On Israel-Wilson-Perjés black holes

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Abstract

We show, under certain conditions, that regular Israel-Wilson-Perjés black holes necessarily belong to the Majumdar-Papapetrou family.

1 Introduction

A classical result in general relativity is the bound on global charge of regular electro-vacuum space-times, in absolute value, by the ADM mass, with equality holding if and only if the metric is, locally, the Israel-Wilson-Perjés (IWP) metric [13, 15, 27], see Theorem 2.1 below for a precise statement. It is therefore of interest to classify all non-singular IWP solutions. A long standing conjecture asserts that those necessarily belong to the Majumdar-Papapetrou family; for partial results see [14], compare [8]. We prove that this is indeed the case in electro-vacuum under supplementary hypotheses.

A key feature which singles out the IWP metrics is the existence of a "super-covariantly constant" spinor field ψ [13, 27]. Each such spinor leads to a Killing vector field X, which can only be *timelike* or *null*. Recall that in a regular black hole space-time a Killing vector cannot be timelike on the event horizon. In IWP space-times ergoregions[1] do not exist, but null orbits away from the event horizon could occur in principle. Assuming that there are no such orbits, we prove that the metric has to be static.

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[1]By *ergoregion* we mean the set where a "stationary" Killing vector is spacelike.

More precisely, our first main result is the following (see Section 2 for definitions):

THEOREM 1.1 *Let $(\mathscr{M}, {}^4g, F)$ be a solution of the Einstein–Maxwell equations with a non-trivial spinor field ψ which is parallel with respect to an F-modified spinor connection as in (2.10)-(2.11). Suppose that $\mathscr{M}$ contains a connected and simply connected space-like hypersurface*[2] *$\mathscr{S}$ (with boundary), which is the union of an asymptotically flat end and of a compact set, such that:*

1. *The Killing vector field X associated with ψ is timelike on the interior $\operatorname{int}\mathscr{S}$ of $\mathscr{S}$.*

2. *The topological boundary $\partial\mathscr{S} \equiv \mathscr{S} \setminus \operatorname{int}\mathscr{S}$ of $\mathscr{S}$ is a nonempty, two-dimensional, topological manifold, with ${}^4g_{\mu\nu}X^\mu X^\nu = 0$ on $\partial\mathscr{S}$.*

Then, performing a duality rotation of the Maxwell field if necessary, there exists a neighborhood of $\mathscr{S}$ in $\mathscr{M}$ which is isometrically diffeomorphic to an open subset of a standard Majumdar-Papapetrou space-time.

For a complete understanding of the problem it is of some interest to look for a corresponding result without the electro-vacuum condition. This is done in Section 6.

Let us give an outline of the proof of Theorem 1.1; this also serves as a guide to the structure of this paper. In Sections 3.1 and 3.2 we examine in detail the space-time geometry near the event horizon. This allows us to show, in Section 3.3, that horizons correspond to isolated singularities in space in the usual local coordinate representation of IWP metrics. This part of our work is purely local, except for the hypothesis of compactness of cross-sections of the horizons; it follows closely the calculations in [10] and is inspired by the analysis of supersymmetric black holes in [22]. In Section 4.1 we analyse the asymptotic behavior of the fields involved in defining the metric. In Section 4.2 we show that the local coordinates are global. In Section 4.3 we establish staticity of the solutions.

The hypothesis in Theorem 1.1 that the set $\{g_{\mu\nu}X^\mu X^\nu = 0\}$ is a topological manifold of co-dimension one is, essentially, the condition that all null orbits of the Killing vector field X lie on the event horizon. This restriction is not needed if we assume instead that there exists a maximal hypersurface in $\mathscr{M}$; this is the second main result of our work:

THEOREM 1.2 *Let $(\mathscr{M}, {}^4g, F)$ be an electrovacuum space-time with a super-covariantly constant spinor field $\psi \not\equiv 0$. Suppose that $\mathscr{M}$ contains a simply connected maximal hypersurface $\mathscr{S}$ which is the union of a compact set with an asymptotically flat region and with a finite number of "weakly cylindrical" regions as in (5.12) below. Then $\mathscr{S}$ is totally geodesic and, up to a duality rotation of the Maxwell field, there exists a neighborhood of $\mathscr{S}$ isometrically diffeomorphic to a subset of a standard Majumdar–Papapetrou space-time.*

[2] We use the geometers' convention that a hypersurface with boundary contains its boundary as a point set. The signature is $(+,-,-,-)$.

Theorem 1.2 is proved in Section 5.

The hypotheses of Theorems 1.1 and 1.2 do not suffice to obtain more information about the size of the set on which the metric is that of a standard Majumdar–Papapetrou space–time (in the sense defined in Section 2). The following version of Theorem 1.1 can be obtained when reasonably mild supplementary hypotheses are made:

THEOREM 1.3 *Let $(\mathscr{M}, {}^4g, F)$ be a solution of the Einstein–Maxwell equations containing a connected space-like hypersurface $\mathscr{S}$, with non-empty topological boundary, which is the union of a finite number of asymptotically flat ends and of a compact interior. Denote by $\mathcal{D}_{oc} \equiv \mathcal{D}_{oc}(\mathscr{M}_{\rm ext})$ the domain of outer communications in $(\mathscr{M}, g)$ associated with one of the asymptotically flat ends of $\mathscr{S}$. Let ψ be a non-trivial super-covariantly constant spinor field on $\mathscr{M}$ and suppose that the associated Killing vector field X is timelike on $\mathcal{D}_{oc}$. Assume morever that*

1. *The interior* $\operatorname{int}\mathscr{S}$ *of $\mathscr{S}$ is a subset of the domain of outer communications $\mathcal{D}_{oc}$.*
2. *The topological boundary $\partial\mathscr{S} \equiv \mathscr{S} \setminus \operatorname{int}\mathscr{S}$ of $\mathscr{S}$ is a nonempty, two-dimensional, topological manifold such that $\partial\mathscr{S} = \mathscr{S} \cap \partial\mathcal{D}_{oc}$.*
3. *X has complete orbits in $\mathcal{D}_{oc}$.*
4. *$(\mathcal{D}_{oc}, g|_{\mathcal{D}_{oc}})$ is globally hyperbolic.*

Then $\mathcal{D}_{oc}$ is isometrically diffeomorphic to a domain of outer communications of a standard extension of a standard Majumdar–Papapetrou space–time.

