AN INTEGRAL EQUATION FOR SPACETIME CURVATURE IN GENERAL RELATIVITY

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ABSTRACT. A key step in the proof of global existence for Yang-Mills fields, propagating in curved, 4-dimensional, globally hyperbolic, background spacetimes, was the derivation and reduction of an integral equation satisfied by the curvature of an arbitrary solution to the Yang-Mills field equations. This article presents the corresponding derivation of an integral equation satisfied by the curvature of a vacuum solution to the Einstein field equations of general relativity. The resultant formula expresses the curvature at a point in terms of a 'direct' integral over the past light cone from that point, a so-called 'tail' integral over the interior of that cone and two additional integrals over a ball in the initial data hypersurface and over its boundary. The tail contribution and the integral over the ball in the initial data surface result from the breakdown of Huygens' principle for waves propagating in a general curved, 4-dimensional spacetime.

By an application of Stokes' theorem and some integration by parts lemmas, however, one can re-express these 'Huygens-violating' contributions purely in terms of integrals over the cone itself and over the 2-dimensional intersection of that cone with the initial data surface. Furthermore, by exploiting a generalization of the parallel propagation, or Cronström, gauge condition used in the Yang-Mills arguments, one can explicitly express the frame fields and connection one-forms in terms of curvature. While global existence is certainly false for general relativity one anticipates that the resulting integral equation may prove useful in analyzing the propagation, focusing and (sometimes) blow up of curvature during the course of Einsteinian evolution and thereby shed light on the natural alternative conjecture to global existence, namely Penrose's *cosmic censorship conjecture*.

I. Introduction

Global existence fails to hold for many, otherwise reasonable solutions to the Einstein field equations. Examples of finite-time blowup include solutions developing black holes and solutions evolving to form cosmological big bang or big crunch singularities. The singularities that arise in such examples often, but not always, involve the blowup of

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certain spacetime curvature invariants. More subtle types of singular behavior include the formation of Cauchy horizons, at which the curvature can remain bounded, but across which global hyperbolicity, and hence classical determinism, is lost.

Examples of this latter phenomenon are provided by the Kerr and Kerr-Newman rotating black hole spacetimes and by non-isotropic, cosmological models of Taub-NUT-type wherein violations of strong causality (as signaled by the occurrence of closed timelike curves or the appearance of naked curvature singularities) develop beyond the Cauchy horizons arising in these solutions. On the other hand a variety of arguments and calculations strongly suggest that such Cauchy horizons, when they occur, are highly unstable-giving way, under generic perturbations, to the formation of strong curvature singularities that block the extension of such perturbed solutions beyond their maximal Cauchy developments.

Considerations such as these led Roger Penrose to propose the so-called (strong) *cosmic censorship conjecture* [1] according to which (in a here deliberately loosely stated form):

> *globally hyperbolic solutions to the Einstein field equations evolving from non-singular Cauchy data are generically inextendible beyond their maximal Cauchy developments.*

For the non-vacuum cases of this conjecture it is natural to consider only those matter sources which exhibit, in the absence of gravitational coupling, the global existence property at least in Minkowski space but perhaps also (being somewhat more cautious) in generic globally hyperbolic 'background' spacetimes. Otherwise, rather straightforward counterexamples can be presented involving, for instance, self-gravitating perfect fluids that evolve to blow up in a nakedly singular but stable fashion [2, 3]. But Penrose's conjecture was never intended to suggest that Einsteinian gravity should miraculously hide the defects of inadequate models of matter inside black holes or cause their singularities to harmlessly merge with big bang or big crunch cosmological

singularities.

There are a number of known types of relativistic matter sources that do exhibit the desired global existence property, but one is currently so far from a proof of cosmic censorship that their inclusion into the picture only presents an unwanted distraction from the more essential issues. Thus it seems natural to set these complications aside until genuine progress can be made in the vacuum special case.

On the other hand there is one particular class of matter fields whose study seems to be directly relevant to the analysis of the vacuum gravitational equations-namely the class of Yang-Mills fields propagating in a given, 4-dimensional, globally hyperbolic, background spacetime. First of all, these are examples of sources for which global existence results (for the case of compact Yang-Mills gauge groups) have already been established in both flat [4, 5] and curved [6] background spacetimes. Secondly however, the vacuum Einstein equations, when expressed in the Cartan formalism and combined with the Bianchi identities, imply that the spacetime curvature tensor, written as a matrix of two-forms, satisfies a propagation equation of precisely (curved-space) Yang-Mills type.

But in contrast to the case of 'pure' Yang-Mills fields this Einsteinian curvature propagation equation is coupled to another equation (the vanishing torsion condition) which links the connection one-form field to the (orthonormal) frame field and thus reinstates that frame (or metric) as the fundamental dynamical variable of general relativity. An additional, related distinction from conventional Yang-Mills theory is that the effective Yang-Mills gauge group for Einsteinian gravity, when formulated in this way, is the *non-compact* group of Lorentz transformations which acts (locally) to generate automorphisms of the bundle of orthonormal frames while leaving the metric invariant.

An initially disconcerting consequence of this non-compactness of the effective gauge

group is that the associated, canonical Yang-Mills stress-energy tensor (a symmetric, second rank tensor quadratic in the curvature) need no longer have a positive definite energy density (as it always does in conventional, compact gauge group, Yang-Mills theory) and indeed this tensor vanishes identically in the gravitational case. Fortunately however the Bel-Robinson tensor (a fourth rank, totally symmetric tensor quadratic in curvature and having positive definite energy density) is available to take over its fundamental role [7].

The proofs of flat and curved space global existence for conventional (compact gauge group) Yang-Mills fields given, respectively, in References [5] and [6] use a combination of light cone estimates and energy arguments that exploit, on the one hand, an integral equation satisfied by the curvature of the Yang-Mills connection and, on the other, the properties of the associated, canonical stress-energy tensor mentioned above. For the case of curved, globally hyperbolic, background spacetimes the proof guarantees only that the Yang-Mills connection, expressed in a suitable gauge, cannot blow up until the background spacetime itself blows up, for example by evolving to form a black hole or cosmological singularity or by developing a Cauchy horizon. But even linear Maxwell fields typically blow up at such singular boundaries or Cauchy horizons, so one could hardly expect better regularity in the nonlinear case.

Of course in general relativity there is no given, 'background' geometry at all and global existence is much too strong a conjecture for the gravitational field as the afore-mentioned examples and arguments show. Spacetime curvature does indeed blow up in many otherwise reasonable instances of Einsteinian evolution and this blowup is anticipated to be a stable feature of such solutions and not merely the artifact of, say, some special symmetry or other 'accidental' property of the spacetime under study. Cosmological solutions may only persist for a finite (proper) time in one or both temporal directions whereas timelike geodesics falling into a black hole may encounter divergent

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curvature, representing unbounded tidal 'forces', in a finite proper time.

But if Penrose's conjecture is true then global hyperbolicity is at least a generic feature of maximally extended Einstein spacetimes that evolve from non-singular Cauchy data and general relativity is thereby effectively rescued from an otherwise seemingly fatal breakdown of classical determinism. If, on the other hand, cosmic censorship is false then the implied breakdown of determinism may well render Einstein's equations inadequate as a classical theory of the gravitational field.

There is currently no clear-cut strategy for trying to prove the cosmic censorship conjecture but it nevertheless seems evident that a better understanding of how space-time curvature propagates, focuses and (in some circumstances) blows up in the course of Einsteinian evolution will be essential for progress on this fundamental problem. For that reason one might hope that a further development of the "Yang-Mills analogy", wherein the parallel issues of curvature propagation, focusing and blowup for 'pure' Yang-Mills fields have already been somewhat successfully analyzed, could yield significant insights for understanding the still-wide-open gravitational problem.

One of the key steps in the 'pure' Yang-Mills analysis was the derivation of an integral equation satisfied by the curvature of an arbitrary solution to the field equations. This integral equation resulted from combining the Yang-Mills equations and their Bianchi identities in a well-known way to derive a wave equation satisfied by curvature and by then applying the fundamental solution of the associated wave operator to derive an integral expression for the curvature at an arbitrary point (within the domain of local existence for the solution in question) in terms of integrals over the past light cone of that point to the initial, Cauchy hypersurface. An additional key step was the transformation of this integral formula through the use of the parallel propagation, or Cronström, gauge condition [5, 6, 8] to eliminate the connection one-form explicitly in favor of the curvature itself. Certain resulting integrals over the light cone, from its

vertex back to the initial data surface, could be bounded in terms of the Yang-Mills energy flux, defined via the aforementioned, canonical stress-energy tensor, and thence in terms of the actual energy on the initial hypersurface.

In the simplest, flat space setting of Ref. [5] a Gronwall lemma argument was employed to prove that the natural (gauge-invariantly-defined) L^∞-norm of curvature is always bounded in terms of the (equally gauge-invariant) conserved total energy, with all reference to the artifice of the Cronström or parallel propagation gauge, used in the intermediate steps, effectively eliminated. Thus equipped with an a priori pointwise bound on curvature one completed the proof of global existence by showing that an appropriate Sobolev norm of the connection one-form, when evolved in the so-called 'temporal gauge', cannot blow up in finite time by a straightforward, higher order energy argument. A more elaborate argument was needed for the case of the curved backgrounds treated in Ref. [6] but the essential role played by the corresponding integral equation for curvature remained unaltered.

In the flat space argument one avoided certain complications, resulting from the breakdown of Huygens' principle for the complete gauge-covariant wave operator appearing in the curvature propagation equation, by splitting that operator into a pure flat-space wave operator (which does of course obey Huygens' principle in four-dimensional Minkowski space) and a collection of lower order, Huygens-violating, connection terms which were moved over and included with the 'source' terms in the full, inhomogeneous wave equation for curvature. One then derived the integral formula for curvature by applying the well-known fundamental solution for the flat space wave operator to the redefined source terms and then eliminating the connection terms in the redefined source, in favor of curvature, through an application of the Cronström gauge argument mentioned above.

This same operator splitting technique was also employed for curved backgrounds

in Ref. [6] but there, since the ordinary tensor wave operator itself violates Huygens' principle (in a generic background), new terms in the resulting 'representation formula' for Yang-Mills curvature arose which had no direct analogue in the operator-split, flat space argument. These new, so-called *tail* terms appeared as integrals over the *interior* of the past light cone from an arbitrary point to the initial hypersurface and over the interior of the three ball in the initial hypersurface bounded by the intersection of the past light cone with this initial surface. Fortunately, however, these tail terms produced only a slight complication in the argument for the curved-space 'pure' Yang-Mills problem because all of the Huygen's-violating, tail contributions to the fundamental solution for the residual tensor wave operator (remaining after the aforementioned operator splitting is carried out) are functionals only of the given, *background* metric and thus are independent of the Yang-Mills field under study. Their contributions can therefore always be bounded by constants dependent only upon the background geometry but independent of the solution in question.

In this article we derive an integral equation satisfied by the curvature tensor of a vacuum solution to Einstein's equations by applying the fundamental solution of the associated, curved-space tensor wave operator to the source terms in the curvature propagation equation defined after an analogous operator splitting, within the Cartan formulation for the field equations, has been carried out. For this purpose we exploit the general theory of such wave operators developed over the years by Hadamard, Sobolev, Reisz, Choquet-Bruhat, Friedlander and others [9]. We then transform the resulting expression, by an application of Stokes' theorem and some integration-by-parts arguments, to rewrite the Huygen's-violating tail contribution integrals in terms of other integrals over the past light cone itself. A generalization of Cronström's argument is given which shows that not only the connection but also the frame field can be explicitly expressed in terms of curvature by exploiting a natural parallel propagation

gauge condition in conjunction with the standard Hadamard/Friedlander constructions.

While the aforementioned calculations exploit an operator split version of the curvature propagation equation (written as an evolution equation for a matrix of two-forms), we also show how the same result can be derived, without using the Cartan formalism or associated operator splitting, by applying the Hadamard/Friedlander fundamental solution for the wave operator acting on a fourth rank tensor to the purely (fourth rank) tensorial form of the curvature propagation equation. At the other extreme one could presumably arrive at the same result in still another way by converting all the indices on the curvature tensor to frame indices, carrying out a maximal operator splitting to include the connection terms with the source and then applying the fundamental solution for the purely scalar wave operator to the wave equation for each component. We have not performed this latter derivation but strongly suspect that it leads to the same, 'canonical' result obtained in the other two ways.

In view of the foregoing remarks it may seem that we have gained little in emphasizing the use of the Cartan formalism and its associated 'Yang-Mills analogy' in analyzing the field equations but one should keep in mind that the derivation of this integral equation for curvature is only the first step in a proposed sequence of arguments wherein one hopes to exploit the Cronström-type formulas to re-express all the fundamental variables in terms of the curvature (written in Cartan fashion as a matrix of two-forms) and derive estimates for curvature by analogy with those obtained in Refs. [5] and [6]. Until such arguments are carried out it will not be evident whether the Cartan formulation is actually essential for the analysis or only a convenience for those familiar with the 'pure' Yang-Mills derivations.

Of course one cannot simply expect to copy the pattern of the 'pure' Yang-Mills arguments and thereby derive a global existence result for the Einstein equations. First of all we know that any such conclusion must be false but it is worth recalling here

that the Yang-Mills arguments did not imply unqualified regularity of the Yang-Mills field but only implied that the field could not blow up until the background spacetime itself blew up. In general relativity though there is of course no *background* spacetime and the vanishing torsion condition, which links the metric to the connection, has no analogue in pure Yang-Mills theory.

One rather explicit obstruction to simply copying the 'pure', curved-space Yang-Mills argument is that one cannot simply bound the Bel-Robinson energy fluxes (which fortunately do bound certain relevant light cone integrals) in terms of the Bel-Robinson energy defined on the initial data hypersurface. While the Bel-Robinson tensor does in fact obey the vanishing divergence condition whose analogue, in the case of the canonical stress energy tensor, permitted the derivation of such a bound in the pure Yang-Mills problem, the Christoffel symbols occurring as coefficients in this equation are no longer background quantities and thus no longer a priori under control as they were in the arguments of Ref. [6].

However the full definition of a Bel-Robinson energy expression (and its associated fluxes) depends upon the additional choice of a timelike vector field on spacetime. If one had the luxury of choosing a timelike Killing or even conformal Killing field in defining these quantities then the corresponding Bel-Robinson energy would be a strictly conserved quantity and a significant portion of the needed arguments would revert to the simple form available in the flat space (or conformally stationary curved space) 'pure' Yang-Mills problem wherein the canonical (positive definite, gauge invariant) energy is strictly conserved. But such an assumption is absurdly restrictive in the case of Einstein's equations for which the small set of vacuum solutions admitting a globally defined timelike conformal Killing field is essentially known explicitly [10].

But whereas the presence of a conformal Killing field is out of the question for generic Einstein spacetimes there is nevertheless a potential utility in identifying what

we might call *quasi-local, approximate Killing and conformal Killing fields* and trying to exploit these in a 'quasi-local, approximate' variant of the arguments that assume a strict Killing or conformal Killing field. The idea we have in mind is spelled out more explicitly in the concluding technical section of this article wherein we show that the parallel propagated frame fields (determined by parallel propagation of a frame chosen at the vertex of each light cone) satisfy Killing's equations approximately with an error term that is explicitly computable in terms of curvature and that tends to zero at a well-defined rate as one approaches the vertex of the given cone. The flux of the corresponding quasi-local energy (built from the chosen vector field and the Bel-Robinson tensor) will of course not be strictly equal (as it would for a truly conserved energy) to the energy contained on an initial data slice but the error will be estimable in terms of an integral involving the (undifferentiated) curvature tensor. The question of how best to use this observation to obtain optimal estimates from the integral equation for curvature is one we hope to address in future work.

