ANALYTIC REDUCTIONS OF SELF-FORCE CALCULATIONS IN CURVED SPACETIMES

Vincent Moncrief

Department of Mathematics
and Department of Physics
Yale University

Abstract. For the case of an electric or scalar charge traveling along a timelike geodesic in a curved, background spacetime we apply a Green's theorem argument to transform the tail contribution to the particle's self force at a point p along its trajectory to a form which involves only an integral over the past light cone from p. One potential advantage of this reformulation is that the tail contribution to the fundamental solution for a (self-adjoint) tensor wave equation in an arbitrary spacetime is (at least within so-called causal domains) explicitly computable on the past light cone of an arbitrary point through the integration of certain linear transport equations defined along the null generators of the cone. The conventional approach to computing the self-force requires this tail field throughout the interior of the past light cone and extracts it from the particle's (numerically generated) total field through an intricate mode-by-mode numerical decomposition. By contrast our approach requires that the particle's *total* field be paired with the (explicitly computable) tail field on the light cone itself and then integrated over this cone. We thus avoid the need for the aforementioned mode-by-mode decomposition of the particle's total field into direct $\oplus$ tail contributions. We speculate that in some circumstance it may be sufficient to truncate the particle's total field, as needed in our calculation, to its (also explicitly computable), *direct* Liénard Wiechert approximation.

I. Introduction

The self-force acting on a relativistic charged particle moving in a flat or curved spacetime has been the subject of numerous investigations beginning with the fundamental work of Dirac [1] for the flat case and DeWitt and Brehme [2] for the case of a curved background. The gravitational analogue of this problem has received much attention recently [3, 4, 5] because of its expected importance for the projected LISA spacecraft observations to search for gravitational waves. A principal goal of this mission will be to detect the gravitational radiation emitted by a relatively small star or black hole orbiting in the vicinity of a supermassive black hole which can be treated as a stationary attractor. As a first approximation, one computes the linearized metric perturbation produced by a point-like distributional source which traces out a timelike geodesic of the background gravitational field (typically a Schwarzschild or Kerr black hole) and the technology for doing such calculations numerically is well under control. Unfortunately though, it is now generally accepted that this first approximation will be inadequate to give the precision needed for comparison with the anticipated spacecraft observations, and that it will be necessary to calculate the correction to the test body's motion due to its (self-force) interaction with its own radiation field and ultimately to then compute the radiation resulting from this corrected motion.

That a massive test body can be acted upon by its own gravitational radiation field is a consequence of the breakdown of Huygen's principle in a general curved spacetime. Rather than merely propagating out sharply along the light cone as would happen in a flat background, this test body's radiation field can in effect scatter off the background curvature and reintercept the particle's timelike trajectory giving rise to what is normally called (even for the gravitational case) a self-force on the particle. More properly the scattered gravitational radiation is modifying the metric in which the particle is moving in such a way that geodesic motion in the perturbed metric can sometimes be usefully viewed as forced motion in the background spacetime. An important complication in this gravitational radiation problem (which is essentially absent in its otherwise rather similar electromagnetic analogue) is that, as always in general relativity, the geometrical meaning of the coordinates that are used to describe the particle's motion depends upon what metric the particle is moving in and this metric is itself undergoing a non-negligible perturbation due to the very presence of the moving particle. This complication shows up as an effective "gauge dependence" of the particle's corrected equation of motion which requires knowledge of the perturbed metric itself to resolve and properly interpret.

By contrast to the above, a charged particle moving in a background gravitational field but subject to its own electromagnetic self-force may very well generate such a negligibly small perturbation to the metric that this latter can be safely ignored in computing corrections to the particle's motion. This corrected motion could then simply be used as a new and better approximation to the particle's actual trajectory and thus as a satisfactory basis for an improved computation of its resultant electromagnetic radiation field.

On the other hand, much of the mathematical machinery available for computing the radiation fields and their reaction effects is somewhat similar for the two problems mentioned above as well as for a third problem involving the radiation of a hypothetical massless scalar field which is slightly easier to handle technically. For this reason investigators often study the electromagnetic and scalar self-force problems as a prelude to dealing with the far more intricate gravitational one.

For any of these problems the radiative contribution to the self-force results from the so-called "tail term" in the retarded field of the moving particle. This tail, because of the breakdown of Huygen's principle, is non-vanishing inside the future light cone of the emitting particle and so can influence the particle's motion at a later time. Conversely, the radiative self-force that a particle "feels" at a given instant is the cumulative effect of the tail contributions (to the instantaneous field at the particle's location) from the entire past history of the moving particle's trajectory. As such, it is normally expressible (in the linearized approximation to which we are working) as an integral over that past history.

Our aim in this note is to sketch how the tail field's contribution to the self-force (at some arbitrary event along the particle's trajectory) can be transformed into an integral over the past light cone of this event. One potential advantage of this transformation lies in the fact that the tail field is explicitly computable on the light cone (through the integration of certain first order, linear "transport equations" defined on the cone) wherein its expression provides the first term in Hadamard's formal series expansion for the tail field inside the light cone. Even for analytic background metrics, for which Hadamard's series actually converges, the domain of convergence may be artifically limited by the presence of poles in the expression for the complexified metric and, in any case, the practical evaluation of the tail field inside the cone normally requires direct numerical calculation of the particle's *total* radiation field from which the tail term must then be extracted by further extensive numerical work.

In our approach the integral of the tail field, paired with the particle's distributional source (which results in the integral over the particle's past history alluded to above), is replaced, through an application of Green's theorem, by an integral of the tail field, paired with the particle's full Liénard-Wiechert field, over the past light cone of the event at which the self-force is being evaluated.

