A Note on Static Metrics

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Abstract

Conditions are given which, subject to a genericity condition on the Ricci tensor, are both necessary and sufficent for a 3-metric to arise from a static space-time metric.

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The vacuum field equations of General Relativity reduce for a static solution to a coupled system involving a (Riemannian) spatial metric g_{ij} and a potential function V equal to the square-root of the norm of the Killing vector. The system is over-determined and one can ask whether a given metric allows the existence of a V for which the equations are satisfied. This is the question: *when is a 3-metric static?* which in turn is the simplest case of a larger question: *when are Cauchy data for a vacuum solution actually data for a static solution?*

In this note we give necessary and sufficient conditions on a 3-metric for it to be static, in the case when the Ricci tensor viewed as an endomorphism of vectors has distinct eigenvalues, which is the generic case. In this case, our method gives an explicit candidate for the gradient dV in terms of the curvature. If the Ricci tensor is degenerate, in the sense of having a repeated eigenvalue, the method fails. One can then ask if it is possible to have two different potentials with the same metric, a problem solved in [1]. Our methods can also be used to give information on a number of related geometrical equations, which we shall note below. The larger question above can also be solved, and will be considered elsewhere.

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In terms of a metric g_{ij} with Ricci tensor R_{ij} and a potential V, the static vacuum field equations are as follows:

$$\begin{aligned} \mathrm{R}_{ij} &= V^{-1}\nabla_i\nabla_j V &(1)\\ \Delta_g V &= 0, &(2) \end{aligned}$$

where Δ_g is the Laplacian for g (for these equations see, for example, [2]). It follows from (1) and (2) that the scalar curvature vanishes. Define the Cotton-York tensor, as usual, by

$$\mathrm{C}_{ij} = \epsilon_j{}^{pq}(\nabla_q \mathrm{R}_{ip} - \tfrac{1}{4} g_{ip}\nabla_q \mathrm{R}), \qquad (3)$$

then C_{ij} is symmetric, trace-free and divergence-free in the sense that $\nabla^j \mathrm{C}_{ij} = 0$. Differentiating (1) and applying (3) and the Ricci identity appropriate for dimension 3 leads to

$$V\mathrm{C}_{ij} = -\epsilon_j{}^{pq}(2\mathrm{R}_{ip}\delta_q^k + g_{ip}\mathrm{R}_q^k)\nabla_k V, \qquad (4)$$

which gives a system of five equations (since both sides are symmetric and trace-free) for the three components of $U_i := V^{-1}\nabla_i V$. For this system to be solvable for U_i two linear constraints must hold on the five components of the tensor C_{ij}. These constraints are

$$\mathrm{C}^{ij}\mathrm{R}_{ij} = 0, \quad \mathrm{C}^i_j \mathrm{R}^j_k \mathrm{R}^k_i = 0, \qquad (5)$$

which, for brevity, we may write as $\mathrm{C}\cdot\mathrm{R} = 0$ and $\mathrm{C}\cdot\mathrm{R}^2 = 0$. The necessity of these conditions follows readily from (4); sufficiency can be seen by calculating in the Ricci eigenframe. In a frame which diagonalises the Ricci tensor, say $\mathrm{R} = \mathrm{diag}(\lambda, \mu, \nu)$ with $\lambda + \mu + \nu = 0$, we see from (4) that C is purely off-diagonal,

$$\mathrm{C} \simeq \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix} \qquad (6)$$

and for U, provided λ, μ and ν are distinct, we obtain the expression

$$U = [a/(\mu-\nu), b/(\nu-\lambda), c/(\lambda-\mu)]. \qquad (7)$$

Thus, provided the eigenvalues of the Ricci tensor are distinct, we have an explicit expression for the gradient of the (logarithm of the) static potential. All that remains is to see whether (1) and (2) are satisfied, which we do below.

Note that the eigenvalues of the Ricci tensor are distinct exactly when the *discriminant* $\mathcal{D}$ satisfies

$$\mathcal{D} := 4\sigma_2^3 + 27\sigma_3^2 < 0, \qquad (8)$$

where $\sigma_1 = \lambda + \mu + \nu = \mathrm{tr}_g \mathrm{R} = 0$, $\sigma_2 = -\frac{1}{2}|\mathrm{R}|^2 = \lambda\mu + \mu\nu + \nu\lambda < 0$ and $\sigma_3 = \det \mathrm{R} = \lambda\mu\nu$ are the elementary symmetric functions of the eigenvalues of

R. Note the alternative expressions

$$
\begin{aligned}
\mathcal{D} &= -\tfrac{1}{2}|\mathrm{R}|^6 + 3(\mathrm{tr}_g \mathrm{R}^3)^2 \\
&= -\tfrac{1}{2}(\mathrm{R}^i_j \mathrm{R}^j_i)^3 + 3(\mathrm{R}^i_j \mathrm{R}^j_k \mathrm{R}^k_i)^2 \\
&= -(\lambda-\mu)^2(\mu-\nu)^2(\nu-\lambda)^2.
\end{aligned}
$$