The idea of exploiting the 'Yang-Mills analogy' to analyze Einstein's equations is certainly nothing new and has been proposed previously by Eardley and van Putten, for example, with a view towards numerical applications [11]. Furthermore the global existence of Yang-Mills fields propagating in Minkowski space has been proven by a completely independent argument, which avoids light cone estimates, in a paper by Klainerman and Machedon [12]. During a visit to the Erwin Schrödinger Institute in the summer of 2004 the author described the preliminary results for this paper with Sergiu Klainerman who then, together with Igor Rodnianski, independently succeeded to derive an integral equation for curvature using a significantly different approach from that described herein [13]. Since the two formulations are quite dissimilar (in that, for example they do not use the frame formalism, the Hadamard/Friedlander analysis or the parallel propagation gauge condition) it is not yet clear whether the

resultant integral equations are ultimately equivalent or perhaps genuinely different. Klainerman and Rodnianski trace the origins of their approach back through some fundamental papers by Choquet-Bruhat [14] and Sobolev [15] whereas the sources for our approach, as we have indicated, trace more directly back through the work of Friedlander [9] and Hadamard [16]

II. Propagation Equations for Spacetime Curvature

In this section we rederive the familiar wave equation satisfied by the curvature tensor of a vacuum spacetime and then reexpress that equation in a form which parallels the one satisfied by the Yang-Mills curvature in a vacuum background. One could generalize both forms by allowing the spacetime to be non-vacuum but since we shall not deal with sources for Einstein's equations in this paper, we simplify the presentation by setting

$$R^{\alpha}{}_{\mu\alpha\nu} := R_{\mu\nu} = 0. \tag{2.1}$$

The Bianchi identities give

$$R^{\alpha}{}_{\beta\gamma\delta;\mu} + R^{\alpha}{}_{\beta\delta\mu;\gamma} + R^{\alpha}{}_{\beta\mu\gamma;\delta} = 0 \tag{2.2}$$

so that, upon contracting and exploiting the algebraic symmetries of the curvature tensor, one gets

$$R_{\gamma\delta\beta\alpha}{}^{;\alpha} = R_{\beta\gamma;\delta} - R_{\beta\delta;\gamma}. \tag{2.3}$$

Imposing the vacuum field equations this yields

$$D_{\alpha}R_{\gamma\delta\beta}{}^{\alpha} := R_{\gamma\delta\beta\alpha}{}^{;\alpha} = 0 \tag{2.4}$$

where we have introduced D_{α} as an alternative to $;\alpha$ to symbolize covariant differentiation.

Taking a divergence of the Bianchi identity (2.2) yields

$$R^{\alpha}{}_{\beta\gamma\delta;\mu}{}^{;\mu} = R^{\alpha}{}_{\beta\gamma\mu;\delta}{}^{;\mu} - R^{\alpha}{}_{\beta\delta\mu;\gamma}{}^{;\mu}. \tag{2.5}$$

Commuting covariant derivatives on the right hand side and exploiting the field equations (2.1) together with Eq. (2.4), which follows from them, and using the algebraic Bianchi identity

$$R^{\alpha}{}_{[\beta\gamma\delta]} = 0 \tag{2.6}$$

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to simplify the resulting expression finally gives

$$D^\mu D_\mu R^\alpha{}_{\beta\gamma\delta} := R^\alpha{}_{\beta\gamma\delta;\mu}{}^{;\mu}$$

$$= -R_{\gamma\delta}{}^{\rho\sigma} R^\alpha{}_{\beta\rho\sigma} \qquad (2.7)$$

$$+ 2R^\alpha{}_{\rho\delta\sigma} R_\beta{}^\rho{}_\gamma{}^\sigma - 2R^\alpha{}_{\rho\gamma\sigma} R_\beta{}^\rho{}_\delta{}^\sigma.$$

This is the fundamental wave equation satisfied by the curvature tensor of a vacuum spacetime.

Now, following the notation of the appendix we set

$$R^{\hat{a}}{}_{\hat{b}\mu\nu;\alpha} = \theta^{\hat{a}}_\lambda h^\sigma_{\hat{b}} R^\lambda{}_{\sigma\mu\nu;\alpha} \qquad (2.8)$$

and expand out the right hand side of this expression to get

$$D_\alpha R^{\hat{a}}{}_{\hat{b}\mu\nu} := R^{\hat{a}}{}_{\hat{b}\mu\nu;\alpha}$$

$$= \theta^{\hat{a}}_\lambda h^\sigma_{\hat{b}} R^\lambda{}_{\sigma\mu\nu;\alpha} \qquad (2.9)$$

$$= \nabla_\alpha R^{\hat{a}}{}_{\hat{b}\mu\nu} + \omega^{\hat{a}}{}_{\hat{c}\alpha} R^{\hat{c}}{}_{\hat{b}\mu\nu}$$

$$- \omega^{\hat{c}}{}_{\hat{b}\alpha} R^{\hat{a}}{}_{\hat{c}\mu\nu}$$

where we have defined

$$\nabla_\alpha R^{\hat{a}}{}_{\hat{b}\mu\nu} := (R^{\hat{a}}{}_{\hat{b}\mu\nu})_{,\alpha}$$

$$- \Gamma^\delta_{\mu\alpha} R^{\hat{a}}{}_{\hat{b}\delta\nu} - \Gamma^\delta_{\nu\alpha} R^{\hat{a}}{}_{\hat{b}\mu\delta}. \qquad (2.10)$$

The operator ∇_α captures only that part of the full spacetime covariant derivative operator D_α that acts on the coordinate basis indices μ and ν of $R^{\hat{a}}{}_{\hat{b}\mu\nu}$ and ignores the contributions arising from the frame indices $\hat{a}$ and $\hat{b}$. These latter contributions are explicitly added back in Eq. (2.9) for the full spacetime covariant derivative of $R^{\hat{a}}{}_{\hat{b}\mu\nu}$ where they appear as the terms containing the Lorentz connection $\omega^{\hat{a}}{}_{\hat{b}\nu}$. We

extend the definitions of D_α and ∇_α to operators on tensors of arbitrary type in the obvious way; D_α is the full spacetime covariant derivative operator while ∇_α ignores frame indices and acts only on spacetime coordinate indices.

This splitting of the full covariant derivative into a spacetime coordinate contribution and a frame or "internal space" contribution is parallel to what one has in Yang-Mills theory wherein the Yang-Mills connection $A^{\hat{a}}{}_{\hat{b}\nu}$ plays the role of the Lorentz connection $\omega^{\hat{a}}{}_{\hat{b}\nu}$ but in which the internal space Lie algebra indices refer to the chosen gauge group and not to the Lorentz group. In Yang-Mills theory of course the spacetime metric and its Christoffel connection are prescribed a priori and have no relation to the internal space connection $A^{\hat{a}}{}_{\hat{b}\nu}$.

Rewriting the Bianchi identity (2.2) in this notation one gets

$$D_\mu R^{\hat{a}}{}_{\hat{b}\gamma\delta} + D_\gamma R^{\hat{a}}{}_{\hat{b}\delta\mu} + D_\delta R^{\hat{a}}{}_{\hat{b}\mu\gamma} = 0 \tag{2.11}$$

or more explicitly, using the aforementioned splitting of D_α

$$
\begin{aligned}
& \nabla_\mu R^{\hat{a}}{}_{\hat{b}\gamma\delta} + \omega^{\hat{a}}{}_{\hat{c}\mu} R^{\hat{c}}{}_{\hat{b}\gamma\delta} - R^{\hat{a}}{}_{\hat{c}\gamma\delta} \omega^{\hat{c}}{}_{\hat{b}\mu} \\
& + \nabla_\gamma R^{\hat{a}}{}_{\hat{b}\delta\mu} + \omega^{\hat{a}}{}_{\hat{c}\gamma} R^{\hat{c}}{}_{\hat{b}\delta\mu} - R^{\hat{a}}{}_{\hat{c}\delta\mu} \omega^{\hat{c}}{}_{\hat{b}\gamma} \\
& + \nabla_\gamma R^{\hat{a}}{}_{\hat{b}\mu\gamma} + \omega^{\hat{a}}{}_{\hat{c}\delta} R^{\hat{c}}{}_{\hat{b}\mu\gamma} - R^{\hat{a}}{}_{\hat{c}\mu\gamma} \omega^{\hat{c}}{}_{\hat{b}\delta} \\
& = 0
\end{aligned}
\tag{2.12}
$$

wherein one sees the internal space (frame) contributions arising as a set of matrix commutators of the Lorentz connection and curvature. This has exactly the structure of the corresponding Bianchi identity for Yang-Mills theory and reproduces that formula if one makes the substitutions of $F^{\hat{a}}{}_{\hat{b}\mu\nu}$ for $R^{\hat{a}}{}_{\hat{b}\mu\nu}$ and $A^{\hat{a}}{}_{\hat{b}\mu}$ for $\omega^{\hat{a}}{}_{\hat{b}\mu}$ with the "spacetime" covariant derivative ∇_μ playing the same role in each equation. The full spacetime/gauge covariant derivative bears the same relation to the pure "spacetime" covariant derivative as D_α does to ∇_α in Eq. (2.9) when the same substitutions are made.

On the other hand, a Yang-Mills curvature does not have the full algebraic symmetries of the Riemann curvature and, for closely related reasons, one cannot form the analogue of the Ricci tensor from $F^{\hat{a}}{}_{\hat{b}\mu\nu}$. Thus equation (2.1) has no analogue in Yang-Mills theory. If Eq. (2.4) however is first reexpressed as

$$D^{\alpha}R^{\hat{a}}{}_{\hat{b}\beta\alpha} := g^{\alpha\gamma}D_{\gamma}R^{\hat{a}}{}_{\hat{b}\beta\alpha} = 0 \tag{2.13}$$

then it corresponds precisely to the (source-free) Yang-Mills equation which, by definition, is

$$
\begin{aligned}
D^{\alpha}F^{\hat{a}}{}_{\hat{b}\beta\alpha} &:= g^{\alpha\gamma}D_{\gamma}F^{\hat{a}}{}_{\hat{b}\beta\alpha} \\
&:= g^{\alpha\gamma}\{\nabla_{\gamma}F^{\hat{a}}{}_{\hat{b}\beta\alpha} + A^{\hat{a}}{}_{\hat{c}\gamma}F^{\hat{c}}{}_{\hat{b}\beta\alpha} \\
&\quad - F^{\hat{a}}{}_{\hat{c}\beta\alpha}A^{\hat{c}}{}_{\hat{b}\gamma}\} \\
&= 0.
\end{aligned} \tag{2.14}
$$

In addition, $F^{\hat{a}}{}_{\hat{b}\mu\nu}$ is defined in terms of $A^{\hat{a}}{}_{\hat{b}\mu}$ by the precise analogue of the equation (A.17) which expresses $R^{\hat{a}}{}_{\hat{b}\mu\nu}$ in terms of $\omega^{\hat{a}}{}_{\hat{b}\mu}$, namely

$$
\begin{aligned}
F^{\hat{a}}{}_{\hat{b}\mu\nu} &= \partial_{\mu}A^{\hat{a}}{}_{\hat{b}\nu} - \partial_{\nu}A^{\hat{a}}{}_{\hat{b}\mu} \\
&\quad + A^{\hat{a}}{}_{\hat{d}\mu}A^{\hat{d}}{}_{\hat{b}\nu} - A^{\hat{a}}{}_{\hat{d}\nu}A^{\hat{d}}{}_{\hat{b}\mu}.
\end{aligned} \tag{2.15}
$$

Note that this formula does not involve the spacetime metric or its Christoffel symbols. In fact, the Christoffel symbols entering into the definition of ∇_{α} also cancel in Eq. (2.12) which entails only the exterior derivatives of the two-forms $F^{\hat{a}}{}_{\hat{b}\mu\nu}dx^{\mu} \wedge dx^{\nu}$ when the aforementioned substitutions are made there. On the other hand, Eq. (2.14) involves the metric and its Christoffel symbols explicitly and these quantities enter thereby into the wave equation for Yang-Mills curvature which played a central role in the Chruściel-Shatah analysis [6] of Yang-Mills fields on a curved background spacetime.

Returning to the wave equation for space time curvature (2.7), we now write it in the Cartan formalism which is, for us, motivated by the rather close analogy with Yang-Mills theory. Setting

$$R^{\hat{a}}{}_{\hat{b}\mu\nu;\lambda}{}^{;\lambda} = \theta^{\hat{a}}_{\rho}\, h^{\sigma}_{\hat{b}}\, R^{\rho}{}_{\sigma\mu\nu;\alpha\beta}\, g^{\alpha\beta} \tag{2.16}$$

and expanding out the right hand side using the notation introduced above one now gets

$$
\begin{aligned}
g^{\alpha\beta}\{&\nabla_{\beta}[\nabla_{\alpha}R^{\hat{a}}{}_{\hat{b}\mu\nu} + \omega^{\hat{a}}{}_{\hat{c}\alpha}R^{\hat{c}}{}_{\hat{b}\mu\nu} \\
&- R^{\hat{a}}{}_{\hat{c}\mu\nu}\omega^{\hat{c}}{}_{\hat{b}\alpha}] \\
&+ \omega^{\hat{a}}{}_{\hat{c}\beta}[\nabla_{\alpha}R^{\hat{c}}{}_{\hat{b}\mu\nu} + \omega^{\hat{c}}{}_{\hat{d}\alpha}R^{\hat{d}}{}_{\hat{b}\mu\nu} \\
&- R^{\hat{c}}{}_{\hat{d}\mu\nu}\omega^{\hat{d}}{}_{\hat{b}\alpha}] \\
&- [\nabla_{\alpha}R^{\hat{a}}{}_{\hat{c}\mu\nu} + \omega^{\hat{a}}{}_{\hat{d}\alpha}R^{\hat{d}}{}_{\hat{c}\mu\nu} - R^{\hat{a}}{}_{\hat{d}\mu\nu}\omega^{\hat{d}}{}_{\hat{c}\alpha}]\omega^{\hat{c}}{}_{\hat{b}\beta}\} \\
=& -R_{\mu\nu}{}^{\rho\sigma}R^{\hat{a}}{}_{\hat{b}\rho\sigma} \\
&+ 2R^{\hat{a}}{}_{\hat{c}\mu\sigma}R^{\hat{c}}{}_{\hat{b}\nu}{}^{\sigma} - 2R^{\hat{a}}{}_{\hat{c}\nu\sigma}R^{\hat{c}}{}_{\hat{b}\mu}{}^{\sigma}.
\end{aligned}
\tag{2.17}
$$

Rearranging this slightly, one can write it in the form

$$
\begin{aligned}
\nabla^{\alpha}&\nabla_{\alpha}R^{\hat{a}}{}_{\hat{b}\mu\nu} + R_{\mu\nu}{}^{\rho\sigma}R^{\hat{a}}{}_{\hat{b}\rho\sigma} \\
=& \ 2R^{\hat{a}}{}_{\hat{c}\mu\sigma}R^{\hat{c}}{}_{\hat{b}\nu}{}^{\sigma} - 2R^{\hat{a}}{}_{\hat{c}\nu\sigma}R^{\hat{c}}{}_{\hat{b}\mu}{}^{\sigma} \\
&- g^{\alpha\beta}\{\nabla_{\beta}[\omega^{\hat{a}}{}_{\hat{c}\alpha}R^{\hat{c}}{}_{\hat{b}\mu\nu} - R^{\hat{a}}{}_{\hat{c}\mu\nu}\omega^{\hat{c}}{}_{\hat{b}\alpha}] \\
&+ \omega^{\hat{a}}{}_{\hat{c}\beta}[\nabla_{\alpha}R^{\hat{c}}{}_{\hat{b}\mu\nu} + \omega^{\hat{c}}{}_{\hat{d}\alpha}R^{\hat{d}}{}_{\hat{b}\mu\nu} \\
&\quad\ - R^{\hat{c}}{}_{\hat{d}\mu\nu}\omega^{\hat{d}}{}_{\hat{b}\alpha}] \\
&- [\nabla_{\alpha}R^{\hat{a}}{}_{\hat{c}\mu\nu} + \omega^{\hat{a}}{}_{\hat{d}\alpha}R^{\hat{d}}{}_{\hat{c}\mu\nu} - R^{\hat{a}}{}_{\hat{d}\mu\nu}\omega^{\hat{d}}{}_{\hat{c}\alpha}]\omega^{\hat{c}}{}_{\hat{b}\beta}\}
\end{aligned}
\tag{2.18}
$$

where we have put $\nabla^{\alpha} = g^{\alpha\beta}\nabla_{\beta}$. The operator acting on $R^{\hat{a}}{}_{\hat{b}\mu\nu}$ on the left hand side of this equation ignores the frame indices and has exactly the same form as the wave operator that acts on the Faraday tensor $F_{\mu\nu}$ of a solution to Maxwell's equation on a vacuum background spacetime.