This full Liénard-Wiechert field of the moving particle is normally calculated numerically by solving the relevant wave equation, with distributional source, on the chosen background spacetime. In our formulation, however, there is no additional necessity to decompose this field (as is done in the conventional approach by an intricate mode-by-mode numerical analysis) into direct $\oplus$ tail contributions. On the other hand there is now the possibility that one might obtain a sufficiently accurate approximation to the desired self-force result by truncating the particle's full Liénard-Wiechert field to its (explicitly computable) *direct*, leading term and dropping its (presumably subdominant) tail contribution. This would amount to calculating analytically the direct $\otimes$ tail contributions but ignoring the tail $\otimes$ tail contributions to the light cone integrals in question. Since an analytical justification for such a truncation is currently lacking it would seem that an estimate of its validity, or lack thereof, must hinge upon detailed numerical comparisons.

One should perhaps clarify and qualify the foregoing remarks by saying that when we use the term *explicitly computable*, as we have already done several times, we only mean this in the sense of 'reducible to quadratures', i.e., integrals and algebraic operations that in general may not themselves be elementary. In much the same sense one says that the geodesics problem for the Schwarzschild or Kerr spacetime is solvable even though the solution is not available in closed form. The integrals and inversions needed in the aforementioned calculations would normally themselves require numerical evaluation.

As we have mentioned above, the gravitational self-force calculation is fraught with additional conceptual difficulties not present in the electromagnetic or scalar one but the purely mathematical aspects of the three problems are nevertheless somewhat similar. Since we are primarily proposing a new mathematical technique, it is natural to test it first in the conceptually simplest setting where subtle issues of coordinate covariance can be (temporarily at least) put aside. For this reason, we shall concentrate on the electromagnetic and scalar self-force problems and leave the gravitational one to a future study.

The idea for transforming the tail contributions to the self-force problem to light cone integrals arose out of an independent project on "light cone estimates for Einstein's equations" wherein the author applied similar techniques to transform tail contributions to the wave equation for spacetime curvature to light cone integrals where their influence (upon the value of curvature at the vertex of the cone) could be more easily estimated [6, 7]. There, as here, the idea was to exploit the fact that the tail field (a summand in the fundamental solution to the appropriate curved space wave equation) is computable via first order, linear transport equations along the light cone whereas evaluation of this field off the cone requires the solution of an associated characteristic initial value problem.

Our analysis is fundamentally based upon the treatment of curved space, linear wave equations developed in F. G. Friedlander's book *The Wave Equation in a Curved Space-time* [8]. To avoid introducing excessive complications in the present paper we have simply cited the relevant sections of this book for standard arguments given in detail there. Our only major deviation from Friedlander's conventions is that we employ the Lorentzian signature $(-+++)$ instead of the opposite one that he uses. This introduces some sign differences into the formulas that we present relative to their antecedents in [8]. A more complete treatment of the reductions discussed here (which do not appear in Friedlander's book) will be presented in [6, 7] devoted to the "light cone estimates for Einsteins equations" project.

II. The wave equation for a Maxwell field on a curved background

Maxwell's equation on a background spacetime $({}^{(4)}V, g)$ can be expressed in local coordinates as

$$F^{\mu\nu}{}_{;\nu} = 4\pi j^{\mu} \tag{1}$$

and

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \tag{2}$$

where $F_{\mu\nu} = -F_{\nu\mu}$ is the Faraday 2-form field and ; signifies covariant diferentiation with respect to $g_{\mu\nu}$. Normally one solves the second equation by setting $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu;\mu} - A_{\mu;\nu}$ and then substitutes this expression for the Faraday tensor into the first equation to derive a second order equation for the "vector potential" A_μ. The latter takes on a manifestly hyperbolic form if one imposes, for example, the Lorentz gauge condition $A_\mu{}^{;\mu} = 0$ as is conventionally done.

Here however we prefer to work with the (equally well-known) second order hyperbolic equation for F itself. Taking the covariant divergence of Eq. (2) above, suitably commuting covariant derivatives through the use of the identity

$$\begin{aligned} F_{\alpha\beta;\gamma\delta} - F_{\alpha\beta;\delta\gamma} = {} & R^{\mu}{}_{\alpha\gamma\delta} F_{\mu\beta} \\ & + R^{\mu}{}_{\beta\gamma\delta} F_{\alpha\mu} \end{aligned} \tag{3}$$

where $R^{\mu}{}_{\alpha\beta\gamma}$ is the Riemann tensor of $g_{\mu\nu}$, and using Eq. (1) above to replace the covariant divergence of F with the current, one arrives at

$$\begin{aligned} & F_{\alpha\beta;\gamma}{}^{;\gamma} + R_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu} + R^{\mu}_{\alpha} F_{\beta\mu} \\ & - R^{\mu}_{\beta} F_{\alpha\mu} = 4\pi (j_{\alpha;\beta} - j_{\beta;\alpha}). \end{aligned} \tag{4}$$

The last two terms on the left hand side can be dropped for vacuum backgrounds but, in any case, the linear operator acting on F defined by the left hand side of this equation is hyperbolic and self-adjoint in the sense defined by Friedlander [9]. In deriving this equation and verifying its self adjoincy, we have used the algebraic Bianchi identities

$$\begin{aligned} &R^{\mu}{}_{[\alpha\beta\gamma]} \equiv 0, \quad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}, \\ &R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]}. \end{aligned} \tag{5}$$

The theory developed in Friedlander's book [8] (which builds on the fundamental work of Hadamard, Riesz, Sobolev, Choquet-Bruhat and others) applies to Eq. (4) and allows one to write a representation formula for the solution of the corresponding Cauchy problem on so-called causal domains of the given spacetime (i.e., on geodesically convex domains which are also globally hyperbolic in a suitable sense [10]).

