$.

2.6 Trace Formula

Recall that the *fixed-point subspace* of the action of G on V is defined by

$$\mathrm{Fix}(G,V) = \{v \in V : T(g)v = v, \ \forall g \in G\}.$$

Theorem 2.3 (Trace Formula) *Let T be any representation of a compact group G over a vector space V. Then*

$$\dim \mathrm{Fix}(G,V) = \int_G \chi_V(g) = \langle \chi_V, 1 \rangle \tag{2.11}$$

where χ_V is the character of T, 1 is the character of the trivial representation of G and $\int_G$ denotes the normalised Haar integral on G.

Proof: See for example [12, Theorem XIII 2.3]. □

3 Steady-State and Hopf Bifurcations

Let G act on a real vector space V and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,\lambda)$$

be a G-equivariant bifurcation problem on V, that is, $f : V \times \mathbf{R} \to V$ is an one-parameter family of smooth maps satisfying

$$f(T(g)v,\lambda) = T(g)f(v,\lambda)$$

for all $g \in G$,

$$f(0,0) = 0\,.$$

For steady-state bifurcation in the presence of a symmetry group G, generically we can assume that G acts absolutely irreducibly on a real vector space V [12, Proposition XIII 3.2]. Moreover, if the action of G on V is non trivial it follows that $\mathrm{Fix}(G,V) = \{0\}$ and $f(0,\lambda) \equiv 0$.

When studying symmetric Hopf bifurcation, generically we can assume that V is a G-simple real vector space and thus there is a complex structure J on V [12, Proposition XVI 1.4]. The complex structure J induces a natural action of the circle group $\mathbf{S}^1$ on V that commutes with the action of G. Then one is naturally led to consider the representation theory of $G \times \mathbf{S}^1$.

More precisely, suppose that V is of the form $V = U \oplus U$ where U is an absolutely irreducible representation of G. Then $V = U \otimes \mathbf{C}$ and $G \times \mathbf{S}^1$ acts by

$$(g,\theta)(w \otimes z) = (gw) \otimes (e^{i\theta} z)$$

for $w \in W,\ z \in \mathbf{C},\ g \in G,\ \theta \in \mathbf{S}^1$. If we choose coordinates on U and identify it with $\mathbf{R}^m$ then $V = U \otimes \mathbf{C} \cong \mathbf{C}^m$. Now, G acts on $\mathbf{R}^m$ through $m \times m$ matrices with real entries hence this action can be extended to an action of G on $\mathbf{C}^m$; the circle $\mathbf{S}^1$ acts on $\mathbf{C}^m$ by

$$\theta \cdot (z_1,\ldots,z_m) = \left(e^{i\theta} z_1,\ldots,e^{i\theta} z_m\right)\,.$$

See Golubitsky and Stewart [11] (see also Golubitsky *et al.* [12, Chapter XVI]). If V carries a non-absolutely irreducible representation of G then there are two possibilities: V may be of *complex* or *quaternionic type* according to whether the G-endomorphism algebra $\mathcal{D} = \{A \in L(V) : gA = Ag \text{ for all } g \in G\}$ of V is isomorphic to $\mathbf{C}$ or $\mathbf{H}$. If V is of complex type there are two possible actions of $\mathbf{S}^1$, one is identified with multiplication by $e^{i\theta}$ and the other with

$e^{-i\theta}$. If V is of quaternionic type then any circle subgroup of the unit quaternions will do (all of them are conjugate). In other words, if V is a G-simple representation then V is also a complex irreducible representation of $G \times \mathbf{S}^1$.

We also consider the action of $G \times \mathbf{S}^1$ on $V \oplus \overline{V}$ where V is now regarded as a complex vector space. Note that the action of $\mathbf{S}^1$ on $V \oplus \overline{V}$ is given by

$$\theta \cdot (v_1, \overline{v_2}) \;=\; (e^{i\theta} v_1, e^{-i\theta}\overline{v_2})$$

for $v_1, v_2 \in V$ and $\theta \in \mathbf{S}^1$.

4 Counting Invariants and Equivariants

In this section we develop formulas for the dimensions of the vector spaces of polynomial functions of degree k that are invariant or equivariant with respect to the action of G or $G \times \mathbf{S}^1$.

Let V be a finite dimensional vector space over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. A function $f : V \to \mathbf{K}$ is called a *homogeneous polynomial of degree k* on V if there exists a $\mathbf{K}$-valued symmetric k-multi-linear function

$$\hat{f} : \underbrace{V \times \cdots \times V}_{k \text{ times}} \longrightarrow \mathbf{K}$$

such that $f(v) = \hat{f}(v, \ldots, v)$ for all $v \in V$. Denote by $L_s(V)$ the space of all $\mathbf{K}$-valued symmetric multi-linear functions and $\mathcal{P}_V^k$ the vector space of all homogeneous polynomials of degree k on V. Define

$$\mathcal{P}_V \;=\; \bigoplus_{k=0}^{\infty} \mathcal{P}_V^k \,.$$

Under the point-wise product $\mathcal{P}_V$ becomes a graded commutative algebra over $\mathbf{K}$. The mapping $\hat{f} \mapsto f$ is a natural isomorphism of graded commutative algebras $L_s(V) \to \mathcal{P}_V$; the inverse mapping which associates to each polynomial a $\mathbf{K}$-valued symmetric k-multi-linear function is called *polarisation*. Using the isomorphism (2.8) we have

$$\mathcal{P}_V^k \;\cong\; (S^k V)^* \,. \tag{4.1}$$

A map $F : V \to V$ is called a *homogeneous polynomial map of degree k* on V if there exists a V-valued symmetric k-multi-linear map

$$\hat{F} : \underbrace{V \times \cdots \times V}_{k \text{ times}} \longrightarrow V$$

such that $F(v) = \hat{F}(v, \ldots, v)$ for all $v \in V$. The vector space of all homogeneous polynomial maps of degree k on V is denoted by $\vec{\mathcal{P}}_V^k$. Define

$$\vec{\mathcal{P}}_V \;=\; \bigoplus_{k=0}^{\infty} \vec{\mathcal{P}}_V^k \, .$$

Since the product of a polynomial map by a polynomial function is again a polynomial map it follows that $\vec{\mathcal{P}}_V$ is a module over the ring of polynomial functions $\mathcal{P}_V$. The mapping $\hat{F} \mapsto F$ is a natural isomorphism of $\mathbf{K}$-vector spaces $L_s(V, V) \to \vec{\mathcal{P}}_V$ which is compatible with the homomorphism $L_s(V) \to \mathcal{P}_V$. Here $L_s(V, V)$ denotes the space of all V-valued symmetric multi-linear maps. Using the isomorphism (2.8) we have

$$\vec{\mathcal{P}}_V^k \;\cong\; (S^k V)^* \otimes V \, . \tag{4.2}$$

Example 4.1 There is a simple example to help understanding the three identifications

$$L_s^k(V) \;\cong\; \mathcal{P}_V^k \;\cong\; (S^k V)^* \, .$$

Let $k = 2$ and $\{e_i : i = 1, \ldots, n\}$ be a basis of V. Then for $x \in V$ we have

$$x \;=\; \sum_i x^i e_i \, .$$

The elements of $L_s^2(V)$ are the symmetric bilinear forms on V, that is,

$$B(x, y) \;=\; \sum_{i,j} a_{ij}\, x^i y^j \, ,$$

where $a_{ij} = a_{ji}$. The elements of $\mathcal{P}_V^2$ are the quadratic forms on V, that is,

$$Q(x) \;=\; \sum_i a_{ii}\, (x^i)^2 + \sum_{i<j} 2a_{ij}\, x^i x^j \, .$$

The isomorphism between $L_s^2(V)$ and $\mathcal{P}_V^2$ is given by $Q(x) = B(x, x)$ with inverse $B(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y))$.