III. Normal Charts and Parallel Propagated Frames

In any Riemannian or pseudo-Riemannian (e.g., Lorentzian) manifold (V, g) one can construct, using the exponential map, a normal coordinate chart on some neighborhood of an arbitrary point in that manifold. Within our framework let $q \, \epsilon \, V$ be an arbitrary point of V and choose an orthonormal frame $\{\tilde{e}_\mu\}$ at the point q. Tangent vectors $\tilde{v} \, \epsilon \, T_q V$ can then be expressed as $\tilde{v} = x^\mu \tilde{e}_\mu$ and, for each such $\tilde{v}$, one can construct the affinely parameterized geodesic of (V, g) which begins (with parameter value zero) at the point q with initial tangent vector $\tilde{v}$. If the components $\{x^\mu\}$ are constrained to a sufficiently small neighborhood of the origin in the relevant real number space each such geodesic will extend (at parameter value unity) to a uniquely defined point $p \, \epsilon \, V$ in some (normal) neighborhood of the point q. More precisely one proves that this (exponential) mapping determines a diffeomorphism between a neighborhood of the origin in the relevant real number space and a corresponding neighborhood of the point q in the manifold V. As usual, such neighborhoods are called normal neighborhoods and the corresponding coordinates $\{x^\mu\}$ normal coordinates. This construction breaks down only when distinct geodesics emerging from q begin to intersect away from q.

Note that by construction one has $\tilde{e}_\mu = \frac{\partial}{\partial x^\mu} \big|_q$ though of course away from q the (normal) coordinate basis fields $\{\frac{\partial}{\partial x^\mu}\}$ will no longer be orthonormal. It is not difficult to show that when the metric and Christoffel connection are expressed in normal coordinates about q (with $x^\mu(q) = 0$) they obey

$$g_{\mu\nu}(0) = \eta_{\mu\nu}, \ \Gamma^\alpha_{\mu\nu}(0) = 0 \tag{3.1}$$

at the point q. More remarkable are the formulas

$$g_{\mu\nu}(x)x^\nu = g_{\mu\nu}(0)x^\nu = \eta_{\mu\nu}x^\nu \tag{3.2}$$

and

$$\Gamma^\alpha_{\mu\nu}(x)x^\mu x^\nu = 0 \tag{3.3}$$

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satisfied throughout an arbitrary normal coordinate chart [17]. We shall give an alternative proof of these equations later in this section.

An important feature of normal coordinates based at q is that the geodesics through q are expressed simply as straight lines in such coordinates. In other words the curves defined by

$$x^\mu(\lambda) = x^\mu \cdot \lambda \ , \ \ \lambda \, \epsilon \, [0, 1] \tag{3.4}$$

are all geodesics beginning at q for any $\{x^\mu(p)\}$ lying in the range of the chosen chart.

The geodesic with $x^\mu = x^\mu(p)$ connects q (at $\lambda = 0$) to p (at $\lambda = 1$) and is the unique geodesic, lying entirely within the chart domain, to have this property. Note that the tangent vector to this geodesic at the point p is given by $\tilde{v}_p = x^\mu(p)\frac{\partial}{\partial x^\mu}\mid_p$. Thus the vector field $\tilde{v} = x^\mu \frac{\partial}{\partial x^\mu}$ is, away from q, everywhere tangent to the geodesic from q which determines that arbitrary point p via the exponential map.

On any such normal coordinate chart domain we now introduce a preferred orthonormal frame field $\{h_{\hat{a}}\}$ as follows. Choose $h_{\hat{a}} \mid_q = \delta^\mu_{\hat{a}} \tilde{e}_\mu$ at the point q and extend each such frame field to a normal neighborhood of q by parallel propagation along the geodesics emerging from q in the construction of the normal chart. Such parallel propagation automatically preserves orthonormality and thus yields an orthonormal frame field $\{h_{\hat{a}}\}$ defined throughout the chart domain. The dual, co-frame field $\{\theta^{\hat{a}}\}$ can either be obtained algebraically by computing $\theta^{\hat{a}}_\mu = \eta^{\hat{a}\hat{b}} g_{\mu\nu} h_{\hat{b}}$ in the normal coordinate system or, equivalently, from parallel propagation of the co-frame field $\{\theta^{\hat{a}}\} \mid_q$ defined at q along the geodesics emerging from q. This works naturally since parallel propagation of both $\{\theta^{\hat{a}}\}$ and $\{h_{\hat{a}}\}$ along these geodesics automatically preserves the duality relations

$$< \theta^{\hat{a}}, h_{\hat{b}} > := \theta^{\hat{a}}_\mu h^\mu_{\hat{b}} = \delta^{\hat{a}}_{\hat{b}}. \tag{3.5}$$

Here and below we let $<,>$ signify the natural pairing of a one-form and a vector.

From the foregoing construction it follows that $\nabla_{\tilde{v}} h_{\hat{a}} = 0$ where $\tilde{v} = x^\mu \frac{\partial}{\partial x^\mu}$ is the geodesic tangent field previously defined and $\nabla_{\tilde{v}}$ is the directional covariant derivative

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operator. More explicitly this yields

$$(\nabla_{\tilde{v}} h_{\hat{a}})^{\mu} = v^{\nu}(h^{\mu}_{\hat{a},\nu} + \Gamma^{\mu}_{\gamma\nu}h^{\gamma}_{\hat{a}}) = x^{\nu}(h^{\mu}_{\hat{a},\nu} + \Gamma^{\mu}_{\gamma\nu}h^{\gamma}_{\hat{a}}) = 0. \tag{3.6}$$

Contracting with $\theta^{\hat{c}}_{\mu}$ one gets the equivalent equation

$$\theta^{\hat{c}}_{\mu}(\nabla_{\tilde{v}} h_{\hat{a}})^{\mu} = v^{\nu}(\theta^{\hat{c}}_{\mu}h^{\mu}_{\hat{a},\nu} + \theta^{\hat{c}}_{\mu}h^{\gamma}_{\hat{a}}\Gamma^{\mu}_{\gamma\nu}) = x^{\nu}(\theta^{\hat{c}}_{\mu}h^{\mu}_{\hat{a},\nu} + \theta^{\hat{c}}_{\gamma}h^{\mu}_{\hat{a}}\Gamma^{\gamma}_{\mu\nu}) = x^{\nu}\omega^{\hat{c}}{}_{\hat{a}\nu} = 0. \tag{3.7}$$

In other words parallel propagation of the orthonormal frame $\{h_{\hat{a}}\}$ along $\tilde{v}$ corresponds to the equation

$$< \omega^{\hat{c}}{}_{\hat{a}}, \tilde{v} >= \omega^{\hat{c}}{}_{\hat{a}\nu}x^{\nu} = 0 \tag{3.8}$$

holding throughout the normal coordinates chart where, as before $\omega^{\hat{c}}{}_{\hat{a}} = \omega^{\hat{c}}{}_{\hat{a}\nu}dx^{\nu}$ is the connection one-form defined by this choice of chart and frame.

Equation (3.8) is completely analogous to the Cronström gauge condition for a Yang-Mills connection $A^{\hat{a}}{}_{\hat{b}} = A^{\hat{a}}{}_{\hat{b}\nu}dx^{\nu}$ introduced in [8] and exploited in [5] and [6] to establish global existence for solutions to the Yang-Mills equation in flat and curved spacetimes respectively. In Yang-Mills theory the gauge condition,

$$A^{\hat{a}}{}_{\hat{b}\nu}x^{\nu} = 0 \tag{3.9}$$

(again imposed throughout a normal coordinate chart on spacetime) results from parallel propagation in the internal space whereas here it results from parallel propagation in the space of orthonormal frames tangent to spacetime. As in Yang-Mills theory one can exploit this choice of gauge to compute the connection one-forms $\omega^{\hat{c}}{}_{\hat{a}}$ directly from the curvature two-forms $R^{\hat{c}}{}_{\hat{a}}$, reversing the order of the usual calculation. In the chosen gauge Eq. (A17) gives immediately

$$x^{\nu}R^{\hat{c}}{}_{\hat{a}\mu\nu} = -x^{\nu}\frac{\partial}{\partial x^{\nu}}\omega^{\hat{c}}{}_{\hat{a}\mu} - \omega^{\hat{c}}{}_{\hat{a}\mu} \tag{3.10}$$

or, equivalently, along the geodesic curve $x^{\mu}(\lambda) = x^{\mu} \cdot \lambda$, that

$$-\frac{d}{d\lambda}[\lambda\omega^{\hat{c}}{}_{\hat{a}\mu}(x(\lambda)] = \lambda x^{\nu}R^{\hat{c}}{}_{\hat{a}\mu\nu}(x(\lambda)). \tag{3.11}$$

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Integrating this from $\lambda = 0$ to $\lambda = 1$ gives

$$\omega^{\hat{c}}{}_{\hat{a}\mu}(x) = -\int_0^1 d\lambda \ \lambda x^\nu R^{\hat{c}}{}_{\hat{a}\mu\nu}(x \cdot \lambda) \tag{3.12}$$

in exact parallel to Cronström's formula for $A^{\hat{c}}{}_{\hat{a}\mu}$ in terms of $F^{\hat{c}}{}_{\hat{a}\mu\nu}$.

In general relativity however, one can go further and compute the (co-) frame field $\{\theta^a\}$ (which has no analogue in Yang-Mills theory) directly in terms of the connection and hence in terms of curvature. To see this first note that the tangent vector to any of the (normal) geodesics through q is given by

$$\frac{dx^\mu(\lambda)}{d\lambda} = \frac{d}{d\lambda}(x^\mu \lambda) = x^\mu \tag{3.13}$$

and thus is independent of λ. Since this tangent vector is (by the definition of geodesics) parallel propagated along the geodesic its natural pairing with a parallel propagated one-form such as $\theta^{\hat{a}}$ is necesarily independent of the curve parameter λ. Equating these pairings at $\lambda = 0$ and $\lambda = 1$ gives

$$\theta_\nu^{\hat{a}}(0)x^\nu = \theta_\nu^{\hat{a}}(x)x^\nu \tag{3.14}$$

$\forall \{x^\nu\}$ within the normal neighborhood. Squaring this formula gives immediately

$$\eta_{\hat{a}\hat{b}}\theta_\nu^{\hat{a}}(0)\theta_\mu^{\hat{b}}(0)x^\nu x^\mu = g_{\mu\nu}(0)x^\mu x^\nu \tag{3.15}$$
$$= \eta_{\hat{a}\hat{b}}\theta_\nu^{\hat{a}}(x)\theta_\mu^{\hat{b}}(x)x^\nu x^\mu = g_{\mu\nu}(x)x^\mu x^\nu$$

which is related to, but weaker than, Equation (3.2). We shall reproduce the strong form momentarily.

The zero torsion condition is given by

$$\partial_\nu \theta_\mu^{\hat{c}}(x) - \partial_\mu \theta_\nu^{\hat{c}}(x)$$
$$+ \omega^{\hat{c}}{}_{\hat{a}\nu}(x)\theta_\mu^{\hat{a}}(x) - \omega^{\hat{c}}{}_{\hat{a}\mu}(x)\theta_\nu^{\hat{a}}(x)$$
$$= 0. \tag{3.16}$$

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Contracting this with x^ν and using Eq. (3.8) one obtains

$$x^\nu \partial_\nu(\theta_\mu^{\hat{c}}(x)) - \partial_\mu[x^\nu \theta_\nu^{\hat{c}}(x)]$$

$$+ \theta_\mu^{\hat{c}}(x) - \omega^{\hat{c}}{}_{\hat{a}\mu}(x)(x^\nu \theta_\nu^{\hat{a}}(x))$$

$$= 0. \tag{3.17}$$

But making use of the result in Eq. (3.14) we can reexpress this as

$$x^\nu \partial_\nu(\theta_\mu^{\hat{c}}(x)) - \partial_\mu[x^\nu \theta_\nu^{\hat{c}}(0)]$$

$$+ \theta_\mu^{\hat{c}}(x) - \omega^{\hat{c}}{}_{\hat{a}\mu}(x)[x^\nu \theta_\nu^{\hat{a}}(0)]$$

$$= x^\nu \partial_\nu(\theta_\mu^{\hat{c}}(x)) + \theta_\mu^{\hat{c}}(x) - \theta_\mu^{\hat{c}}(0) \tag{3.18}$$

$$- \omega^{\hat{c}}{}_{\hat{a}\mu}(x)[x^\nu \theta_\nu^{\hat{a}}(0)]$$

$$= 0$$

which can be written as

$$x^\nu \partial_\nu[\theta_\mu^{\hat{c}}(x) - \theta_\mu^{\hat{c}}(0)]$$

$$+ [\theta_\mu^{\hat{c}}(x) - \theta_\mu^{\hat{c}}(0)]$$

$$= \omega^{\hat{c}}{}_{\hat{a}\mu}(x)[\theta_\nu^{\hat{c}}(0)x^\nu], \tag{3.19}$$

a transport equation for the quantity $\theta_\mu^{\hat{c}}(x) - \theta_\mu^{\hat{c}}(0)$. Along a geodesic $x^\mu(\lambda) = x^\mu \cdot \lambda$. Through q one thus has

$$\frac{d}{d\lambda}[\lambda(\theta_\mu^{\hat{c}}(x(\lambda)) - \theta_\mu^{\hat{c}}(0))] \tag{3.20}$$

$$= \omega^{\hat{c}}{}_{\hat{a}\mu}(x(\lambda))[x^\nu \cdot \lambda \, \theta_\nu^{\hat{c}}(0)].$$

Integrating this form $\lambda = 0$ to $\lambda = 1$ one gets

$$\theta_\mu^{\hat{c}}(x) = \theta_\mu^{\hat{c}}(0) \tag{3.21}$$

$$+ \int_0^1 d\lambda[\omega^{\hat{c}}{}_{\hat{a}\mu}(\lambda x)(\lambda x^\nu \theta_\nu^{\hat{a}}(x))]$$

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which is the desired expression for $\theta_\mu^{\hat{c}}(x)$.

Combined with Eq. (3.12) this allows us to express both the connection and the frame one-forms directly in terms of curvature by explicit integral formulas. Given the (co-) frame $\{\theta^{\hat{a}}\}$ one can of course compute the frame fields $\{h_{\hat{a}}\}$ and the metric algebraically.