Of course not every solution of Eq. (4) corresponds to a solution of the original Maxwell equations. It is necessary, in order to avoid introducing spurious solutions, to restrict the Cauchy data for Eq. (4) by imposing Eqs. (1) and (2) on the data given for Eq. (4) on the initial Cauchy slice. The Friedlander formalism applies to all solutions of Eq. (4) and hence in particular to the solutions of physical interest. One need only remember that the allowed Cauchy data for a Maxwell solution is subject to the restriction mentioned above.

With reference to Fig. 5.3.1 of Friedlander's book, let p be a point in some causal domain of $({}^{(4)}V, g)$ and S be a spacelike hypersurface within this domain such that every past-directed causal geodesic from p meets S. Further, let C_p be the mantle of the (truncated) past light cone from p to S, σ_p be the (two-dimensional) intersection of C_p with S and let D_p be the interior of this truncated cone and designate by S_p the (three-dimensional) intersection of D_p with S. Finally, let T_p designate the expanding lightlike hypersurface which intersects S in σ_p.

Friedlander's representation formula for the Faraday field at point p is given in local coordinates by [11]:

$$
\begin{aligned}
F_{\alpha\beta}(x) &= \frac{1}{2\pi}\int\limits_{C_p} U^{\mu'\nu'}_{\alpha\beta}(x,x') f_{\mu'\nu'}(x')\mu_\Gamma(x') \\
&+ \frac{1}{2\pi}\int\limits_{D_p} (V^+)^{\mu'\nu'}_{\alpha\beta}(x,x') f_{\mu'\nu'}(x')\mu(x') \\
&+ \frac{1}{2\pi}\int\limits_{S_p} *[(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')\nabla^{\gamma'} F_{\mu'\nu'}(x') \\
&\quad - F_{\mu'\nu'}(x')\nabla^{\gamma'}(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')] \qquad (6) \\
&+ \frac{1}{2\pi}\int\limits_{\sigma_p} \{U^{\mu'\nu'}_{\alpha\beta}(x,x')[2(\nabla^{\gamma'} t(x'))(\nabla_{\gamma'} F_{\mu'\nu'}(x')) \\
&\quad + F_{\mu'\nu'}(x') \,\Box\, t(x')] \\
&- \langle \nabla t(x'), \nabla'\Gamma(x,x')\rangle (V^+)^{\mu'\nu'}_{\alpha\beta}(x,x') F_{\mu'\nu'}(x')\}\mu_{t,\Gamma}(x').
\end{aligned}
$$

Here $U^{\mu'\nu'}_{\alpha\beta}(x,x') = \kappa(x,x')\tau^{\mu'\nu'}_{\alpha\beta}(x,x')$ where κ is the transport biscalar defined by Eq. (4.2.17) of [8] and given in local coordinates by Eq. (4.2.18) or (4.2.19) of that reference and $\tau^{\mu'\nu'}_{\alpha\beta}(x,x')$ is the transport bitensor (or propagator) defined in Section (5.5) of Friedlander. The latter is expressible explicitly in terms of an orthonormal frame parallel propagated from p along the geodesic issuing from that point (c.f., Refs. [6,7]). The field $f_{\mu\nu}(x)$ represents the source term for the inhomogeneous wave equation and is here given by the distribution

$$f_{\mu\nu}(x) = 4\pi(j_{\mu;\nu}(x) - j_{\nu;\mu}(x)) \qquad (7)$$

where $j_\mu(x)$ is the current of the moving charged particle. The measure $\mu(x')$ is the standard spacetime volume measure given in local coordinates by $\sqrt{-\det g_{\mu\nu}(x')}d^4x'$ whereas the measure on the light cone $\mu_\Gamma(x')$ is a Leray form defined such that

$$d_{x'}\Gamma(x,x') \wedge \mu_\Gamma(x') = \mu(x') \qquad (8)$$

where $\Gamma(x,x')$ is the optical function (squared geodesic distance within a causal domain) introduced in Sect. (1.2) of Friedlander (c.f., Theorem 1.2.3). Leray forms are introduced in Sect. (2.9) and developed further in Sect. (4.5) of this same reference and the coordinate expression for the dual $*v$ of a vector v is given there by Eq. (2.9.3). This is needed in the boundary integral over S_p whereas μ_Γ arises in that over C_p. The two-dimensional Leray form $\mu_{t,\Gamma}(x')$ needed for the integral over σ_p, is defined such that (c.f., Lemma 5.3.3. of [8])

$$dt(x') \wedge d_{x'}\Gamma(x,x') \wedge \mu_{t,\Gamma}(x') = \mu(x') \tag{9}$$

where $t(x')$ is the null field defined by Lemma 5.3.2 of Friedlander. Note also in this reference the needed expressions for $(\Box\, t)\mu_{t,\Gamma}$ and $\langle \nabla t, \nabla\Gamma\rangle$ given respectively by Eqs. (5.3.20) and (5.3.19) of this same section.