The elements of $(S^2 V)^* = S^2(V^*)$ are the symmetric 2-tensors on V^*, that is,

$$T \;=\; \sum_{i,j} a_{ij}\, e^i \otimes e^j \, ,$$

where $a_{ij} = a_{ji}$ and $\{e^i : i = 1, \ldots, n\}$ is the dual basis on V^*. The isomorphism between $L_s^2(V)$ and $S^2(V^*)$ is given by $B(x, y) = \langle x \otimes y, T \rangle$. $\diamond$

Now suppose that T is a unitary or orthogonal representation of a compact Lie group G on V. A polynomial function $f : V \to \mathbf{K}$ is *invariant* under G if $f(T(g)v) = f(v)$ for all $g \in G, v \in V$. A polynomial mapping $F : V \to V$ is *equivariant* under G if $F(T(g)v) = T(g)F(v)$ for all $g \in G$, $v \in V$. The vector space $\mathcal{P}_V(G)$ of G-invariant polynomials is a sub-algebra of the algebra of all polynomial functions $\mathcal{P}_V$ on V and $\mathcal{P}_V^k(G) = \mathcal{P}_V(G) \cap \mathcal{P}_V^k$ is the vector space of homogeneous G-invariant polynomials of degree k. Under the isomorphism (4.1) we have

$$\mathcal{P}_V^k(G) \;\cong\; \mathrm{Fix}(G, (S^kV)^*)\,. \tag{4.3}$$

The space of G-equivariant polynomial mappings from V to V is a module over the ring $\mathcal{P}_V(G)$, and we denote it by $\vec{\mathcal{P}}_V(G)$. Similarly, the space of homogeneous G-equivariant polynomial maps from V to V of degree k is $\vec{\mathcal{P}}_V^k(G) = \vec{\mathcal{P}}_V(G) \cap \vec{\mathcal{P}}_V^k$. Under the isomorphism (4.2) we have

$$\vec{\mathcal{P}}_V^k(G) \;\cong\; \mathrm{Fix}(G, (S^kV)^* \otimes V)\,. \tag{4.4}$$

Now, we recall the use of character theory to compute the dimension of $\mathcal{P}_V^k(G)$ and $\vec{\mathcal{P}}_V^k(G)$ for an arbitrary representation of a compact Lie group G.

Theorem 4.2 *Let T be a unitary or orthogonal representation of a compact Lie group G on a finite-dimensional vector space V, and denote by χ the corresponding character and by $\chi_{(k)}$ the character of the induced action of G on the k-th symmetric power S^kV. Then:*

(i)

$$\dim \mathcal{P}_V^k(G) \;=\; \int_G \chi_{(k)}(g) \;=\; \langle \chi_{(k)}, 1 \rangle \tag{4.5}$$

where 1 *is the character of the trivial representation of G.*

(ii)

$$\dim_{\mathbf{R}} \vec{\mathcal{P}}_V^k(G) \;=\; \int_G \chi_{(k)}(g)\,\chi(g) \;=\; \langle \chi_{(k)}, \chi \rangle\,, \tag{4.6}$$

if V is a real vector space and G acts orthogonally and

(iii)

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}_V^k(G) \;=\; \int_G \chi_{(k)}(g)\,\overline{\chi(g)} \;=\; \langle \chi_{(k)}, \overline{\chi} \rangle\,, \tag{4.7}$$

if V is a complex vector space and G acts unitarilly.

Proof: See for example Sattinger [21, Theorem 5.10.]. The Trace Formula (2.11) applied to the isomorphism (4.3) yields

$$\dim \mathcal{P}_V^k(G) \;=\; \dim \mathrm{Fix}(G, (S^kV)^*) \;=\; \int_G \chi_{(k)}(g^{-1}) \;=\; \int_G \chi_{(k)}(g)$$

where the last equality follows from one of the properties (2.1) of the Haar integral.

For the equivariants, we use the isomorphism (4.4) and apply the Trace Formula (2.11):

$$\dim \vec{\mathcal{P}}^k_V(G) \;=\; \dim \mathrm{Fix}(G, (S^kV)^* \otimes V) \;=\; \int_G \chi_{(k)}(g^{-1})\, \chi(g)\,.$$

If V is real and the representation of G on V is orthogonal, it follows that $\chi(g^{-1}) = \chi(g)$ and then

$$\dim_{\mathbf{R}} \vec{\mathcal{P}}^k_V(G) \;=\; \int_G \chi_{(k)}(g)\, \chi(g)\,.$$

On the other hand, if V is complex and the representation of G on V is unitary, it follows that $\chi(g^{-1}) = \overline{\chi(g)}$ and then

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}^k_V(G) \;=\; \int_G \overline{\chi_{(k)}(g)}\, \chi(g) \;=\; \int_G \chi_{(k)}(g)\, \overline{\chi(g)}\,,$$

where the last equality follows from (2.9). □

There is a recursive formula for the character $\chi_{(k)}$ of G acting on S^kV:

$$k\, \chi_{(k)}(g) \;=\; \sum_{i=0}^{k-1} \chi(g^{k-i})\chi_{(i)}(g). \tag{4.8}$$

To prove this, let G act unitarilly on a finite-dimensional complex vector space, say $W \equiv \mathbf{C}^n$. Denote by T the representation, χ the corresponding character and $\chi_{(k)}$ the character of the induced action of G on the kth symmetric power S^kW. Fix $g \in G$. We have that $T(g)$ is diagonalisable. Suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $T(g)$. It follows then that

$$\chi_{(k)}(g) \;=\; \sum \lambda_1^{m_1} \ldots \lambda_n^{m_n}$$

where the sum is over all non-negative integers m_j satisfying $m_1 + \cdots + m_n = k$. We introduce the generating function

$$f(t) \;=\; \frac{1}{(1-\lambda_1 t)\cdots(1-\lambda_n t)}.$$

Observe that f is well defined for t in a sufficiently small neighbourhood of $t = 0$. Moreover, all the kth derivatives at $t = 0$ exist, and

$$\chi_{(k)}(g) \;=\; \frac{f^k(0)}{k!}.$$

By induction, it can be shown that

$$k\frac{f^k(0)}{k!} \;=\; \sum_{j=0}^{k-1} (\lambda_1^{k-j} + \cdots + \lambda_n^{k-j}) \frac{f^j(0)}{j!}$$

where $f^0(0) = 1$ and observe that

$$\chi(g^j) \;=\; \lambda_1^j + \cdots + \lambda_n^j.$$

Therefore

$$k\chi_{(k)}(g) \;=\; \sum_{j=0}^{k-1} \chi(g^{k-j})\frac{f^j(0)}{j!} \;=\; \sum_{j=0}^{k-1} \chi(g^{k-j})\chi_{(j)}(g).$$

For $k = 2, 3$ we obtain:

$$\chi_{(2)}(g) \;=\; \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right),$$

$$\chi_{(3)}(g) \;=\; \frac{1}{6}\left(\chi(g)^3 + 3\chi(g)\chi(g^2) + 2\chi(g^3)\right),$$

$$\chi_{(4)}(g) \;=\; \frac{1}{4!}\left(\chi(g)^4 + 3\chi(g^2)^2 + 6\chi(g)^2\chi(g^2) + 8\chi(g)\chi(g^3) + 6\chi(g^4)\right).$$

There is also an explicit but rather unwieldy expression for $\chi_{(k)}(g)$:

$$\chi_{(k)}(g) \;=\; \sum \frac{\chi^{i_1}(g)\chi^{i_2}(g^2)\cdots\chi^{i_k}(g^k)}{1^{i_1}i_1!\,2^{i_2}i_2!\cdots k^{i_k}i_k!}$$

where the sum is over all non-negative integers i_j satisfying $\sum_{j=1}^k j i_j = k$. See [21, p. 110].

Remark 4.3 A number of well known results regarding the number of invariants and equivariants in the absolutely (and nontrivial) irreducible case follow from Theorem 4.2 and the properties of characters in Section 2.5. There can be no invariant of degree 1, and there is only one equivariant of first degree, which is simply the identity mapping. There is a unique independent quadratic invariant, which in the orthogonal case is $\sum_{k=1}^m x_k^2$. This last result follows from the fact that $\int_G \chi_{(2)}(g) = 1$, which in turn follows from Theorem 2.2. ◇

The character formulas for the dimensions of invariants and equivariants are very convenient when G is a finite group, since they can be explicitly evaluated using GAP [9]. Examples of the application of these formulas are given in Section 7.

As a corollary of Theorem 4.2, we obtain:

Corollary 4.4 *Let V be a G-simple real representation and denote by χ the character of the complex representation of $G \times \mathbf{S}^1$ on V. Then:*

(i) Denoting by $\chi_{(k)}$ the character of k-th symmetric tensor power of V, we have

$$\dim_{\mathbf{C}} \mathcal{P}_V^k(G \times \mathbf{S}^1) \;=\; \int_{G\times\mathbf{S}^1} \chi_{(k)}(g,\theta), \qquad \dim_{\mathbf{C}} \vec{\mathcal{P}}_V^k(G \times \mathbf{S}^1) \;=\; \int_{G\times\mathbf{S}^1} \chi_{(k)}(g,\theta)\,\chi(g,\theta),$$

(ii) Denoting by $\chi^{\mathbf{R}}$ the character of $V^{\mathbf{R}}$ and by $\chi^{\mathbf{R}}_{(k)}$ the character of k-th symmetric tensor power of $V^{\mathbf{R}}$, we have

$$\dim_{\mathbf{R}} \mathcal{P}_{V^{\mathbf{R}}}^k(G \times \mathbf{S}^1) \;=\; \int_{G\times\mathbf{S}^1} \chi^{\mathbf{R}}_{(k)}(g,\theta), \qquad \dim_{\mathbf{R}} \vec{\mathcal{P}}_{V^{\mathbf{R}}}^k(G \times \mathbf{S}^1) \;=\; \int_{G\times\mathbf{S}^1} \chi^{\mathbf{R}}_{(k)}(g,\theta)\,\chi^{\mathbf{R}}(g,\theta)$$

When $V = U \oplus U$ for some absolutely irreducible representation U of G – which is the most common case in applications – the following observation leads to a simplification in the formulas. Let G act absolutely irreducibly on U and $\chi(g)$ be the corresponding character. If we write $\chi^R(g,\theta)$ for the real character of the action of $G \times \mathbf{S}^1$ on $V^R = U \oplus U$ then from (2.10) we have

$$\chi^R(g,\theta) \;=\; e^{i\theta}\chi(g) + e^{-i\theta}\chi(g) \;=\; 2\cos(\theta)\,\chi(g)\ . \tag{4.9}$$

It follows that the formulas of Corollary 4.4 can be written as an integral over G of an expression depending only on χ, the character of G.