The proof of Theorem 1.3 follows from Theorem 1.1 by standard arguments (compare [7]) and will be omitted. We simply note that the properties that $\mathscr{S}$ is simply connected and has only one asymptotically flat end follow from [11]. The hypothesis of timelikeness of X in $\mathcal{D}_{oc}$ can be replaced by that of existence of a maximal surface with a finite number of asymptotically flat ends and of weakly cylindrical ends, invoking Theorem 1.2.

2 Preliminaries

As we will be using two-index spinor fields it is convenient to use the signature $(+---)$. The space-time metric will be denoted by ${}^4g$, the associated *Riemannian* metric induced on space-like hypersurfaces by g. In adapted coordinated in which $\mathscr{S} = \{t = 0\}$ one thus has $g_{ij} = -{}^4g_{ij}$. The symbol ∇ denotes the space-time covariant derivative operator associated with ${}^4g$.

Recall that the Majumdar–Papapetrou (MP) metrics are, locally, of the form [19, 21]

$$ {}^4g = u^{-2}dt^2 - u^2(dx^2 + dy^2 + dz^2)\,, \tag{2.1} $$

$$ A = u^{-1}dt\,, \tag{2.2} $$

where A is the Maxwell potential, $F = dA$, with some nowhere-vanishing, say positive, function u. A space–time will be called a *standard MP space–time* if the coordinates x^μ of (2.1)–(2.2) are global with range $\mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_i\})$ for a finite set of points $\vec{a}_i \in \mathbb{R}^3$, $i = 1, \dots, I$, and if the function u has the form

$$u = 1 + \sum_{i=1}^{I} \frac{m_i}{|\vec{x} - \vec{a}_i|} , \qquad (2.3)$$

for some positive constants m_i. It has been shown by Hartle and Hawking [14] that every standard MP space–time can be analytically extended to an electro–vacuum space–time with a non–empty black hole region, and with a domain of outer communication which is non–singular in the sense of the theorems proved here. Those extensions will be called the *standard extensions* of the standard Majumdar–Papapetrou space–times.

A data set $(\mathscr{S}_{\rm ext}, g, K)$ with Maxwell initial data $F = (E, B)$ will be called an *asymptotically flat end* if $\mathscr{S}_{\rm ext}$ is diffeomorphic to $\mathbb{R}^3$ minus a ball and if the fields (g_{ij}, K_{ij}) satisfy the fall–off conditions (ρ is the radial coordinate in $\mathbb{R}^3$)

$$|g_{ij} - \delta_{ij}| + \rho|\partial_\ell g_{ij}| + \cdots + \rho^k|\partial_{\ell_1\cdots\ell_k} g_{ij}| + \rho|K_{ij}| + \cdots + \rho^k|\partial_{\ell_1\cdots\ell_{k-1}} K_{ij}| \le C_{k,\alpha}\rho^{-\alpha} , \qquad (2.4)$$

for some constants $C_{k,\alpha}$, $\alpha > 0$, $k \ge 1$. We shall further require that in the local coordinates as above on $\mathscr{S}_{\rm ext}$ the Maxwell field F satisfies the fall–off conditions

$$|E_i| + \rho|\partial_\ell E_i| + \cdots + \rho^k|\partial_{\ell_1\cdots\ell_k} E_i| + |B_i| + \rho|\partial_\ell B_i| + \cdots + \rho^k|\partial_{\ell_1\cdots\ell_k} B_i| \le \hat{C}_{k,\alpha}\rho^{-\alpha-1} , \qquad (2.5)$$

for some constants $\hat{C}_{k,\alpha}$, $\alpha > 0$, $k \ge 0$. We will always assume $\alpha > 1/2$, which makes the ADM mass well defined in electro-vacuum.[3] A hypersurface will be said to be *asymptotically flat* if it contains an asymptotically flat end $\mathscr{S}_{\rm ext}$.

A two-dimensional surface $S \subset \mathscr{S}$ will be called *weakly outer trapped* if it separates $\mathscr{S}$ into two components, and if $\lambda + h_{ab}K^{ab} \le 0$, *or* if $\lambda - h_{ab}K^{ab} \le 0$, where h_{ab} is the metric induced on S, and where λ is the mean curvature of S within $\mathscr{S}$, as measured with respect to a field of unit normals pointing towards the component of $\mathscr{S} \setminus S$ which contains $\mathscr{S}_{\rm ext}$.

We note a precise version of the charge bound mentioned in the Introduction (compare [12, 13, 15]):

THEOREM 2.1 *Let $(\mathscr{S}, g, K)$ be a smooth three-dimensional initial data set, with $(\mathscr{S}, g)$ complete, and with an asymptotically flat end $\mathscr{S}_{\rm ext}$ (in the sense of Equation (2.4) with $k \ge 4$ and $\alpha > 1/2$), and with $\partial\mathscr{S}$ weakly outer trapped, if not empty. Suppose, further, that we are given on $\mathscr{S}$ two smooth vector fields E and B satisfying*

$$4\pi\rho_B := D_i B^i \in L^1(\mathscr{S}) , \quad 4\pi\rho_E := D_i E^i \in L^1(\mathscr{S}) .$$

Set

$$4\pi Q^E := \lim_{R\to\infty} \int_{r=R} E^i dS_i , \quad 4\pi Q^B := \lim_{R\to\infty} \int_{r=R} B^i dS_i .$$

[3] It follows in any case from [6, Section 1.3] or from [17] that in stationary electro–vacuum space–times there is no loss of generality in assuming $\alpha = 1$, k – arbitrary.

Let R be the Ricci scalar of g and assume

$$0 \le R - |K|^2 + (trK)^2 - 2g(E,E) - 2g(B,B) =: 16\pi\rho_m \in L^1(\mathscr{S}_{\rm ext}) \ . \quad (2.6)$$

If

$$\rho_E^2 + \rho_B^2 + |J|_g^2 \le \rho_m^2 \ , \quad (2.7)$$

where

$$16\pi J^i = 2D_j(K^{ij} - {\rm tr}K g^{ij}) - 4\epsilon^i{}_{k\ell}E^k B^\ell \ , \quad (2.8)$$

then the ADM mass m of $\mathscr{S}_{\rm ext}$ satisfies

$$m \ge \sqrt{|\vec{p}|^2 + Q_E^2 + Q_B^2} \ . \quad (2.9)$$

*If the equality is attained in (2.9) then (2.7) is also an equality, and there exists on $\mathscr{S}$ a spinor field satisfying (2.10)-(2.11) below. Furthermore, the associated space-time metric is, locally, an IWP (*not *necessarily electro-vacuum) metric.*

The conditions (2.6) and (2.7) are clearly satisfied in electro-vacuum. This theorem justifies the interest in IWP space-times; its hypotheses further serve as a guiding principle for the hypotheses of our remaining results in this paper.