The tail field $(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')$ is the solution of a characteristic initial value problem for the homogeneous wave equation. By virtue of the self-adjoincy of our Eq. (4) and the reciprocity relations derived by Friedlander in Sect. (5.2) (which apply as well to the tensor case as disucssed in Sect. (5.5)) the tail bitensor V^+ satisfies the wave equation

$$\begin{aligned}
&(V^+)^{\mu'\nu';\gamma'}_{\alpha\beta;\gamma'}(x,x') + R^{\mu'\nu'}{}_{\delta'\gamma'}(x')(V^+)^{\delta'\gamma'}_{\alpha\beta}(x,x') \\
&+ R^{\mu'}_{\delta'}(x')(V^+)^{\nu'\delta'}_{\alpha\beta}(x,x') - R^{\nu'}_{\delta'}(x')(V^+)^{\mu'\delta'}_{\alpha\beta}(x,x') \\
&= 0
\end{aligned} \tag{10}$$

wherein the indices $\alpha\beta$ and coordinates x^μ play inert roles. In the foregoing formulas, as well as below, the notation ∇_γ and $;\gamma$ are used interchangeably. The initial data for V^+ is computable on the light cone C_p where it reduces to the bitensor field that Friedlander expresses as V_0. The transport equation determining V_0 is provided by Friedlander's Eq. (5.5.23) and its explicit solution is given in his Eq. (5.5.25).

II. Transformations of the tail field integrals

Define the tail field contributions to $F_{\alpha\beta}(x)$ by

$$
\begin{aligned}
F^{\rm tail}_{\alpha\beta}(x) &:= \frac{1}{2\pi}\int\limits_{D_p} (V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')f_{\mu'\nu'}(x')\mu(x') \\
&+ \frac{1}{2\pi}\int\limits_{S_p} *[(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')\nabla^{\gamma'}F_{\mu'\nu'}(x') \\
&\quad - F_{\mu'\nu'}(x')\nabla^{\gamma'}(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')] \\
&- \frac{1}{2\pi}\int\limits_{\sigma_p} \langle \nabla t(x'), \nabla'\Gamma(x,x')\rangle (V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')\mu_{t,\Gamma}(x').
\end{aligned}
\tag{11}
$$

This consists of all the terms that vanish when Huygen's principle is valid since in that case $V^+ = 0$ but, in a curved spacetime, these terms are generally non-zero and yield a contribution to the self-force if the source particle passes through the point x. This latter is simply the Lorentz force due to $F^{\rm tail}_{\alpha\beta}(x)$ acting on the particle at that point. In many discussions of the self-force problem it is assumed that if the initial data surface S_p is pushed back sufficiently far into the past the contributions to the tail field coming from the initial surface integrals over S_p and its boundary σ_p will be negligibly small. In that case the expression for $F^{\rm tail}_{\alpha\beta}(x)$ effectively reduces to the integral over D_p which, for the case of a point particle, simplifies to an integral over the past history of the particle. We shall need a similar assumption later but, for now, find it more natural to retain all the potential contributions to the tail field since the form of these integrals will be modified by the transformations we carry out. In particular, the integrals over S_p and σ_p will be exactly canceled.

Let us reexpress the source f through the use of the wave equation for F as

$$
f_{\mu'\nu'}(x') = (PF)_{\mu'\nu'}(x') \tag{12}
$$

where P is the second order linear, self-adjoint operator defined by the left hand side of Eq. (4). Recalling Eq. (10) which can be written as

$$
(PV^+)^{\mu'\nu'}_{\alpha\beta}(x,x') = 0 \tag{13}
$$

where P acts at x' and the indices α, β and x are inert, one finds that the integrand $(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')f_{\mu'\nu'}(x')$ can be expressed as

$$
\begin{aligned}
&(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')f_{\mu'\nu'}(x') = \\
&(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')(PF)_{\mu'\nu'}(x') - (PV^+)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x') \\
&= \nabla_{\gamma'}\{(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')(\nabla^{\gamma'}F_{\mu'\nu'}(x')) \\
&\qquad - (\nabla^{\gamma'}(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x'))F_{\mu'\nu'}(x')\}
\end{aligned}
\tag{14}
$$

where the curvature terms have canceled from the final expression by virtue of the self-adjoint structure of the wave operator P. Thus the integrand in the volume integral over D_p can be reexpressed as a total divergence. It is worth noting that the scalar field analogue to the above observation is given at the end of p.187 in Friedlander's book.

Using Eq. (14) to reexpress the integral over D_p in the equation for $F_{\alpha\beta}^{\text{tail}}(x)$ and using Stokes' theorem to rewrite this volume integral as a boundary integral over $\partial D_p = C_p \cup S_p$, one arrives at the result that

$$
\begin{aligned}
F_{\alpha\beta}^{\text{tail}}(x) = \frac{1}{2\pi}\int\limits_{C_p} &*[(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')\nabla^{\gamma'}F_{\mu'\nu'}(x') \\
&- F_{\mu'\nu'}(x')\nabla^{\gamma'}(V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')] \\
- \frac{1}{2\pi}\int\limits_{\sigma_p} &\langle \nabla t(x'), \nabla'\Gamma(x,x')\rangle (V^+)_{\alpha\beta}^{\mu'\nu'}(x,x')F_{\mu'\nu'}(x')\mu_{t,\Gamma}(x')
\end{aligned}
\tag{15}
$$

where the orientation chosen for the integral over the null cone C_p corresponds to a normal field directed towards the vertex p. The cancelation of the two boundary integrals over S_p parallels that shown by Friedlander for the scalar case in his Eq. (5.3.14) (wherein however it was assumed that the support of the scalar field did not meet C_p). One can also think of deriving Eq. (15) from Eq. (11) by pushing the surface S_p forward, holding its boundary σ_p fixed, until it merges in the limit with

C_p. Friedlander remarks in his Section (5.4) that the representation formula for the characteristic initial value problem can be derived in a similar manner wherein, however, one pushes $S_p$towards the past rather than towards the future.