Example 4.5 Applying the above formula for $\chi_{(2)}$ and using (4.9) we obtain

$$\begin{aligned}\dim_{\mathbf{R}} \mathcal{P}^2_{V^{\mathbf{R}}}(G \times \mathbf{S}^1) &= \dim \mathrm{Fix}_{\mathbf{R}}\left(G \times \mathbf{S}^1, S^2(V^{\mathbf{R}})\right)\\ &= \int_{G\times\mathbf{S}^1} \chi_{(2)}(g,\theta)\\ &= \frac{1}{2}\int_{\mathbf{S}^1} 4\cos^2(\theta)\int_G \chi(g)^2 + \frac{1}{2}\int_{\mathbf{S}^1} 2\cos(2\theta)\int_G \chi(g^2)\\ &= \int_G \chi(g)^2.\end{aligned}$$

The last equality follows from the fact that $\int_{\mathbf{S}^1}\cos(2\theta) = 0$ and $\int_{\mathbf{S}^1}\cos^2(\theta) = 1/2$. ◇

The above result was obtained by Montaldi *et al.* [19], together with formulas for the cases $k = 4$ and $k = 6$. In Theorem 4.6 below, we generalise these results to the case of arbitrary k, and also give the corresponding formula for equivariants.

The most convenient way to write invariants and equivariants for Hopf bifurcation is to use coordinates adapted to the circle action. Let $\{v_1, \ldots, v_m\}$ be a basis of V (over $\mathbf{C}$).

Then $\{v_1, \dots, v_m; iv_1, \dots, iv_m\}$ is a basis of V over $\mathbf{R}$. Denote by $\{x_1, \dots, x_m; y_1, \dots, y_m\}$ the coordinates of a vector $v \in V$ relative to this basis and let $z_j = x_j + iy_j$ for $j = 1, \dots, m$. Then $x_j = (z_j + \overline{z}_j)/2,\ y_j = -i(z_j - \overline{z}_j)/2$ and

$$v = \sum_{j=1}^{m} (x_j + iy_j) v_j = \sum_{j=1}^{m} z_j v_j .$$

Thus any polynomial f on V can be written either as a linear combination of monomials which are products of powers of the real coordinates x_j and y_j, or as a linear combination of monomials which are products of powers of the complex coordinates z_j and $\overline{z}_j$. To be more precise, let us write $\overline{z} = (\overline{z}_1, \dots, \overline{z}_m)$ and using multi-indices, any polynomial function $f : V^{\mathbf{R}} \to \mathbf{C}$ can be written as

$$f(z, \overline{z}) = \sum_{\alpha,\beta} a_{\alpha\beta} z^\alpha \overline{z}^\beta \tag{4.10}$$

where $\alpha, \beta \in (\mathbf{Z}_0^+)^m$, $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_m^{\alpha_m}$ and the coefficients $a_{\alpha\beta}$ may be required to be complex.

Consider the action of $\mathbf{S}^1$ on $V \oplus \overline{V}$ as before, given by

$$\theta\,(v_1, \overline{v_2}) = \left(e^{i\theta} v_1, e^{-i\theta} \overline{v_2}\right) .$$

Note that $V^{\mathbf{R}}$ is the subspace of $V \oplus \overline{V}$ such that $v_1 = v_2$ and it is invariant under the action of $G \times \mathbf{S}^1$. Now it follows that if f is $\mathbf{S}^1$-invariant then for each α, β such that $a_{\alpha\beta} \neq 0$ we must have $|\alpha| = |\beta|$ (where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$). Therefore f has even degree in $z, \overline{z}$. Similarly, if $g : V^{\mathbf{R}} \to V^{\mathbf{R}}$ has components

$$g_j(z, \overline{z}) = \sum_{\alpha,\beta} b_{\alpha\beta} z^\alpha \overline{z}^\beta$$

then the $\mathbf{S}^1$-equivariance is equivalent to have $|\alpha| = |\beta| + 1$ if $b_{\alpha\beta} \neq 0$. This is [12, Lemma XVI 9.3]. Therefore g has odd degree components in $z, \overline{z}$.

Theorem 4.6 *Let U be a finite-dimensional absolutely irreducible representation of G and denote by χ the corresponding character. Let $\mathbf{S}^1$ act on $\mathbf{C}$ by*

$$\theta \cdot z = e^{i\theta} z \quad (\theta \in \mathbf{S}^1,\ z \in \mathbf{C}).$$

and $V = U^{\mathbf{C}}$. Then

$$\dim_{\mathbf{R}} \mathcal{P}_{V^{\mathbf{R}}}^{2k}(G \times \mathbf{S}^1) = \int_G \chi_{(k)}(g)^2, \qquad \dim_{\mathbf{C}} \vec{\mathcal{P}}_{V^{\mathbf{R}}}^{2k+1}(G \times \mathbf{S}^1) = \int_G \chi_{(k+1)}(g)\, \chi_{(k)}(g)\, \chi(g) .$$

Proof: We know that the space $\mathcal{P}^k_{V^{\mathbf{R}}}(G\times\mathbf{S}^1)$ can be identified with the real vector space $\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^k\left(V^{\mathbf{R}}\right)\right)^*\right)$. Since $V\oplus\overline{V}=U^{\mathbf{C}}\oplus\overline{U^{\mathbf{C}}}$ and $V^{\mathbf{R}}=U\oplus U$ we have

$$V\oplus\overline{V} \;=\; (U\oplus U)^{\mathbf{C}}$$

as representations of $G\times\mathbf{S}^1$. Therefore,

$$\begin{aligned}\dim_{\mathbf{R}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^k\left(V^{\mathbf{R}}\right)\right)^*\right) &= \dim_{\mathbf{R}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^k(U\oplus U)\right)^*\right)\\ &= \dim_{\mathbf{C}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^k\left(V\oplus\overline{V}\right)\right)^*\right).\end{aligned}$$

Recall that a polynomial on V is invariant under $G\times\mathbf{S}^1$ if and only if it is invariant under G and $\mathbf{S}^1$. Note that

$$S^{2k}(V\oplus\overline{V}) \;\cong\; \bigoplus_{a=0}^{2k} S^aV\otimes S^{2k-a}\overline{V}$$

and from the $\mathbf{S}^1$ action, it follows that

$$\mathrm{Fix}\left(\mathbf{S}^1,\left(S^{2k}\left(V\oplus\overline{V}\right)\right)^*\right) \;\cong\; \left(S^kV\otimes S^k\overline{V}\right)^*.$$

Therefore

$$\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^{2k}\left(V\oplus\overline{V}\right)\right)^*\right) \;\cong\; \mathrm{Fix}\left(G,\left(S^kV\otimes S^k\overline{V}\right)^*\right).$$

Using the Trace Formula (2.11) we have

$$\dim_{\mathbf{C}}\mathrm{Fix}\left(G,\left(S^kV\otimes S^k\overline{V}\right)^*\right) \;=\; \int_G \overline{\chi}_{(k)}(g)\chi_{(k)}(g)\;. \tag{4.11}$$

Now, χ is also the character of the representation of G on U and hence it is real valued. Thus

$$\dim_{\mathbf{R}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^kV^{\mathbf{R}}\right)^*\right) \;=\; \int_G \chi_{(k)}(g)^2\;. \tag{4.12}$$

For the equivariants, we have

$$\vec{\mathcal{P}}^{2k+1}_{V^{\mathbf{R}}}(G\times\mathbf{S}^1) \;\cong\; \mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^{2k+1}\left(V^{\mathbf{R}}\right)\right)^*\otimes V\right)\;.$$

As before

$$\begin{aligned}\dim_{\mathbf{C}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^{2k+1}\left(V^{\mathbf{R}}\right)\right)^*\otimes_{\mathbf{R}}V\right) &= \dim_{\mathbf{C}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^{2k+1}(U\oplus U)\right)^*\otimes_{\mathbf{R}}V\right)\\ &= \dim_{\mathbf{C}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^{2k+1}\left(V\oplus\overline{V}\right)\right)^*\otimes_{\mathbf{C}}V\right)\end{aligned}$$

and

$$S^{2k+1}\left(V\oplus\overline{V}\right) \;\cong\; \bigoplus_{a=0}^{2k+1} S^aV\otimes_{\mathbf{C}} S^{2k+1-a}\overline{V}\;.$$