We sketch a proof of Theorem 2.1 in Appendix B.

2.1 IWP metrics

Consider a space-time that admits a "super-covariantly constant spinor" given in two-component spinor notation by $\psi = (\alpha_A, \beta_{A'})$ where the constituent spinors α_A and $\beta_{A'}$ satisfy the coupled system of equations:

$$\nabla_{AA'}\alpha_B + \sqrt{2}\phi_{AB}\beta_{A'} = 0 \quad (2.10)$$

$$\nabla_{AA'}\beta_{B'} - \sqrt{2}\overline{\phi}_{A'B'}\alpha_A = 0. \quad (2.11)$$

Here ϕ_{AB} is the Maxwell spinor and $\overline{\phi}_{A'B'}$ is its complex conjugate, related to the Maxwell tensor F_{ab} in the standard way by

$$F_{ab} = \phi_{AB}\epsilon_{A'B'} + \overline{\phi}_{A'B'}\epsilon_{AB}.$$

(Strictly speaking, the system (2.10)-(2.11) has a two-complex-dimensional vector space of solutions, but we shall normalise the solution by choices made later.)

As is well known (and will be shown in any case below), the metric is invariant under the flow of the following vector field[4]

$$X = \frac{1}{\sqrt{2}}(\alpha^A\overline{\alpha}^{A'} + \overline{\beta}^A\beta^{A'})\frac{\partial}{\partial x^{AA'}}, \quad (2.12)$$

It is known [27] that near any point at which X is timelike the metric can locally be written in the IWP form

$$ds^2 = V\overline{V}(dt + \omega\cdot dx)^2 - (V\overline{V})^{-1}dx\cdot dx \quad (2.13)$$

[4]The vector field X here equals $K/\sqrt{2}$ in [27]. The value of the normalisation constant k of that last reference is a matter of convention, and here we take it to be equal to $\sqrt{2}$. Note a factor of 2 missing in the first term at the right-hand-side of [27, (2.13)].

To derive (2.13) from the system (2.10)-(2.11) one makes a sequence of definitions. First, introduce

$$V = \alpha_A \overline{\beta}^A ,$$

note that V is a smooth function on space-time, vanishing only at those points at which X is null or zero.[5] Then introduce the coordinates $x = (x^1, x^2, x^3)$ by solving:

$$dx^1 + idx^2 = \sqrt{2}\alpha_A \beta_{A'} dx^{AA'} , \tag{2.14}$$

$$dx^3 = \frac{1}{\sqrt{2}}(\alpha_A \overline{\alpha}_{A'} - \overline{\beta}_A \beta_{A'}) dx^{AA'} , \tag{2.15}$$

where the one-forms on the right are closed by virtue of (2.10)-(2.11). Using a time coordinate adapted to the vector field X,

$$X = \frac{\partial}{\partial t} , \tag{2.16}$$

one writes

$$X_\mu dx^\mu = V\overline{V}(dt + \omega \cdot dx)$$

where ω is a one-form, to be determined. Here, by definition

$${}^4g(X, X) = V\overline{V}.$$

From (2.10)-(2.11) and definitions made so far one readily obtains

$$\nabla_{AA'} V = -2\phi_{AB} X_{A'}{}^B \tag{2.17}$$

$$\nabla_\mu X_\nu = \overline{V}\phi_{AB}\epsilon_{A'B'} + V\overline{\phi}_{A'B'}\epsilon_{AB}. \tag{2.18}$$

This leads to an Einstein-Maxwell space-time with sources which may be calculated from the following:

$$\nabla_\mu F^{\nu\mu} = 4\pi\chi^E X^\nu , \tag{2.19}$$

$$\nabla_\mu (*_4 F^{\nu\mu}) = 4\pi\chi^B X^\nu , \tag{2.20}$$

$$T_{\mu\nu} = \chi X_\mu X_\nu + \tfrac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\,{}^4g_{\alpha\beta}\right) , \tag{2.21}$$

$$\chi^E + i\chi^B = \chi V \tag{2.22}$$

It follows from (2.17) and (2.18) that X is a Killing vector, while V^{-1} is related to χ as follows

$$\Delta(V^{-1}) := \left(\left(\frac{\partial}{\partial x^1}\right)^2 + \left(\frac{\partial}{\partial x^2}\right)^2 + \left(\frac{\partial}{\partial x^3}\right)^2\right)(V^{-1}) = -\frac{4\pi\chi}{V}, \tag{2.23}$$

We note that it is easy to construct an infinite-dimensional family of solutions of (2.23) with smooth *real strictly positive* V, and prescribed positive smooth

[5] It follows from (2.10)-(2.11) that, if α_A and $\beta_{A'}$ both vanish at a point p then they vanish everywhere, so we may assume that they have no common zero. Consequently the Killing vector X has no zeroes: X will be null where V vanishes but is time-like at all other points. Space-times with V identically zero are *pp*-waves [27], which do not concern us here.

χV^{-1}. Thus, there exist many singularity-free non-vacuum solutions with positive energy density which saturate the bound of Theorem 2.1.

More generally, (2.22) and the physical requirement that there are no magnetic currents imposes the non-trivial restriction that V should be real on the support of χ. This leads to $\Delta\Im(V^{-1}) = 0$. Similarly to our main Theorem 1.1, we expect that this forces the space-time to be static (either with or without black holes), but we have not attempted to prove that.

Finally ω satisfies the equation

$$\mathrm{curl}\,\omega = \mathrm{i}(\overline{\mathrm{V}}^{-1}\nabla \mathrm{V}^{-1} - \mathrm{V}^{-1}\nabla\overline{\mathrm{V}}^{-1}). \tag{2.24}$$

Locally, the integrability condition for (2.24) is satisfied by virtue of (2.23) since χ is real. However, there are global conditions if ω is to be well-defined and we shall return to this point. We write Ω for the set of points in $\mathbb{R}^3$ at which V^{-1} is singular.