To show how Eq. (3.21) implies Eq. (3.2) we use the former to evaluate

$$
\begin{aligned}
g_{\mu\nu}(x)x^\nu &= \eta_{\hat{c}\hat{d}}\theta_\mu^{\hat{c}}(x)\theta_\nu^{\hat{d}}(x)x^\nu \\
&= \eta_{\hat{c}\hat{d}}(\theta_\mu^{\hat{c}}(0) + \int_0^1 d\lambda[\omega^{\hat{c}}{}_{\hat{a}\mu}(\lambda x)(\lambda x^\gamma \theta_\gamma^{\hat{a}}(0))]) \\
&\quad \times (\theta_\nu^{\hat{d}}(0)x^\nu + \int_0^1 d\sigma[\omega^{\hat{d}}{}_{\hat{b}\nu}(\sigma x)x^\nu(\sigma x^\delta \theta_\delta^{\hat{b}}(0))]) \\
&= (\eta_{\hat{c}\hat{d}}\theta_\mu^{\hat{c}}(0) + \int_0^1 d\lambda[\omega_{\hat{d}\hat{a}\mu}(\lambda x)(\lambda x^\gamma \theta_\gamma^{\hat{a}}(0))]) \\
&\quad \times \theta_\nu^{\hat{d}}(0)x^\nu \\
&= \eta_{\hat{c}\hat{d}}\theta_\mu^{\hat{c}}(0)\theta_\nu^{\hat{d}}(0)x^\nu + \int_0^1 d\lambda[\omega_{\hat{d}\hat{a}\mu}(\lambda x)\lambda x^\gamma \theta_\gamma^{\hat{a}}(0) \cdot x^\nu \theta_\nu^{\hat{d}}(0)] \\
&= g_{\mu\nu}(0)x^\nu
\end{aligned}
\tag{3.22}
$$

where we have used the parallel propagation condition, $\omega^{\hat{a}}{}_{\hat{b}\nu}(x)x^\nu = 0$, and the metric compatibility condition, $\omega_{\hat{a}\hat{b}\nu}(x) = -\omega_{\hat{b}\hat{a}\nu}(x)$, to simplify the intermediate expressions.

Equation (3.3) is normally proven directly from the geodesic equation specialized to normal coordinates. Using the duality relations $\theta_\mu^{\hat{a}} h_{\hat{c}}^\mu = \delta^{\hat{a}}{}_{\hat{c}}$ and $\theta_\mu^{\hat{a}} h_{\hat{a}}^\nu = \delta_\mu{}^\nu$ however, we can reexpress Eq. (A.13) in the equivalent form

$$
\Gamma_{\delta\nu}^\lambda = h_{\hat{a}}^\lambda \, \theta_\delta^{\hat{c}} \, \omega^{\hat{a}}{}_{\hat{c}\nu} + h_{\hat{a}}^\lambda \, \theta_{\delta,\nu}^{\hat{a}}.
\tag{3.23}
$$

Thus since $\omega^{\hat{a}}{}_{\hat{c}\nu}(x)x^\nu = 0$ we get

$$
\Gamma_{\delta\nu}^\lambda(x)x^\delta x^\nu = h_{\hat{a}}^\lambda(x)\theta_{\delta,\nu}^{\hat{a}}(x)x^\delta x^\nu.
\tag{3.24}
$$

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But using Eq. (3.14), one gets

$$
\begin{aligned}
x^\nu x^\delta (\theta^{\hat{a}}_{\delta,\nu}(x)) &= x^\nu \{\partial_\nu [\theta^{\hat{a}}_{\delta}(x) x^\delta] \\
&\quad - \theta^{\hat{a}}_{\nu}(x)\} = x^\nu \{\partial_\nu [\theta^{\hat{a}}_{\delta}(0) x^\delta] - \theta^{\hat{a}}_{\nu}(x)\} \\
&= x^\nu \{\theta^{\hat{a}}_{\nu}(0) - \theta^{\hat{a}}_{\nu}(x)\} = 0
\end{aligned}
\tag{3.25}
$$

where the last step follows from Eq. (3.21) and the parallel propagation condition $\omega^{\hat{c}}{}_{\hat{a}\nu}(x)x^\nu = 0$. Thus $\Gamma^{\lambda}_{\delta\nu}(x)x^\delta x^\nu = 0$ in normal coordinates.

IV. An Integral Equation for the Curvature Tensor

In Section II we rederived the fundamental wave equation satisfied by the curvature tensor of a vacuum spacetime and expressed this, via the Cartan formalism, as a curved space Yang-Mills equation coupled to the vanishing torsion condition. The latter equation, which relates the frame field determining the spacetime metric to the connection, has no analogue in a "pure" Yang-Mills problem but here of course provides the fundamental link between the metric and its curvature.

In the Cartan formalism wherein one regards the curvature tensor as a matrix of two-forms, $R^{\hat{a}}{}_{\hat{b}\mu\nu}dx^{\mu} \wedge dx^{\nu}$, or equivalently as a two-form with values in the matrix Lie algebra for the Lorenz group $SO(3,1)$, the wave operator (defined by the left-hand side of Eq. (2.18)) takes the form (for each separate matrix element) of the same wave operator that acts on the Faraday tensor $F_{\mu\nu}dx^{\mu} \wedge dx^{\nu}$ of a solution to Maxwell's equations. In particular, the frame indices play completely inert roles on the left-hand side of Eq. (2.18) which leaves the different matrix elements uncoupled.

We want to derive an integral equation satisfied by curvature by applying the fundamental solution for this wave operator to the "source" term defined by the right hand side of Eq. (2.18), using Eqs. (3.12) and (3.21) to eliminate the connection and frame in favor of curvature in much the same way that one previously used Cronström's formula to eliminate the Yang-Mills connection in favor of its curvature in studies of the flat and curved space pure Yang-Mills fields. The theory developed in Friedlander's book [9] (which builds on the fundamental work of Hadamard, Riesz, Sobolev, Choquet-Bruhat and others) applies to this wave operator (as well as to others we shall consider later) and allows one to write an integral formula for the solution of the corresponding Cauchy problem on so-called causal domains of the spacetime (i.e., on geodesically convex domains which are also globally hyperbolic in a suitable sense [18]). For Friedlander, who treats only linear problems, the integral formula in question is a genuine representation

formula for the solution of the associated wave equation whereas for us it only yields an integral equation satisfied by the relevant solution to the Cauchy problem.

Of course not every solution to Eq. (2.18) corresponds to a solution of Einstein's field equations. It is necessary, in order to avoid introducing spurious solutions, to restrict the Cauchy data appearing in the Friedlander formula by imposing those first order equations upon the curvature which results from the Bianchi identities when the Ricci tensor vanishes (the vacuum condition). The Friedlander formalism applies to all solutions of the relevant wave equation and hence in particular to the solutions of physical interest.

To simplify the notation, let us write $F_{\mu\nu}$ for any particular matrix element $R^{\hat{a}}{}_{\hat{b}\mu\nu}$ of curvature (surpressing the inert frame indices $\hat{a}, \hat{b}$) and $f_{\mu\nu}$ for the corresponding source term so that Eq. (2.18) now takes the form

$$F_{\mu\nu;\gamma}{}^{;\gamma} + R_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta} = f_{\mu\nu}. \tag{4.1}$$

With reference to Fig. 5.3.1 of Friedlander's book, let p be a point in some causal domain of $(^{(4)}V, g)$ and S be a spacelike hypersurface within this domain such that every past-directed causal geodesic from p meet S. Further, let C_p be the mantle of the (truncated) past light cone from p to S, σ_p be the (two-dimensional) intersection of C_p with S and let D_p be the interior of this truncated cone and designate by S_p the (three-dimensional) intersection of D_p with S. Finally, let T_p designate the expanding lightlike hypersurface which intersects S in σ_p.

Friedlander's representation formula for the field at point p is given in local coordi-

nates by [19]:

$$F_{\alpha\beta}(x) = \frac{1}{2\pi} \int\limits_{C_p} U^{\mu'\nu'}_{\alpha\beta}(x,x') f_{\mu'\nu'}(x') \mu_\Gamma(x')$$

$$+ \frac{1}{2\pi} \int\limits_{D_p} (V^+)^{\mu'\nu'}_{\alpha\beta}(x,x') f_{\mu'\nu'}(x') \mu(x')$$

$$+ \frac{1}{2\pi} \int\limits_{S_p} *[(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x') \nabla^{\gamma'} F_{\mu'\nu'}(x') \tag{4.2}$$

$$- F_{\mu'\nu'}(x') \nabla^{\gamma'} (V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')]$$

$$+ \frac{1}{2\pi} \int\limits_{\sigma_p} \{ U^{\mu'\nu'}_{\alpha\beta}(x,x') [2(\nabla^{\gamma'} t(x'))(\nabla_{\gamma'} F_{\mu'\nu'}(x'))$$

$$+ F_{\mu'\nu'}(x') \,\Box\, t(x')]$$

$$- \langle \nabla t(x'), \nabla' \Gamma(x,x') \rangle (V^+)^{\mu'\nu'}_{\alpha\beta}(x,x') F_{\mu'\nu'}(x') \} \mu_{t,\Gamma}(x').$$

Here $U^{\mu'\nu'}_{\alpha\beta}(x,x') = \kappa(x,x') \tau^{\mu'\nu'}_{\alpha\beta}(x,x')$ where κ is the transport biscalar defined by Eq. (4.2.17) of Ref. [9] and given in local coordinates by Eq. (4.2.18) or (4.2.19) of that reference and $\tau^{\mu'\nu'}_{\alpha\beta}(x,x')$ is the transport bitensor (or propagator) defined in Section (5.5) of Friedlander. The latter is expressible explicitly in terms of an orthonormal frame parallel propagated from p along the geodesic issuing from that point.

The measure $\mu(x')$ is the standard spacetime volume measure given in local coordinates by $\sqrt{-\det g_{\mu\nu}(x')} d^4 x'$ whereas the measure on the light cone $\mu_\Gamma(x')$ is a Leray form defined such that

$$d_{x'} \Gamma(x,x') \wedge \mu_\Gamma(x') = \mu(x') \tag{4.3}$$

where $\Gamma(x,x')$ is the optical function (squared geodesic distance within a causal domain) introduced in Sect. (1.2) of Friedlander (c.f., Theorem 1.2.3). Leray forms are introduced in Sect. (2.9) and developed further in Sect. (4.5) of this same reference and the coordinate expression for the dual $*v$ of a vector v is given there by Eq. (2.9.3). This is needed in the boundary integral over S_p whereas μ_Γ arises in that over C_p. The

two-dimensional Leray form $\mu_{t,\Gamma}(x')$ needed for the integral over σ_p, is defined such that (c.f., Lemma 5.3.3. of Ref. [9])

$$dt(x') \wedge d_{x'}\Gamma(x, x') \wedge \mu_{t,\Gamma}(x') = \mu(x') \tag{4.4}$$

where $t(x')$ is the null field defined by Lemma 5.3.2 of Friedlander. Note also in this reference the needed expressions for $(\Box\, t)\mu_{t,\Gamma}$ and $\langle \nabla t, \nabla \Gamma \rangle$ given respectively by Eqs. (5.3.20) and (5.3.19) of this same section.

The tail field $(V^+)^{\mu'\nu'}_{\alpha\beta}(x, x')$ is the solution of a characteristic initial value problem for the homogeneous wave equation. By virtue of the self-adjoincy of our Eq. (4.1) and the reciprocity relations derived by Friedlander in Sect. (5.2) (which apply as well to the tensor case as discussed in Sect. (5.5)) the tail bitensor V^+ satisfies the wave equation

$$
\begin{aligned}
&(V^+)^{\mu'\nu';\gamma'}_{\alpha\beta;\gamma'}(x, x') + R^{\mu'\nu'}{}_{\delta'\gamma'}(x')(V^+)^{\delta'\gamma'}_{\alpha\beta}(x, x') \\
&+ R^{\mu'}_{\delta'}(x')(V^+)^{\nu'\delta'}_{\alpha\beta}(x, x') - R^{\nu'}_{\delta'}(x')(V^+)^{\mu'\delta'}_{\alpha\beta}(x, x') \\
&= 0
\end{aligned}
\tag{4.5}
$$

wherein the indices $\alpha\beta$ and coordinates x^μ play inert roles. In the foregoing formulas, as well as below, the notation ∇_γ and $;\gamma$ are used interchangeably. The initial data for V^+ is computable on the light cone C_p where it reduces to the bitensor field that Friedlander expresses as V_0. The transport equation determining V_0 is provided by Friedlander's Eq. (5.5.23) and its explicit solution is given in his Eq. (5.5.25).

V. Transformations of the Tail Field Integrals

Define the tail field contributions to $F_{\alpha\beta}(x)$ by

$$F_{\alpha\beta}^{\text{tail}}(x) := \frac{1}{2\pi} \int\limits_{D_p} (V^+)_{\alpha\beta}^{\mu'\nu'}(x,x') f_{\mu'\nu'}(x')\mu(x')$$

$$+ \frac{1}{2\pi} \int\limits_{S_p} *[(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')\nabla^{\gamma'} F_{\mu'\nu'}(x') \tag{5.1}$$

$$- F_{\mu'\nu'}(x')\nabla^{\gamma'}(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')]$$

$$- \frac{1}{2\pi} \int\limits_{\sigma_p} \langle \nabla t(x'), \nabla'\Gamma(x,x')\rangle (V^+)_{\alpha\beta}^{\mu'\nu'}(x,x') F_{\mu'\nu'}(x')\mu_{t,\Gamma}(x').$$

This consists of all the terms that would vanish if Huygen's principle were valid since in that case $V^+ = 0$ but, in a curved spacetime, these terms are generally non-zero.

Let us reexpress the source f through the use of the wave equation for F as

$$f_{\mu'\nu'}(x') = (PF)_{\mu'\nu'}(x') \tag{5.2}$$

where P is the second order linear, self-adjoint operator defined by the left hand side of Eq. (4.1). Recalling Eq. (4.5) which can be written as

$$(PV^+)_{\alpha\beta}^{\mu'\nu'}(x,x') = 0 \tag{5.3}$$

where P acts at x' and the indices α, β and x are inert, one finds that the integrand $(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x') f_{\mu'\nu'}(x')$ can be expressed as

$$(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x') f_{\mu'\nu'}(x') =$$

$$(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')(PF)_{\mu'\nu'}(x') - (PV^+)_{\alpha\beta}^{\mu'\nu'}(x,x') F_{\mu'\nu'}(x')$$

$$= \nabla_{\gamma'}\{(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')(\nabla^{\gamma'} F_{\mu'\nu'}(x')) \tag{5.4}$$

$$- (\nabla^{\gamma'}(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')) F_{\mu'\nu'}(x')\}$$

where the curvature terms have canceled from the final expression by virtue of the self-adjoint structure of the wave operator P. Thus the integrand in the volume integral

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over D_p can be reexpressed as a total divergence. It is worth noting that the scalar field analogue to the above observation is given at the end of p.187 in Friedlander's book.

Using Eq. (5.4) to reexpress the integral over D_p in the equation for $F_{\alpha\beta}^{\text{tail}}(x)$ and using Stokes' theorem to rewrite this volume integral as a boundary integral over $\partial D_p = C_p \cup S_p$, one arrives at the result that

$$
\begin{aligned}
F_{\alpha\beta}^{\text{tail}}(x) = \frac{1}{2\pi} \int\limits_{C_p} &* [(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')\nabla^{\gamma'} F_{\mu'\nu'}(x') \\
& - F_{\mu'\nu'}(x')\nabla^{\gamma'}(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')] \\
& - \frac{1}{2\pi} \int\limits_{\sigma_p} \langle \nabla t(x'), \nabla'\Gamma(x,x')\rangle (V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x')\mu_{t,\Gamma}(x')
\end{aligned}
\tag{5.5}
$$

where the orientation chosen for the integral over the null cone C_p corresponds to a normal field directed towards the vertex p. The cancelation of the two boundary integrals over S_p parallels that shown by Friedlander for the scalar case in his Eq. (5.3.14) (wherein however it was assumed that the support of the scalar field did not meet C_p). One can also think of deriving Eq. (5.5) from Eq. (5.1) by pushing the surface S_p forward, holding its boundary σ_p fixed, until it merges in the limit with C_p. Friedlander remarks in his Section (5.4) that the representation formula for the characteristic initial value problem can be derived in a similar manner wherein, however, one pushes $S_p$towards the past rather than towards the future.