Moreover, from the $\mathbf{S}^1$-equivariance it follows that

$$\mathrm{Fix}\left(\mathbf{S}^1, \left(S^{2k+1}\left(V \oplus \overline{V}\right)\right)^* \otimes_{\mathbf{C}} V\right) \;\cong\; (S^{k+1}V^* \otimes_{\mathbf{C}} S^k\overline{V}^*) \otimes_{\mathbf{C}} V$$

and so by (2.7)

$$\mathrm{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V \oplus \overline{V}\right)\right)^* \otimes_{\mathbf{C}} V\right) \;\cong\; \mathrm{Fix}\left(G, S^{k+1}\overline{V} \otimes_{\mathbf{C}} S^kV \otimes_{\mathbf{C}} V\right)$$

Again, by the Trace Formula we obtain

$$\dim_{\mathbf{C}} \mathrm{Fix}(G, S^{k+1}\overline{V} \otimes_{\mathbf{C}} S^kV \otimes_{\mathbf{C}} V) \;=\; \int_G \overline{\chi}_{(k+1)}(g)\chi_{(k)}(g)\chi(g)\;. \tag{4.13}$$

Since χ is also the character of the representation of G on U, it is real valued and we have

$$\dim_{\mathbf{C}} \mathrm{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V^{\mathbf{R}}\right)\right)^* \otimes_{\mathbf{R}} V\right) \;=\; \int_G \chi_{(k+1)}(g)\,\chi_{(k)}(g)\,\chi(g) \tag{4.14}$$

since the polynomials are complex valued. □

See Table 1 for the dimensions of the spaces of invariants and equivariants of lower degrees for $G \times \mathbf{S}^1$ on $V^{\mathbf{R}}$.

k	$\dim_{\mathbf{R}} \mathcal{P}^k_{V^{\mathbf{R}}}(G \times \mathbf{S}^1)$	k	$\dim_{\mathbf{C}} \vec{\mathcal{P}}^k_{V^{\mathbf{R}}}(G \times \mathbf{S}^1)$
2	$\int_G \chi(g)^2$	1	$\int_G \chi(g)^2$
4	$\int_G \chi_{(2)}(g)^2$	3	$\int_G \chi_{(2)}(g)\chi(g)^2$
6	$\int_G \chi_{(3)}(g)^2$	5	$\int_G \chi_{(3)}(g)\chi_{(2)}(g)\chi(g)$

Table 1: Dimension of space of invariants and equivariants of degree k for $G \times \mathbf{S}^1$ on $V^{\mathbf{R}}$.

Proposition 4.7 *Let U be an absolutely irreducible representation of G with corresponding character χ and $V = U \otimes \mathbf{C}$ the irreducible representation of $G \times \mathbf{S}^1$. Then for each non-trivial linear character λ of G the dimensions of the spaces of invariants and equivariants of the irreducible representation corresponding to $\lambda\chi$ are equal to $\dim_{\mathbf{R}} \mathcal{P}^k_{V^{\mathbf{R}}}(G \times \mathbf{S}^1)$ and $\dim_{\mathbf{C}} \vec{\mathcal{P}}^k_{V^{\mathbf{R}}}(G \times \mathbf{S}^1)$, respectively.*

Proof: Let λ be a nontrivial linear character of G and let $\phi = \lambda\chi$. Let $k \geq 1$. Since λ is a linear character, we have

$$\phi_{(k)}(g) \;=\; \lambda(g)^k \chi_{(k)}(g).$$

Thus

$$\phi_{(k)}^2 \;=\; \lambda^{2k}\chi_{(k)}^2 \;=\; \chi_{(k)}^2, \qquad \phi_{(k+1)}\,\phi_{(k)}\,\phi \;=\; \lambda^{2k+2}\chi_{(k+1)}\,\chi_{(k)}\,\chi \;=\; \chi_{(k+1)}\,\chi_{(k)}\,\chi$$

since $\lambda(g) = \pm 1$ and so λ^{2k} is the trivial character. Hence we have shown that

$$\int_G \chi_{(k)}(g)^2 \;=\; \int_G \phi_{(k)}(g)^2, \quad \int_G \chi_{(k+1)}(g)\,\chi_{(k)}(g)\,\chi(g) \;=\; \int_G \phi_{(k+1)}(g)\,\phi_{(k)}(g)\,\phi(g)\;.$$

□

5 Hilbert-Poincaré Series

We review Hilbert-Poincaré series for the rings of invariants and modules of equivariants for general compact Lie groups, before presenting new results for Hopf bifurcation in Section 6. The original definition of Hilbert-Poincaré series is for complex representations. In this paper we are interested in real representations. As we explain (see Remark 5.1 below) the 'real' and 'complex' Hilbert-Poincaré series are the same.

Let G be a compact Lie group acting on $V = \mathbf{R}^m$. Without loss of generality, we can assume that G acts orthogonally and linearly on V, so that any $g \in G$ acts as an orthogonal matrix M_g with real entries. Moreover, we can view it as a matrix acting on $V^{\mathbf{C}} = \mathbf{C}^m$. If $(x_1, \ldots, x_m)$ denote real coordinates on $\mathbf{R}^m$, $x_j \in \mathbf{R}$, then we obtain complex coordinates on $\mathbf{C}^m$ by permitting the x_j to be complex. Note that there is a natural inclusion

$$\mathbf{R}[x_1, \ldots, x_m] \;\subseteq\; \mathbf{C}[x_1, \ldots, x_m]$$

where these are the rings of polynomials in the x_j with coefficients in $\mathbf{R}$, $\mathbf{C}$ respectively.

Remark 5.1 Every real-valued G-invariant in $\mathbf{R}[x_1, \ldots, x_m]$ is also a complex-valued G-invariant in $\mathbf{C}[x_1, \ldots, x_m]$. Conversely, the real and imaginary parts of a complex valued invariant are real invariants (because the matrices M_g have real entries). Therefore a basis over $\mathbf{R}$ for the real vector space of degree k real-valued invariants is also a basis over $\mathbf{C}$ for the complex vector space of degree k $\mathbf{C}$-valued invariants. Similar remarks apply to the equivariants. ◇

We suppose now that V is a m-dimensional vector space over $\mathbf{C}$, where $x_1, \ldots, x_m$ denote coordinates relative to a basis for V, and $G \subseteq \mathrm{GL}(V)$ is a compact Lie group acting on V. Let $\mathcal{P}_V(G)$ denote the sub-algebra of $\mathbf{C}[x_1, \ldots, x_m]$ formed by the invariant polynomials under G (over $\mathbf{C}$). Note that $\mathbf{C}[x_1, \ldots, x_m]$ is graded:

$$\mathbf{C}[x_1, \ldots, x_m] \ = \ R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

where R_k consists of all homogeneous polynomials of degree k. Now observe that if $f(x) \in R_k$ for some k then $f(gx) \in R_k$ for all $g \in G$. Therefore the space $\mathcal{P}_V(G)$ has the structure

$$\mathcal{P}_V(G) \ = \ \mathcal{P}_V^0(G) \oplus \mathcal{P}_V^1(G) \oplus \mathcal{P}_V^2(G) \oplus \cdots$$

of a graded $\mathbf{C}$-algebra given by $\mathcal{P}_V^k(G) = R_k \cap \mathcal{P}_V(G)$.

The *Hilbert-Poincaré series* of the graded algebra $\mathcal{P}_V(G)$ is a generating function for the dimension of the vector space of invariants at each degree defined by

$$\Phi_G(t) \ = \ \sum_{d=0}^{\infty} (\dim \mathcal{P}_V^d(G)) t^d.$$

Throughout we denote by g the linear transformation corresponding to the action of $g \in G$ on V.

Consider the normalised Haar measure μ_G defined on G and denote by $\int_G f$ the integral with respect to μ_G of a continuous function f defined on G. Molien's Theorem gives an explicit formula for Φ_G:

Theorem 5.2 (Molien's Theorem) *Let G be a compact Lie group acting on V. Then the Hilbert-Poincaré series of $\mathcal{P}_V(G)$ is*

$$\Phi_G(t) \ = \ \int_G \frac{1}{\det(1 - gt)}$$

Proof: See Molien [18] for the original proof of the finite case, and Sattinger [21] for the extension to a compact group. □

If G is finite, the Molien formula for the Hilbert-Poincaré series of $\mathcal{P}_V(G)$ is

$$\Phi_G(t) \ = \ \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gt)}.$$

Equivariants can be interpreted as invariants with respect to a different action of G on a different space. See Section 4 for details. The *Hilbert series* for the graded module $\vec{\mathcal{P}}_V(G)$ over the ring $\mathcal{P}_V(G)$ is the generating function

$$\Psi_G(t) \ = \ \sum_{d=0}^{\infty} \dim(\vec{\mathcal{P}}_V^d(G)) t^d$$

and an explicit formula for Ψ_G is given by:

Theorem 5.3 (Equivariant Molien Theorem) *Let G be a compact Lie group with an action defined on V. Then the Hilbert-Poincaré series of the module $\vec{\mathcal{P}}_V(G)$ over $\mathcal{P}_V(G)$ is*

$$\Psi_G(t) \;=\; \int_G \frac{\chi(g^{-1})}{\det(1-gt)}$$

where χ is the character for the G action on V.