In fact, an appealing symmetric alternative to the formulation outlined above would result from replacing the solution of the Cauchy initial value problem with that for the characteristic initial value problem and then transforming it, via Green's theorem, in the analogous way. For example let p' be a point on the particle's trajectory lying to the past of p and designate by $C_{p'}$ the future light cone issuing from p'. $C_{p'}$ will intersect the past directed cone C_p issuing from p in a 2-surface which we now take to be σ_p. The region D_p is here understood to include the entire interior region bounded by the two cones (each truncated at σ_p) and the initial data originally prescribed on S_p is herein replaced by characteristic initial data defined on $C_{p'}$. If however, we again use Green's theorem to transform this representation formula in the analogous way, we again arrive at the expression given by Eq. (15) above except that now σ_p is the surface defined by $C_p \cap C_{p'}$ instead of being the boundary to S_p.

IV. Reduction of the tail contributions

To simplify the notation slightly let us write Eq. (15) in the form

$$F^{\text{tail}}_{\alpha\beta}(x) = {}^{I}F^{\text{tail}}_{\alpha\beta}(x) + {}^{II}F^{\text{tail}}_{\alpha\beta}(x) \tag{16}$$

where ${}^{I}F^{\text{tail}}_{\alpha\beta}(x)$ is the integral over C_p and ${}^{II}F^{\text{tail}}_{\alpha\beta}(x)$ that over σ_p. Reexpressing the dual $*v$ to a vector v via Eq. (2.9.3) of Ref. [8] (see also p. 194 of this reference)

$$*v(x') = \langle v(x'), \text{grad}'\Gamma(x,x')\rangle \mu_\Gamma(x') \tag{17}$$

one gets the more explicit formula for ${}^{I}F^{\text{tail}}_{\alpha\beta}(x)$

$$\begin{aligned} {}^{I}F^{\text{tail}}_{\alpha\beta}(x) = \frac{1}{2\pi}\int\limits_{C_p} \mu_\Gamma(x')\{\nabla^{\gamma'}\Gamma(x,x')[(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')\nabla_{\gamma'}F_{\mu'\nu'}(x') \\ - F_{\mu'\nu'}(x')\nabla_{\gamma'}(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')]\}. \end{aligned} \tag{18}$$

The key point here is that only derivatives tangential to the null generators of the cone C_p appear in the integrand. This allows one to integrate by parts to eliminate derivatives of $F_{\mu'\nu'}$ in favor of (tangential) derivatives of $(V^+)^{\mu'\nu'}_{\alpha\beta}$ which, in turn, may be evaluated from the transport equation (c.f. Eq. (5.5.23) of Ref. [8]) which determines this quantity along C_p. Carrying out these operations and writing $(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')$ for the restriction of $(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')$ to C_p one arrives at

$$\begin{aligned} {}^{I}F^{\text{tail}}_{\alpha\beta}(x) = \frac{1}{2\pi}\int\limits_{C_p} \mu_\Gamma(x')\{(\nabla^{\gamma'}\Gamma(x,x'))\nabla_{\gamma'}((V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')) \\ + F_{\mu'\nu'}(x')[PU^{\mu'\nu'}_{\alpha\beta}(x,x') + (\Box'\Gamma(x,x') - 4)(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')]\} \end{aligned} \tag{19}$$

where P is the wave operator defined in Eq. (10) above and where, as mentioned above, we can write

$$U^{\mu'\nu'}_{\alpha\beta}(x,x') = \kappa(x,x')\tau^{\mu'\nu'}_{\alpha\beta}(x,x') \tag{20}$$

with the parallel transport "propagator" $\tau^{\mu'\nu'}_{\alpha\beta}$, expressible in terms of an orthonormal frame $\{h_{\hat{a}}\}$ and co-frame $\{\theta^{\hat{a}}\}$ parallel propagated from point p, as

$$\tau^{\mu'\nu'}_{\alpha\beta}(x, x') = h^{\mu'}_{\hat{e}}(x')\theta^{\hat{e}}_{\alpha}(x)h^{\nu'}_{\hat{f}}(x')\theta^{\hat{f}}_{\beta}(x). \tag{21}$$

One can evaluate the first integral in the above expression for ${}^{I}F^{\rm tail}_{\alpha\beta}(x)$ by first transforming from normal coordinates $\{x^{\mu'}\}$ to spherical null coordinates defined by

$$\begin{aligned} x^{1'} &= r' \sin\ \theta\ \cos\varphi \\ x^{2'} &= r'\ \sin\ \theta \sin\varphi \\ x^{3'} &= r'\ \cos\theta \\ t' = x^{0'} &= \frac{u+v}{2},\ r' = \frac{v-u}{2} \\ r' &= \sqrt{\Sigma(x^{i'})^2} \end{aligned} \tag{22}$$

so that

$$u = t' - r',\ v = t' + r' \tag{23}$$

with $\Gamma = -uv$ everywhere and $v = 0$ on C_p. In terms of these coordinates it is straightforward to show that

$$\Gamma^{;\alpha}\frac{\partial}{\partial x^{\alpha}} = 2v\frac{\partial}{\partial v} + 2u\frac{\partial}{\partial u} \tag{24}$$

and that the Leray form

$$\mu_{\Gamma} = \frac{\sqrt{-\det(g_{\mu\nu})}}{u}du \wedge d\theta \wedge d\varphi \tag{25}$$

satisfies

$$\mu = d\Gamma \wedge \mu_{\Gamma} = \sqrt{-\det(g_{\mu\nu})}du \wedge dv \wedge d\theta \wedge d\varphi \tag{26}$$

as required by its definitions (where $\det(g_{\mu\nu})$ is the determinant of g in the spherical null coordinates). Substituting these expressions into the integral in question one easily arrives at