Though we have succeeded to reexpress the tail contributions in terms of integrals only over C_p and σ_p the resulting formula is still not in a satisfactory state from the point of view of the ultimate applications we have in mind. This is so, in large measure, because Eq. (5.5) contains derivatives of the unknown curvature and it would be hopeless to try to derive estimates for the undifferentiated curvature from an integral equation involving the derivatives of this same quantity.

Fortunately, however, in the integral over C_p in Eq. (5.5) for $F_{\alpha\beta}^{\text{tail}}(x)$ only deriva-

tives of $(V^+)^{\mu'\nu'}_{\alpha\beta}(x, x')$ and $F_{\mu'\nu'}(x')$ tangential to the null generators of the light cone are involved. The point is that since C_p is a null surface its normal ($\nabla^{\gamma'}\Gamma(x, x')$ in Friedlander's notation) is in fact tangential to the cone and hence the dual operator ($*$ in Eq. (5.5)) produces only these tangential derivatives in the integrand. Thus one is at liberty to integrate by parts and throw the directional derivative onto V^+ for example and thereby remove it from F. In effect, Friedlander exploited this freedom (though in the opposite way) in recasting the integral over $C_{p'}$ in his representation formula for the characteristic initial value problem into a form in which only tangential derivatives of F were involved. For our purposes, though it is essential to avoid the necessity of computing tangential derivatives to F and to recall that the tangential derivative of V^+ is given rather explicitly by Friedlander's Eq. (5.5.23) for this latter quantity (which coincides with V_0 on C_p). On the other hand, this integration by parts produces an additional contribution to the integral over C_p (since $\nabla^{\gamma'}\Gamma(x, x')$ gets differentiated) and a boundary contribution which modifies the integral over σ_p. We shall carry out these further reductions in the following section and thereby arrive at our final integral equation for curvature within the framework of the Cartan formalism.

The reader may be wondering though why it should be possible, as we have argued, to transform the tail contributions, which result from the failure of Huygen's principle to hold in a general spacetime, into a form (involving only integrals over C_p and σ_p) which seems to have miraculously restored Huygen's principle. The resolution of this seeming paradox results from noting that even for a truly linear problem (where the meaning of Huygen's principle is clearly defined) the transformed "representation" formula requires knowledge of the unknown field $F_{\mu\nu}$, on the light cone C_p and not merely on σ, the intersection of the cone with the initial hypersurface. Thus the transformed equation is not really a representation formula at all, even in the linear case, whereas initially (in Eq. (4.2)) it was. For the non-linear problems that we are interested in however, a

genuine representation formula (for the solution of the Cauchy problem) is out of the question and it is far more convenient to have the tail contributions transformed, as we have done, to integrals over C_p and σ_p alone.

VI. Reduction of the Tail Contributions

To simplify the notation slightly let us write Eq. (5.5) in the form

$$F_{\alpha\beta}^{\text{tail}}(x) = {}^{I}F_{\alpha\beta}^{\text{tail}}(x) + {}^{II}F_{\alpha\beta}^{\text{tail}}(x) \tag{6.1}$$

where ${}^{I}F_{\alpha\beta}^{\text{tail}}(x)$ is the integral over C_p and ${}^{II}F_{\alpha\beta}^{\text{tail}}(x)$ that over σ_p. Reexpressing the dual $*v$ to a vector v via Eq. (2.9.3) of Ref. [9] (see also p. 194 of this reference)

$$*v(x') = \langle v(x'), \text{grad}' \Gamma(x, x') \rangle \mu_\Gamma(x') \tag{6.2}$$

one gets the more explicit formula for ${}^{I}F_{\alpha\beta}^{\text{tail}}(x)$

$$
{}^{I}F_{\alpha\beta}^{\text{tail}}(x) = \frac{1}{2\pi} \int\limits_{C_p} \mu_\Gamma(x') \{ \nabla^{\gamma'} \Gamma(x, x') [(V^+)_{\alpha\beta}^{\mu'\nu'}(x, x') \nabla_{\gamma'} F_{\mu'\nu'}(x') \tag{6.3}
$$
$$
- F_{\mu'\nu'}(x') \nabla_{\gamma'} (V^+)_{\alpha\beta}^{\mu'\nu'}(x, x')] \}.
$$

The key point here is that only derivatives tangential to the null generators of the cone C_p appear in the integrand. This allows one to integrate by parts to eliminate derivatives of $F_{\mu'\nu'}$ in favor of (tangential) derivatives of $(V^+)_{\alpha\beta}^{\mu'\nu'}$ which, in turn, may be evaluated from the transport equation (c.f. Eq. (5.5.23) of Ref. [9]) which determines this quantity along C_p. Carrying out these operations and writing $(V_0)_{\alpha\beta}^{\mu'\nu'}(x, x')$ for the restriction of $(V^+)_{\alpha\beta}^{\mu'\nu'}(x, x')$ to C_p one arrives at

$$
{}^{I}F_{\alpha\beta}^{\text{tail}}(x) = \frac{1}{2\pi} \int\limits_{C_p} \mu_\Gamma(x') \{ (\nabla^{\gamma'} \Gamma(x, x')) \nabla_{\gamma'} ((V_0)_{\alpha\beta}^{\mu'\nu'}(x, x') F_{\mu'\nu'}(x')) \tag{6.4}
$$
$$
+ F_{\mu'\nu'}(x') [PU_{\alpha\beta}^{\mu'\nu'}(x, x') + (\Box' \Gamma(x, x') - 4)(V_0)_{\alpha\beta}^{\mu'\nu'}(x, x')] \}
$$

where P is the wave operator defined in Eq. (5.2) above and where, as mentioned above, we can write

$$U_{\alpha\beta}^{\mu'\nu'}(x, x') = \kappa(x, x') \tau_{\alpha\beta}^{\mu'\nu'}(x, x') \tag{6.5}$$

with the parallel transport "propagator" $\tau^{\mu'\nu'}_{\alpha\beta}$ expressible in terms of our orthonormal frame as

$$\tau^{\mu'\nu'}_{\alpha\beta}(x,x') = h^{\mu'}_{\hat{e}}(x')\theta^{\hat{e}}_{\alpha}(x)h^{\nu'}_{\hat{f}}(x')\theta^{\hat{f}}_{\beta}(x). \tag{6.6}$$

One can evaluate the first integral in the above expression for ${}^I F^{\text{tail}}_{\alpha\beta}(x)$ by first transforming from normal coordinates $\{x^{\mu'}\}$ to spherical null coordinates defined by

$$x^{1'} = r' \sin\theta \cos\varphi$$

$$x^{2'} = r' \sin\theta \sin\varphi$$

$$x^{3'} = r' \cos\theta \tag{6.7}$$

$$t' = x^{0'} = \frac{u+v}{2},\ r' = \frac{v-u}{2}$$

$$r' = \sqrt{\Sigma(x^{i'})^2}$$

so that

$$u = t' - r',\ v = t' + r' \tag{6.8}$$

with $\Gamma = -uv$ everywhere and $v = 0$ on C_p. In terms of these coordinates it is straightforward to show that

$$\Gamma^{;\alpha}\frac{\partial}{\partial x^\alpha} = 2v\frac{\partial}{\partial v} + 2u\frac{\partial}{\partial u} \tag{6.9}$$

and that the Leray form

$$\mu_\Gamma = \frac{\sqrt{-\det(g_{\mu\nu})}}{u}du \wedge d\theta \wedge d\varphi \tag{6.10}$$

satisfies

$$\mu = d\Gamma \wedge \mu_\Gamma = \sqrt{-\det(g_{\mu\nu})}du \wedge dv \wedge d\theta \wedge d\varphi \tag{6.11}$$

as required by its definitions (where $\det g_{\mu\nu}$ is the determinant of g in the spherical null coordinates). Substituting these expressions into the integral in question one easily

arrives at

$$\frac{1}{2\pi}\int\limits_{C_p} \mu_\Gamma(x')(\nabla^{\gamma'}\Gamma(x,x'))\nabla_{\gamma'}[(V_0)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x')]$$

$$=\frac{1}{2\pi}\int\limits_{C_p} du\wedge d\theta\wedge d\varphi[\frac{\partial}{\partial u}[2\sqrt{-\det(g_{\gamma\delta})}(V_0)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x')]]$$

$$+\frac{1}{2\pi}\int\limits_{C_p} \mu_\Gamma(x')[(4-\nabla_{\gamma'}\nabla^{\gamma'}\Gamma(x,x'))(V_0)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x')] \qquad (6.12)$$

$$=-\frac{1}{2\pi}\int\limits_{\sigma_p} d\theta\wedge d\varphi\{2\sqrt{-\det(g_{\gamma\delta})}[(V_0)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x')]\}$$

$$+\frac{1}{2\pi}\int\limits_{C_p} \mu_\Gamma(x')[(4-\Box'\Gamma(x,x'))(V_0)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x')].$$

Evaluating the metric form restricted to C_p one gets

$$ds^2\big|_{C_p}= -dudv +\ ^{(2)}V_\theta dvd\theta +\ ^{(2)}V_\varphi dvd\varphi \qquad (6.13)$$

$$^{(2)}g_{AB}dx^A dx^B + (-\frac{1}{4}\ ^{(4)}g^{uu} + \frac{1}{4}\ ^{(2)}g_{AB}\ ^{(2)}V^A\ ^{(2)}V^B)dv^2$$

where $\{x^A; A = 1, 2\} = \{\theta, \varphi\}$ and where $^{(2)}g_{AB}dx^A dx^B$ and

$^{(2)}V_A dx^A =\ ^{(2)}g_{AB}\ ^{(2)}V^B dx^A$ are (at each fixed u on the hypersurface C_p defined by

$v = 0$) a 2-dimensional Riemannian metric and one-form respectively. Thus, on C_p

$$2\sqrt{-\det g_{\gamma\delta}}\big|_{C_p}= \sqrt{\det\ ^{(2)}g_{AB}}\big|_{C_p} \qquad (6.14)$$

so that

$$\frac{1}{2\pi}\int\limits_{C_p} \mu_\Gamma(x')\{(\nabla^{\gamma'}\Gamma(x,x'))\nabla_{\gamma'}[(V_0)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x')]\} \qquad (6.15)$$

$$=-\frac{1}{2\pi}\int\limits_{\sigma_p} \sqrt{\det\ ^{(2)}g_{AB}}d\theta\wedge d\varphi[(V_0)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x')]$$

$$+\frac{1}{2\pi}\int\limits_{C_p} \mu_\Gamma(x')[(4-\Box'\Gamma(x,x'))(V_0)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x')]$$

It is easy to see from the metric form (6.13) that $\sqrt{\det {}^{(2)}g_{AB}}d\theta \wedge d\varphi$ is just the invariant 2-surface area element induced on σ_p (defined in coordinates by $v = 0$, $u = u(\theta, \varphi)$) by the spacetime metric. Writing this as $d\sigma_p$ and combining Eqs. (6.4) and (6.15) we get

$$
{}^I F_{\alpha\beta}^{\text{tail}}(x) = -\frac{1}{2\pi} \int\limits_{\sigma_p} d\sigma_p [(V_0)_{\alpha\beta}^{\mu'\nu'}(x, x')F_{\mu'\nu'}(x')] \tag{6.16}
$$

$$
+ \frac{1}{2\pi} \int\limits_{C_p} \mu_\Gamma(x')F_{\mu'\nu'}(x')(PU_{\alpha\beta}^{\mu'\nu'}(x, x'))
$$

where the terms involving $(\Box'\Gamma(x, x') - 4)$ have cancelled. Adding this result to the expression for ${}^{II}F_{\alpha\beta}^{\text{tail}}(x)$ and recalling Friedlander's formula for the measure $\mu_{t,\Gamma}(x')$ given by his Eq. (5.3.19),

$$
\langle \nabla t, \nabla \Gamma \rangle \mu_{t,\Gamma} = -d\sigma_p \tag{6.17}
$$

one finds that the two remaining integrals in $F_{\alpha\beta}^{\text{tail}}(x)$ involving the non-local quantity $(V_0)_{\alpha\beta}^{\mu'\nu'}(x, x')$ also cancel leaving

$$
F_{\alpha\beta}^{\text{tail}}(x) = \frac{1}{2\pi} \int\limits_{C_p} \mu_\Gamma(x')(F_{\mu'\nu'}(x')PU_{\alpha\beta}^{\mu'\nu'}(x, x')) \tag{6.18}
$$

so that our expression for $F_{\alpha\beta}(x)$ (c.f. Eq. (5.1)) now becomes

$$
F_{\alpha\beta}(x) = \frac{1}{2\pi} \int\limits_{C_p} U_{\alpha\beta}^{\mu'\nu'}(x, x')f_{\mu'\nu'}(x')\mu_\Gamma(x')
$$

$$
\frac{1}{2\pi} \int\limits_{\sigma_p} \{U_{\alpha\beta}^{\mu'\nu'}(x, x')[2(\nabla^{\gamma'}t(x'))(\nabla_{\gamma'}F_{\mu'\nu'}(x')) \tag{6.19}
$$

$$
+ F_{\mu'\nu'}(x')\Box't(x')]\}\mu_{t,\Gamma}(x')
$$

$$
+ \frac{1}{2\pi} \int\limits_{C_p} \mu_\Gamma(x')(F_{\mu'\nu'}(x')PU_{\alpha\beta}^{\mu'\nu'}(x, x')).
$$

The integral over σ_p in the above formula involves first derivatives of the unknown field $F_{\alpha\beta}$ but only on the initial, Cauchy hypersurface where these quantities must be given.

Upon substituting the explicit form for the source terms $f_{\mu'\nu'}(x')$ into Eq. (6.19) we shall encounter integrals of the type

$$I = \frac{1}{2\pi} \int_{C_p} \mu_\Gamma(x')(\nabla^{\gamma'}\Omega_{\gamma'}) \tag{6.20}$$

where $\Omega_{\gamma'}$ is a one-form which (thanks to its explicit dependence upon $\omega^{\hat{a}}{}_{\hat{b}\gamma'}$ which satisfies the Cronström gauge condition) obeys $\Gamma^{;\gamma'}\Omega_{\gamma'} = 0$ everywhere throughout the causal domain containing C_p. This special fact allows us to successfully integrate the 4-divergence over the 3-manifold C_p and obtain a boundary integral over σ_p. In deriving this result, we must compute derivatives of the equation $\Gamma^{;\gamma'}\Omega_{\gamma'} = 0$ in directions transversal to the cone C_p so it is essential that this equation hold not just on C_p but (at least to first order) off the cone as well.

By introducing coordinates $\{\overline{x^\mu}\} = \{\bar{u}, \bar{v}, \bar{\theta}, \bar{\varphi}\}$ of the form

$$\bar{u} = \bar{u}(u, \theta, \varphi), \ \bar{v} = v, \ \bar{\theta} = \theta, \ \bar{\varphi} = \varphi \tag{6.21}$$

adapted to the domain of integration so that σ_p coincides with a surface $\bar{u} = $ constant lying in C_p one can carry out the integration explicitly to find that

$$I = \frac{1}{2\pi} \int_{\sigma_p} d\sigma_p(\xi^\mu\Omega_\mu) \tag{6.22}$$

where, as before, $d\sigma_p$ is the invariant surface area element induced upon σ_p by the spacetime metric and in which $\xi^\mu\partial_\mu$ is a future pointing null vector, orthogonal to σ_p and normalized such that

$$\xi^\mu\Gamma_{;\mu} = 1. \tag{6.23}$$

In Friedlander's terminology, this vector is tangent to the null generators of the null surface T_p which contains σ_p. As we shall see, the boundary term arising in this way will combine naturally with the integral over σ_p in Eq. (6.19).