Proof: See Sattinger [21]. □

Observe that if the action of G on V is orthogonal then $g^{-1} = g^t$ and $\chi(g^{-1}) = \chi(g)$.

6 Hilbert-Poincaré Series for Hopf Bifurcation

In Section 5 we reviewed Hilbert-Poincaré series that consist of a generating function for counting the number of invariant real polynomials in a real representation or the number of invariant complex polynomials in a complex representation, as a function of their degree. For the equivariants, the situation is analogous.

We are now interested in counting G-invariant real polynomials in a complex representation V of a group G. Therefore, the application of the Molien Theorem of Section 5 supposes the choice of a basis for V (as a real vector space) and to take the corresponding action of G on this basis. The disadvantage of this approach is that one no longer have coordinates adapted to the $\mathbf{S}^1$-action.

In order to avoid this problem, we modify the Poincaré series and the Molien function. That is, we complexify the problem as described in Section 3. Taking $z, \overline{z}$ coordinates, we obtain that any polynomial p on V can be written as a linear combination of monomials which are products of powers of the complex coordinates z_j and $\overline{z}_j$. For $z = (z_1, \ldots, z_m) \in \mathbf{C}^m$, and $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_m)$, using multi-indices, any polynomial function $p: \mathbf{C}^m \to \mathbf{R}$ can be written as

$$p(z, \overline{z}) \;=\; \sum_{\alpha,\beta} a_{\alpha\beta} z^\alpha \overline{z}^\beta \tag{6.1}$$

where the coefficients $a_{\alpha\beta}$ may be required to be complex. Here $\alpha, \beta \in (\mathbf{Z}_0^+)^m$. Moreover,

$$p(z, \overline{z}) \;=\; \sum_{k\geq 0} p_k(z, \overline{z})$$

where

$$\begin{array}{rcl} p_k(z,\overline{z}) & = & \sum_{q+r=k} p_{q,r}(z,\overline{z}) \\ p_{q,r}(z,\overline{z}) & = & \sum_{|\alpha|=q,\,|\beta|=r} a_{\alpha\beta}\, z^{\alpha}\overline{z}^{\beta}\ . \end{array}$$

The polynomial $p_{q,r}(z,\overline{z})$ is homogeneous of degree k (if $q+r=k$) and of *bidegree* (q,r).

These decompositions preserve the invariance under the group: the polynomial function p is invariant if and only if each p_k is invariant. Moreover, p_k is invariant if and only if each $p_{q,r}$ with $q+r=k$ is invariant. Denoting by $c_{q,r}$ the dimension of the space of *real* G-invariants of bidegree (q,r), Forger [7] defines the generating function of two variables

$$\Phi_G\,(z,\overline{z}) \;=\; \sum_{q,r\,=0}^{\infty} c_{q,r} z^q \overline{z}^r$$

and obtains the following integral form:

Theorem 6.1 ([7]) *Let G be a compact Lie group acting on a complex vector space V. Then the bigraded Hilbert-Poincaré series of $\mathcal{P}_V(G)$ is*

$$\Phi_G\,(z,\overline{z}) \;=\; \int_G \frac{1}{\det(1-gz)\det(1-\overline{gz})}$$

where $\int_G$ is the normalised Haar integral on G.

Proof: See Forger [7]. □

Denoting by $e_{q,r}$ the *complex* dimension of the space of G-equivariants with homogeneous polynomial components of bidegree (q,r), we define the generating function of two variables

$$\Psi_G\,(z,\overline{z}) \;=\; \sum_{q,r\,=0}^{\infty} e_{q,r} z^q \overline{z}^r$$

and we obtain the following integral form:

Theorem 6.2 *Let G be a compact Lie group acting on a complex vector space V. Then the bigraded Hilbert-Poincaré series of $\vec{\mathcal{P}}_V(G)$ is*

$$\Psi_G\,(z,\overline{z}) = \int_G \frac{\chi\big(g^{-1}\big)}{\det(1-gz)\det\big(1-\overline{gz}\big)}$$

where $\int_G$ is the normalised Haar integral on G and χ is the character of the (complex) representation of G on V.

Proof: The proof is similar to that for the usual Molien function [21, pages 109-111]. Consider the space $V \oplus \overline{V}$ and since

$$\dim_{\mathbf{C}} V \oplus \overline{V} \;=\; \dim_{\mathbf{R}} V^{\mathbf{R}}$$

we have

$$\dim_{\mathbf{C}} \left(S^q V \otimes S^r \overline{V}\right)^* \otimes V \;=\; \dim_{\mathbf{C}} \vec{\mathcal{P}}_V^{q,r}.$$

that is, the complex valued mappings from V to V with polynomial components homogeneous of bidegree (q,r) are in one-to-one correspondence with $\left(S^q V \otimes S^r \overline{V}\right)^* \otimes V$. Moreover, the action of the group G on V induces an action of G on $\left(S^q V \otimes S^r \overline{V}\right)^* \otimes V$ and therefore the G-equivariant mappings from V to V with polynomial components homogeneous of bidegree (q,r) are in one-to-one correspondence with the G-invariant elements of $\left(S^q V \otimes S^r \overline{V}\right)^* \otimes V$ under this induced action.

By the Trace Formula we have

$$\dim_{\mathbf{C}} \mathrm{Fix}\left(G, \left(S^q V \otimes S^r \overline{V}\right)^* \otimes V\right) \;=\; \int_G \overline{\chi}_{(q,r)}(g)\chi(g) \;=\; \int_G \chi_{(q,r)}(g)\overline{\chi}(g)$$

where $\chi_{(q,r)}$ is the character of the induced action of G on $S^q V \otimes S^r \overline{V}$.

The rest of the proof consists in calculating the character $\chi_{(q,r)}$ and we follow [7]. Fix $g \in G$ and as before let g denote the linear transformation corresponding to the action of $g \in G$ on V. Since g is an unitary matrix it can be diagonalized. Suppose that V has complex dimension m, and let $w_1, \ldots, w_m$ be a basis of V consisting of eigenvectors of g, with eigenvalues $\lambda_1, \ldots, \lambda_m$. The monomials $z^\alpha \overline{z}^\beta$ where $|\alpha| = q$ and $|\beta| = r$ form a basis of the space of homogeneous polynomials on $V \oplus \overline{V}$ of bidegree (q,r). Moreover, they correspond to the eigenvectors associated with the eigenvalues $\lambda^\alpha \overline{\lambda}^\beta$ of the induced action of G. Here we use multi-indice notation for λ and $\overline{\lambda}$. Recall that $\chi_{(p,q)}$ is the character of the representation of G on the space of polynomials on $V \oplus \overline{V}$ of bidegree (q,r). We obtain

$$\chi_{(q,r)}(g) \;=\; \sum_{|\alpha|=q,\, |\beta|=r} \lambda^\alpha \overline{\lambda}^\beta$$

In what follows, we use $z\lambda$ to denote $(z\lambda_1, \ldots, z\lambda_m)$ and $\overline{z\,\lambda}$ to denote $\left(\overline{z\,\lambda}_1, \ldots, \overline{z\,\lambda}_m\right)$. Multiplying by $z^q \overline{z}^r$ and summing over q and r, we obtain the formal power series

$$\begin{aligned}
\sum_{q,r=0}^{\infty} \chi_{q,r}(g) z^q \overline{z}^r \;&=\; \sum_{q,r=0}^{\infty} \sum_{|\alpha|=q,\, |\beta|=r} (z\lambda)^\alpha \left(\overline{z\,\lambda}\right)^\beta \\
&=\; \prod_{j=1}^{m} \frac{1}{(1 - z\lambda_j)} \prod_{j=1}^{m} \frac{1}{\left(1 - \overline{z\,\lambda_j}\right)} \\
&=\; \frac{1}{\det(1 - zg)} \frac{1}{\det\left(1 - \overline{z\,g}\right)}\,.
\end{aligned}$$

Finally, multiplying by $\overline{\chi}$ and using the Trace Formula we obtain the result. □

Theorem 6.3 *Let G act absolutely irreducibly on a finite-dimensional real vector space V, and let $\mathbf{S}^1$ act on $\mathbf{C}$ by*

$$\theta \cdot z \;=\; e^{i\theta} z \quad (\theta \in \mathbf{S}^1,\; z \in \mathbf{C})$$

