ON EXISTENCE OF NON-SINGULAR, VACUUM, STATIONARY SPACE-TIMES WITH A NEGATIVE COSMOLOGICAL CONSTANT

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Abstract. We construct infinite dimensional families of non-singular stationary space times, solutions of the vacuum Einstein equations with a negative cosmological constant.

Contents

1. Introduction

A class of space-times of interest is that of vacuum metrics with a negative cosmological constant admitting a smooth conformal completion at infinity. It is natural to seek for stationary solutions with this property. In this paper we show that a large class of such solutions can be constructed by prescribing the conformal class of a stationary Lorentzian metric on the conformal boundary $\partial\mathscr{M}$, provided that the boundary data are sufficiently close to, e.g., those of anti-de Sitter space-time.

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We mention the recent papers [4, 5], where we have constructed infinite dimensional families of *static*, singularity free solutions of the vacuum Einstein equations with a negative cosmological constant. The main point of the current work is to remove the staticity restriction. This leads to new, infinite dimensional families of non-singular, *stationary* solutions of those equations.

We thus seek to construct Lorentzian metrics ${}^{n+1}g$ in any space-dimension $n \geq 2$, with Killing vector $X = \partial/\partial t$. In adapted coordinates those metrics can be written as

$$ {}^{n+1}g = -V^2(dt + \underbrace{\theta_i dx^i}_{=\theta})^2 + \underbrace{g_{ij}dx^i dx^j}_{=g} , \tag{1.1} $$

$$ \partial_t V = \partial_t \theta = \partial_t g = 0 . \tag{1.2} $$

Our main result reads as follows (see below for the definition of non-degeneracy; the function ρ in (1.3) is a coordinate near ∂M that vanishes at ∂M):

THEOREM 1.1. *Let $n = \dim M \geq 2$, $k \in \mathbb{N} \setminus \{0\}$, $\alpha \in (0,1)$, and consider a static Lorentzian Einstein metric of the form* (1.1)-(1.2) *with strictly positive $V = \mathring{V}$, $g = \mathring{g}$, and $\theta = 0$, such that the associated Riemannian metric $\widetilde{g} = \mathring{V}^2 d\varphi^2 + \mathring{g}$ on $\mathbb{S}^1 \times M$ is C^2 compactifiable and non-degenerate, with smooth conformal infinity. For every smooth $\widehat{\theta}$, sufficiently close to zero in $C^{k+2,\alpha}(\partial M, \mathcal{T}_1)$, there exists a unique, modulo diffeomorphisms which are the identity at the boundary, nearby stationary vacuum metric of the form* (1.1)-(1.2) *such that, in local coordinates near the conformal boundary ∂M,*

$$ V - \mathring{V} = O(\rho) , \quad \theta_i = \widehat{\theta}_i + O(\rho), \quad g_{ij} - \mathring{g}_{ij} = O(1) . \tag{1.3} $$

Theorem 1.1 is more or less a rewording of Theorem 5.3 below, taking into account the discussion of uniqueness in Section 6.

The $(n+1)$-dimensional anti-de Sitter metric is non-degenerate in the sense above, so Theorem 1.1 provides in particular an infinite dimensional family of solutions near that metric.

The requirement of strict positivity of $\mathring{V}$ excludes black hole solutions, it would be of interest to remove this condition.

The decay rates in (1.3) have to be compared with the leading order behavior ρ^{-2} both for $\mathring{V}$ and $\mathring{g}_{ij}$. A precise version of (1.3) in terms of weighted function spaces (as defined below) reads

$$ (V - \mathring{V}) \in C_1^{k+2,\alpha}(\mathbb{S}^1 \times M) , \ (g - \mathring{g}) \in C_2^{k+2,\alpha}(\mathbb{S}^1 \times M, \mathcal{S}_2) , \tag{1.4} $$

$$ \theta - \widehat{\theta} \in C_2^{k+2,\alpha}(\mathbb{S}^1 \times M, \mathcal{T}_1) , \tag{1.5} $$

and the norms of the differences above are small in those spaces.

Note that our hypothesis that the metric $\widetilde{g}$ is conformally C^2 implies that $\widetilde{g}$ is $C^{n-1,\alpha} \cap C^{3,\alpha}$–conformally compactifiable and polyhomogeneous [9]. We show in Section 7 that our solutions have complete polyhomogeneous expansions near the conformal boundary, see Theorem 7.1 for a precise statement. Since the Fefferman-Graham expansions are valid regardless of the signature of the boundary metric, the solutions are smooth in even space-time dimensions. In odd space-time dimensions the obstruction to smoothness is the non-vanishing of the Fefferman-Grahm obstruction tensor [13, 15] of the (Lorentzian) metric obtained by restricting $-(dt+\theta)^2 + V^{-2}g$ to the conformal boundary at infinity.

Theorem 1.1 is proved by an implicit-function argument. This requires the proof of isomorphism properties of an associated linearised operator. This operator turns out to be rather complicated, its mapping properties being far from evident. We overcome this by reinterpreting this operator as the Lichnerowicz operator $\tilde{\Delta}_L + 2n$ in one-dimension higher. Our non-degeneracy condition above is then precisely the condition that $\tilde{\Delta}_L + 2n$ has no L^2–kernel. While this is certainly a restrictive condition, large classes of Einstein metrics satisfying this condition are known [2, 3, 5, 18].

Because of the V^2 multiplicative factor in front of θ in (1.1), for distinct $\widehat{\theta}$'s the resulting space-time metrics have distinct conformal metrics at the conformal boundary at infinity. This makes it problematic to determine the energy of the new solutions relative e.g. to the anti-de Sitter solution; similarly for angular momentum. Now, each of our solutions ${}^{n+1}g$ comes associated with a family of non-stationary solutions, which asymptote to ${}^{n+1}g$, and which can be constructed using e.g. a technique of Friedrich [14]. To each member of such a family one can then associate global Hamiltonian charges relative to ${}^{n+1}g$ as in [8, 11]. In this approach our solutions define the zero point of energy for each family, and there is no natural way of comparing relative energies, angular momenta, and so on, of members of distinct families.

2. Definitions, notations and conventions

Let $\overline{N}$ be a smooth, compact $(n+1)$-dimensional manifold with boundary ∂N. Let $N := \overline{N}\backslash\partial N$, a non-compact manifold without boundary. In our context the boundary ∂N will play the role of a *boundary at infinity* of N. Let g be a Riemannian metric on N, we say that (N, g) is *conformally compact* if there exists on $\overline{N}$ a smooth defining function ρ for ∂N (that is $\rho \in C^\infty(\overline{N})$, $\rho > 0$ on N, $\rho = 0$ on ∂N and $d\rho$ nowhere vanishing on ∂N) such that $\overline{g} := \rho^2 g$ is a $C^{2,\alpha}(\overline{N}) \cap C^\infty_0(N)$ Riemannian metric on $\overline{N}$, we will denote by $\widehat{g}$ the metric induced on ∂N. Our definitions of function spaces follow [18]. Now if $|d\rho|_{\overline{g}} = 1$ on ∂N, it is well known (see [19] for instance) that g has asymptotically sectional curvature -1 near its boundary at infinity, in that case we say that (N, g) is *asymptotically hyperbolic*. If we assume moreover than (N, g) is Einstein, then asymptotic hyperbolicity enforces the normalisation

$$\mathrm{Ric}(g) = -ng \ , \tag{2.1}$$

where $\mathrm{Ric}(g)$ is the Ricci curvature of g.

We recall that the Lichnerowicz Laplacian acting on a symmetric two-tensor field is defined as [7, § 1.143]

$$\Delta_L h_{ij} = -\nabla^k\nabla_k h_{ij} + R_{ik}h^k{}_j + R_{jk}h^k{}_j - 2R_{ikjl}h^{kl} \ .$$

The operator $\Delta_L + 2n$ arises naturally when linearising (2.1). We will say that g is *non-degenerate* if $\Delta_L + 2n$ has no L^2-kernel.

While we seek to construct metrics of the form (1.1), for the purpose of the proofs we will often work with manifolds N of the form

$$N = \mathbb{S}^1 \times M,$$

equipped with a warped product, asymptotically hyperbolic metric

$$V^2 d\varphi^2 + g,$$

where V is a positive function on M and g is a Riemannian metric on M. By an abuse of terminology, such metrics will be said *static*.

The basic example of a non-degenerate, asymptotically hyperbolic, static Einstein space is the Riemannian counterpart of the AdS space-time. In that case M is the unit ball of $\mathbb{R}^n$, with the hyperbolic metric

$$g_0 = \rho^{-2}\delta \ ,$$

δ is the Euclidean metric, $\rho(x) = \frac{1}{2}(1 - |x|^2_\delta)$, and

$$V_0 = \rho^{-1} - 1 \ .$$

We denote by $\mathcal{T}_p^q$ the set of rank p covariant and rank q contravariant tensors. When $p = 2$ and $q = 0$, we denote by $\mathcal{S}_2$ the subset of symmetric tensors. We use the summation convention, indices are lowered and raised with g_{ij} and its inverse g^{ij}.