Except for Section 6, from now on we assume that $\chi \equiv 0$, so that V^{-1} is harmonic in the flat three-metric:

$$\Delta(V^{-1}) = 0, \tag{2.25}$$

The source-free Maxwell equations, which we assume hold, are equivalent to the equation

$$\nabla^{AA'}\phi_{AB} = 0.$$

The space-time is electro-vacuum, so that the Ricci spinor is related to the Maxwell spinor by Einstein's equations which in this formalism take the form

$$\Phi_{ABA'B'} = 2\phi_{AB}\overline{\phi}_{A'B'} \tag{2.26}$$

in units with $G = c = 1$.

3 Local considerations

3.1 The near horizon geometry of IWP metrics

Now concentrate on one component of Ω which by assumption corresponds to a component of the Killing horizon.

As in [9, 10, 22] we introduce Gaussian null coordinates near a component $\mathcal{N}$ of the event horizon, with the signature chosen so that the metric is

$$^4g = r\phi du^2 - 2dudr - 2rh_a dy^a du - h_{ab}dy^a dy^b, \tag{3.1}$$

where a, b range over $\{1, 2\}$.

In these coordinates, the Killing vector X is $\partial/\partial u$ with norm

$$^4g(X, X) = r\phi$$

and $\mathcal{N}$ is located at $r = 0$, where also $V = 0$. Using the metric (3.1) to lower the index on X we find

$$X_\mu dx^\mu = r\phi du - dr - rh_a dy^a$$

whence, at $\mathcal{N}$,

$$d(X_\mu dx^\mu) = \phi dr \wedge du - dr \wedge (h_a dy^a).$$

However, from (2.18), we see that $\nabla_\mu X_\nu$ vanishes at $V = 0$. Two things follow from this: the surface gravity $\kappa = -\partial_r(r\phi)$ at $r = 0$ vanishes, and so the horizon is degenerate; and h_a vanishes at $r = 0$. It follows that

$$\phi = rA(r, y^b) \ ; \ h_a = rH_a(r, y^b)$$

for some function A and covector field H_a. We shall often use a circle over a quantity to indicate its value at $r = 0$, e.g. $\mathring{A} = A|_{r=0}$.

We shall show the following:

Proposition 3.1 *(i) The metric h_{ab} on the spheres $\mathcal{S} = (r = 0,\ u = u_0)$ has constant Gauss curvature K.*

(ii) On $\mathcal{N}$, $A = K > 0$, so that $Ah_{ab}|_{r=0}$ is the unit round metric on S^2.

(iii) The function V satisfies $\partial_r V = 2Q$ at $\mathcal{N}$ where Q is a complex constant related to K by $K = 4|Q|^2$.

Proof: $V = 0$ at $r = 0$ implies that

$$V = 2rQ + O(r^2). \tag{3.2}$$

for some smooth function Q independent of r, with $\mathring{A} = 4|Q|^2$. We start by showing that Q is not identically zero. It is useful to invoke the *near horizon limit*, as in [22], obtained by introducing new coordinates $(\hat{r}, \hat{u}, \hat{y}^a)$ defined by the formula

$$r = \epsilon\hat{r}\ , \quad u = \epsilon^{-1}\hat{u}\ , \quad \hat{y}^a = y^a\ ,$$

and letting ϵ go to zero. Assume, for contradiction, that Q vanishes identically. By (2.17) the Maxwell field of an IWP solution satisfies

$$X^\mu(F_{\mu\nu}+i{*_4}F_{\mu\nu})dx^\nu = (F_{u\nu}+i{*_4}F_{u\nu})dx^\nu = -dV = O(r)dr+O(r^2)dy^a\ . \tag{3.3}$$

This shows that $\frac{1}{\epsilon}(F_{u\nu} + i *_4 F_{u\nu})d\hat{x}^\nu$ vanishes in the limit $\epsilon \to 0$. Also the term $F_{ra}dr \wedge dy^a = \epsilon F_{ra}d\hat{r} \wedge d\hat{y}^a$, vanishes in the limit, and it is now simple to check that the whole F vanishes in the near-horizon limit. The near-horizon geometry of such a solution is therefore a vacuum solution, with metric

$${}^4g = -2d\hat{u}d\hat{r} - \mathring{h}_{ab}d\hat{y}^a d\hat{y}^b. \tag{3.4}$$

Since ${}^4g$ is Ricci flat, and $-2d\hat{u}d\hat{r}$ is flat, one obtains that $\mathring{h}_{ab}d\hat{y}^a d\hat{y}^b$ is flat, contradicting the fact that the horizon must have S^2 topology. Hence Q cannot vanish identically.

The IWP metric (2.13) includes a flat 3-metric $dx \cdot dx$. This is invariantly defined where V is non-zero by projecting the 4-metric orthogonally to X and multiplying by $V\overline{V}$. Calculating this in the Gaussian null coordinates gives

$$dx \cdot dx = \left(dr + r^2 H_a dy^a\right)^2 + r^2 A h_{ab} dy^a dy^b. \tag{3.5}$$

The left hand side is flat so if we multiply the right hand side by $1/\epsilon^2$ we should get another flat metric. Set $r = \epsilon r'$ in this new metric and now let $\epsilon \to 0$ to obtain the metric

$$dr'^2 + 4r'^2|Q|^2\mathring{h}_{ab}dy^a dy^b. \tag{3.6}$$

This must also be flat away from $r' = 0$. By calculating the Ricci scalar of this metric one finds that $4|Q|^2\mathring{h}_{ab}$ has Ricci scalar equal to two, so it is the unit round metric on S^2. But $\mathring{h}_{ab}$ is also a metric on S^2. Therefore Q cannot vanish anywhere.

Finally, consider the equation $\Delta V^{-1} = 0$. Writing this out using the metric (3.5) we find that

$$\tilde{\Delta} Q^{-1} = \lim_{r\to 0}\left(8r^3|Q|^2\Delta V^{-1}\right) = 0, \tag{3.7}$$

where $\tilde{\Delta}$ is the Laplacian associated with the 2-metric $\mathring{h}_{ab}$. Since Q^{-1} is globally defined, this equation implies that Q must be constant. It then follows that $\mathring{h}_{ab}$ must be a metric on S^2 of constant curvature $K = 4|Q|^2 = \mathring{A}$. □

3.2 The supercovariantly constant spinors near the horizon

In this section, our aim is to obtain the supercovariantly constant spinors near the horizon in order to relate the two coordinate systems (t, x^i) and (u, r, y^a). Following [10], in the metric (3.1), choose the coordinates y^a so that they are isothermal on $\mathcal{N}$ and then introduce $\zeta = y^1 + iy^2$. Choosing m to be proportional to $d\bar{\zeta}$ at $r = 0$, the metric becomes

$$^4g = r^2 A du^2 - 2dudr - 2r^2(Hd\zeta + \overline{H}d\bar{\zeta})du - 2m\overline{m} \tag{3.8}$$

where

$$m = -\mathring{Z}d\bar{\zeta} + O(r) ,$$

in terms of a complex function $\mathring{Z}$ of ζ and $\bar{\zeta}$.