$$\begin{aligned}
&\frac{1}{2\pi}\int\limits_{C_p}\mu_\Gamma(x')(\nabla^{\gamma'}\Gamma(x,x'))\nabla_{\gamma'}[(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')] \\
&= \frac{1}{2\pi}\int\limits_{C_p} du\wedge d\theta\wedge d\varphi[\frac{\partial}{\partial u}[2\sqrt{-\det(g_{\gamma\delta})}(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')]] \\
&+ \frac{1}{2\pi}\int\limits_{C_p}\mu_\Gamma(x')[(4-\nabla_{\gamma'}\nabla^{\gamma'}\Gamma(x,x'))(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')] \\
&= -\frac{1}{2\pi}\int\limits_{\sigma_p} d\theta\wedge d\varphi\{2\sqrt{-\det(g_{\gamma\delta})}[(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')]\} \\
&+ \frac{1}{2\pi}\int\limits_{C_p}\mu_\Gamma(x')[(4-\Box'\Gamma(x,x'))(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')].
\end{aligned} \tag{27}$$

Evaluating the metric form rstricted to C_p one gets

$$\begin{aligned}
ds^2\,\Big|_{C_p} &= -du dv + {}^{(2)}V_\theta dv d\theta + {}^{(2)}V_\varphi dv d\varphi \\
&{}^{(2)}g_{AB}dx^A dx^B + (-\frac{1}{4}\,{}^{(4)}g^{uu} + \frac{1}{4}\,{}^{(2)}g_{AB}\,{}^{(2)}V^A\,{}^{(2)}V^B)dv^2
\end{aligned} \tag{28}$$

where $\{x^A; A=1,2\} = \{\theta,\varphi\}$ and where ${}^{(2)}g_{AB}dx^A dx^B$ and ${}^{(2)}V_A dx^A = {}^{(2)}g_{AB}\,{}^{(2)}V^B dx^A$ are (at each fixed u on the hypersurface C_p defined by $v=0$) a 2-dimensional Riemannian metric and one-form respectively. Thus, on C_p

$$2\sqrt{-\det g_{\gamma\delta}}\,\Big|_{C_p} = \sqrt{\det\,{}^{(2)}g_{AB}}\,\Big|_{C_p} \tag{29}$$

so that

$$\begin{aligned}
&\frac{1}{2\pi}\int\limits_{C_p}\mu_\Gamma(x')\{(\nabla^{\gamma'}\Gamma(x,x'))\nabla_{\gamma'}[(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')]\} \\
&= -\frac{1}{2\pi}\int\limits_{\sigma_p}\sqrt{\det\,{}^{(2)}g_{AB}}\,d\theta\wedge d\varphi[(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')] \\
&+ \frac{1}{2\pi}\int\limits_{C_p}\mu_\Gamma(x')[(4-\Box'\Gamma(x,x'))(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')]
\end{aligned} \tag{30}$$

It is easy to see from the metric form (28) that $\sqrt{\det {}^{(2)}g_{AB}}d\theta \wedge d\varphi$ is just the invariant 2-surface area element induced on σ_p (defined in coordinates by $v = 0$, $u = u(\theta, \varphi)$) by the spacetime metric. Writing this as $d\sigma_p$ and combining Eqs. (19) and (30) we get

$$
{}^{I}F^{\text{tail}}_{\alpha\beta}(x) = -\frac{1}{2\pi}\int\limits_{\sigma_p} d\sigma_p[(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')F_{\mu'\nu'}(x')] \qquad (31)
$$

$$
+\frac{1}{2\pi}\int\limits_{C_p} \mu_\Gamma(x')F_{\mu'\nu'}(x')(PU^{\mu'\nu'}_{\alpha\beta}(x,x'))
$$

where the terms involving $(\Box'\Gamma(x,x') - 4)$ have cancelled. Adding this result to the expression for ${}^{II}F^{\text{tail}}_{\alpha\beta}(x)$ and recalling Friedlander's formula for the measure $\mu_{t,\Gamma}(x')$ given by his Eq. (5.3.19),

$$
\langle \nabla t, \nabla \Gamma \rangle \mu_{t,\Gamma} = -d\sigma_p \qquad (32)
$$

one finds that the two remaining integrals in $F^{\text{tail}}_{\alpha\beta}(x)$ involving the non-local quantity $(V_0)^{\mu'\nu'}_{\alpha\beta}(x,x')$ also cancel leaving

$$
F^{\text{tail}}_{\alpha\beta}(x) = \frac{1}{2\pi}\int\limits_{C_p} \mu_\Gamma(x')(F_{\mu'\nu'}(x')PU^{\mu'\nu'}_{\alpha\beta}(x,x')). \qquad (33)
$$

Useful formulas for the evaluation of the "source" term $PU^{\mu'\nu'}_{\alpha\beta}(x,x')$ may be found in Ref. [6].

Returning to the main stream of the argument we want to evaluate $F^{\text{tail}}_{\alpha\beta}(x)$ as given by Eq. (33). Notice especially that this expression for $F^{\text{tail}}_{\alpha\beta}(x)$ involves an integral of the source's total field, $F_{\mu'\nu'}(x')$ (rather than just its tail contribution) paired with a computable background quantity, $PU^{\mu'\nu'}_{\alpha\beta}(x,x')$. The integral extends over the light cone from point p back to the "initial" Cauchy hypersurface which can, in principle, be pushed into the distant past where, one presumes, the evaluation of $F^{\text{tail}}_{\alpha\beta}(x)$ is insensitive to this hypersurface's precise location. Our simplified formula for $F^{\text{tail}}_{\alpha\beta}(x)$ has resulted, in effect, from trading the source distribution $j^\mu(x)$ for its (total) field $F_{\mu\nu}(x)$ and the tail function $(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')$, after an application of Green's theorem, for its restriction to C_p where it is determined by a transport equation. In the final simplifying step, we have been able to eliminate all direct reference to the tail function by using this transport equation to reexpress the tangential derivative, $\Gamma^{;\gamma'}(x,x')\nabla_{\gamma'}(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')$, of this tail function in terms of its "source", $PU^{\mu'\nu'}_{\alpha\beta}(x,x')$.