3. Isomorphism theorems

Some of the isomorphism theorems we will use are consequences of Lee's theorems [18], it is therefore convenient to follow his notation for the weighted Hölder spaces $C^{k,\alpha}_\delta$. As described in the second paragraph before proposition B of [18], a tensor in this space corresponds to ρ^δ times a tensor in the usual $C^{k,\alpha}$ space as defined using the norm of the conformally compact metric. This implies that, in local coordinates near the conformal boundary, a function in $C^{k,\alpha}_\delta$ is $O(\rho^\delta)$, a one-form in $C^{k,\alpha}_\delta$ has components which are $O(\rho^{\delta-1})$, and a covariant two-tensor in $C^{k,\alpha}_\delta$ has components which are $O(\rho^{\delta-2})$.

We will often appeal to isomorphism theorems of [18] in weighted $C^{k,\alpha}$ spaces, for $k \in \mathbb{N}$. Under the regularity conditions on the metric in our definition of asymptotically hyperbolic metric, those theorems apparenly only apply to low values of k. However, under our hypotheses, one can use those theorems for $k = 2$, and use scaling estimates to obtain the conclusion for any value of k.

3.1. An isomorphism on two-tensors.

We first recall a result of Lee (see Theorem C(c) and proposition D of [18], there is no L^2-kernel here by hypothesis):

Theorem 3.1. *Let $\mathbb{S}^1 \times M$ be equipped with a non-degenerate asymptotically hyperbolic metric $\widetilde{g}$. For $0 < k + \alpha \notin \mathbb{N}$ and $\delta \in (0, n)$ the operator $\widetilde{\Delta}_L + 2n$ is an isomorphism from $C^{k+2,\alpha}_\delta(\mathbb{S}^1 \times M, \mathcal{S}_2)$ to $C^{k,\alpha}_\delta(\mathbb{S}^1 \times M, \mathcal{S}_2)$.*

When the metric is static of the form $\widetilde{g} = V^2 d\varphi^2 + g$ we deduce

Corollary 3.2. *On (M,g) we consider the operator*

$$(W,h)\mapsto (l(W,h),L(W,h))\ ,$$

where

$$\begin{aligned} l(W,h) &= V\Big[(\nabla^*\nabla+2n+V^{-1}\nabla^*\nabla V+V^{-2}|dV|^2)W+V^{-1}\nabla_jV\nabla^jW \\ &\quad -V^{-1}\nabla^jV\nabla^kVh_{kj}+\langle \mathrm{Hess}_g\,V,h\rangle_g\Big]\,. \end{aligned}$$

and

$$\begin{aligned} L_{ij}(W,h) &= \frac12\Delta_Lh_{ij}+nh_{ij}-\frac12V^{-1}\nabla^kV\nabla_kh_{ij} \\ &\quad +\frac12V^{-2}(\nabla_iV\nabla^kVh_{kj}+\nabla_jV\nabla^kVh_{ki}) \\ &\quad -\frac12V^{-1}(\nabla_i\nabla^kVh_{kj}+\nabla_j\nabla^kVh_{ki}) \\ &\quad +2V^{-2}W(\mathrm{Hess}_g\,V)_{ij}-2V^{-3}\nabla_iV\nabla_jVW. \end{aligned}$$

Then (l,L) is an isomorphism from $C^{k+2,\alpha}_{\delta-1}(M)\times C^{k+2,\alpha}_{\delta}(M,\mathcal{S}_2)$ to $C^{k,\alpha}_{\delta-2}(M)\times C^{k,\alpha}_{\delta}(M,\mathcal{S}_2)$ when $\delta\in(0,n)$.

Proof. First, it is easy to see that the Laplacian commutes with the Lie derivative operator in the Killing direction, so the operator $\widetilde\Delta_L+2n$ restricted to φ-independent tensor field is again an isomorphism. Now, from Lemma A.2 below, if we define $\mathcal{P}$ to be the set of symmetric covariant two tensors of the form

$$\widetilde h = 2VWd\varphi^2+h_{ij}dx^idx^j\ ,$$

and if we let $\mathcal{T}$ denote the collection of tensors of the form

$$\widetilde h = 2\xi_idx^id\varphi,$$

then the Lichnerowicz Laplacian preserves the decomposition $\mathcal{P}\oplus\mathcal{T}$. In particular the operator $\frac12\widetilde\Delta_L+n$ restricted to $\mathcal{P}$ is an isomorphism, and this operator is (l,L). □

3.2. **Two isomorphisms on one-forms.** The proof of Corollary 3.2 also shows the following (note a shift in the rates of decay, as compared to the previous section, due to the fact that a tensor field $\xi_idx^id\varphi$ is in $C^{m,\sigma}_\rho$ if and only if the one-form ξ_idx^i is in $C^{m,\sigma}_{\rho-1}$):

Corollary 3.3. *The operator on one-forms defined as*

$$\begin{aligned} \mathcal{L}:\xi_i &\mapsto -\nabla^k\nabla_k\xi_i+V^{-1}\nabla^kV\nabla_k\xi_i+3V^{-2}\nabla_iV\nabla^kV\xi_k \\ &\quad +R^l{}_i\xi_l-3V^{-1}\nabla_i\nabla_jV\xi^j+2n\xi_i\ , \end{aligned}$$

is an isomorphism from $C^{k+2,\alpha}_{\delta-1}(M,\mathcal{T}_1)$ to $C^{k,\alpha}_{\delta-1}(M,\mathcal{T}_1)$ when $\delta\in(0,n)$. If we let $\xi=V^2\theta$, we therefore obtain that the operator $\mathcal{Q}$ on one-forms defined as $V^{-2}\mathcal{L}(V^2\theta_i)$

$$\begin{aligned} \mathcal{Q}:\theta_i &\mapsto -\nabla^k\nabla_k\theta_i-3V^{-1}\nabla^kV\nabla_k\theta_i-2V^{-1}\nabla^k\nabla_kV\theta_i+3V^{-2}\nabla_iV\nabla^kV\theta_k \\ &\quad +R^l{}_i\theta_l-3V^{-1}\nabla_i\nabla_jV\theta^j+2n\theta_i\ , \end{aligned}$$

is an isomorphism from $C^{k+2,\alpha}_{\delta+1}(M,\mathcal{T}_1)$ to $C^{k,\alpha}_{\delta+1}(M,\mathcal{T}_1)$ when $\delta\in(0,n)$.

We will appeal to yet another result of Lee (see [18] Theorem C(c), Proposition F and Corollary 7.4, there is again no L^2-kernel here because of the Ricci curvature condition):

THEOREM 3.4. *On $\mathbb{S}^1 \times M$ equipped with an asymptotically hyperbolic metric $\widetilde{g}$ with negative Ricci curvature, the operator $\widetilde{\nabla}^*\widetilde{\nabla} - \widetilde{\mathrm{Ric}}$ acting on one-forms is an isomorphism from $C^{k+2,\alpha}_\delta(\mathbb{S}^1 \times M, \mathcal{T}_1)$ to $C^{k,\alpha}_\delta(\mathbb{S}^1 \times M, \mathcal{T}_1)$ when $|\delta - \frac{n}{2}| < \sqrt{\frac{n^2}{4}+1}$.*

When the metric is static of the form $\widetilde{g} = V^2 d\varphi^2 + g$ we deduce:

COROLLARY 3.5. *Under the hypotheses of the preceding theorem, on (M, g) consider the operator*

$$\Omega_i \mapsto B(\Omega)_i + R_{ij}\Omega^j - V^{-1}\nabla_i\nabla^j V\Omega_j =: \mathcal{B}(\Omega)_i \ ,$$

where

$$B(\Omega)_i := \nabla^k\nabla_k\Omega_i + V^{-1}\nabla^k V\nabla_k\Omega_i - V^{-2}\nabla_i V\nabla^k V\Omega_k \ .$$

Then $\mathcal{B}$ is an isomorphism from $C^{k+2,\alpha}_\delta(M, \mathcal{T}_1)$ to $C^{k,\alpha}_\delta(M, \mathcal{T}_1)$ when $|\delta - \frac{n}{2}| < \sqrt{\frac{n^2}{4}+1}$.