Consider the tensor product representation of $G \times \mathbf{S}^1$ on $V \otimes_{\mathbf{R}} \mathbf{C}$. Then the bigraded Hilbert-Poincaré series for $\mathcal{P}_{V\otimes_{\mathbf{R}}\mathbf{C}}(G \times \mathbf{S}^1)$ and for $\vec{\mathcal{P}}_{V\otimes\mathbf{C}}(G \times \mathbf{S}^1)$ are given by

$$\Phi_{G\times\mathbf{S}^1}(z,\overline{z}) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \Phi_G\left(e^{i\theta}z, e^{-i\theta}\overline{z}\right) \mathrm{d}\theta$$

and

$$\Psi_{G\times\mathbf{S}^1}(z,\overline{z}) \;=\; \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}\Psi_G\left(e^{i\theta}z, e^{-i\theta}\overline{z}\right) \mathrm{d}\theta \;.$$

where Φ_G and Ψ_G are the bigraded Hilbert-Poincaré series for $\mathcal{P}_{V\otimes_{\mathbf{R}}\mathbf{C}}(G)$ and $\vec{\mathcal{P}}_{V\otimes_{\mathbf{R}}\mathbf{C}}(G)$, respectively.

Proof: Given $(g,\theta) \in G \times \mathbf{S}^1$, we have

$$\det(1-(g,\theta)z) \;=\; \det\left(1 - g\left(e^{i\theta}z\right)\right)$$

and

$$\chi\big((g,\theta)^{-1}\big) \;=\; e^{-i\theta}\chi\left(g^{-1}\right) \;.$$

Applying Theorems 6.1, 6.2 and using the fact that the normalised Haar measure on the circle group $\mathbf{S}^1$ is $\frac{1}{2\pi}\mathrm{d}\theta$, we obtain the above formulas. □

Example 6.4 Consider the symmetry group $\mathbf{D}_4$ of the square. The group is generated by the permutations $g = (1234)$ and $\kappa = (12)(34)$ and the conjugacy classes are $\{e\}$, $\{g^2\}$, $\{g, g^3\}$, $\{\kappa, g^2\kappa\}$ and $\{g\kappa, g^3\kappa\}$.

A two-dimensional representation of $\mathbf{D}_4$ is obtained by considering the standard action of $\mathbf{D}_4$ as rotations and reflections in the plane: let $T(g)$ denote the matrix for the rotation through $2\pi/4$ and $T(\kappa)$ the matrix of the reflection in the y-axis. Thus

$$T(g) \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T(\kappa) \;=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that T acts on $V = \mathbf{R}^2$ irreducibly. The Hilbert-Poincaré series for $\mathcal{P}_V(\mathbf{D}_4)$ and $\vec{\mathcal{P}}_V(\mathbf{D}_4)$, are:

$$\begin{aligned}\Phi_{\mathbf{D}_4}(t) &= \frac{1}{8}\left(\frac{1}{(1-t)^2}+\frac{1}{(1+t)^2}+\frac{2}{1+t^2}+\frac{4}{1-t^2}\right)\\ \Psi_{\mathbf{D}_4}(t) &= \frac{1}{8}\left(\frac{2}{(1-t)^2}-\frac{2}{(1+t)^2}\right)\end{aligned}$$

Consider now the action of $\mathbf{D}_4 \times \mathbf{S}^1$ on $V = \mathbf{C}$ where $\theta \in \mathbf{S}^1$ acts on $\mathbf{C}$ by multiplication by $e^{i\theta}$. The bigraded Hilbert-Poincaré series for $\mathcal{P}_{V\otimes\mathbf{C}}(\mathbf{D}_4)$ and $\vec{\mathcal{P}}_{V\otimes\mathbf{C}}(\mathbf{D}_4)$ are:

$$\begin{aligned}\Phi_{\mathbf{D}_4}(z,\overline{z}) &= \frac{1}{8}\left(\frac{1}{(1-z)^2(1-\overline{z})^2}+\frac{1}{(1+z)^2(1+\overline{z})^2}\right.\\ &\quad\left.+\frac{2}{(1+z^2)(1+\overline{z}^2)}+\frac{4}{(1-z^2)(1-\overline{z}^2)}\right)\\ \Psi_{\mathbf{D}_4}(z,\overline{z}) &= \frac{1}{8}\left(\frac{2}{(1-z)^2(1-\overline{z})^2}-\frac{2}{(1+z)^2(1+\overline{z})^2}\right)\end{aligned}$$

Then the bigraded Hilbert-Poincaré series for $\mathbf{D}_4 \times \mathbf{S}^1$ are:

$$\begin{aligned}\Phi_{\mathbf{D}_4\times\mathbf{S}^1}(z,\overline{z}) &= \int_{\mathbf{S}^1}\Phi_{\mathbf{D}_4}\left(e^{i\theta}z, e^{-i\theta}\overline{z}\right)\\ &= 1+z\overline{z}+3z^2\overline{z}^2+4z^3\overline{z}^3+7z^4\overline{z}^4+\cdots\\ \Psi_{\mathbf{D}_4\times\mathbf{S}^1}(z,\overline{z}) &= \int_{\mathbf{S}^1}e^{-i\theta}\Psi_{\mathbf{D}_4}\left(e^{i\theta}z, e^{-i\theta}\overline{z}\right)\\ &= z+3z^2\overline{z}+6z^3\overline{z}^2+10z^4\overline{z}^3+\cdots\end{aligned}$$

$\diamond$

Hence, the numbers of independent invariants of degree $(2,4,6,8)$ are $(1,3,4,7)$, and the numbers of independent equivariants of degree $(3,5,7)$ are $(3,6,10)$. These are all in agreement with the results obtained using the character formulas (4.12), (4.14) given in Table 4 below.

7 Examples: Finite Groups

In this section we first go through an example of the calculation of the dimensions of the spaces of invariants and equivariants using characters for a particular group action, and then summarise the results obtained computationally for several other groups.

Class	1	(12)	(123)	(12)(34)	(1234)
|Class|	1	6	8	3	6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

Table 2: Character table of S_4. The rows are indexed by the irreducible characters of S_4 and the columns are indexed by the conjugacy class representatives.

Class	1	(12)	(123)	(12)(34)	(1234)
|Class|	1	6	8	3	6
$\chi(g)$	3	1	0	-1	-1
$\chi(g^2)$	3	3	0	3	-1
$\chi(g^3)$	3	1	3	-1	-1
$\chi(g^4)$	3	3	0	3	3
$\chi_{(2)}(g)$	6	2	0	2	0
$\chi_{(3)}(g)$	10	2	1	-2	0
$\chi_{(4)}(g)$	15	3	0	3	1

Table 3: Irreducible character $\chi(g)$ and derived characters $\chi(g^k)$ and $\chi_{(k)}(g)$ for the group S_4, for the natural character labelled χ_4 in Table 2.

Example 7.1 Consider the permutation group S_4, which has five conjugacy classes containing elements of the same cycle type. The character table for S_4 is given in Table 2. The group acts on $\mathbf{R}^3$ via the 'natural' irreducible representation in which the character $\chi(g)$ for each class is obtained by subtracting one from the number of elements fixed by each permutation; this character is denoted by χ_4 in Table 2. From this character we can find the characters $\chi(g^k)$ and $\chi_{(k)}(g)$. These are listed in Table 3 for $k \leq 4$. From the information in that table it is then possible to calculate the dimensions of the spaces of invariants and equivariants for stationary and Hopf bifurcation, by finding the appropriate sums using Theorems 4.2 and 4.6 respectively.