We now reinstate the heretofore inert indices on the curvature and its source by letting $F_{\mu\nu} \to R^{\hat{a}}{}_{\hat{b}\mu\nu}$ and $f_{\mu\nu} \to f^{\hat{a}}{}_{\hat{b}\mu\nu}$ so that Eq. (6.19) becomes

$$
R^{\hat{a}}{}_{\hat{b}\alpha\beta}(x) = -\frac{1}{2\pi} \int\limits_{C_p} (U^{\mu'\nu'}_{\alpha\beta}(x,x') f^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x')) \mu_\Gamma(x')
$$

$$
+ \frac{1}{2\pi} \int\limits_{\sigma_p} \{U^{\mu'\nu'}_{\alpha\beta}(x,x')[2(\nabla^{\gamma'} t(x'))(\nabla_{\gamma'} R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x')) \tag{6.24}
$$

$$
+ R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x') \Box' t(x')]\} \mu_{t,\Gamma}(x')
$$

$$
+ \frac{1}{2\pi} \int_{C_p} \mu_\Gamma(x')(R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x')(PU^{\mu'\nu'}_{\alpha\beta}(x,x'))).
$$

Upon inserting the explicit formula for $f^{\hat{a}}{}_{\hat{b}\mu'\nu'}$ from Eq. (2.18) and rewriting it slightly one finds that it contains the divergence integral

$$
\mathcal{D}^{\hat{a}}{}_{\hat{b}\alpha\beta}(x) := \frac{1}{2\pi} \int\limits_{C_p} \mu_\Gamma(x')\{\nabla^{\sigma'}[2\omega^{\hat{a}}{}_{\hat{c}\sigma'}(x') U^{\mu'\nu'}_{\alpha\beta}(x,x') R^{\hat{c}}{}_{\hat{b}\mu'\nu'}(x')
$$

$$
- 2\omega^{\hat{c}}{}_{\hat{b}\sigma'} U^{\mu'\nu'}_{\alpha\beta}(x,x') R^{\hat{a}}{}_{\hat{c}\mu'\nu'}(x')]\} \tag{6.25}
$$

which includes the only terms in the integrals over C_p which contain derivatives of curvature. Exploiting the argument above to reduce this expression to an integral over σ_p one finds that

$$
\mathcal{D}^{\hat{a}}{}_{\hat{b}\alpha\beta}(x) = \frac{1}{2\pi} \int\limits_{\sigma_p} d\sigma_p \{\xi^{\sigma'}[2\omega^{\hat{a}}{}_{\hat{c}\sigma'}(x') U^{\mu'\nu'}_{\alpha\beta}(x,x') R^{\hat{c}}{}_{\hat{b}\mu'\nu'}(x') \tag{6.26}
$$

$$
- 2\omega^{\hat{c}}{}_{\hat{b}\sigma'}(x') U^{\mu'\nu'}_{\alpha\beta}(x,x') R^{\hat{a}}{}_{\hat{c}\mu'\nu'}(x')]\}.
$$

The remaining integral over σ_p in Eq. (6.24) can be reexpressed, thanks to Eq. (6.17) as

$$
\mathcal{S}^{\hat{a}}{}_{\hat{b}\alpha\beta}(x) := \frac{1}{2\pi} \int\limits_{\sigma_p} \{U^{\mu'\nu'}_{\alpha\beta}(x,x')[2(\nabla^{\gamma'} t(x'))(\nabla_{\gamma'} R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x'))
$$

$$
+ R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x') \Box' t(x')]\} \mu_{t,\Gamma}(x')
$$

$$
= \frac{1}{2\pi} \int\limits_{\sigma_p} d\sigma_p \{2 U^{\mu'\nu'}_{\alpha\beta}(x,x')(\xi^{\sigma'} \nabla_{\sigma'} R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x')) \tag{6.27}
$$

$$
+ U^{\mu'\nu'}_{\alpha\beta}(x,x') R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x,x') \left(\frac{\Box' t(x')}{(\langle\nabla' t, \nabla'\Gamma\rangle(x'))}\right)\}.
$$

Defining (via Friedlander's Eqs. (5.3.7) and (5.3.20)) the dilation θ of $d\sigma_p$ along the bicharacteristics of T_p by

$$\theta(x') = \frac{\Box't(x')}{\langle \nabla't, \nabla'\Gamma\rangle(x')} \tag{6.28}$$

and combining the integrals $\mathcal{D}^{\hat{a}}{}_{\hat{b}\alpha\beta}(x)$ and $\mathcal{S}^{\hat{a}}{}_{\hat{b}\alpha\beta}(x)$ one gets

$$
\begin{aligned}
\mathcal{D}^{\hat{a}}{}_{\hat{b}\alpha\beta}&(x) + \mathcal{S}^{\hat{a}}{}_{\hat{b}\alpha\beta}(x) \\
&= \frac{1}{2\pi} \int_{\sigma_p} d\sigma_p \{ 2 U^{\mu'\nu'}_{\alpha\beta}(x,x')(\xi^{\sigma'} D_{\sigma'} R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x')) \\
&\quad + U^{\mu'\nu'}_{\alpha\beta}(x,x') R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x')\theta(x') \}
\end{aligned} \tag{6.29}
$$

where now $D_{\sigma'}$ is the *total* spacetime covariant derivative defined in Section II. The addition of $\mathcal{D}^{\hat{a}}{}_{\hat{b}\alpha\beta}$ to $\mathcal{S}^{\hat{a}}{}_{\hat{b}\alpha\beta}$ has contributed precisely the terms needed to convert $\nabla_{\sigma'}$ to $D_{\sigma'}$ in the formula above.

Writing out the factor $PU^{\mu'\nu'}_{\alpha\beta}(x,x')$ more explicitly as

$$
\begin{aligned}
PU^{\mu'\nu'}_{\alpha\beta}(x,x') &= \nabla^{\gamma'}\nabla_{\gamma'} U^{\mu'\nu'}_{\alpha\beta}(x,x') \\
&\quad + R^{\mu'\nu'}{}_{\delta'\gamma'}(x') U^{\delta'\gamma'}_{\alpha\beta}(x,x') \\
&= \left(\frac{\nabla^{\gamma'}\nabla_{\gamma'}\kappa(x,x')}{\kappa(x,x')}\right) U^{\mu'\nu'}_{\alpha\beta}(x,x') + R^{\mu'\nu'}{}_{\delta'\gamma'}(x') U^{\delta'\gamma'}_{\alpha\beta}(x,x') \\
&\quad + 2(\nabla^{\gamma'}\kappa(x,x'))(\nabla_{\gamma'}\tau^{\mu'\nu'}_{\alpha\beta}(x,x')) \\
&\quad + \kappa(x,x')(\nabla^{\gamma'}\nabla_{\gamma'}(\tau^{\mu'\nu'}_{\alpha\beta}(x,x'))),
\end{aligned} \tag{6.30}
$$

where $\tau^{\mu'\nu'}_{\alpha\beta}(x,x')$ is defined via Eq. (6.6), one can evaluate the derivatives of $\tau^{\mu'\nu'}_{\alpha\beta}(x,x')$ using Eqs. (A.10) - (A.13) which yield

$$h^{\gamma}{}_{\hat{a};\nu} = h^{\gamma}{}_{\hat{c}}\omega^{\hat{c}}{}_{\hat{a}\nu} \tag{6.31}$$

so that

$$
\begin{aligned}
\nabla_{\gamma'}\tau^{\mu'\nu'}_{\alpha\beta}(x,x') &= \omega^{\hat{d}}{}_{\hat{e}\gamma'} h^{\mu'}_{\hat{d}}(x')\theta^{\hat{e}}_{\alpha}(x) h^{\nu'}_{\hat{f}}(x')\theta^{\hat{f}}_{\beta}(x) \\
&\quad + \omega^{\hat{d}}{}_{\hat{f}\gamma'} h^{\mu'}_{\hat{e}}(x')\theta^{\hat{e}}_{\alpha}(x) h^{\nu'}_{\hat{d}}(x')\theta^{\hat{f}}_{\beta}(x)
\end{aligned} \tag{6.32}
$$

with a similar expanded formula for $\nabla^{\gamma'}(\nabla_{\gamma'}\tau^{\mu'\nu'}_{\alpha\beta}(x,x'))$. The latter will clearly entail factors of the type $(\nabla^{\gamma'}\omega^{\hat{d}}{}_{\hat{e}\gamma'})$ as well as factors quadratic in the connection coefficients $\omega^{\hat{d}}{}_{\hat{e}\gamma'}$. Written out explicitly it becomes:

$$
\begin{aligned}
\nabla^{\gamma'}\nabla_{\gamma'}\tau^{\mu'\nu'}_{\alpha\beta}(x,x') &= (\nabla^{\gamma'}\omega^{\hat{d}}{}_{\hat{e}\gamma'})h^{\mu'}_{\hat{d}}(x')\theta^{\hat{e}}_{\alpha}(x)h^{\nu'}_{\hat{f}}(x')\theta^{\hat{f}}_{\beta}(x) \\
&+ (\nabla^{\gamma'}\omega^{\hat{d}}{}_{\hat{f}\gamma'})h^{\mu'}_{\hat{e}}(x')\theta^{\hat{e}}_{\alpha}(x)h^{\nu'}_{\hat{d}}(x')\theta^{\hat{f}}_{\beta}(x) \\
&\omega^{\hat{d}}{}_{\hat{e}\gamma'}g^{\gamma'\sigma'}(x')[h^{\mu'}_{\hat{c}}(x')\omega^{\hat{c}}{}_{\hat{d}\sigma'}(x')h^{\nu'}_{\hat{f}}(x') \\
&+ h^{\mu'}_{\hat{d}}(x')\omega^{\hat{c}}{}_{\hat{f}\sigma'}(x')h^{\nu'}_{\hat{c}}(x')]\theta^{\hat{e}}_{\alpha}(x)\theta^{\hat{f}}_{\beta}(x) \\
&+ \omega^{\hat{d}}{}_{\hat{f}\gamma'}g^{\gamma'\sigma'}(x')[h^{\mu'}_{\hat{c}}(x')\omega^{\hat{c}}{}_{\hat{e}\sigma'}(x')h^{\nu'}_{\hat{d}}(x') \\
&+ h^{\mu'}_{\hat{e}}(x')\omega^{\hat{c}}{}_{\hat{d}\sigma'}(x')h^{\nu'}_{\hat{c}}(x')]\theta^{\hat{e}}_{\alpha}(x)\theta^{\hat{f}}_{\beta}(x).
\end{aligned}
\tag{6.33}
$$

Assembling the various pieces of the formula for $R^{\hat{a}}{}_{\hat{b}\alpha\beta}(x)$ we thus get:

$$
\begin{aligned}
R^{\hat{a}}{}_{\hat{b}\alpha\beta}(x) &= \frac{1}{2\pi}\int_{C_p}\mu_\Gamma(x')\{U^{\delta'\gamma'}_{\alpha\beta}(x,x')[-2R^{\hat{a}}{}_{\hat{c}\delta'\sigma'}(x')R^{\hat{c}}{}_{\hat{b}\gamma'}{}^{\sigma'}(x') \\
&+ 2R^{\hat{a}}{}_{\hat{c}\gamma'\sigma'}(x')R^{\hat{c}}{}_{\hat{b}\delta'}{}^{\sigma'}(x') + R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x')R_{\delta'\gamma'}{}^{\mu'\nu'}(x')]\} \\
&+ \frac{1}{2\pi}\int_{C_p}\mu_\Gamma(x')\{U^{\mu'\nu'}_{\alpha\beta}(x,x')[g^{\lambda'\sigma'}(x')\{\omega^{\hat{a}}{}_{\hat{c}\sigma'}[\omega^{\hat{c}}{}_{\hat{d}\lambda'}R^{\hat{d}}{}_{\hat{b}\mu'\nu'}(x') - R^{\hat{c}}{}_{\hat{d}\mu'\nu'}\omega^{\hat{d}}{}_{\hat{b}\lambda'}(x')] \\
&- [\omega^{\hat{a}}{}_{\hat{d}\lambda'}R^{\hat{d}}{}_{\hat{c}\mu'\nu'}(x') - R^{\hat{a}}{}_{\hat{d}\mu'\nu'}\omega^{\hat{d}}{}_{\hat{c}\lambda'}(x')]\omega^{\hat{c}}{}_{\hat{b}\sigma'}(x')\} \\
&- [(\nabla^{\sigma'}\omega^{\hat{a}}{}_{\hat{c}\sigma'})R^{\hat{c}}{}_{\hat{b}\mu'\nu'}(x') - R^{\hat{a}}{}_{\hat{c}\mu'\nu'}(\nabla^{\sigma'}\omega^{\hat{c}}{}_{\hat{b}\sigma'})]]\} \\
&+ \frac{1}{2\pi}\int_{C_p}\mu_\Gamma(x')\{-2(\nabla^{\sigma'}U^{\mu'\nu'}_{\alpha\beta}(x,x'))\omega^{\hat{a}}{}_{\hat{c}\sigma'}(x')R^{\hat{c}}{}_{\hat{b}\mu'\nu'}(x') \\
&+ 2(\nabla^{\sigma'}U^{\mu'\nu'}_{\alpha\beta}(x,x'))R^{\hat{a}}{}_{\hat{c}\mu'\nu'}(x')\omega^{\hat{c}}{}_{\hat{b}\sigma'}(x')\} \\
&+ \frac{1}{2\pi}\int_{\sigma_p}d\sigma_p\{2U^{\mu'\nu'}_{\alpha\beta}(x,x')(\xi^{\sigma'}D_{\sigma'}R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x'))
\end{aligned}
\tag{6.34}
$$

$$+ U^{\mu'\nu'}_{\alpha\beta}(x,x')R^{\hat{a}}{}_{\hat{b}\mu'\nu'}(x')\theta(x')\}$$

$$+ \frac{1}{2\pi}\int\limits_{C_p} \mu_\Gamma(x')\{R^{\hat{a}}{}_{\hat{b}\mu'\nu'}[(\nabla^{\gamma'}\nabla_{\gamma'}\kappa(x,x'))\tau^{\mu'\nu'}_{\alpha\beta}(x,x')$$

$$+ 2(\nabla^{\gamma'}\kappa(x,x'))(\nabla_{\gamma'}\tau^{\mu'\nu'}_{\alpha\beta}(x,x'))$$

$$+ \kappa(x,x')(\nabla^{\gamma'}\nabla_{\gamma'}\tau^{\mu'\nu'}_{\alpha\beta}(x,x'))]\}$$

where, of course, the factors involving

$$\nabla^{\sigma'}U^{\mu'\nu'}_{\alpha\beta}(x,x') = (\nabla^{\sigma'}\kappa(x,x'))\tau^{\mu'\nu'}_{\alpha\beta}(x,x') + \kappa(x,x')\nabla^{\sigma'}\tau^{\mu'\nu'}_{\alpha\beta}(x,x') \qquad (6.35)$$

can be expanded out as in the foregoing paragraph.