Proof. The argument is identical to the proof of Corollary 3.2 using Lemma A.3 and the fact that, in the notation of Lemma A.3,

$$\widetilde{R}_{ic}\widetilde{\Omega}^c = R_{ij}\Omega^j - V^{-1}\nabla_i\nabla^j V\Omega_j \ .$$

□

3.3. **An isomorphism on functions in dimension n.** If we assume that $V^2 d\varphi^2 + g$ is a static asymptotically hyperbolic metric on $\mathbb{S}^1 \times M$, then it is easy to check that at infinity $V^{-2}|dV|^2 = 1$ and $V^{-1}\nabla^i\nabla_i V = n$. In dimension n, we will need an isomorphism property for the following operator acting on functions:

$$\sigma \mapsto \mathcal{T}\sigma := V^{-3}\nabla^i(V^3\nabla_i\sigma) = \nabla^i\nabla_i\sigma + 3V^{-1}\nabla^i V\nabla_i\sigma \ .$$

From [6, Theorem 7.2.1 (ii) and Remark (i), p. 77] we obtain:

THEOREM 3.6. *Let (V, g) be close in $C^{k+2,\alpha}_{-1}(M) \times C^{k+2,\alpha}_0(M, \mathcal{S}_2)$ to an asymptotically hyperbolic static metric. Then $\mathcal{T}$ is an isomorphism from $C^{k+2,\alpha}_\delta(M)$ to $C^{k,\alpha}_\delta(M)$ when $0 < \delta < n+2$.*

REMARK 3.7. Theorem 3.6 will be used with $\sigma = O(\rho^2)$, note that $\delta = 2$ verifies the inequality above since $n \geq 2$.

3.4. **An isomorphism on functions in dimension 3.** In dimension $n = 3$, we will also be interested in the following operator acting on functions:

$$\omega \mapsto \mathcal{Z}\omega := V^3\nabla^i(V^{-3}\nabla_i\omega) = \nabla^i\nabla_i\omega - 3V^{-1}\nabla^i V\nabla_i\omega \ .$$

The indicial exponents for this equation are $\mu_- = -1$ and $\mu_+ = 0$ (see [6, Remark (i), p. 77]). As $\mu_+ \not> 0$ we cannot invoke [6, Theorem 7.2.1] to conclude. Instead we appeal to the results of Lee [18]. For this we need to have a formally self-adjoint operator, so we set $\omega = V^{\frac{3}{2}} f$, thus

$$\mathcal{Z}\omega = V^{\frac{3}{2}}\left[\nabla^i\nabla_i f - \left(\frac{15}{4}V^{-2}|dV|^2 - \frac{3}{2}V^{-1}\nabla^i\nabla_i V\right) f\right] =: V^{\frac{3}{2}} Z f \ . \tag{3.1}$$

At infinity $V^{-2}|dV|^2 = 1$ and $V^{-1}\nabla^i\nabla_i V = 3$, leading to the following indicial exponents

$$\delta = \frac{1}{2}, \frac{3}{2} .$$

We want to show that Z satisfies condition (1.4) of [18],

$$\|u\|_{L^2} \leq C\|Zu\|_{L^2} , \tag{3.2}$$

for smooth u compactly supported in a sufficiently small open set $\mathcal{U} \subset M$ such that $\overline{\mathcal{U}}$ is a neighborhood of ∂M. We will need the following, well known result; we give the proof for completeness:

LEMMA 3.8. *On an asympotically hyperbolic manifold (M, g) with boundary definining function ρ we have, for all compactly supported C^2 functions,*

$$\int u\nabla^*\nabla u \geq \left(\frac{n-1}{2}\right)^2 \int (1 + O(\rho))u^2.$$

Proof. Let f be a smooth function to be chosen later, then

$$\int |f^{-1}d(fu)|^2 = \int |du|^2 + f^{-2}|df|^2u^2 + 2f^{-1}u\langle df, du\rangle \geq 0$$

An integration by parts shows that

$$\int 2f^{-1}u\langle df, du\rangle = \int u^2f^{-2}|df|^2 + u^2f^{-1}\nabla^*\nabla f.$$

This leads to

$$\int u\nabla^*\nabla u = \int |du|^2 \geq \int (-f^{-1}\nabla^*\nabla f - 2f^{-2}|df|^2)u^2 .$$

When $f = \rho^{-\frac{n-1}{2}}$ the last term equals $(n-1)^2\|u(1 + O(\rho))\|^2_{L^2}/4$, which concludes the proof. □

Lemma 3.8 combined with the fact that $V^{-2}|dV|^2 = 1 + O(\rho)$ and that $V^{-1}\nabla^*\nabla V = -3 + O(\rho)$ shows that

$$\|u\|_{L^2}\|Zu\|_{L^2} \geq -\int uZu \geq \Big(\frac{(3-1)^2}{4} + \frac{15}{4} - \frac{9}{2}\Big)\int (1 + O(\rho))u^2,$$

which shows that Z satisfies the condition (3.2) with

$$C = \Big(\frac{(3-1)^2}{4} + \frac{15}{4} - \frac{9}{2}\Big)^{-1/2} = 2 .$$

We recall that the critical weight to be in L^2 is $O(\rho^1)$ so the function $f = V^{-3/2} = O(\rho^{3/2})$, corresponding to $\omega = 1$, is in the L^2-kernel of Z. We prove now that this kernel equals

$$\ker Z = V^{\frac{-3}{2}}\mathbb{R} .$$

Assume f is in the L^2-kernel of $\mathcal{Z}$, by elliptic regularity f is smooth on M. Let $\varphi_k \in W^{1,\infty}$ be any function on M such that $\varphi_k = 1$ on the geodesic ball $B_p(k)$ of radius k centred at p, with $\varphi_k = 0$ on $M \setminus B_p(k+1)$, and $|\nabla\varphi_k| \leq C$ independently of k. Such functions can be constructed by composing the geodesic distance from p with a test function on $\mathbb{R}$. Integrating by parts one has

$$\begin{aligned} 0 &= -\int V^3\varphi_k^2 f\mathcal{Z}f = -\int \varphi_k^2 f\nabla^i(V^{-3}\nabla_i f) \\ &= \int \varphi_k^2 V^{-3}|\nabla f|^2 + 2V^{-3}f\varphi_k\nabla^i\varphi_k\nabla_i f \end{aligned}$$

Using Hölder's inequality, the second integral can be estimated from below by

$$-2\left(\int \varphi_k^2 V^{-3}|\nabla f|^2\right)^{1/2}\left(\int f^2 V^{-3}|\nabla\varphi_k|^2\right)^{1/2} ,$$

leading to

$$\int \varphi_k^2 V^{-3}|\nabla f|^2 \le 4\int f^2 V^{-3}|\nabla\varphi_k|^2 .$$

By Lebesgue's dominated convergence theorem, the right-hand side converges to zero as k tends to infinity because $f \in L^2$, while V^{-1} is uniformly bounded, and $\nabla\varphi_k$ is supported in $B_p(k+1) \setminus B_p(k)$. So f is a constant. Using [18], Theorem C(c), we thus obtain

Theorem 3.9. *Let (V,g) be close in $C^{k+2,\alpha}_{-1}(M) \times C^{k+2,\alpha}_0(M,\mathcal{S}_2)$ to an asymptotically hyperbolic static metric. Then Z is an isomorphism from $C^{k+2,\alpha}_\delta(M)/V^{-3/2}\mathbb{R}$ to*

$$\left\{f \in C^{k,\alpha}_\delta(M) : \int_M V^{-3/2}f = 0\right\} .$$

when $1/2 < \delta < \frac{3}{2}$. Equivalently, $\mathcal{Z}$ is an isomorphism from $C^{k+2,\alpha}_\delta(M)/\mathbb{R}$ to

$$\left\{f \in C^{k,\alpha}_\delta(M) : \int_M V^{-3}f = 0\right\} . \tag{3.3}$$

when $-1 < \delta < 0$.

4. The equations

Rescaling the metric to achieve a convenient normalisation of the cosmological constant, the vacuum Einstein equations for a metric satisfying (1.1)-(1.2) read (see, e.g., [12])

$$\left\{\begin{array}{l} V(\nabla^*\nabla V + nV) = \frac{1}{4}|\lambda|^2_g , \\ \mathrm{Ric}(g) + ng - V^{-1}\,\mathrm{Hess}_g\, V = \frac{1}{2V^2}\lambda\circ\lambda , \\ \mathrm{div}(V\lambda) = 0 , \end{array}\right. \tag{4.1}$$

where

$$\lambda_{ij} = -V^2(\partial_i\theta_j - \partial_j\theta_i) , \quad (\lambda\circ\lambda)_{ij} = \lambda_i{}^k\lambda_{kj} .$$

In dimension $n = 3$ an alternative set of equations can be obtained by introducing the *twist potential* ω. Writing $d\omega = \omega_i dx^i$ one sets

$$\omega_i = \frac{V}{2}\varepsilon_{ijk}\lambda^{jk} \quad\Longleftrightarrow\quad \lambda_{jk} = \frac{1}{V}\varepsilon_{jk\ell}\omega^\ell .$$

This leads to (compare [17])

$$\begin{cases} V(\nabla^*\nabla V + 3V) = \frac{1}{2V^2}|d\omega|^2 \,, \\ \mathrm{Ric}(g) + 3g - V^{-1}\,\mathrm{Hess}_g\, V = \frac{1}{2V^4}(d\omega\otimes d\omega - |d\omega|^2 g) \,, \\ \nabla^*(V^{-3}\nabla\omega) = 0 \,. \end{cases} \tag{4.2}$$