We shall investigate the metric (3.8) in the spin-coefficient formalism [20]. We introduce the null tetrad $(l^\mu, n^\mu, m^\mu, \overline{m}^\mu)$ by

$$\begin{aligned} l^\mu\partial_\mu &= D = \partial_u + \tfrac{r^2A}{2}\partial_r , \\ n^\mu\partial_\mu &= \Delta = -\partial_r , \\ m^\mu\partial_\mu &= \delta = \tfrac{1}{\overline{Z}}\partial_\zeta + \tfrac{r}{\overline{Y}}\partial_{\bar{\zeta}} - \left(\tfrac{r^2H}{\overline{Z}} + \tfrac{r^3\overline{H}}{\overline{Y}}\right)\partial_r , \end{aligned}$$

where $Z = \mathring{Z} + O(r)$.

We follow the numbering of [20] to calculate the spin-coefficients and curvature quantities. We may take the results from [10] by replacing h there by rH, to give

$$\alpha = -\frac{1}{2\mathring{Z}\overline{\mathring{Z}}}\frac{\partial\overline{\mathring{Z}}}{\partial\bar{\zeta}} + O(r) , \tag{3.9}$$

$$\beta = \frac{1}{2\mathring{Z}\overline{\mathring{Z}}}\frac{\partial\mathring{Z}}{\partial\zeta} + O(r) , \tag{3.10}$$

$$\gamma = -\frac{1}{4}\frac{\partial}{\partial r}\log\left(\frac{\mathring{Z}}{\overline{\mathring{Z}}}\right) + O(r) \,, \tag{3.11}$$

$$\epsilon = \frac{1}{2}r\mathring{A} + O(r^2) \,, \tag{3.12}$$

$$\mu = -\frac{1}{2}\frac{\partial}{\partial r}\log(\mathring{Z}\overline{\mathring{Z}}) + O(r) \,, \tag{3.13}$$

together with $\pi = -\overline{\tau} = O(r), \nu = 0, \lambda = 1 + O(r), \rho = O(r^2), \kappa = O(r^2)$, and $\sigma = O(r^2)$. In Appendix A we give an alternative proof of Proposition 3.1, based on the above.

Expanding the spinor fields α_A and $\beta_{A'}$ in the spinor dyad as

$$\alpha_A = -\alpha_0\iota_A + \alpha_1 o_A \,; \quad \beta_{A'} = -\beta_{0'}\iota_{A'} + \beta_{1'}o_{A'} \,, \tag{3.14}$$

and substituting into (2.10) and (2.11), we obtain eight equations from (3.14) as follows (compare [24] p. 219)

$$D\alpha_0 - \epsilon\alpha_0 + \kappa\alpha_1 + \sqrt{2}\phi_0\beta_{0'} = 0 \tag{3.15}$$
$$D\alpha_1 + \epsilon\alpha_1 - \pi\alpha_0 + \sqrt{2}\phi_1\beta_{0'} = 0 \tag{3.16}$$
$$\delta\alpha_0 - \beta\alpha_0 + \sigma\alpha_1 + \sqrt{2}\phi_0\beta_{1'} = 0 \tag{3.17}$$
$$\delta\alpha_1 + \beta\alpha_1 - \mu\alpha_0 + \sqrt{2}\phi_1\beta_{1'} = 0 \tag{3.18}$$
$$\overline{\delta}\alpha_0 - \alpha\alpha_0 + \rho\alpha_1 + \sqrt{2}\phi_1\beta_{0'} = 0 \tag{3.19}$$
$$\overline{\delta}\alpha_1 + \alpha\alpha_1 - \lambda\alpha_0 + \sqrt{2}\phi_2\beta_{0'} = 0 \tag{3.20}$$
$$\Delta\alpha_0 - \gamma\alpha_0 + \tau\alpha_1 + \sqrt{2}\phi_1\beta_{1'} = 0 \tag{3.21}$$
$$\Delta\alpha_1 + \gamma\alpha_1 - \nu\alpha_0 + \sqrt{2}\phi_2\beta_{1'} = 0 \tag{3.22}$$

and the corresponding eight equations for $\beta_{A'}$, which can be obtained from (3.15)-(3.22) by complex conjugation followed by the substitution of $-\alpha_A$ for $\overline{\beta}_A$ and $\beta_{A'}$ for $\overline{\alpha}_{A'}$.

We also have two expressions for the Killing vector (2.12):

$$X^\mu = \frac{1}{\sqrt{2}}(\alpha^A\overline{\alpha}^{A'} + \overline{\beta}^A\beta^{A'}) = \delta^\mu_u = l^\mu + \frac{r^2 A}{2}n^\mu \,.$$

Substituting from (3.14) into this, we obtain

$$|\alpha_0|^2 + |\beta_{0'}|^2 = \frac{\sqrt{2}}{2}r^2 A \,, \tag{3.23}$$
$$\alpha_0\overline{\alpha}_{1'} + \beta_{1'}\overline{\beta}_0 = 0 \,, \tag{3.24}$$
$$|\alpha_1|^2 + |\beta_{1'}|^2 = \sqrt{2} \,. \tag{3.25}$$