Since χ is irreducible of real type there is only one quadratic invariant (we refer to 'number of invariants' as an abbreviation for the dimension of the space of invariants). From (4.5), the number of cubic invariants is

$$I(3) \;=\; \frac{1}{24}\sum_g \chi_{(3)}(g) \;=\; \frac{1}{24}(10-6+12+8) \;=\; 1$$

and the number of quartic invariants is

$$I(4) \;=\; \frac{1}{24}\sum_g \chi_{(4)}(g) \;=\; \frac{1}{24}(15+6+9+18) \;=\; 2\;.$$

The number of quadratic equivariants is, using (4.6),

$$E(2) \;=\; \frac{1}{24}\sum_g \chi_{(2)}(g)\chi(g) \;=\; \frac{1}{24}(18-6+12) \;=\; 1$$

and the numbers of cubic and quartic equivariants are

$$E(3) \;=\; \frac{1}{24}\sum_g \chi_{(3)}(g)\chi(g) \;=\; 2, \qquad E(4) \;=\; \frac{1}{24}\sum_g \chi_{(4)}(g)\chi(g) \;=\; 2\;.$$

For the case of Hopf bifurcation, there is one quadratic invariant and the numbers of invariants of degree 4, 6, 8 are found from (4.12) to be

$$I_H(4) \;=\; \frac{1}{24}\sum_g \chi_{(2)}(g)^2 \;=\; 3, \quad I_H(6) \;=\; \frac{1}{24}\sum_g \chi_{(3)}(g)^2 \;=\; 6,$$

$$I_H(8) \;=\; \frac{1}{24}\sum_g \chi_{(4)}(g)^2 \;=\; 13\;.$$

The numbers of equivariants of degree 3, 5, 7 are, using (4.14),

$$E_H(3) \;=\; \frac{1}{24}\sum_g \chi_{(2)}(g)\chi(g)^2 \;=\; 3, \quad E_H(5) \;=\; \frac{1}{24}\sum_g \chi_{(3)}(g)\chi_{(2)}(g)\chi(g) \;=\; 9,$$

$$E_H(7) \;=\; \frac{1}{24}\sum_g \chi_{(4)}(g)\chi_{(3)}(g)\chi(g) \;=\; 21\;.$$

$\diamond$

Note that the group S_4 has another three-dimensional irreducible representation in which the character $\chi_5(g)$ is the same as $\chi(g)$ except for a sign change of the elements (1234) and (12). See Table 2 where $\chi = \chi_4$ and $\chi_5 = \chi_2\chi_4$. Hence by Proposition 4.7, the numbers of invariants and equivariants for Hopf bifurcation in this representation are the same as those given above. Hopf bifurcation in this representation, which arises from the symmetries of

G	D	$I(3)$	$I(4)$	$I(5)$	$E(2)$	$E(3)$	$E(4)$	$E(5)$	$I_H(4)$	$I_H(6)$	$I_H(8)$	$E_H(3)$	$E_H(5)$	$E_H(7)$
S_3	2	1	1	1	1	1	2	2	2	3	5	2	4	7
S_4	3	1	2	1	1	2	2	4	3	6	13	3	9	21
S_5	4	1	2	2	1	2	3	4	3	7	19	3	11	33
S_6	5	1	2	2	1	2	3	5	3	8	24	3	12	41
A_4	3	1	2	1	1	3	3	6	4	10	21	5	16	39
A_5	3	0	1	0	0	1	0	2	2	3	6	2	4	9
A_5	4	1	2	2	1	2	3	4	3	8	24	3	14	48
A_5	5	2	2	4	2	3	8	12	6	24	92	7	46	210
D_4	2	0	2	0	0	2	0	3	3	4	7	3	6	10
D_5	2	0	1	1	0	1	1	1	2	2	3	2	3	4
D_6	2	0	1	0	0	1	0	2	2	3	5	2	4	7

Table 4: Dimensions of vector spaces of invariants $I(k)$ and equivariants $E(k)$ of degree k for stationary bifurcation, and invariants $I_H(k)$ and equivariants $E_H(k)$ for Hopf bifurcation, for several symmetric, alternating and dihedral groups. D denotes the dimension of the irreducible representation.

rotations of the cube, was investigated by Ashwin and Podvigina [1], who also pointed out this equivalence.

In Table 4 we show the numbers of invariants and equivariants for stationary and Hopf bifurcation for a number of finite groups. These results were obtained by adapting an existing computer program originally written to obtain isotropy subgroups using characters and trace formulae [17]. The program is written in the GAP [9] language, where $\chi(g^k)$ can be computed using the k-th power map of the conjugacy classes, and so $\chi_{(k)}$ can be found using the recursive formula (4.8). The formulas (4.5), (4.6), (4.12) and (4.14) can then be implemented as inner products in GAP.

For the symmetric groups S_n we consider the natural irreducible representation of dimension $n-1$, as in the example above. In this case it is known that for the stationary bifurcation, for $n > 3$ there is one equivariant quadratic and two equivariant cubic terms [5, 6]. Since $E(2) = I(3)$ and $E(3) = I(4)$, the dynamics truncated to cubic order is variational. But since $E(4) > I(5)$ for $n = 5, 6$ the quartic equivariants are non-variational. For the case of the Hopf bifurcation with S_n symmetry we see that there are three cubic equivariants for $n = 4, 5, 6$.

For the alternating group A_5, which is isomorphic to the group I of rotations of the icosahedron, there are unique faithful irreducible representations of dimension 3, 4 and 5, up to quasi-equivalence, that is, equivalence composed with an outer automorphism of A_5. From Table 4 we can see that for stationary bifurcation, the quadratic and cubic terms are variational in the 3- and 4-dimensional representations but not in the 5-dimensional one. This is consistent with the work of Hoyle [15] who found heteroclinic cycles in the cubic truncation for the 5-dimensional representation.

For the dihedral groups D_n, for $n = 4, 5, 6$, we consider the standard irreducible representation of dimension 2 where the n-cycle $(12\ldots n)$ acts as rotation through $2\pi/n$.

8 Examples: Continuous Groups

In this section we apply our results to the calculation of the dimensions of the spaces of invariants and equivariants for the groups $\mathbf{O}(3)$ and $\mathbf{SO}(3)$.

We first recall some facts about these groups and their representations, see [12, XIII, §7] for details. For each $l \geqslant 0$ there is only one (absolutely) irreducible representation of $\mathbf{SO}(3)$ of dimension $2l+1$ denoted by V_l. Each of these spaces carry two representations of $\mathbf{O}(3)$ called *plus* and *minus* representations: on the first one $-I$ acts trivially and on the second $-I$ acts non-trivially. In applications, the usual way that $\mathbf{O}(3)$ acts is induced from the natural action on $\mathbf{R}^3$. This leads to the representation *plus* for l even and *minus* for l odd which is called *natural representation* of $\mathbf{O}(3)$ on V_l.

The character afforded by the irreducible representation of $\mathbf{SO}(3)$ on V_l is given by

$$\chi_l(R_\theta) = \sum_{m=-l}^{l} e^{im\theta} = 1 + 2\sum_{m=1}^{l} \cos(m\theta) = \frac{\cos(l\theta) - \cos((l+1)\theta)}{1 - \cos(\theta)} \tag{8.1}$$

where $\theta \in [0, \pi]$ parametrises the conjugacy classes of $\mathbf{SO}(3)$ and represents the rotation R_θ. The Haar integral of a class function f on $\mathbf{SO}(3)$ is (see [22, p. 156])

$$\frac{1}{\pi}\int_0^\pi f(R_\theta)(1-\cos\theta)\,\mathrm{d}\theta\ . \tag{8.2}$$

The conjugacy classes of $\mathbf{O}(3)$ are also parametrised by $\theta \in [0, \pi]$, however there are two classes for each θ: one class is represented by the rotation R_θ and the other is represented by $-I \circ R_\theta = -R_\theta$. In this case the Haar integral of a class function f on $\mathbf{O}(3)$ is

$$\frac{1}{2\pi}\int_0^\pi [f(R_\theta) + f(-R_\theta)](1-\cos\theta)\,\mathrm{d}\theta\ . \tag{8.3}$$

Observe that for any representation of $\mathbf{O}(3)$ on V_l we have that

$$\begin{aligned} \dim_{\mathbf{R}} \mathcal{P}^{2k}_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1) &= \dim_{\mathbf{R}} \mathcal{P}^{2k}_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{SO}(3)\times\mathbf{S}^1) \\ \dim_{\mathbf{C}} \vec{\mathcal{P}}^{2k+1}_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1) &= \dim_{\mathbf{C}} \vec{\mathcal{P}}^{2k+1}_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{SO}(3)\times\mathbf{S}^1) \end{aligned} \tag{8.4}$$

To see this note the following. Let χ_l be the character of an irreducible representation V_l of $\mathbf{O}(3)$. If $-I$ acts trivially on V_l and $k \geq 1$, we have

$$\chi_{l,(k)}(R_\theta) = \chi_{l,(k)}(-R_\theta)$$

and so

$$\int_{\mathbf{O}(3)} \chi^2_{l,(k)} = \int_{\mathbf{SO}(3)} \chi^2_{l,(k)}, \quad \int_{\mathbf{O}(3)} \chi_{l,(k+1)}\,\chi_{l,(k)}\,\chi_l = \int_{\mathbf{SO}(3)} \chi_{l,(k+1)}\,\chi_{l,(k)}\,\chi_l.$$

By Theorem 4.6 we have the equalities (8.4).

Now, if $-I$ acts non-trivially on V_l, then the function $\lambda : \mathbf{O}(3) \to \mathbf{R}$ defined by

$$\lambda(g) = \begin{cases} 1 & \text{if} \quad g \in \mathbf{SO}(3), \\ -1 & \text{if} \quad g \in \mathbf{O}(3)\setminus\mathbf{SO}(3)\ , \end{cases}$$

is a linear character of $\mathbf{O}(3)$. Moreover, we have that $\lambda\chi_l$ is an irreducible character of $\mathbf{O}(3)$ where $-I$ acts trivially on V_l. Also, $\lambda\chi_l(g) = \chi_l(g)$ for $g \in \mathbf{SO}(3)$. By Proposition 4.7 and the above observation we have the equalities (8.4).