4.1. **The linearised equation.** We first consider the operator from the set of functions times symmetric two tensor fields to itself, defined as

$$\begin{pmatrix} V \\ g \end{pmatrix} \mapsto \begin{pmatrix} V(\nabla^*\nabla V + nV) \\ \mathrm{Ric}(g) + ng - V^{-1}\,\mathrm{Hess}_g\, V \end{pmatrix} .$$

The two components of its linearisation at (V,g) are

$$p(W,h) = V\left[(\nabla^*\nabla + 2n + V^{-1}\nabla^*\nabla V)W + \langle \mathrm{Hess}_g\, V, h\rangle_g - \langle \mathrm{div}\,\mathrm{grav}\, h, dV\rangle_g\right] \,,$$

$$\begin{aligned} P_{ij}(W,h) &= \frac{1}{2}\Delta_L h_{ij} + nh_{ij} + \frac{1}{2}V^{-1}\nabla^k V(\nabla_i h_{kj} + \nabla_j h_{kj} - \nabla_k h_{ij}) - (\mathrm{div}^*\,\mathrm{div}\,\mathrm{grav}\, h)_{ij} \\ &\quad + V^{-2}W(\mathrm{Hess}_g\, V)_{ij} - V^{-1}(\mathrm{Hess}_g\, W)_{ij} \,. \end{aligned}$$

We let Tr denote the trace and we set

$$\mathrm{grav}\, h = h - \frac{1}{2}\mathrm{Tr}_g\, hg, \quad (\mathrm{div}\, h)_i = -\nabla^k h_{ik}, \quad (\mathrm{div}^*\, w)_{ij} = \frac{1}{2}(\nabla_i w_j + \nabla_j w_i) \,,$$

(note the geometers' convention to include a minus in the definition of divergence). It turns out to be convenient to introduce the one-form

$$w_j = V^{-1}\nabla^k V h_{kj} + \nabla^k h_{kj} - \frac{1}{2}\nabla_j(\mathrm{Tr}\, h) - V^{-1}\nabla_j W - V^{-2}\nabla_j V W \,,$$

which allows us to rewrite $P(W,h)$ as

$$P(W,h) \;=\; L(W,h) + \mathrm{div}^*\, w \,,$$

where L is as in Corollary 3.2. Similarly, $p(W,h)$ can be rewritten as

$$p(W,h) \;=\; l(W,h) + V\langle w, dV\rangle_g \,.$$

4.2. **The modified equation.** We want to use the implicit function theorem to construct our solutions. As is well known, the linearisation of the Ricci tensor does not lead to well behaved equations, and one adds "gauge fixing terms" to take care of this problem. Our choice of those terms arises from harmonic coordinates for the vacuum Einstein equations in one dimension higher.

In dimension 3, we start by solving the following system of equations

$$\begin{cases} q(V,g) := V(\nabla^*\nabla V + 3V + \langle\Omega, dV\rangle) - \frac{1}{2V^2}|d\omega|^2 = 0 \,, \\ Q(V,g) := \mathrm{Ric}(g) + 3g - V^{-1}\,\mathrm{Hess}_g\, V + \mathrm{div}^*\,\Omega \\ \qquad\qquad - \frac{1}{2V^4}(d\omega d\omega - |d\omega|^2 g) = 0 \,, \\ \qquad \nabla^*(V^{-3}\nabla\omega) = 0 \,, \end{cases} \tag{4.3}$$

with

$$\begin{aligned} -\Omega_j &\equiv -\Omega(V,g,U,b)_j \\ &:= \widehat{g}_{j\mu}\widehat{g}^{\alpha\beta}(\widehat{\Gamma}^\mu_{\alpha\beta} - \widetilde{\Gamma}^\mu_{\alpha\beta}) \\ &= g_{jk}g^{\ell m}(\Gamma^k_{\ell m} - \mathring{\Gamma}^k_{\ell m}) + V^{-2}g_{jk}(U\mathring{\nabla}^j U - V\nabla^j V) \\ &= g^{\ell m}(\mathring{\nabla}_m g_{j\ell} - \frac{1}{2}\mathring{\nabla}_j g_{\ell m}) + V^{-2}g_{jk}(U\mathring{\nabla}^j U - V\nabla^j V) \end{aligned} \tag{4.4}$$

where $\mathring{\nabla}$-derivatives are relative to a fixed metric b with Christoffel symbols $\mathring{\Gamma}^\alpha_{\beta\gamma}$, U is a fixed positive function, latin indices run from 0 to n, and $\widehat{g} := V^2(dx^0)^2+g$ with Christoffel symbols $\widehat{\Gamma}^\alpha_{\beta\gamma}$, while the $\widetilde{\Gamma}^\alpha_{\beta\gamma}$'s are the Christoffel symbols of the metric $U^2(dx^0)^2+b$, compare (A.1) below. The co-vector field Ω has been chosen to contain terms which cancel the "non-elliptic terms" in the Ricci tensor, together with some further terms which will ensure bijectivity of the operators involved. The first line of the equation above makes clear the relation of Ω to the $n+1$-dimensional metric $\widehat{g}$ and its (U,b)-equivalent.

In dimension n, as a first step we will solve the system

$$\left\{\begin{array}{l} q(V,g) := V(\nabla^*\nabla V + nV + \langle\Omega, dV\rangle) - \frac{1}{4}|\lambda|^2_g = 0\ ,\\ Q(V,g) := \mathrm{Ric}(g) + ng - V^{-1}\,\mathrm{Hess}_g V + \mathrm{div}^*\,\Omega - \frac{1}{2V^2}\lambda\circ\lambda = 0\ ,\\ \mathrm{div}(V\lambda) = -V^3 d\sigma\ ,\end{array}\right. \tag{4.5}$$

where Ω is as in dimension 3, while the "Lorenz-gauge fixing function" σ equals

$$\sigma = V^{-3}\nabla^i(V^3\theta_i)\ .$$

A calculation shows

$$\mathrm{div}(V\lambda) + V^3 d\sigma = V^3[-\mathcal{Q} + 2(V^{-1}\nabla^*\nabla V + n)](\theta)\ ,$$

where $\mathcal{Q}$ is as in Corollary 3.3, which makes clear the elliptic character of the third equation in (4.5).

The derivative of Ω with respect to (V,g) at (U,b) is

$$D_{(V,g)}\Omega(U,b)(W,h) = -w,$$

where w is the one-form defined in Section 4.1 with (V,g) replaced with (U,b). Thus, the linearisation of (q,Q) at (U,b) is

$$D(q,Q)(U,b) = (l,L)\ ,$$

where (l,L) is the operator defined in Section 4.1 with (V,g) replaced with (U,b). We will show that, under reasonable conditions, solutions of (4.3) (resp. (4.5)) are solutions of (4.2) (resp. (4.1)). If (ω,V,g) solves (4.3) (resp. if (θ,V,g) solves (4.5)), we set

$$\Phi := \mathrm{div}^*\,\Omega\ ,$$

$$a := \left\{\begin{array}{ll} \frac{1}{2V^4}|d\omega|^2 & \text{in the context of (4.3),}\\ \frac{1}{4V^2}|\lambda|^2_g & \text{when studying (4.5),}\end{array}\right.$$

$$A := \left\{\begin{array}{ll} \frac{1}{2V^4}(d\omega d\omega - |d\omega|^2 g) & \text{when studying (4.3),}\\ \frac{1}{2V^2}\lambda\circ\lambda & \text{when analysing (4.5).}\end{array}\right.$$

With this notation, the first two equations in both (4.3) and (4.5) take the form

$$\left\{\begin{array}{l} \nabla^*\nabla V + nV + \langle\Omega, dV\rangle = Va\ ,\\ \mathrm{Ric}(g) + ng - V^{-1}\,\mathrm{Hess}_g V + \Phi = A\ ,\end{array}\right. \tag{4.6}$$

If we take the trace of the second equation in (4.6) we obtain

$$\begin{array}{rcl} 0 &=& R(g) + n^2 + V^{-1}\nabla^*\nabla V + \mathrm{Tr}\,\Phi - \mathrm{Tr}\,A\\ &=& R(g) + n^2 - n - V^{-1}\langle\Omega(V,g), dV\rangle + \mathrm{Tr}\,\Phi + a - \mathrm{Tr}\,A\ .\end{array}$$

Then

$$\begin{aligned}
E(g) &:= \operatorname{grav}_g \operatorname{Ric}(g) \\
&= -ng + V^{-1}\operatorname{Hess} V - \Phi - \frac{1}{2}(-n(n-1) + V^{-1}\langle \Omega(V,g), dV\rangle - \operatorname{Tr}\Phi)g \\
&\quad + \operatorname{grav}_g A + \frac{a}{2}g \;.
\end{aligned}$$