Thus, near the horizon, α_0 and $\beta_{0'}$ are $O(r)$ while α_1 and $\beta_{1'}$ are $O(1)$. We write

$$\alpha_1 = \mathring{P} + O(r) \,;\ \alpha_0 = r\mathring{S} + O(r^2) \,;\ \beta_{1'} = \mathring{W} + O(r) \,;\ \beta_{0'} = r\mathring{T} + O(r^2) \,, \tag{3.26}$$

and proceed to analyse the system (3.15)-(3.22), using what we know of the spin-coefficients and curvature components. From Appendix A we have $\phi_0 = O(r)$. Equation (3.15) is already $O(r^2)$; equations (3.16), (3.17) and (3.19) are $O(r)$; from the rest, (3.18) and (3.20) and the corresponding equations for $\beta_{A'}$ yield the system:

$$\begin{aligned}
(\delta+\beta)\alpha_1 + \sqrt{2}\phi_1\beta_{1'} &= O(r), &(3.27)\\
(\overline{\delta}+\overline{\beta})\beta_{1'} - \sqrt{2}\overline{\phi_1}\alpha_1 &= O(r), &(3.28)\\
(\overline{\delta}+\alpha)\alpha_1 &= O(r), &(3.29)\\
(\delta+\overline{\alpha})\beta_{1'} &= O(r). &(3.30)
\end{aligned}$$

Substituting from (3.26) into (3.29) and (3.30) we obtain equations which can be readily solved to give

$$\mathring{P} = (\overline{\mathring{Z}})^{\frac{1}{2}} f(\zeta)\ , \quad \mathring{W} = (\mathring{Z})^{\frac{1}{2}} g(\overline{\zeta})\ , \tag{3.31}$$

for some holomorphic functions f and g. It is convenient now to take an explicit form for $\mathring{Z}$. From Proposition 3.1 we know that $2\mathring{A}\mathring{Z}\overline{\mathring{Z}}d\zeta d\overline{\zeta}$ is the unit round metric, thus if we introduce $\mathring{L}$ by $\mathring{A} = \mathring{L}^{-2}$ then we may choose

$$\mathring{Z} = \overline{\mathring{Z}} = \frac{\mathring{L}\sqrt{2}}{(1+\zeta\overline{\zeta})}. \tag{3.32}$$

The relation in Proposition 3.1 (*iii*) can now be written $4|Q|^2\mathring{L}^2 = 1$, so that

$$2Q\mathring{L} = e^{i\kappa} \tag{3.33}$$

for some real constant κ. Appendix A shows that $\phi_1 = Q + O(r)$. With the choice (3.32) for $\mathring{Z}$, we substitute (3.31) into (3.27) and (3.28) to obtain

$$\begin{aligned}
\mathring{Z}^{-2}\partial_\zeta(\mathring{Z}f) &= -\sqrt{2}Qg\ ,\\
\mathring{Z}^{-2}\partial_{\bar{\zeta}}(\mathring{Z}g) &= \sqrt{2}\bar{Q}f\ .
\end{aligned}$$

Differentiating the first equation with respect to ζ, and expanding, one obtains an equation which can be solved for f; inserting the result into the second, one is led to

$$f = a + b\zeta\ \ ; \quad g = (a\overline{\zeta} - b)e^{-i\kappa}\ ,$$

for constant complex a and b. We can exploit a residual freedom in the choice of ζ to simplify these expressions. Indeed, in view of the choice (3.32), the ζ's are defined now only up to a rigid $SO(3)$ rotation of S^2, which will transform a and b. Without loss of generality, we may assume that $b = 0$ and then, to satisfy (3.25), that $a = \mathring{L}^{-\frac{1}{2}}$.

For $\mathring{S}$ and $\mathring{T}$ we go to (3.21), and the corresponding equation for $\beta_{0'}$, and solve to find

$$\mathring{S} = \sqrt{2}Q\mathring{W}\ \ ; \quad \mathring{T} = -\sqrt{2}\overline{Q}\mathring{P}.$$

Now we have the supercovariantly constant spinors to the desired order, and may proceed to define the coordinates x^i from (2.14) and (2.15). This is a mechanical process, and we simply record the results. Introduce polar angles via

$$\zeta = \tan(\theta/2)e^{-i\phi}$$

then

$$\begin{aligned} dx^1 + idx^2 &= d(-r\sin\theta e^{i\phi - i\kappa}) + O(r)dr + O(r^2)(d\zeta, d\bar{\zeta}, du) \quad (3.34) \\ dx^3 &= d(-r\cos\theta) + O(r)dr + O(r^2)(d\zeta, d\bar{\zeta}, du). \quad (3.35) \end{aligned}$$

If we introduce a set of Cartesian coordinates z^i for $i = 1, 2, 3$ related to the polar coordinates (r, θ, ϕ) by

$$z^1 + iz^2 = -r\sin\theta e^{i\phi - i\kappa} \;, \quad z^3 = -r\cos\theta \;, \tag{3.36}$$

then the system (3.34) and (3.35) implies, for $z \neq 0$,

$$\frac{\partial x^i}{\partial z^j} = \delta^i_j + O(|z|) \;, \tag{3.37}$$

which will be important in the next section.

3.3 Horizons are isolated singularities

We equip $\mathscr{S}$ with the *orbit space metric* γ defined as

$$\gamma(Y, Z) = -{}^4g(Y, Z) + \frac{{}^4g(X, Y)\;{}^4g(X, Z)}{{}^4g(X, X)} \;, \tag{3.38}$$

where X is the Killing vector field (2.12). Our main local result is the following:

THEOREM 3.2 *Every connected component of the horizon corresponds to an isolated singular point*[6] x_0 *of the orbit space metric (3.38). Furthermore, there exists* $\rho > 0$*, a smooth (perhaps* $\mathbb{C}$*-valued) harmonic function* $U_0 \in C^\infty(B(x_0, \rho))$*, and real constants* m_0*,* n_0 *such that in the small punctured coordinate ball* $B^*(x_0, \rho)$ *near* x_0 *we have*

$$\frac{1}{V} = \frac{m_0 + in_0}{|x - x_0|} + U_0 \;. \tag{3.39}$$

REMARK 3.3 This result provides an alternative proof of Proposition 2 of [8], which is the key step of the argument there.

PROOF: By the previous section there exists $\epsilon > 0$ such that, in the punctured z–cordinate ball $B^*(0, \epsilon)$, the following holds

$$\frac{\partial x^i}{\partial z^j} = \delta^i_j + O(|z|) \;. \tag{3.40}$$

[6] By "isolated singular point" we mean here a point in local coordinates as in (2.13), with an isolated singularity of V^{-1} there. We will see in Section 5 that the correct geometric interpretation is that of "cylindrical ends".