In this explicit form the result seems quite complicated but it is straightforward to reexpress it as

$$R^{\alpha}{}_{\beta\gamma\delta}(x) =$$

$$\frac{1}{2\pi}\int\limits_{C_p}\mu_\Gamma(x')\{\nabla^{\kappa'}\nabla_{\kappa'}(U^{\alpha\nu'\rho'\sigma'}_{\mu'\beta\gamma\delta}(x,x'))R^{\mu'}{}_{\nu'\rho'\sigma'}(x')$$

$$+ U^{\alpha\nu'\rho'\sigma'}_{\mu'\beta\gamma\delta}(x,x')[R^{\lambda'\xi'}{}_{\rho'\sigma'}(x')R_{\lambda'\xi'}{}^{\mu'}{}_{\nu'}(x') \qquad (6.36)$$

$$- 2R^{\mu'}{}_{\lambda'\sigma'\xi'}(x')R_{\nu'}{}^{\lambda'}{}_{\rho'}{}^{\xi'}(x')$$

$$+ 2R^{\mu'}{}_{\lambda'\rho'\xi'}(x')R_{\nu'}{}^{\lambda'}{}_{\sigma'}{}^{\xi'}(x')]\}$$

$$+ \frac{1}{2\pi}\int\limits_{\sigma_p}d\mu_\sigma\{U^{\alpha\nu'\rho'\sigma'}_{\mu'\beta\gamma\delta}(x,x')[2\xi^{\lambda'}(\nabla_{\lambda'}R^{\mu'}{}_{\nu'\rho'\sigma'}(x'))$$

$$+ R^{\mu'}{}_{\nu'\rho'\sigma'}(x')\theta(x')]\}$$

where

$$U^{\alpha\,\nu'\rho'\sigma'}_{\mu'\,\beta\gamma\delta}(x,x') \qquad (6.37)$$

$$= \kappa(x,x')\theta^{\hat{a}}_{\mu'}(x')h^{\alpha}_{\hat{a}}(x)h^{\nu'}_{\hat{b}}(x')\theta^{\hat{b}}_{\beta}(x)h^{\rho'}_{\hat{c}}(x')\theta^{\hat{c}}_{\gamma}(x)h^{\sigma'}_{\hat{d}}(x')\theta^{\hat{d}}_{\delta}(x)$$

the parallel propagator for tensors of type $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Equation (6.36) can be derived much more directly by simply applying the Friedlander formalism to the wave equation (2.7)

for curvature treated as a 4-th rank tensor and then proceeding as above to recast the tail terms in the representation formula in terms of integrals over C_p which can in turn be simplified by the methods of the present section.

However, we have already emphasized the potential usefulness of the Cartan formulation in carrying out the sought-after light cone estimates for curvature because of its close resemblance to the integral equation for curvature arising in Yang-Mills theory. In references [5] and [6] it was necessary to express the integral equation for (Yang-Mills) curvature in the form analogous to Eq. (6.34) above in order to exploit the Cronström gauge conditions and derive bounds on the curvature tensor. Thus we anticipate that the expanded form of the integral expression for gravitational curvature, given by Eq. (6.34), will play an important role in subsequent work to derive estimates for the spacetime curvature of a solution to Einstein's equations.

VII. Approximate Quasi-Local Killing and Conformal Killing Fields

As is well-known the Bel-Robinson tensor for a vacuum spacetime can be used to construct a conserved positive definite "energy" (essentially an L^2-norm of spacetime curvature) for any timelike Killing or conformal Killing field admitted by the metric. This follows from exploiting its total symmetry as a 4-th rank tensor and the vanishing of its divergence and trace in much the same way that one can use the (trace-free) stress energy tensor of a matter field to construct the conserved energy associated to a Killing or conformal Killing field of the "background". Except for "test" matter fields propagating on a stationary or self-similar background however this observation is of little value in practice since the imposition of a Killing or conformal Killing symmetry is far too restrictive a condition to enforce on physically interesting gravitational fields.

On the other hand it may not be necessary to have a strictly conserved energy in order to get adequate analytical control of some mathematically relevant energy norm. For example, in their treatment of Yang-Mills fields propagating in a background space-time, Chruściel and Shatah exploited the observation that the (gauge invariant, positive definite) L^2-norm of Yang-Mills curvature cannot blow up until the spacetime itself blows up (through becoming singular or developing a Cauchy horizon at its boundary) [6]. This fact, which follows from the vanishing of the divergence of the Yang-Mills stress energy tensor and the fact that its components are pointwise bounded by the energy density, was essentially as useful in practice as a fully conserved energy would have been had it existed. When the spacetime itself though is the object of dynamical study this argument (applied to the Bel-Robinson tensor) is of less interest since it requires pointwise control of the connection to yield a mere L^2 bound on the curvature and there is no a priori reason for the Christoffel components to be so bounded.

For this reason it seems potentially useful, especially in the gravitational case, to look for approximate Killing or conformal Killing fields, in a general spacetime, that could in

turn be employed to construct corresponding approximately conserved energies. With this in mind we show below that the orthonormal frame fields $\{h_{\hat{a}}^{\mu} \frac{\partial}{\partial x^{\mu}}\}$ defined, as in Section III, by parallel propagation of a fixed frame at a point p along the radial geodesics issuing from that point, satisfy Killing's equations in an approximate sense that becomes more and more exact (at a well-defined rate) as one approaches the point p along an arbitrary radial geodesic. The error term, or so-called deformation tensor, which measures precisely the failure of Killing's equations to be satisfied, will be shown to be explicitly expressible in terms of radial integrals of spacetime curvature which vanish linearly (in normal coordinates centered at point p) as one approaches this vertex radially.

In a similar way we shall show that the gradient, $\nabla\Gamma$, of the "optical function" Γ (representing squared geodesic distance from the vertex p) satisfies an approximate form of the conformal (in fact homothetic) Killing equations with an error term that vanishes quadratically (in terms of normal coordinates) as one approaches p radially. Both $\nabla\Gamma$ and any timelike linear combination of the $\{h_{\hat{a}}^{\mu} \frac{\partial}{\partial x^{\mu}}\}$ provide timelike vector fields inside the past lightcone from point p (and restricted to a causal domain of p) and thus allow the definition of corresponding positive definite and approximately conserved energy expressions for curvature inside this past lightcone. The timelike character of a frame field such as $\{h_{\hat{0}}^{\mu} \frac{\partial}{\partial x^{\mu}}\}$ is of course not confined to the interior of the cone and its associated energy is therefore positive definite throughout the causal domain in which it remains well-defined.

These approximate Killing and conformal Killing fields should perhaps (for lack of a better term) be called *quasi-local* since they only approach satisfaction of the relevant Killing equations as one approaches the preferred vertex that was used in their construction. The potential (quasi-local) application that we have in mind for such objects can be described loosely as follows. Suppose that some future directed timelike

geodesic γ approaches a singular boundary point for the spacetime under study and that we wish to derive bounds on the rate at which curvature can blow up as γ nears its (singular) endpoint. For each point p lying on γ we can construct the past lightcone from p and parallel propagate the (unit, timelike) tangent to γ at p throughout a causal domain for p to get a timelike, approximate Killing field of the type described above (which will however vary with the choice of the "moving" point p). By exploiting the associated approximately conserved energy we might reasonably hope to estimate (with some controllable error) the energy flux through the past light cone from p, back to some "initial" hypersurface, in terms on the energy defined (by an integral over the ball bounded by the intersection of the light cone with this surface) on this initial "slice". Since control of these (Bel-Robinson) energy fluxes is sure to play a vital role in carrying out the light cone estimates we propose to derive later, the possibility of bounding them in terms of initial data is sure to provide a key step in the hoped-for argument to bound curvature pointwise in terms of its L^2-norms.

If, as in Section III, $\{h_{\hat{b}}^{\nu} \frac{\partial}{\partial x^{\nu}}\}$ is an orthonormal frame field constructed by parallel propagation of a fixed frame at p along the radial geodesics spraying out from p to fill out a causal domain of this point, then the corresponding co-frame field $\{\theta_{\mu}^{\hat{a}} dx^{\mu}\}$ is given by $\theta_{\mu}^{\hat{a}} = \eta^{\hat{a}\hat{b}} g_{\mu\nu} h_{\hat{b}}^{\nu}$. Using the defining formula for the connection coefficients $\omega^{\hat{a}}{}_{\hat{b}\nu}$,

$$\theta_{\mu,\nu}^{\hat{a}} - \Gamma_{\mu\nu}^{\lambda} \theta_{\lambda}^{\hat{a}} = \theta_{\mu;\nu}^{\hat{a}} = -\omega^{\hat{a}}{}_{\hat{b}\nu} \theta_{\mu}^{\hat{b}} \qquad (7.1)$$

one computes the Killing form of $\{\theta_{\mu}^{\hat{a}}\}$ to find

$$\theta_{\mu;\nu}^{\hat{a}} + \theta_{\nu;\mu}^{\hat{a}} = -\omega^{\hat{a}}{}_{\hat{b}\nu} \theta_{\mu}^{\hat{b}} - \omega^{\hat{a}}{}_{\hat{b}\mu} \theta_{\nu}^{\hat{b}} \qquad (7.2)$$

with the right hand side representing the error for Killing's equation. The frame fields approach a fixed orthonormal (co-) frame at the vertex point p but the connection

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components satisfy the "Cronström" formula given (taking $x^\mu(p) = 0$) by

$$\omega^{\hat{c}}{}_{\hat{a}\mu}(x) = -\int_0^1 d\lambda \ \lambda x^\nu R^{\hat{c}}{}_{\hat{a}\mu\nu}(\lambda x) \tag{7.3}$$

and thus vanish to order $0(x)$, for any metric with pointwise bounded curvature, as one approaches the vertex along a radial geodesic. A key observation, from our point of view, is that only undifferentiated curvature enters into this equation for the error. By contrast one can show that the coordinate basis fields $\{\frac{\partial}{\partial x^\mu}\}$ (of a normal coordinate system based at p, with $x^\mu(p) = 0$) also satisfy Killing's equations approximately, with an error that vanishes linearly with the $\{x^\mu\}$, but, in this case, we do not have a formula for the error that depends only upon undifferentiated curvature (though it is conceivable that one exists). Thus we are inclined to strongly prefer the parallel propagated frame fields as natural candidates for our quasi-local approximate Killing fields. Though not commuting in general (as the coordinate basis fields would of course do) these fields nevertheless satisfy an approximate Lie algebra relation, with linearly vanishing error terms, since their commutator is given by

$$h^\mu_{\hat{a}} h^\nu_{\hat{b};\mu} - h^\mu_{\hat{b}} h^\nu_{\hat{a};\mu} = [h_{\hat{a}}, h_{\hat{b}}]^\nu = h^\nu_{\hat{f}} [h^\mu_{\hat{a}} \ \omega^{\hat{f}}{}_{\hat{b}\mu} - h^\mu_{\hat{b}} \ \omega^{\hat{f}}{}_{\hat{a}\mu}] \ . \tag{7.4}$$

Now, consider the "optical" function Γ, introduced in Section IV, and its gradient $\nabla\Gamma$ which, in normal coordinates, satisfies

$$(\nabla\Gamma)^\beta = \Gamma^{;\beta} = g^{\alpha\beta}\Gamma_{,\alpha} = 2g^{\alpha\beta}(x)g_{\alpha\nu}(x)x^\nu = 2x^\beta \tag{7.5}$$

One expects that $\nabla\Gamma = 2x^\beta \frac{\partial}{\partial x^\beta}$ should generalize the well-known, corresponding homothetic Killing field of Minkowski space and indeed, by construction, this vector field is timelike inside the lightcone from p, null on the cone itself and spacelike outside since, in general we have

$$\langle \nabla\Gamma, \nabla\Gamma \rangle_g = g^{\alpha\beta}\Gamma_{,\alpha}\Gamma_{,\beta} = 4\Gamma \tag{7.6}$$

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and Γ represents the squared geodesic distance from the cone vertex p.

Computing the Killing and conformal Killing forms for $\nabla\Gamma$ one gets

$$\Gamma_{;\alpha\beta} + \Gamma_{;\beta\alpha} = 4g_{\alpha\beta} + 2x^\nu g_{\alpha\beta,\nu},$$

$$\Gamma_{;\alpha\beta} + \Gamma_{;\alpha\beta} - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}\Gamma_{;\mu\nu} \tag{7.7}$$

$$= 2x^\nu g_{\alpha\beta,\nu} - \frac{1}{2}g_{\alpha\beta}g^{\gamma\delta}(x^\nu g_{\gamma\delta,\nu})$$

where the error term on the right hand side of the last equation is simply the trace free part of $2x^\nu g_{\alpha\beta,\nu}$ (evaluated in normal coordinates). This latter quantity can be calculated using the same transport formula (derived from the zero torsion equation) that we used in Section III to express the frame field in terms of the connection. The result is

$$x^\beta g_{\mu\nu,\beta}(x) = \eta_{\hat{a}\hat{b}}\{\theta^{\hat{b}}_\nu(x)[\omega^{\hat{a}}{}_{\hat{f}\mu}(x)(x^\gamma\theta^{\hat{f}}_\gamma(0))$$

$$- \int_0^1 d\lambda[\omega^{\hat{a}}{}_{\hat{f}\mu}(\lambda x)(\lambda x^\gamma\theta^{\hat{f}}_\gamma(0))]] \tag{7.8}$$

$$+ \theta^{\hat{a}}_\mu(x)[\omega^{\hat{b}}{}_{\hat{f}\nu}(x)(x^\gamma\theta^{\hat{f}}_\gamma(0)) - \int_0^1 d\lambda[\omega^{\hat{b}}{}_{\hat{f}\nu}(\lambda x)(\lambda x^\gamma\theta^{\hat{f}}_\gamma(0))]]\}$$

wherein $\theta^{\hat{b}}_\nu(x)$ and $\omega^{\hat{b}}{}_{\hat{f}\nu}(x)$ are given explicitly in terms of integrals of curvature by Eqs. (3.21) and (3.12). Thus in this case the error term vanishes quadratically with the normal coordinates as one approaches the vertex at $x^\mu(p) = 0$ though here of course the vector field itself, $\nabla\Gamma = 2x^\beta\frac{\partial}{\partial x^\beta}$ vanishes linearly. The divergence of this approximate conformal Killing field is given, through the trace of the first Eqs. (7.7), by

$$\Gamma^{;\alpha}{}_{;\alpha} = 8 + x^\nu(g^{\alpha\beta}g_{\alpha\beta,\nu}) \tag{7.9}$$

$$= 8 + 2x^\nu\frac{(\sqrt{-\det g}),_\nu}{\sqrt{-\det g}}$$

which coincides with a well-known equation for the d'Alembertian of Γ given by Friedlander [20]. Thus the divergence is constant up to a quadratically vanishing error which suggests that we regard $\nabla\Gamma$ as approximately homothetic.

In Minkowski space, the vector fields $\{h_{\hat{a}}^{\mu}\frac{\partial}{\partial x^{\mu}}, \Gamma^{;\mu}\frac{\partial}{\partial x^{\mu}}\}$ form a Lie sub-algebra of the algebra of generators of the conformal group. Here of course this algebra can at most be approximate but, for completeness, we compute the remaining commutators of $\nabla\Gamma$ with the frame fields $\{h_{\hat{a}}^{\mu}\frac{\partial}{\partial x^{\mu}}\}$. The Lie brackets are given initially by

$$[h_{\hat{a}}, \nabla\Gamma]^{\nu} = h_{\hat{a}}^{\mu}\Gamma^{;\nu}{}_{;\mu} - \Gamma^{;\mu}h_{\hat{a};\mu}^{\nu} \tag{7.10}$$

but we can simplify this by noting that the equations

$$h_{\hat{a};\mu}^{\nu} = \omega^{\hat{f}}{}_{\hat{a}\mu}h_{\hat{f}}^{\nu}, \tag{7.11}$$

$$\Gamma^{;\beta} = 2x^{\beta}$$

together with the parallel propagation gauge condition, $x^{\mu}\omega^{\hat{f}}{}_{\hat{a}\mu} = 0$, imply that,

$$\Gamma^{;\mu}h_{\hat{a};\mu}^{\nu} = 0 \tag{7.12}$$

and thus (using Eq. (7.9)) that

$$[h_{\hat{a}}, \nabla\Gamma]^{\nu} = h_{\hat{a}}^{\mu}g^{\lambda\nu}\Gamma_{;\lambda\mu}$$
$$= h_{\hat{a}}^{\mu}g^{\lambda\nu}(2g_{\lambda\mu} + x^{\beta}g_{\lambda\mu,\beta}) \tag{7.13}$$
$$= 2h_{\hat{a}}^{\nu} + h_{\hat{a}}^{\mu}g^{\lambda\nu}(x^{\beta}g_{\lambda\mu,\beta}).$$

Hence we recover the flat space result up to a quadratically vanishing "error" in the would-be Lie algebra. Though we did not need it to derive the foregoing results, it is useful to note that

$$x^{\nu}\Gamma_{\mu\nu}^{\beta} = \frac{1}{2}g^{\alpha\beta}(x^{\nu}g_{\mu\alpha,\nu}) \tag{7.14}$$

which follows from the normal coordinate identity $g_{\mu\nu}(x)x^{\nu} = g_{\mu\nu}(0)x^{\nu}$ by differentiating to get

$$x^{\nu}g_{\mu\nu,\alpha}(x) = g_{\mu\alpha}(0) - g_{\mu\alpha}(x) \tag{7.15}$$

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and then antisymmetrizing in μ and α to arrive at

$$x^\nu (g_{\mu\nu,\alpha}(x) - g_{\alpha\nu,\mu}(x)) = 0. \tag{7.16}$$

Without this result, the direct calculation of $\Gamma_{;\alpha\beta}$, beginning with $\Gamma_{;\alpha} = 2g_{\alpha\nu}(x)x^\nu$, would not yield a symmetric formula in α and β as it must.