As usual, we will use the vanishing of the divergence of E to obtain an equation for Ω. For a solution to the modified equation, the negative of the divergence of $E(g)$ equals

$$\begin{aligned}
\operatorname{div} E(g)_j &= V^{-2}\nabla^i V \nabla_i \nabla_j V - V^{-1}\nabla^i \nabla_j \nabla_i V + \nabla^i \Phi_{ij} - \frac{1}{2}\nabla_j \operatorname{Tr}\Phi \\
&\quad + \frac{1}{2}\nabla_j (V^{-1}\langle \Omega(V,g), dV\rangle) + \Big(\operatorname{div}(\operatorname{grav}_g A + \frac{a}{2}g)\Big)_j \\
&= V^{-1}\nabla^i V(R_{ij} + ng_{ij} + \Phi_{ij} - A_{ij}) - V^{-1}\nabla^i V R_{ij} \\
&\quad - V^{-1}\nabla_j (nV + \langle \Omega(V,g), dV\rangle + Va) + \nabla^i \Phi_{ij} - \frac{1}{2}\nabla_j \operatorname{Tr}\Phi \\
&\quad + \frac{1}{2}\nabla_j (V^{-1}\langle \Omega(V,g), dV\rangle) + \Big(\operatorname{div}(\operatorname{grav}_g A + \frac{a}{2}g)\Big)_j \\
&= \nabla^i \Phi_{ij} + V^{-1}\nabla^i V \Phi_{ij} - \frac{1}{2}\nabla_j \operatorname{Tr}\Phi - \frac{1}{2}V^{-1}\nabla_j \langle \Omega(V,g), dV\rangle \\
&\quad - \frac{1}{2}V^{-2}\nabla_j V\langle \Omega(V,g), dV\rangle + \underbrace{V^{-1}\nabla_j (Va) - V^{-1}\nabla^i V A_{ij} + \Big(\operatorname{div}(\operatorname{grav}_g A + \frac{a}{2}g)\Big)_j}_{=:\beta_j} \\
&= \nabla^i \Phi_{ij} + V^{-1}\nabla^i V \Phi_{ij} - \frac{1}{2}\nabla_j \operatorname{Tr}\Phi - \frac{1}{2}V^{-2}\nabla_j (V\langle \Omega(V,g), dV\rangle) + \beta_j \\
&= \frac{1}{2}[\nabla^k \nabla_k \Omega_j + V^{-1}\nabla^i V \nabla_i \Omega_j - V^{-2}\nabla_j V \nabla^i V \Omega_i + R_{ij}\Omega^i - V^{-1}\nabla_j \nabla^i V \Omega_i] + \beta_j \\
&= \frac{1}{2}[B(\Omega)_j + R_{ij}\Omega^i - V^{-1}\nabla_j \nabla^i V \Omega_i] + \beta_j \\
&= \frac{1}{2}\mathcal{B}(\Omega)_j + \beta_j \;.
\end{aligned}$$

We now claim that β_j vanishes when σ does. For (4.3) this is a straightforward computation. For (4.5) we have

$$\begin{aligned}
-\beta_j &= \frac{1}{2}V\lambda_j{}^k \nabla_k \sigma - \frac{1}{2}V^{-3}\nabla_j V|\lambda|^2 + \frac{1}{8}V^{-2}\nabla_j |\lambda|^2 \\
&\quad + \frac{1}{2}V^{-2}\lambda^{ik}\nabla_i \lambda_{kj} + V^{-3}\nabla^i V \lambda_i{}^k \lambda_{jk} \;.
\end{aligned}$$

From the definition of λ_{ij} one has

$$\nabla_{[i}(V^{-2}\lambda_{kj]}) = 0 \;.$$

This gives

$$\lambda^{ik}(2\nabla_i \lambda_{kj} + \nabla_j \lambda_{ik} - 6V^{-1}(\nabla_{[i}V)\lambda_{kj]}) = 0 \;,$$

which can also be rewritten as

$$\lambda^{ik}\nabla_i\lambda_{kj} = -\frac{1}{4}\nabla_j|\lambda|^2 + 2V^{-1}\nabla_i V\lambda^i{}_k\lambda^k{}_j + V^{-1}\nabla_j V|\lambda|^2 \ ,$$

and our claim follows.

We will see during the construction to follow that solutions of the third equation in (4.5) which decay sufficiently fast satisfy $\sigma = 0$. The Bianchi identity $\operatorname{div} E(g) = 0$ shows then that Ω is in the kernel of $\mathcal{B}$. It follows from Corollary 3.5 that the only solution of this equation which decays sufficiently fast is zero.

5. The construction

5.1. **The n-dimensional case.** We consider $\mathring{V}^2 d\varphi^2 + \mathring{g}$, an asymptotically hyperbolic Einstein static metric on $\mathbb{S}^1 \times M$. We prescribe $\widehat{\theta} \in C^{k+2,\alpha}(\partial M, \mathcal{T}_1)$, and we seek a solution

$$\theta = \theta(\widehat{\theta}, V, g) \in C_1^{k+2,\alpha}(M, \mathcal{T}_1)$$

of the problem

$$\left\{ \begin{array}{l} \operatorname{div}(V\lambda) + V^3 d\sigma \equiv V^3[-\mathcal{Q} + 2(V^{-1}\nabla^*\nabla V + n)]\theta = 0 \ , \\ \theta - \widehat{\theta} \in C_2^{k+2,\alpha}(M, \mathcal{T}_1) \ ; \end{array} \right. \tag{5.1}$$

recall that

$$\lambda_{ij} = -V^2(\partial_i\theta_j - \partial_j\theta_i) \text{ and } \sigma = V^{-3}\nabla^i(V^3\theta_i) \ . \tag{5.2}$$

Such solutions can be obtained by solving the following equation for $\theta - \widehat{\theta}$:

$$[-\mathcal{Q} + 2(V^{-1}\nabla^*\nabla V + n)](\theta - \widehat{\theta}) = -[-\mathcal{Q} + 2(V^{-1}\nabla^*\nabla V + n)]\widehat{\theta} \ .$$

When $V = \mathring{V}$ in (5.1), then the term $V^{-1}\nabla^*\nabla V + n$ vanishes and the operator is an isomorphism by Corollary 3.3 with $\delta = 1$. Thus, the operator appearing there is an isomorphism for all nearby V's. In fact, for any Riemannian metric g on M, close to $\mathring{g}$ in $C_0^{k+2,\alpha}(M, \mathcal{S}_2)$, with $g - \mathring{g} \in C_1^{k+2,\alpha}(M, \mathcal{S}_2)$, and for any function V on M, close to $\mathring{V}$ in $C_{-1}^{k+2,\alpha}(M)$, with $V - V_0 \in C_0^{k+2,\alpha}(M)$ a unique solution exists. Moreover the map $(\widehat{\theta}, V, g) \mapsto \theta - \widehat{\theta}$ is smooth.

Let us denote by $\mathring{\theta}$ the solution of (5.1) with $(V, g) = (\mathring{V}, \mathring{g})$.

Remark 5.1. $\mathring{\theta}$ is polyhomogenous when $\mathring{V}$ and $\mathring{g}$ are by the results in [6]. Applying the second line of (5.1) twice we obtain

$$\theta - \mathring{\theta} = \theta - \widehat{\theta} + \widehat{\theta} - \mathring{\theta} \in C_2^{k+2,\alpha}(M, \mathcal{T}_1) \ .$$

Furthermore, one has directly that $\sigma - \mathring{\sigma} \in C_2^{k+1,\alpha}(M)$; $\mathring{\sigma}$ is in fact also in $C_2^{k+1,\alpha}(M)$ by expanding near the boundary.

Suppose that θ solves $\operatorname{div}(V\lambda) + V^3 d\sigma = 0$, then clearly

$$\operatorname{div}[\operatorname{div}(V\lambda) + V^3 d\sigma] = 0.$$

Since the double divergence of any anti-symmetric tensor vanishes identically it holds that $\operatorname{div}\operatorname{div}(V\lambda) = 0$, so that under (5.1) σ is in $C_2^{k+1,\alpha}(M)$ and verifies

$$\nabla^i(V^3\nabla_i\sigma) = 0.$$

It follows from Theorem 3.6 that $\sigma = 0$ when $n \geq 2$.