The key point is that one can evaluate $F^{\rm tail}_{\alpha\beta}(x)$ without the necessity of splitting the moving charge's field into direct and tail contributions as is needed in the more traditional approaches to this problem which require tedious numerical decompositions.

Though one approach to evaluating $F^{\rm tail}_{\alpha\beta}(x)$ is simply to use the source's (numerically produced) total field $F_{\mu'\nu'}(x')$ in a numerical computation of the integral (33), we want to propose an alternative, though approximate, purely analytical calculation. The idea is to use Friedlander's curved-space Liénard -Wiechert formula for the total field $F_{\mu'\nu'}(x')$ of a moving charge (c.f., Sect. (5.6) of Ref. (8)) but to truncate this by dropping its tail contribution, keeping only the direct part of the total field.

Our proposal thus is to approximate this total field by the direct part of the particle's Liénard-Wiechert field as given by (the curl of) Friedlander's formula for $A_{\mu'}(x')$. In substituting this (truncated) expression for the particle's actual retarded field, we are discarding the contribution from the tail part of the fundamental solution as represented by the integral in Friedlander's formula (5.6.22) over the past history of the moving particle.

Is there any justification for such a truncation, aside from the fact that it yields a computable expression? The part we have dropped, to the contribution to the retarded field at x', consists of an integral of tail contributions from the particle's past history up until the instant where the past light cone from x' intercepts the particle's world-line (at some point $q(x')$). As a superposition of solutions to the homogeneous wave equation (c.f., Eq. (10)) one might anticipate that the pairing of this contribution to

$F^{\rm tail}_{\mu'\nu'}(x')$ with the tail field $(V^+)^{\mu'\nu'}_{\alpha\beta}(x,x')$ should yield a potentially negligible addition to $F^{\rm tail}_{\mu\nu}(x)$ for the following reason. If the reduction argument using Green's theorem is now reversed (but only for the tail$\otimes$tail contribution to $F^{\rm tail}$) then there is no longer a source inside D_p to produce the retarded field on C_p but only "initial" data prescribed (depending upon the setup used) on either S_p or $C_{p'}$ and on the boundary 2-surface σ_p. One's hope is that if these surfaces are pushed sufficiently far to the past, then their contributions will be negligible.

Unfortunately this argument is complicated by the observation that the tail integral contribution to Friedlander's Liénard-Wiechert formula for $F_{\mu'\nu'}(x')$ is not, strictly speaking, a solution to the homogeneous wave equation although it would be if the upper limit of integration ($q(x')$ in the notation introduced above) were independent of x'. Thus Friedlander's formula for the Liénard-Wiechert field does not split simply into a computable, direct term plus a solution of the homogeneous wave equation. It is difficult to assess the impact of this subtlety upon the plausibility of neglecting the tail contribution to the Liénard-Wiechert expression for the particle's retarded field. One's expectation though is that the tail$\otimes$tail contribution to $F^{\rm tail}_{\alpha\beta}(x)$ (which we are suggesting to neglect, is of higher order than the tail$\otimes$direct contribution (which we are retaining) and thus that the latter should provide the dominant part of the calculation.

Before concluding this section, we would like to mention an alternative formulation which may shed some light upon the plausibility of our approximation or its limitations. Suppose, using the characteristic initial value formulation defined above, we endeavor to compute the self-force at p' (rather than at p) from "final" data prescribed on the backwards light cone C_p extending from p to its boundary σ_p and from data defined by the orbiting source. Reversing the roles of p and p' in the argument above, one arrives at a formula analogous to that of (33) above though now with an integral over the cone $C_{p'}$ (rather than over C_p) extending to the future of the point p' where the self-force is desired.

Again we require the particle's retarded field but now along $C_{p'}$ (rather than along C_p where its contribution has been canceled by the Green's theorem argument). A crucial difference is that now the tail contribution to Friedlander's Liénard-Wiechert formula involves a fixed upper limit (namely the point p') and thus this expression for $F_{\mu'\nu'}(x') \mid_{C_{p'}}$ splits into a direct, computable term and a remainder which now does satisfy the homogeneous wave equation.

V. Self-force reduction for scalar radiation fields

Consider a point particle with mass m, scalar charge q and distributional source μ which generates a scalar field Φ via the wave equation

$$\Phi_{;\gamma}{}^{;\gamma} = -4\pi\mu \tag{34}$$

and which reacts to that field according to the force law

$$m\frac{du^{\mu}}{d\tau} = ma^{\mu} = q(g^{\mu\nu} + u^{\mu}u^{\nu})\nabla_{\nu}\Phi \tag{35}$$

where $u^{\mu} = \frac{dx^{\mu}}{d\tau}$ is the particle's normalized four-velocity. It was shown by Quinn [12] that if one requires the equations of motion to follow from a variational principle, the mass m must vary according to

$$\frac{dm}{d\tau} = -qu^{\mu}\nabla_{\mu}\Phi. \tag{36}$$

Since the force on the particle involves the gradient of Φ rather than Φ itself, we find it more convenient to work with the field equation for the gradient. Writing Λ_{α} for $\Phi_{;\alpha}$ one finds, upon differentiating Eq. (34) that Λ_{α} satisfies

$$\Lambda_{\alpha;\gamma}^{;\gamma} - R_{\alpha}^{\beta}\Lambda_{\beta} = -4\pi\mu_{;\alpha}. \tag{37}$$