We want to show that the limits $\lim_{|z|\to 0} x(z)$ exist. In order to do that, consider any ray $[\lambda,\mu] \ni s \to sz$, with $0<\lambda\le\mu<\epsilon$, by (3.40) we have

$$x^k(\mu z) - x^k(\lambda z) = \int_\mu^\lambda \frac{d\left(x^k(sz)\right)}{ds} ds = (\mu-\lambda) z^k + O(|\mu z|^2) \ . \tag{3.41}$$

Let μ_n be any sequence converging to zero. Equation (3.41) with $\mu=\mu_n$ and $\lambda=\mu_{n'}$ shows that $x^k(\mu_n z)$ is Cauchy, therefore there exist numbers $x_0^k(z)$ such that $\lim_{n\to\infty} x^k(\mu_n z) = x_0^k(z)$. We similarly have, for any two points $z_1, z_2 \in B^*(0,\epsilon)$, with $z_1 \ne -z_2$,

$$x^k(\mu_n z_1) - x^k(\mu_n z_2) = \int_0^1 \frac{d\left(x^k(s\mu_n z_1 + (1-s)\mu_n z_2)\right)}{ds} ds = \mu_n (z_2^k - z_1^k) + O(\mu_n^2) \ , \tag{3.42}$$

and passing to the limit $n\to\infty$ we obtain

$$x_0^k(z_1) = x_0^k(z_2) \ .$$

Thus, the limits $x_0^k(z)$ are in fact z-independent, we will write x_0^k for those limits from now on. Passing to the limit $n\to\infty$ in (3.41) with $\mu=1$ and $\lambda=\mu_n$ we obtain now

$$x^k(z) - x_0^k = z^k + O(|z|^2) \ . \tag{3.43}$$

This shows that $\lim_{|z|\to 0} x^k(z) = x_0^k$, as claimed. It also follows from this equation that the map

$$z \mapsto x \tag{3.44}$$

is differentiable at the origin, with (3.40) holding now both at the origin and in $B^*(0,\epsilon)$. Consequently, the map (3.44) is continuously differentiable on $B(0,\epsilon)$.

The implicit function theorem shows that, decreasing ϵ if necessary, the map (3.44) is a diffeomorphism near x_0. Therefore $1/V$ is well-defined, as a function of x, in a small punctured ball $B^*(x_0,\rho)$ near x_0, and is harmonic with respect to the Euclidean metric there by (2.25). By (3.43) we have $|z| = |x-x_0| + O(|x-x_0|^2)$, and (A.6) gives, away from x_0,

$$\frac{1}{V} = \frac{1}{2Q|x-x_0|} + O(1) \ .$$

Write $1/2Q = m_0 + i n_0$ and set

$$U_0 = \frac{1}{V} - \frac{m_0 + i n_0}{|x-x_0|} \ ,$$

then U_0 is a bounded harmonic function on $B^*(x_0,\rho)$. By Serrin's removable singularity theorem [28, Theorem 1.19, p. 30] U_0 can be extended through x_0 to a smooth harmonic function on $B(x_0,\rho)$. □

For further reference we note the following: It follows from Theorem 3.2 that near each event horizon the metric can be written in the form (2.13), with

V of the form (3.39). From (3.8) we have

$$\begin{aligned} {}^4g &= r^2Adu^2 - 2dudr - 2r^2H_a dy^a du - h_{ab}dy^a dy^b \\ &= r^2A\Big(\underbrace{du - \frac{dr}{r^2A} - \frac{H_a}{A}dy^a}_{dt+\omega_i dx^i}\Big)^2 \\ &\quad -\frac{1}{r^2A}\Big(\underbrace{dr^2 + 2r^2H_a dy^a dr + r^2(Ah_{ab} + r^2H_aH_b)dy^a dy^b}_{dx^i dx^i}\Big) , \end{aligned} \tag{3.45}$$

with $r^2A = V\bar{V}$. The function t is defined up to the addition of a function $f = f(x^i)$, which corresponds to the "gauge transformation" $\omega_i \to \omega_i + \partial_i f$. The simplest choice for t suggested by (3.45) is

$$dt = du - \frac{dr}{r^2\mathring{A}} . \tag{3.46}$$

An important consequence of (3.45) is that $\omega_i dx^i$ is a well-defined one form in a neighbourhood of each connected component of the horizon.

4 Global arguments

4.1 Asymptotics for large r

Consider an asymptotically flat end $\mathscr{S}_{\rm ext}$. The decay of the Maxwell field and a standard analysis of (2.10)-(2.11) show that the components of α and β approach constant values as $|x|$ tends to infinity in a δ-parallel spin frame associated with the flat euclidean metric δ on $\mathscr{S}_{\rm ext}$.

The global hypotheses of Theorem 1.1 show that the ADM four-momentum of $\mathscr{S}_{\rm ext}$ is timelike. By [4, Section 3] the Killing vector X defined by (2.12) is strictly timelike, $g(X,X) > \epsilon > 0$. By boosting $\mathscr{S}_{\rm ext}$ within the Killing development of $\mathscr{S}$ one can assume that X is asymptotically normal to $\mathscr{S}_{\rm ext}$, and a multiplicative normalisation of the spinors (α,β) leads to $X \to \partial_t$ as r goes to infinity.

Again a straightforward asymptotic analysis of (2.14)-(2.15) shows that, performing a rigid coordinate rotation if necessary, the functions x^i asymptote to the asymptotically flat coordinates of (2.4), and in fact provide a coordinate system outside a large compact set on $\mathscr{S}_{\rm ext}$, for $|x|$ large enough.

Now V, and therefore also V^{-1}, tends to a pure phase at infinity, but V^{-1} is harmonic so without loss of generality this phase is constant. Again adjusting our choice of ψ we may suppose the phase to be zero. Now we have, for large $|x|$,

$$V = 1 + \frac{C}{|x|} + O(|x|^{-2}) \tag{4.1}$$

for a (complex) constant C. The usual asymptotic expansion of stationary initial data [23] gives

$$\omega = O(|x|^{-2}) , \quad \partial\omega = O(|x|^{-3}) . \tag{4.2}$$

Inserting (4.1) into (2.24), we see that this decay of the derivatives of ω is possible only if C is real.