Let us define a map F, from one-forms on $\partial_\infty M$ times functions on M times symmetric two-tensor fields on M to functions on M times symmetric two-tensor fields, which to $(\widehat{\theta}, V, g)$ associates

$$\begin{pmatrix} V(\nabla^*\nabla V + nV + \langle \Omega(V, g, \mathring{V}, \mathring{g}), dV\rangle) - \frac{1}{4}|\lambda|^2_g \\ \mathrm{Ric}(g) + ng - V^{-1}\,\mathrm{Hess}_g\, V + \mathrm{div}^*\,\Omega(V, g, \mathring{V}, \mathring{g}) + \frac{1}{2V^2}\lambda \circ \lambda \end{pmatrix} .$$

PROPOSITION 5.2. *Let $\mathring{V}^2 d\varphi^2 + \mathring{g}$ be an asymptotically hyperbolic static Einstein metric on $\mathbb{S}^1 \times M$, $k \in \mathbb{N}$, $\alpha \in (0,1)$. The map $\mathcal{F}$ defined as*

$$\begin{array}{ccc} C^{k+2,\alpha}(\partial M, \mathcal{T}_1) \times C^{k+2,\alpha}_1(M) \times C^{k+2,\alpha}_2(M, \mathcal{S}_2) & \longrightarrow & C^{k,\alpha}_0(M) \times C^{k,\alpha}_2(M, \mathcal{S}_2) \\ (\widehat{\theta}, W, h) & \longmapsto & F(\widehat{\theta}, \mathring{V} + W, \mathring{g} + h) \end{array}$$

is smooth in a neighborhood of zero.

Proof. The function $\mathring{V} \in C^{k+2,\alpha}_{-1}(M)$ is strictly positive, so the same is true for $\mathring{V} + W$ if W is sufficiently small in $C^{k+2,\alpha}_1(M) \subset C^{k+2,\alpha}_{-1}(M)$. Similarly, the symmetric two-tensor field $\mathring{g} + h \in C^{k+2,\alpha}_0(M, \mathcal{S}_2)$ is positive definite when h is small in $C^{k+2,\alpha}_2(M, \mathcal{S}_2) \subset C^{k+2,\alpha}_0(M, \mathcal{S}_2)$. The map $(\widehat{\theta}, V, g) \mapsto \theta$ is smooth. Now, for $\theta \in C^{k+2,\alpha}_1(M)$, by (5.2) and by Remark 5.1 we have

$$\lambda_{ij} = O(\rho^{-2}) \ ,$$

which further implies

$$\frac{1}{2V^2}\lambda \circ \lambda \in C^{k,\alpha}_2(M, \mathcal{S}_2) \ .$$

The fact that the remaining terms in $F(\widehat{\theta}, \mathring{V} + v, \mathring{g} + h)$ are in the space claimed, and that the map is smooth is standard (see [18] for instance). $\square$

We can conclude now as follows:

THEOREM 5.3. *Let $n \geq 2$, and let $\mathring{V}^2 d\varphi^2 + \mathring{g}$ be a polyhomogenous non-degenerate asymptotically hyperbolic static Einstein metric on $\mathbb{S}^1 \times M$, $k \in \mathbb{N}\backslash\{0\}$, $\alpha \in (0,1)$. For all $\widehat{\theta}$ close to zero in $C^{k+2,\alpha}(\partial M, \mathcal{T}_1)$, there exists a unique solution*

$$(\theta, V, g) = (\mathring{\theta} + \vartheta, \mathring{V} + W, \mathring{g} + h)$$

to (4.1) with $\mathring{\theta} - \widehat{\theta} \in C^{k+2,\alpha}_2(M)$ and

$$(\vartheta, W, h) \in C^{k+2,\alpha}_3(M) \times C^{k+2,\alpha}_1(M) \times C^{k+2,\alpha}_2(M, \mathcal{S}_2) \ ,$$

close to zero, satisfying the gauge conditions $\Omega = \sigma = 0$. Moreover, the maps $\widehat{\theta} \mapsto \mathring{\theta} - \widehat{\theta}$ and $\widehat{\theta} \mapsto (\vartheta, W, h)$ are smooth maps of Banach spaces near zero.

Proof. As already pointed out, the one-form $\theta = \theta(\widehat{\theta}, V, g)$ exists and is unique when W and h are small. From Proposition 5.2 we know that the map $\mathcal{F}$ is smooth. The linearisation of $\mathcal{F}$ at zero is

$$D_{(W,h)}\mathcal{F}(0,0,0) = D_{(V,g)}F(0, \mathring{V}, \mathring{g}) = (l, L) \ .$$

From Corollary 3.2, with $\delta = 2$, we obtain that $D_{(W,h)}\mathcal{F}(0,0,0)$ is an isomorphism. The implicit function theorem shows that the conclusion of Theorem 5.3 remains valid *for the modified equation* (4.5). Returning to Section 4.2, we see that $\Omega = \Omega(V, g, \mathring{V}, \mathring{g}) \in C_2^{k+1,\alpha}(M, T_1)$ and that $\mathcal{B}(\Omega) = 0$, so from Corollary 3.5, we have $\Omega = 0$, obtaining thus a solution to (4.1). $\square$

5.2. **The three-dimensional case.** In three dimensions an alternative construction can be given, as follows. We consider again an asymptotically hyperbolic Einstein static metric $\mathring{V}^2 d\varphi^2 + \mathring{g}$ on $\mathbb{S}^1 \times M$. We consider Theorem 3.9, with g — a Riemannian metric on M close to $\mathring{g}$ in $C_0^{k+2,\alpha}(M, \mathcal{S}_2)$, and V — a function on M close to $\mathring{V}$ in $C_{-1}^{k+2,\alpha}(M)$. For our purposes there is no preferred value of parameter δ there. It is convenient to set $\delta = s - 1$, and arbitrarily choose some $s \in (0,1)$. For any $\widehat{\omega} \in C^{k+1,\alpha}(\partial M)$ satisfying

$$\int_{\partial M} \widehat{\omega} = 0 \,, \tag{5.3}$$

there exists a unique, modulo constant, solution

$$\omega = \omega(\widehat{\omega}, V, g) \in C_{-1}^{k+2,\alpha}(M)$$

to

$$\begin{cases} \nabla^*(V^{-3}\nabla\omega) = 0, \\ \omega - \widehat{\omega}\rho^{-1} \in C_{s-1}^{k+2,\alpha}(M) \,. \end{cases}$$

(This can be seen by writing $\mathcal{Z}\delta\omega = -\mathcal{Z}(\widehat{\omega}\rho^{-1})$, and checking that the source term in the equation for $\delta\omega$ satisfies the integrability condition (3.3) when (5.3) holds.) Moreover, the map $(\widehat{\omega}, V, g) \mapsto \omega - \widehat{\omega}\rho^{-1}$ is smooth in the $C_{s-1}^{k+2,\alpha}(M)$ topology.

We define a new map F, defined on the set of functions on $\partial_\infty M$ times functions on M times symmetric two-tensor fields, mapping to functions on M times symmetric two-tensor fields, which to $(\widehat{\omega}, V, g)$ associates

$$\begin{pmatrix} V(\nabla^*\nabla V + 3V + \langle \Omega(V, g, \mathring{V}, \mathring{g}), dV\rangle) - \frac{1}{2V^2}|d\omega|^2 \\ \mathrm{Ric}(g) + 3g - V^{-1}\nabla_i\nabla_j V + \mathrm{div}^*\,\Omega(V, g, \mathring{V}, \mathring{g}) - \frac{1}{2V^4}(d\omega d\omega - |d\omega|^2 g) \end{pmatrix} .$$

Proposition 5.4. *Let $\mathring{V}^2 d\varphi^2 + \mathring{g}$ be an asymptotically hyperbolic static Einstein metric on $\mathbb{S}^1 \times M$, $k \in \mathbb{N}$, $\alpha \in (0,1)$. The map $\mathcal{F}$ defined as*

$$\begin{array}{ccc} C^{k+2,\alpha}(\partial M) \times C_1^{k+2,\alpha}(M) \times C_2^{k+2,\alpha}(M, \mathcal{S}_2) & \longrightarrow & C_0^{k,\alpha}(M) \times C_2^{k,\alpha}(M, \mathcal{S}_2) \\ (\widehat{\omega}, W, h) & \longmapsto & F(\widehat{\omega}, \mathring{V} + W, \mathring{g} + h) \end{array}$$

is smooth in a neighborhood of zero.

Proof. The proof is essentialy the same as that of Proposition 5.2. We simply note that for all $s \in (0,1)$ we have, by direct estimations,

$$V^{-4}(d\omega d\omega - |d\omega|^2 g) \in C_2^{k,\alpha}(M, \mathcal{S}_2) \,.$$

$\square$

We are ready to formulate now:

THEOREM 5.5. *Let* $\dim M = 3$ *and let* $\mathring{V}^2 d\varphi^2 + \mathring{g}$ *be a non-degenerate asymptotically hyperbolic static Einstein metric on* $\mathbb{S}^1 \times M$, $k \in \mathbb{N}$, $\alpha \in (0,1)$, $s \in (0,1)$. *For all* $\widehat{\omega}$ *close to zero in* $C^{k+2,\alpha}(\partial M)$ *and satisfying* (5.3) *there exists a unique solution*

$$(\omega, V, g) = (\widehat{\omega}\rho^{-1} + w, \mathring{V} + W, \mathring{g} + h)$$

to (4.2) *with*

$$(w, W, h) \in C^{k+2,\alpha}_{s-1}(M) \times C^{k+2,\alpha}_{1}(M) \times C^{k+2,\alpha}_{2}(M, \mathcal{S}_2)\ ,$$

close to zero, satisfying the gauge condition $\Omega = 0$. *Moreover, the map* $\widehat{\omega} \mapsto (w, W, h)$ *is a smooth map of Banach spaces near zero.*

Proof. The proof is identical to that of Theorem 5.3, making use of Proposition 5.4. □

6. Uniqueness

So far we have shown that solutions are unique in the gauge $\Omega = 0$, together with the condition $\sigma = 0$ in the context of (4.1). We claim that any metrics satisfying the hypotheses of Theorem 1.1 can be brought to this gauge.