While one could continue along the above lines and define approximate Killing and conformal Killing analogues for Lorentz rotation, boost and inversion generators with formulas like $x^1 h^\nu_{\hat{2}} \frac{\partial}{\partial x^\nu} - x^2 h^\nu_{\hat{1}} \frac{\partial}{\partial x^\nu}$, $x^0 h^\nu_{\hat{1}} \frac{\partial}{\partial x^\nu} + x^1 h^\nu_{\hat{0}} \frac{\partial}{\partial x^\nu}$, etc., these would not be timelike throughout the regions (interiors of past light cones from vertices with $x^\mu(p) = 0$) of interest and so would not yield positive energy expressions. While their approximate conformal Lie algebra relations might be of interest to develop, we shall not pursue that issue here.

Appendix

Notation, conventions and basic definitions

Much of our analysis will be carried out in rather specially chosen charts and associated frames. For the moment however, to introduce the notation that we shall use throughout, we consider an arbitrary chart and an arbitrary (orthonormal) frame. In coordinates $\{x^\mu\}$ defined on some domain of our spacetime manifold V we write the Lorentzian metric g in the standard form

$$g = g_{\mu\nu}dx^\mu \otimes dx^\nu \tag{A.1}$$

and introduce an orthonormal frame $\{h_{\hat{a}}\} = \{h_{\hat{a}}^\mu \frac{\partial}{\partial x^\mu}\}$ and dual, coframe $\{\theta^{\hat{a}}\} = \{\theta_\mu^{\hat{a}} dx^\mu\}$ for this (locally expressed) metric. The orthonormality and duality relations satisfied by these fields are summarized as follows:

$$h_{\hat{a}} = h_{\hat{a}}^\mu \frac{\partial}{\partial x^\mu}, \;\; \theta^{\hat{a}} = \theta_\mu^{\hat{a}} dx^\mu \;\; \text{coordinate basis expression}$$

$$g_{\mu\nu}h_{\hat{a}}^\mu h_{\hat{b}}^\nu = \eta_{\hat{a}\hat{b}}, \;\; g^{\mu\nu}\theta_\mu^{\hat{a}}\theta_\nu^{\hat{b}} = \eta^{\hat{a}\hat{b}} \;\; \text{orthonormality relations}$$

$$\theta_\mu^{\hat{a}}h_{\hat{b}}^\mu = \delta_{\hat{b}}^{\hat{a}}, \;\; \theta_\mu^{\hat{a}}h_{\hat{a}}^\nu = \delta_\mu^\nu \;\; \text{duality relations} \tag{A.2}$$

$$g_{\mu\nu} = \eta_{\hat{a}\hat{b}}\theta_\mu^{\hat{a}}\theta_\nu^{\hat{b}}, \;\; g^{\mu\nu} = \eta^{\hat{a}\hat{b}}h_{\hat{a}}^\mu h_{\hat{b}}^\nu \;\; \text{metric formulas}$$

$$\frac{\partial}{\partial x^\nu} = \theta_\nu^{\hat{a}}h_{\hat{a}}, \;\; dx^\nu = h_{\hat{a}}^\nu \theta^{\hat{a}} \;\; \text{coordinate basis vectors and forms}$$

$$\theta_\mu^{\hat{a}} = \eta^{\hat{a}\hat{b}}g_{\mu\nu}h_{\hat{b}}^\nu, \;\; h_{\hat{a}}^\mu = \eta_{\hat{a}\hat{b}}g^{\mu\nu}\theta_\nu^{\hat{b}} \;\; \text{component relations.}$$

Here $(\eta_{\hat{a}\hat{b}})$ is the standard Minkowski metric

$$\eta = (\eta_{\hat{a}\hat{b}}) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \tag{A.3}$$

and $\eta^{-1} = (\eta^{\hat{a}\hat{b}})$ is its inverse. Many of the formulas we shall derive in this section hold true for arbitrary spacetime dimensions and also for Riemannian metrics instead of Lorentzian ones if η is replaced by a Euclidean metric. Tensors are expressed in coordinate and orthonormal bases as

$$S = S^{\mu\nu\cdots}{}_{\gamma\delta\cdots} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \otimes \ldots \otimes dx^\gamma \otimes dx^\delta \otimes \ldots \tag{A.4}$$

$$= S^{\hat{a}\hat{b}\cdots}{}_{\hat{e}\hat{f}\cdots} h_{\hat{a}} \otimes h_{\hat{b}} \otimes \ldots \otimes \theta^{\hat{e}} \otimes \theta^{\hat{f}} \otimes \ldots$$

with components related by

$$S^{\hat{a}\hat{b}\cdots}{}_{\hat{e}\hat{f}} \ldots = S^{\mu\nu\cdots}{}_{\gamma\delta\cdots} \theta^{\hat{a}}_\mu \theta^{\hat{b}}_\nu \ldots h^\gamma_{\hat{e}} h^\delta_{\hat{f}} \ldots \ . \tag{A.5}$$

In particular, the metric g and its inverse g^{-1} take the forms

$$g = \eta_{\hat{a}\hat{b}} \theta^{\hat{a}} \otimes \theta^{\hat{b}}, \quad g^{-1} = \eta^{\hat{a}\hat{b}} h_{\hat{a}} \otimes h_{\hat{b}}. \tag{A.6}$$

For all differentiable tensor fields, we have the conventional (coordinates basis) expressions for the covariant derivatives of scalar, vector and one-form (or co-vector) fields respectively given by

$$\begin{aligned}
\varphi_{;\alpha} &= \varphi_{,\alpha} = \frac{\partial\varphi}{\partial x^\alpha} && \text{scalar} \\
v^\mu_{;\nu} &= v^\mu_{,\nu} + \Gamma^\mu_{\gamma\nu} v^\gamma && \text{vector} \\
\lambda_{\mu;\nu} &= \lambda_{\mu,\nu} - \Gamma^\gamma_{\mu\nu} \lambda_\gamma && \text{co-vector}
\end{aligned} \tag{A.7}$$

where $\{\Gamma^\mu_{\alpha\beta}\}$ are the Christoffel symbols of g given by

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}). \tag{A.8}$$

The frame components of these formulas take the form

$$\begin{aligned}
\varphi_{;\hat{a}} &= h^\mu_{\hat{a}} \varphi_{;\mu} = h^\mu_{\hat{a}} \frac{\partial\varphi}{\partial x^\mu} \\
v^{\hat{a}}_{;\hat{b}} &= \theta^{\hat{a}}_\mu h^\nu_{\hat{b}} v^\mu_{;\nu} = v^{\hat{a}}_{,\hat{b}} + \Gamma^{\hat{a}}_{\hat{c}\hat{b}} v^{\hat{c}} = h^\nu_{\hat{b}} \left(\frac{\partial v^{\hat{a}}}{\partial x^\nu} \right) + \Gamma^{\hat{a}}_{\hat{c}\hat{b}} v^{\hat{c}} \\
\lambda_{\hat{a};\hat{b}} &= h^\mu_{\hat{a}} h^\nu_{\hat{b}} \lambda_{\mu;\nu} = \lambda_{\hat{a},\hat{b}} - \Gamma^{\hat{c}}_{\hat{a}\hat{b}} \lambda_{\hat{c}} = h^\nu_{\hat{b}} \left(\frac{\partial\lambda_{\hat{a}}}{\partial x^\nu} \right) - \Gamma^{\hat{c}}_{\hat{a}\hat{b}} \lambda_{\hat{c}}
\end{aligned} \tag{A.9}$$

where

$$\Gamma^{\hat{a}}_{\hat{c}\hat{b}} = h^{\nu}_{\hat{b}}\ \theta^{\hat{a}}_{\mu}\ \{h^{\gamma}_{\hat{c}}\Gamma^{\mu}_{\gamma\nu} + h^{\mu}_{\hat{c},\nu}\} \tag{A.10}$$

we shall also write

$$\Gamma^{\hat{a}}_{\hat{c}\hat{b}} = h^{\nu}_{\hat{b}}\omega^{\hat{a}}_{\ \hat{c}\nu} \tag{A.11}$$

and express the connection one-forms, $\omega^{\hat{a}}_{\ \hat{c}}$ as

$$\omega^{\hat{a}}_{\ \hat{c}} := \omega^{\hat{a}}_{\ \hat{c}\nu}dx^{\nu} = \omega^{\hat{a}}_{\ \hat{c}\nu}h^{\nu}_{\hat{b}}\theta^{\hat{b}} = \Gamma^{\hat{a}}_{\hat{c}\hat{b}}\theta^{\hat{b}} \tag{A.12}$$

which is equivalent to setting

$$\omega^{\hat{a}}_{\ \hat{c}\nu} = \theta^{\hat{a}}_{\mu}h^{\gamma}_{\hat{c}}\Gamma^{\mu}_{\gamma\nu} + \theta^{\hat{a}}_{\mu}(h^{\mu}_{\hat{c},\nu}). \tag{A.13}$$

Defining

$$\omega_{\hat{b}\hat{c}} = \eta_{\hat{a}\hat{b}}\omega^{\hat{a}}_{\ \hat{c}} = \eta_{\hat{a}\hat{b}}\omega^{\hat{a}}_{\ \hat{c}\nu}dx^{\nu} \tag{A.14}$$

one easily verifies that

$$\omega_{\hat{b}\hat{c}\nu} = -\omega_{\hat{c}\hat{b}\nu} \tag{A.15}$$

which captures the metric compatibility of the chosen connection (i.e., the fact that $g_{\mu\nu;\alpha} = 0$). The vanishing of torsion for the Christoffel connection (i.e., the fact that $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$) takes the form

$$\partial_{\nu}\theta^{\hat{c}}_{\mu} - \partial_{\mu}\theta^{\hat{c}}_{\nu} + \omega^{\hat{c}}_{\ \hat{a}\nu}\theta^{\hat{a}}_{\mu} - \omega^{\hat{c}}_{\ \hat{a}\mu}\theta^{\hat{a}}_{\nu} = 0 \tag{A.16}$$

which can also be regarded as an equation determining the connection one forms, $\omega^{\hat{c}}_{\ \hat{a}\nu}\ dx$ in terms of the (co-) frame fields $\theta^{\hat{c}} = \theta^{\hat{c}}_{\mu}dx^{\mu}$.

In this same notation the Riemann curvature tensor takes the form

$$R^{\hat{c}}_{\ \hat{a}\mu\nu} = \theta^{\hat{c}}_{\gamma}h^{\lambda}_{\hat{a}}R^{\gamma}_{\ \lambda\mu\nu} \tag{A.17}$$

$$= \partial_{\mu}\omega^{\hat{c}}_{\ \hat{a}\nu} - \partial_{\nu}\omega^{\hat{c}}_{\ \hat{a}\mu} + \omega^{\hat{c}}_{\ \hat{d}\mu}\omega^{\hat{d}}_{\ \hat{a}\nu} - \omega^{\hat{c}}_{\ \hat{d}\nu}\omega^{\hat{d}}_{\ \hat{a}\mu}$$

which, since

$$R^{\hat{c}}{}_{\hat{a}\mu\nu} = -R^{\hat{c}}{}_{\hat{a}\nu\mu} \tag{A.18}$$

and

$$R_{\hat{b}\hat{a}\mu\nu} = -R_{\hat{a}\hat{b}\mu\nu} \tag{A.19}$$

where

$$R_{\hat{b}\hat{a}\mu\nu} := \eta_{\hat{b}\hat{c}}R^{\hat{c}}{}_{\hat{a}\mu\nu} \tag{A.20}$$

may be regarded as a two-form which takes values in the space of anti-symmetric Lorentz matrices. In view of Eq. (A.15) the connection one-form can be thought of as taking values in this same space which in turn represents the Lie algebra of (local) Lorentz transformations that can act on the frame fields while leaving the spacetime metric invariant.

Regarding connection and curvature as one and two-forms which take their values in the Lie algebra of some "internal" gauge group (in this case the Lorentz group of frame transformations) is parallel to what one does in Yang-Mills theory. There the principle bundle connection one-form $A_\mu dx^\mu$ and its curvature two-form $F_{\mu\nu}dx^\mu \wedge dx^\nu$ take their values in a matrix representation of the Lie algebra $\mathfrak{g}$ of some gauge "internal" Lie group G. By attaching (in a slightly unconventional way) row and column indices to label the matrix elements of these geometric objects, one could express their components as $A^{\hat{a}}{}_{\hat{b}\mu}$ and $F^{\hat{a}}{}_{\hat{b}\mu\nu}$ respectively, in parallel to the notation we have used above. The expression for $F^{\hat{a}}{}_{\hat{b}\mu\nu}$ in terms of $A^{\hat{a}}{}_{\hat{b}\mu}$ is identical in form to that for $R^{\hat{a}}{}_{\hat{b}\mu\nu}$ in terms of $\omega^{\hat{a}}{}_{\hat{b}\mu}$ given in Eq. (A.17) above. There are numerous other precise correspondences between Yang-Mills theory and Cartan's formulation of general relativity but there are also significant differences. For example in Yang-Mills theory, even if formulated on a curved background spacetime, there is no relationship between the connection one-form $A_\mu dx^\mu$ and the spacetime connection as expressed through the Christoffel symbols $\{\Gamma^\mu_{\alpha\beta}\}$ since the former does not derive from a metric or frame field

whereas the latter does. Furthermore, the gauge groups for physically interesting Yang-Mills theories are normally required to be compact whereas the corresponding "gauge" group for general relativity is the non-compact (local) Lorentz group of orthonormal frame transformations. The compactness normally assumed for a gauge group G allows one to define an energy momentum tensor, quadratic in the Yang-Mills curvature, which has positive definite energy density. The corresponding second rank symmetric tensor, quadratic in the spacetime curvature, vanishes identically in Einstein's theory. Fortunately, the fourth-rank, totally symmetric Bel-Robinson tensor and its associated positive definite "energy" density supply the needed replacements for these important objects.

Acknowledgements

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15. S. Sobolev, *Méthode nouvelle à resoudre le problème de Cauchy pour les équations linéaires hyperboliques normales*, Math. Sb. (N.S.) **1** (1936), 39-71.

16. J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Silliman Lecture Series, Yale University Press (1921).

17. See the discussion of normal coordinate systems and their properties in, for example, Ref. [9], Sect. 1.2.

18. See Sect. 4.4 of Ref. [9] for a definition and discussion of *causal domains*.

19. See Sect. 5.5 of Ref. [9], especially Theorem 5.5.2 for the representation formula in the case of tensor wave equations.

20. See Sect. 4.2 of Ref. [9], in particular the discussion on p. 132.