After some calculation, we may obtain the general form of (7) in a matrix notation as

$$
\begin{aligned}
U &= \mathcal{D}^{-1}(\sigma_2 I + 3\mathrm{R}^2)^2[\mathrm{C},\mathrm{R}]^* \\
&= \tfrac{1}{4}\mathcal{D}^{-1}(|\mathrm{R}|^2 I - 6\mathrm{R}^2)^2[\mathrm{C},\mathrm{R}]^*, \qquad (9)
\end{aligned}
$$

where $|\mathrm{R}|^2 := \mathrm{R}^i_j \mathrm{R}^j_i$ and $[\mathrm{C},\mathrm{R}]^* := \epsilon_{ijk}\mathrm{C}^j_p \mathrm{R}^{pk}$. More explicitly,

$$
U_i = \tfrac{1}{4}\mathcal{D}^{-1}(|\mathrm{R}|^2\delta^j_i - 6\mathrm{R}^p_i \mathrm{R}^j_p)(|\mathrm{R}|^2\delta^k_j - 6\mathrm{R}^q_j \mathrm{R}^k_q)\epsilon_{krs}\mathrm{C}^r_t \mathrm{R}^{ts} \qquad (10)
$$

We shall say that a 3-metric g is *Ricci-non-degenerate* if its Ricci tensor has distinct eigenvalues; equivalently, if $\mathcal{D} \neq 0$.

Theorem 1 *A Ricci-non-degenerate metric g, defined on a simply-connected region, admits a static potential V exactly when the vector field U (defined in terms of $g, \mathrm{R}, \mathrm{C}$ by (9) or (10)) satisfies*

$$
\nabla_i U_j + U_i U_j = \mathrm{R}_{ij}. \qquad (11)
$$

Proof: If g is Ricci-non-degenerate and static with potential V, then $U_i = V^{-1}\nabla_i V$ satisfies (11) by (1), and the above calculations show (9) holds because $\mathcal{D} < 0$. Conversely, if U_i defined by (9) satisfies (11), then $\nabla_{[i}U_{j]} = 0$ and thus U_i is a gradient, $U_i = \nabla_i \log V$, where the potential V is unique up to an arbitrary multiplicative constant. It then follows directly from (11) that V is a static potential for g. ■

Equation (11) with U_i as in (10) is an equation directly on the Ricci tensor, and it is of interest to find the form of the derivatives of highest degree. We have the identities

$$
\begin{aligned}
\nabla_i \mathrm{C}_{jk} &= \nabla_{(i}\mathrm{C}_{jk)} + \frac{2}{3}\nabla_{[i}\mathrm{C}_{j]k} + \frac{2}{3}\nabla_{[i}\mathrm{C}_{k]j} \\
\epsilon^{ijm}\nabla_i \mathrm{C}_{jk} &= \Delta \mathrm{R}^m_k - 3\mathrm{R}^{mn}\mathrm{R}_{nk} + \delta^m_k \mathrm{R}^{pq}\mathrm{R}_{pq}
\end{aligned}
$$

Thus the highest derivatives in (11) are $\nabla_{(i}\mathrm{C}_{jk)}$ and $\Delta \mathrm{R}_{ij}$. However, given a solution of (1) and (2), $\Delta \mathrm{R}_{ij}$ can be expressed as

$$
\Delta \mathrm{R}_{ij} = 6\mathrm{R}^k_i \mathrm{R}_{jk} - 2g_{ij}\mathrm{R}^{km}\mathrm{R}_{km} + U^m(\nabla_i \mathrm{R}_{jm} + \nabla_j \mathrm{R}_{im} - 3\nabla_m \mathrm{R}_{ij})
$$

in terms of R_{ij}, $\nabla_i \mathrm{R}_{jk}$ and U_i, and we obtain a necessary condition on just $\nabla_{(i}\mathrm{C}_{jk)}$ from (11) and (10).

The conditions found here give an algorithm for testing a metric for staticity. Assuming Ricci-nondegeneracy, one proceeds by asking the questions:

1. does the scalar curvature vanish?

2. if so, is C_{ij} purely off-diagonal in the Ricci eigenframe?

3. if so, is U_i from (10) a gradient?

4. if so, is (11) satisfied?

This algorithm can be tested on, for example, spatially homogeneous 3-metrics when it finds all the static cases.

Finally, we note the following two equations which have arisen in the literature:

$$\begin{aligned} \nabla_i \nabla_j F &= \mathrm{R}_{ij} - \lambda g_{ij} & (12) \\ 2\nabla_i \nabla_j F &= F\mathrm{R}_{ij}. & (13) \end{aligned}$$

The first is the *gradient Ricci soliton* equation (see e.g. [3]) while the second has arisen in the study of black holes (see e.g. [4]). For both equations, the interest is typically in global solutions on compact manifolds, but our methods in the 3-dimensional case again deduce candidate dF given the metric.

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