Remark 8.1 Sattinger [20] proved that for $\mathbf{SO}(3)$-symmetric steady-state bifurcations posed on an absolutely irreducible space V_l, the quadratic terms vanish for odd l, and possess a gradient structure for even l. The gradient structure for the cubic truncation for $\mathbf{O}(3)$-symmetric steady-state bifurcation on V_l, for any $l \geq 1$ was proved by Michel (unpublished). See Chossat *et al.* [3]. This can be proven in the following way. It is shown by Chossat and Lauterbach [4] that for $\mathbf{O}(3)$-symmetric steady-state bifurcation the number of cubic equivariants $E(3)$ is equal to $1 + [l/3]$. Using (8.1) and (8.2) we obtain the following expressions for $l \geq 1$:

$$
\begin{aligned}
\int_{\mathbf{SO}(3)} \chi_l^4(R_\theta) &= \frac{1}{\pi}\int_0^\pi \left[\sum_{m=-l}^{l} e^{im\theta}\right]^3 (\cos(l\theta)-\cos((l+1)\theta))\,\mathrm{d}\theta \;=\; 2l+1,\\
\int_{\mathbf{SO}(3)} \chi_l^2(R_{2\theta}) &= \frac{1}{\pi}\int_0^\pi \left[\sum_{m=-l}^{l} e^{i2m\theta}\right]^2 \left(1-\frac{e^{i\theta}}{2}-\frac{e^{-i\theta}}{2}\right)\mathrm{d}\theta \;=\; 2l+1\,,\\
\int_{\mathbf{SO}(3)} \chi_l^2(R_\theta)\chi_l(R_{2\theta}) &= \frac{1}{\pi}\int_0^\pi \sum_{m=-l}^{l} e^{im\theta}\sum_{n=-l}^{l} e^{i2n\theta}\,(\cos(l\theta)-\cos((l+1)\theta))\,\mathrm{d}\theta \;=\; 1,\\
\int_{\mathbf{SO}(3)} \chi_l(R_\theta)\chi_l(R_{3\theta}) &= \frac{1}{\pi}\int_0^\pi \sum_{m=-l}^{l} e^{i3m\theta}\,(\cos(l\theta)-\cos((l+1)\theta))\,\mathrm{d}\theta \;=\; 1-l+3\left[\frac{l}{3}\right],\\
\int_{\mathbf{SO}(3)} \chi_l(R_{4\theta}) &= \frac{1}{\pi}\int_0^\pi \sum_{m=-l}^{l} e^{i4m\theta}\left(1-\frac{e^{i\theta}}{2}-\frac{e^{-i\theta}}{2}\right)\mathrm{d}\theta \;=\; 1\,.
\end{aligned}
\tag{8.5}
$$

Notice that the integral formulas involving $\chi_l^k(R_\theta)$ can be simplified by using the third expression in (8.1) so that the factor $1-\cos(\theta)$ cancels. It follows then by Theorem 4.2 that

$$
I(4) \;=\; \int_{\mathbf{O}(3)} \chi_{l,(4)} \;=\; \int_{\mathbf{SO}(3)} \chi_{l,(4)} \;=\; 1+\left[\frac{l}{3}\right]\;.
$$

Similarly, we can use (8.5) to verify that $E(3)=1+[l/3]$. That is, $E(3)=I(4)=1+[l/3]$. Thus cubic $\mathbf{O}(3)$-equivariants also have a gradient structure. $\diamond$

Proposition 8.2 *Let* $\mathbf{O}(3)$ *act irreducibly on* V_l *and denote by* χ_l *the corresponding character. Then:*

(i) $\dim_{\mathbf{R}} \mathcal{P}^4_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1) \;=\; \dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1)$,

(ii) $\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1) \;=\; l+1$.

Proof: Direct computations using (8.5) give the results. We include an alternative proof using orthogonality of the characters of $\mathbf{O}(3)$.
(i) Denote by $E_H(3)=\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1)$ and $I_H(4)=\dim_{\mathbf{R}} \mathcal{P}^4_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1)$. By Theorem 4.6 we have

$$
E_H(3)-I_H(4) \;=\; \int_{\mathbf{O}(3)} \left(\chi_{l,(2)}\chi_l^2-\chi_{l,(2)}^2\right) \;=\; \langle \chi_{l,(2)},\chi_{l,[2]}\rangle
$$

where $\chi_{l,[2]}$ denotes the character of $\mathbf{O}(3)$ on the antisymmetric tensor square $A^2(V_l)$ of V_l (and as before $\chi_{l,(2)}$ is the character of $\mathbf{O}(3)$ on the symmetric tensor square $S^2(V_l)$ of V_l). Recall that

$$\chi_l^2 \;=\; \chi_{l,(2)} + \chi_{l,[2]} \;.$$

As

$$S^2(V_l) \;=\; \bigoplus_{a=0}^{l} V_{2l-2a} \;=\; V_{2l} \oplus V_{2l-2} \oplus \cdots \oplus V_0$$

(see for example, Fulton and Harris [8, page 159]), it follows then that

$$\chi_{l,(2)} \;=\; \chi_{2l} + \chi_{2l-2} + \cdots + \chi_0 \;.$$

Also,

$$A^2(V_l) \;=\; V_{2l-1} \oplus V_{2l-3} \oplus \cdots \oplus V_1$$

(see for example, Fulton and Harris [8, page 160]), and so

$$\chi_{l,[2]} \;=\; \chi_{2l-1} + \chi_{2l-3} + \cdots + \chi_1 \;.$$

Therefore $\langle \chi_{l,(2)}, \chi_{l,[2]} \rangle \;=\; 0$.

(b) Observe that by Theorem 4.6 we have

$$E_H(3) \;=\; \int_{\mathbf{O}(3)} \chi_{l,(2)} \chi_l^2 \;=\; \langle \chi_{l,(2)}, \chi_l^2 \rangle$$

As χ_l^2 is the character of the $\mathbf{O}(3)$-module $V_l \otimes V_l$ and

$$V_l \otimes V_l \;=\; V_0 \oplus V_1 \oplus \cdots \oplus V_{2l}$$

(see for example Sattinger [21, p. 138, Lemma 5.20]), we obtain

$$\chi_l^2 \;=\; \chi_0 + \chi_l + \cdots + \chi_{2l}$$

and so

$$\langle \chi_{l,(2)}, \chi_l^2 \rangle \;=\; l + 1 \;.$$

□

l	$I(2)$	$I^+(3)$	$E^+(2)$	$E(3)$	$I_H(2)$	$I_H(4)$	$I_H(6)$	$E_H(3)$	$E_H(5)$
1	1	0	0	1	1	2	2	2	3
2	1	1	1	1	1	3	5	3	9
3	1	0	0	2	1	4	10	4	21
4	1	1	1	2	1	5	17	5	40
5	1	0	0	2	1	6	28	6	69
6	1	1	1	3	1	7	43	7	110
7	1	0	0	3	1	8	62	8	164
8	1	1	1	3	1	9	87	9	234
9	1	0	0	4	1	10	118	10	322
10	1	1	1	4	1	11	155	11	429

Table 5: Dimensions of vector spaces of invariants $I(k)$ and equivariants $E(k)$ of degree k for stationary bifurcation, and invariants $I_H(k)$ and equivariants $E_H(k)$ for Hopf bifurcation, for the group $\mathbf{O}(3)$. For the plus representation $\mathbf{O}(3)$ we denote those by by $I^+(k)$ and $E^+(k)$ and omit the values for the minus representation if they are zero.

In Table 5 we show the numbers of invariants and equivariants for stationary and Hopf bifurcation with $\mathbf{O}(3)$-symmetry for $l = 1, \ldots, 10$. For Hopf bifurcation the values are the same for the plus and minus representations of $\mathbf{O}(3)$ on V_l. For steady-state bifurcation the values for the two representations of $\mathbf{O}(3)$ on V_l differ for the number of cubic invariants and quadratic equivariants. For the plus representation we have the values denoted by $I^+(3)$ and $E^+(2)$. For the minus representation these are zero.

As shown above, $E_H(3) = I_H(4) = l+1$, so the cubic equivariants for Hopf bifurcation can be written as gradients of the quartic invariants. However, in the case of Hopf bifurcation, this does not constrain the dynamics in the way that it does for stationary bifurcation. Note that $E_H(5)$ and $I_H(6)$ increase very rapidly with l; this increase appears to be of order l^3 for large l.

Acknowledgements

The authors are grateful for financial support from the Isaac Newton Institute, Cambridge, where this work was carried out. APSD thanks Departamento de Matemática Pura of Faculdade de Ciências of Universidade do Porto for granting leave.

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