First, consider σ; note that the one-form θ of (1.1) is defined modulo the differential of a function f defined on M; indeed, the replacement $t \to t + f$ leads to $\theta \to \theta + df$. We can then use Theorem 3.6 to find a unique f such that the function σ associated with $\theta + df$ vanishes.

The vanishing of Ω requires a smallness hypothesis, as well as some work. Suppose that we are given a couple (V, g) near to $(\mathring{V}, \mathring{g})$. The second line of (4.4) shows that, in the notation of [9], the condition $\Omega = 0$ is exactly the condition $\Delta_{g'\tilde{g}} \operatorname{Id} = 0$, where $g' = V^2 d\varphi^2 + g$. The proof that Ω can be made to vanish is established by inspection of the arguments of Section 4 of [9]. We simply note that the implicit function theorem, as invoked there, can be applied globally on M (rather than in a collar neighborhood of the boundary, as in [9]) if we assume that $(\mathring{V}/V, V^{-2}g)$ is close to $(1, \mathring{V}^{-2}\mathring{g})$ in $C^2(\overline{M})$. Indeed, the linearised operator, denoted by L in [9], is again an isomorphism by the results of [18], as follows from the fact that $\tilde{g}$ is Einstein, with negative scalar curvature. (Actually, the Einstein equations are irrelevant for the question of $\Omega = 0$ gauge, as long as the Ricci tensor of $\tilde{g}$ is negative definite.)

Uniqueness of solutions up-to-diffeomorphism (which is the identity on the boundary) is a direct consequence of the above.

Somewhat more generally, without the smallness hypothesis, a variation of an argument due to M. Anderson [1] applies: Suppose that we are given two C^2-compactifiable solutions, say (V, θ, g) and (V', θ', g'), with identical smooth boundary data. We can then bring θ and θ' to the $\sigma = \sigma' = 0$ gauge as above, and put both $V^2 d\varphi^3 + g$ and $(V')^2 d\varphi^3 + g'$ in the harmonic gauge of Section 4 of [9] near a collar neighborhood of the boundary. Unique continuation [21] shows that (V', θ', g') coincides with (V, θ, g) in this collar neighborhood. But strict positivity of V implies the existence of an analytic atlas for the interior of M in which the solutions are analytic, and as g and g'

are complete we can conclude that the lifts of the solutions to the universal cover of M coincide (compare [16, Vol. I, Corollary 6.4, p.256]).

We note the non-uniqueness of the conformal Dirichlet problem for the (static) Horowitz-Myers metrics, which shows that the last result would be wrong if one did not invoke the universal cover.

7. Polyhomogeneity

Let $U_0 \subset \mathbb{R}^n$ be an open set, and let $U = U_0 \times (0, \varepsilon) \subset \mathbb{H}^{n+1}$. For any $\delta \in \mathbb{R}$, we denote by $\mathscr{C}^\delta$ the space of functions $f \in C^\infty(U)$ that satisfy, on any subset $K \times (0, \varepsilon_0)$ with $K \subset U_0$ compact and $0 < \varepsilon_0 < \varepsilon$, estimates of the following form for all integers $r \geq 0$ and all multi-indices α:

$$|(y\partial_y)^r \partial_x^\alpha f(x, y)| \leq C_{r,\alpha} y^\delta .$$

(We use the multi-index notations $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial_x^\alpha = (\partial_{x^1})^{\alpha_1} \ldots (\partial_{x^n})^{\alpha_n}$.)

A smooth function $f\colon U \to \mathbb{R}$ is said to be *polyhomogeneous* (cf. [6,20]) if there exists a sequence of real numbers $s_i \to +\infty$, a sequence of nonnegative integers $\{q_i\}$, and functions $f_{ij} \in C^\infty(U_0)$ such that

$$f(x, y) \sim \sum_{i=1}^{\infty} \sum_{j=0}^{q_i} y^{s_i} (\log y)^j f_{ij}(x) \tag{7.1}$$

in the sense that for any $\delta > 0$, there exists a positive integer N such that

$$f(x, y) - \sum_{i=1}^{N} \sum_{j=0}^{q_i} y^{s_i} (\log y)^j f_{ij}(x) \in \mathscr{C}^\delta .$$

A function or tensor field on M is said to be polyhomogeneous if its coordinate representation in local coordinates near the conformal boundary is polyhomogeneous. (We refer the reader to [10] for a discussion of equivalence of alternative definitions of polyhomogeneity.)

In this section, we apply the theory of [6] to conclude that solutions to (4.5) are polyhomogeneous. The key step in the proof is a regularity result for the linearised operator $(l, L, \mathcal{L})$. Following [6], we say that an interval $(\delta_-, \delta_+) \subset \mathbb{R}$ is a (*weak*) *regularity interval* for a second-order linear operator P on the spaces $C^{k,\lambda}_\delta(M_R; S_2)$ if whenever u is a locally C^2 section of S_2 such that $u \in C^{0,0}_{\delta_0}(M_R; S_2)$ and $Pu \in C^{0,\lambda}_\delta(M_R; S_2)$ with $\lambda \in (0, 1)$ and $\delta_- < \delta_0 < \delta < \delta_+$, it follows that $u \in C^{2,\lambda}_\delta(M_R; S_2)$. We use the notation of [9].

Theorem 7.1. *Solutions given by Theorems 5.3 and 5.5 are polyhomogeneous. Similarly, solutions of* (4.1) *and* (4.2) *with smooth boundary data such that* θ *and* $\rho^{-2}(V^2 d\varphi^2 + g)$ *are in* $C^2(\overline{M})$ *are polyhomogeneous.*

Proof. We start by noting that metrics such that θ and $\rho^{-2}(V^2 d\varphi^2 + g)$ are in $C^2(\overline{M})$ can be brought, near the boundary, to a gauge in which the equations are elliptic by setting σ to zero as in Section 6, and then applying the results of Section 4 of [9] to the metric $V^2 d\varphi^2 + g$. On the other hand, solutions given by Theorems 5.3 and 5.5 are directly in the closely related gauge $\Omega = 0$; those two gauges do not coincide, but the proof works in both gauges. Alternatively one could use the analysis in Section 4 of [9] to transform a C^2–compactifiable $V^2d\varphi^2 + g$ to the gauge $\Omega = 0$. A polyhomogeneous approximate solution $\mathring{V}^{-2}(dt + \mathring{\theta})^2 + \mathring{g}$ can then be constructed using a Fefferman-Graham expansion up-to-not-including the critical exponent.

For any $\phi = (^0\phi, ^2\phi, ^1\phi)$, function, two-tensor, one-form on M, define

$$F[\phi] := (\rho\mathring{V}^{-1}q, \rho^2 Q, \mathcal{Q})(\mathring{V} + \rho^{-1}\, {}^0\phi, \mathring{g} + \rho^{-2}\, {}^2\phi, \mathring{\theta} + \, {}^1\phi)$$

with $(q, Q, \mathcal{Q})$ as in (4.5), while for the solutions arising from Theorems 5.3 and 5.5 the one-form $\mathring{\theta}$ can be taken as the solution of the third equation in (4.5) with $V = \mathring{V}$ and $g = \mathring{g}$. (F should not be confused with the map F of the previous section.) Then ϕ satisfies $F[\phi] = 0$. One can apply [6, Theorem 5.1.1] to F, and thereby conclude that ϕ is polyhomogeneous. The argument proceeds as in [9, Section 5] and will not be repeated here. We simply mention that the property, that *the interval* $(0, n)$ *is a regularity interval for the operator* $F'[\phi_0]$ *on the spaces* $AC^\delta_{k+\lambda}(M_R)$, is an immediate consequence of Corollaries 3.2 and 3.3. □