To avoid spurious solutions to Eq. (37) one should impose the restrictions that $\Lambda_{\alpha} = \Phi_{;\alpha}$ with $\Lambda_{\alpha}^{;\alpha} = -4\pi\mu$ on the initial Cauchy surface. Note that Eq. (37) is equivalent to the field equation for a (pure gauge) Maxwell vector potential $A_{\alpha} = \Lambda_{\alpha} = \Phi_{;\alpha}$ with (for consistency) vanishing source current ($j_{\mu} = 0$) but satisfying the modified Lorentz gauge condition $A_{\alpha}{}^{;\alpha} = \Lambda_{\alpha}{}^{;\alpha} = \Phi_{;\alpha}{}^{;\alpha} = -4\pi\mu$. Since the effective source in Eq. (37) is not a genuine Maxwell current, it need not be required to have vanishing divergence. No inconsistency results here if $\mu_{;\alpha}{}^{;\alpha} \neq 0$.

Friedlander's formulation of the Cauchy problem now gives a representation formula for the field Λ_α:

$$
\begin{aligned}
\Lambda_\alpha(x) = & \frac{1}{2\pi}\int\limits_{C_p} U_\alpha^{\mu'}(x,x')f_{\mu'}(x')\mu_\Gamma(x') \\
& + \frac{1}{2\pi}\int\limits_{D_p} (V^+)_\alpha^{\mu'}(x,x')f_{\mu'}(x')\mu(x') \\
& + \frac{1}{2\pi}\int\limits_{S_p} *[(V^+)_\alpha^{\mu'}(x,x')\nabla^{\gamma'}\Lambda_{\mu'}(x') \\
& \quad - \Lambda_{\mu'}(x')\nabla^{\gamma'}(V^+)_\alpha^{\mu'}(x,x')] \\
& + \frac{1}{2\pi}\int\limits_{\sigma_p} \{U_\alpha^{\mu'}(x,x')[2(\nabla^{\gamma'}t(x'))(\nabla_{\gamma'}\Lambda_{\mu'}(x')) \\
& \quad + \Lambda_{\mu'}(x')\ \Box\ t(x')] \\
& \quad - \langle\nabla t(x'),\nabla'\Gamma(x,x')\rangle(V^+)_\alpha^{\mu'}(x,x')\Lambda_{\mu'}(x')\}\mu_{t,\Gamma}(x').
\end{aligned}
\tag{38}
$$

Here $U_\alpha^{\mu'}(x,x') = \kappa(x,x')\tau_\alpha^{\mu'}(x,x')$ with $\tau_\alpha^{\mu'}(x,x')$ the transport bitensor (parallel propagator) for one-forms and the source $f_\mu(x)$ is now given by

$$
f_\mu(x) = -4\pi\mu_{;\mu}(x). \tag{39}
$$

The tail field $(V^+)_\alpha^{\mu'}(x,x')$ now satisfies the self-adjoint, homogeneous wave equation

$$
(V^+)_{\alpha;\gamma'}^{\mu';\gamma'}(x,x') - R_{\nu'}^{\mu'}(x')(V^+)_\alpha^{\nu'}(x,x') = 0 \tag{40}
$$

with initial data on the light cone C_p determined by Friedlander's transport equation (5.5.23).

As in the electromagnetic case, we identify the tail contribution to $\Lambda_\alpha(x)$ as the collection of integrals involving $(V^+)_\alpha^{\mu'}(x,x')$,

$$\begin{aligned}\Lambda_\alpha^{\text{tail}}(x) :=& \frac{1}{2\pi}\int\limits_{Dp} (V^+)_\alpha^{\mu'}(x,x') f_{\mu'}(x')\mu(x') \\ &+ \frac{1}{2\pi}\int\limits_{Sp} *[(V^+)_\alpha^{\mu'}(x,x')\nabla^{\gamma'}\Lambda_{\mu'}(x') \\ &\quad - \Lambda_{\mu'}(x')\nabla^{\gamma'}(V^+)_\alpha^{\mu'}(x,x')] \\ &- \frac{1}{2\pi}\int\limits_{\sigma p} \{\langle \nabla t(x'), \nabla'\Gamma(x,x')\rangle (V^+)_\alpha^{\mu'}(x,x')\Lambda_{\mu'}(x')\}\mu_{t,\Gamma}(x').\end{aligned} \tag{41}$$

Reducing this with exactly the same techniques we applied in the previous section, we arrive at the simplified formula

$$\Lambda_\alpha^{\text{tail}}(x) = \frac{1}{2\pi}\int\limits_{Cp} \mu_\Gamma(x')(\Lambda_{\mu'}(x') P U_\alpha^{\mu'}(x,x')) \tag{42}$$

where P is the operator acting (at x') on $(V^+)_\alpha^{\mu'}(x,x')$ on the left hand side of Eq. (40) As in the previous problem, one can imagine evaluating $\Lambda_\alpha^{\text{tail}}(x)$ numerically from the (numerically produced) total field $\Lambda_{\mu'}(x')$ of the moving charge or, alternatively (and approximately), by truncating the analogue of Friedlander's Liénard-Wiechert expression for $\Lambda_{\mu'}(x')$ to its direct, explicitly computable contribution.

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9. See Sect. (5.5) of Ref. [8] for the definition of self-adjoint tensor wave equations.
10. See Sect. (4.4) of Ref. [8] for the definition of "causal domains".
11. See Theorem 5.5.2 (pp. 210-211) of Ref. [8] for the tensor representation formula.
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