Appendix A. "Dimensional reduction" of some operators

A.1. Lichnerowicz Laplacian on two-tensor for a warped product metric. We shall use the following coordinate systems on $S^1 \times M$:

$$(x^a) = (\varphi, x^i) = (x^0, x^i) = (x^0, ..., x^n).$$

Lemma A.1. *Let* (M, g) *be a Riemannian manifold, let* V, W *be two functions on* M*, let* h *be a symmetric covariant two-tensor on* M *and let* θ *be a one-form on* M*. On* $S^1 \times M$ *we consider the Riemannian metric* $\widetilde{g} = V^2d\varphi^2 + g$ *and the symmetric covariant two-tensor*

$$\widetilde{h} = 2VWd\varphi^2 + 2\xi_i d\varphi dx^i + h_{ij}dx^i dx^j$$

satisfying $\mathscr{L}_{\partial_\varphi}\widetilde{h} = 0$*, where* $\mathscr{L}$ *denotes a Lie derivative. Then, in local coordinates, the Laplacian of* $\widetilde{h}$ *has the following components:*

$$\begin{aligned}
\widetilde{\nabla}^c\widetilde{\nabla}_c\widetilde{h}_{00} &= 2[V\nabla^k\nabla_k W - \nabla^k\nabla_k VW - \nabla^k V\nabla_k W - V^{-1}|dV|^2W + \nabla^k V\nabla^l Vh_{kl}]\ , \\
\widetilde{\nabla}^c\widetilde{\nabla}_c\widetilde{h}_{i0} &= \nabla^k\nabla_k\xi_i - V^{-1}\nabla^k\nabla_k V\xi_i - V^{-1}\nabla^k V\nabla_k\xi_i - 3V^{-2}\nabla_i V\nabla^k V\xi_k\ , \\
\widetilde{\nabla}^c\widetilde{\nabla}_c\widetilde{h}_{ij} &= \nabla^k\nabla_k h_{ij} + V^{-1}\nabla^k V\nabla_k h_{ij} - V^{-2}(\nabla_i V\nabla^k Vh_{kj} + \nabla_j V\nabla^k Vh_{ki}) \\
&\quad +4V^{-3}\nabla_i V\nabla_j VW\ .
\end{aligned}$$

Proof. The Christoffel symbols of the metric $\widetilde{g} = V^2d\varphi^2 + g$ are

$$\widetilde{\Gamma}^0_{00} = \widetilde{\Gamma}^0_{ij} = \widetilde{\Gamma}^k_{0j} = 0\ ,\ \ \widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij}\ ,\ \ \widetilde{\Gamma}^0_{i0} = V^{-1}\nabla_i V\ ,\ \ \widetilde{\Gamma}^k_{00} = -V\nabla^k V\ . \tag{A.1}$$

The covariant derivatives of $\widetilde{h}$, in local coordinates, read

$$\begin{aligned}
&\widetilde{\nabla}_0\widetilde{h}_{00} = 2V\nabla^k V\xi_k\ , \\
&\widetilde{\nabla}_0\widetilde{h}_{ij} = -V^{-1}(\nabla_i V\xi_j + \nabla_j V\xi_i)\ , \\
&\widetilde{\nabla}_k\widetilde{h}_{i0} = \nabla_k\xi_i - V^{-1}\nabla_k V\xi_i \\
&\widetilde{\nabla}_0\widetilde{h}_{i0} = V\nabla^k Vh_{ki} - 2\nabla_i VW\ , \\
&\widetilde{\nabla}_k\widetilde{h}_{00} = 2V\nabla_k W - 2\nabla_k VW\ , \\
&\widetilde{\nabla}_k\widetilde{h}_{ij} = \nabla_k h_{ij}\ .
\end{aligned}$$

The result is obtained by substition. $\square$

We recall that the Lichnerowicz Laplacian is

$$\widetilde{\Delta}_L \widetilde{h}_{ab} = -\widetilde{\nabla}^c \widetilde{\nabla}_c h_{ab} + \widetilde{R}_{ac}\widetilde{h}^c{}_b + \widetilde{R}_{bc}\widetilde{h}^c{}_a - 2\widetilde{R}_{acbd}\widetilde{h}^{cd} \; . \tag{A.2}$$

The curvature tensor of the warped product metric $\widetilde{g} = V^2 d\varphi^2 + g$ has the following components [22, Prop. 42, Chap. 7] (note, however, that our curvature tensor is the negative of the one in [22]):

$$\widetilde{R}^l{}_{ijk} = R^l{}_{ijk} \; , \quad \widetilde{R}^l{}_{0j0} = -V\nabla_j\nabla^l V \; , \quad \widetilde{R}^0{}_{ijk} = 0 \; ,$$

$$\widetilde{R}_{ik} = R_{ik} - V^{-1}\nabla_k\nabla_i V, \;\; \widetilde{R}_{0k} = 0, \;\; \widetilde{R}_{00} = -V\nabla^i\nabla_i V.$$

The zero order terms in (A.2) are thus

$$\begin{aligned}
\widetilde{R}_{0c}\widetilde{h}^c{}_0 &= 2\nabla^*\nabla V W \; , \\
2\widetilde{R}_{0c0d}\widetilde{h}^{cd} &= -2V\nabla^i\nabla^j V h_{ij} \; , \\
\widetilde{R}_{ic}\widetilde{h}^c{}_0 + \widetilde{R}_{0c}\widetilde{h}^c{}_i &= R^l{}_i\xi_l - V^{-1}\nabla_i\nabla^l V\xi_l + V^{-1}\nabla^*\nabla V\xi_i \; , \\
2\widetilde{R}_{ic0d}\widetilde{h}^{cd} &= 2V^{-1}\nabla_i\nabla_j V\xi^j \; , \\
\widetilde{R}_{ic}\widetilde{h}^c{}_j + \widetilde{R}_{jc}\widetilde{h}^c{}_i &= R_{ik}h^k{}_j + R_{jk}h^k{}_i - V^{-1}\nabla_i\nabla_k V h^k{}_j - V^{-1}\nabla_j\nabla_k V h^k{}_i \; , \\
2\widetilde{R}_{icjd}\widetilde{h}^{cd} &= 2R_{ikjl}h^{kl} - 4V^{-2}\nabla_i\nabla_j V W \; .
\end{aligned}$$

Lemma A.1 implies now:

LEMMA A.2. *Under the hypotheses of Lemma A.1, the Lichnerowicz Laplacian of $\widetilde{h}$ is*

$$\begin{aligned}
\widetilde{\Delta}_L\widetilde{h}_{00} &= 2\Big[-V\nabla^k\nabla_k W - \nabla^k\nabla_k V W + \nabla^k V\nabla_k W + V^{-1}|dV|^2 W \\
&\quad -\nabla^k V\nabla^l V h_{kl} + V\nabla^i\nabla^j V h_{ij}\Big] \; , \\
\widetilde{\Delta}_L\widetilde{h}_{i0} &= -\nabla^k\nabla_k\xi_i + V^{-1}\nabla^k V\nabla_k\xi_i + 3V^{-2}\nabla_i V\nabla^k V\xi_k \\
&\quad +R^l{}_i\xi_l - 3V^{-1}\nabla_i\nabla_j V\xi^j \; , \\
\widetilde{\Delta}_L\widetilde{h}_{ij} &= \Delta_L h_{ij} - V^{-1}\nabla^k V\nabla_k h_{ij} + V^{-2}(\nabla_i V\nabla^k V h_{kj} + \nabla_j V\nabla^k V h_{ki}) \\
&\quad -4V^{-3}\nabla_i V\nabla_j V W - V^{-1}(\nabla_i\nabla^k V h_{kj} + \nabla_j\nabla^k V h_{ki}) \\
&\quad +4V^{-2}\nabla_i\nabla_j V W \; .
\end{aligned}$$

$\square$

A.2. **The Laplacian on one-forms for a warped product metric.**

LEMMA A.3. *Let (M,g) be a Riemannian manifold, let V, f be two functions on M and let Ω be a one-form on M. Let us consider, on $S^1\times M$, the Riemannian metric $\widetilde{g} = V^2d\varphi^2 + g$ and the one-form*

$$\widetilde{\Omega} = f d\varphi + \Omega_i dx^i \; .$$

Then in local coordinates, the Laplacian of $\widetilde{\Omega}$ equals

$$\widetilde{\nabla}^c\widetilde{\nabla}_c\widetilde{\Omega}_0 = \nabla^k\nabla_k f - V^{-1}f\nabla^k\nabla_k V - V^{-1}\nabla^k V\nabla_k f,$$

$$\widetilde{\nabla}^c\widetilde{\nabla}_c\widetilde{\Omega}_i = \nabla^k\nabla_k\Omega_i + V^{-1}\nabla^k V\nabla_k\Omega_i - V^{-2}\nabla_i V\nabla^k V\Omega_k =: B(\Omega)_i \; .$$

Proof. We have

$$\begin{aligned}\widetilde{\nabla}_0\widetilde{\Omega}_0 = V\nabla^k V\Omega_k \ , \quad \widetilde{\nabla}_0\widetilde{\Omega}_i = -V^{-1}\nabla_i V f \\ \widetilde{\nabla}_i\widetilde{\Omega}_0 = \partial_i f - V^{-1}\nabla_i V f \ , \quad \widetilde{\nabla}_j\widetilde{\Omega}_i = \nabla_j\Omega_i \ ,\end{aligned}$$

and the result easily follows. □

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