4.2 Injectivity

The map which to a point of the space-time $(\mathscr{M}, {}^4g)$ assigns the functions (t, x^i) by solving the equations (2.14)-(2.15) may fail to provide a global coordinate system on $\mathscr{M}$. An example is provided by the usual maximal extension of the degenerate Reissner-Nordström solutions: in this space-time each point (t, x^i) corresponds to an infinite number of distinct points $p_{t,x}$ lying in distinct asymptotically flat regions of $\mathscr{M}$. We emphasise that in this example the functions x^i are smooth and globally defined throughout both $\mathscr{M}$ and $\mathscr{M}/\mathbb{R}$, where $\mathbb{R}$ is the action of the flow of X, but the map which to a point in $\mathscr{M}/\mathbb{R}$ assigns the coordinates x fails to be injective.

Before proceeding further, recall that the Killing development $(\mathscr{M}_{\mathscr{K}}, g_{\mathscr{K}})$ of a hypersurface $(\mathscr{S}, g)$ with Killing initial data (N, Y) is defined [4] as $\mathbb{R} \times \mathscr{S}$ with the metric

$$g_{\mathscr{K}} := N^2 dt^2 - g_{ij}(dx^i + Y^i dt)(dx^j + Y^j dt) \; . \qquad (4.3)$$

If a space-time $(\mathscr{M}, {}^4g)$ contains $\mathscr{S}$ as a hypersurface with unit normal n, and if we decompose a Killing vector X as $X = Nn + Y$, with Y tangent to $\mathscr{S}$, then $(\mathscr{M}_{\mathscr{K}}, g_{\mathscr{K}})$ is isometrically diffeomorphic to the subset of $\mathscr{M}$ obtained by moving $\mathscr{S}$ in the space-time $(\mathscr{M}, {}^4g)$ with the flow of X, when this flow is complete, provided, e.g., that X is causal and $\mathscr{S}$ is acausal in $(\mathscr{M}, {}^4g)$.

We have the following:

THEOREM 4.1 *Under the hypotheses of Theorem 1.1, the coordinate representation (2.13) on the Killing development of $\mathscr{S}$ is global.*

PROOF: Consider the manifold $\widehat{\mathscr{S}}$ defined as follows: as a point set, $\widehat{\mathscr{S}}$ consists of the interior $\operatorname{int}\mathscr{S}$ of $\mathscr{S}$ with an abstract point x_i added for each connected component of the event horizon. The differentiable structure on $\widehat{\mathscr{S}}$ is the one induced from $\operatorname{int}\mathscr{S}$ away from the x_i's, and the one coming from the x coordinates as in Theorem 3.2 around each point x_i.

On $\operatorname{int}\mathscr{S}$, considered as a subset of $\widehat{\mathscr{S}}$, we introduce the metric

$$\widehat{\gamma} := V\bar{V}\gamma \; , \qquad (4.4)$$

where γ is the orbit space metric (3.38). The local coordinate representation (2.13) shows that $\widehat{\gamma}$ is flat. By Theorem 3.2 the metric $\widehat{\gamma}$ extends by continuity to a smooth (flat) metric on $\widehat{\mathscr{S}}$, still denoted by the same symbol $\widehat{\gamma}$. Thus $(\widehat{\mathscr{S}}, \widehat{\gamma})$ is a smooth, flat, Riemannian manifold. By construction $\widehat{\mathscr{S}}$ is the union of a compact set and of an asymptotically flat region, and such manifolds are complete[7]. Again by construction, $\widehat{\mathscr{S}}$ is simply connected. By the Hadamard-Cartan theorem (see, e.g., [18]) $\widehat{\mathscr{S}}$ is diffeomorphic to $\mathbb{R}^3$, with a global manifestly flat coordinate system. This provides the global coordinate representation (2.13). □

As a corollary we obtain

[7]This can be established using, e.g., the arguments of Appendix B of [5].

Corollary 4.2 *Under the hypotheses of Theorem 1.1, there exist constants $m_i, n_i \in \mathbb{R}$ such that*

$$\frac{1}{V} = 1 + \sum_{j=1}^{N} \frac{m_j + i n_j}{|x - x_j|} \ . \tag{4.5}$$

Proof: The hypotheses of Theorem 1.1 imply that $\mathscr{S}$ has a finite numbr of boundary components, say N. By Theorems 3.2 and 4.1, there exist constants $m_i, n_i \in \mathbb{R}$ such that the function

$$U = \frac{1}{V} - 1 - \sum_{j=1}^{N} \frac{m_j + i n_j}{|x - x_j|}$$

approaches zero at infinity, and can be extended by continuity to a smooth harmonic function on $\mathbb{R}^3$. By the maximum principle $U \equiv 0$. □

4.3 Staticity

Suppose that V is given by (4.5). We introduce G and θ by

$$V^{-1} = G e^{i\theta} . \tag{4.6}$$

Since V^{-1} is nowhere zero, θ is well-defined. Since C in (4.1) is real (see Section 4.1) we have, for large $|x|$,

$$G = 1 + \frac{M}{|x|} + O(|x|^{-2}) \ , \tag{4.7}$$

$$\theta = O(|x|^{-2}) \ , \tag{4.8}$$

where M is the ADM mass of $\mathscr{S}_{\mathrm{ext}}$.

From (2.24) we now find

$$\mathrm{curl}\ \omega = 2G^2 \nabla\theta \tag{4.9}$$

which is divergence-free since χ is real.

Set $U = \mathbb{R}^3 \setminus \bigcup_j B_j$, where B_j is a small ball around x_j. By the divergence theorem (with signs appropriately chosen)

$$\int_U G^2 |\nabla\theta|^2 dV = \oint_{S_\infty} G^2 \theta \nabla\theta \cdot d\mathbf{S} + \sum_{j=1}^{N} \oint_{S_j} G^2 \theta \nabla\theta \cdot d\mathbf{S}. \tag{4.10}$$

Here S_∞ is a sphere of large radius, and $S_j = \partial B_j$. We shall show that the integrals at the right-hand-side vanish in the obvious limit, proving vanishing of $\nabla\theta$.

As emphasised at the end of Section 3.3, ω is a well defined one-form near each component of the horizon, thus no "Dirac string" singularities in the form discussed in [14] arise near the punctures. Therefore any small topological two-sphere S around a puncture we have

$$\oint_S \mathrm{curl}\ \omega \cdot d\mathbf{S} = 0 \ . \tag{4.11}$$

Near x_j we have

$$\begin{aligned} G &= \frac{(m_j^2+n_j^2)^{1/2}}{|x-x_j|} + O(1) \\ \sin\theta &= \frac{n_j}{(m_j^2+n_j^2)^{1/2}} + O(|x-x_j|) \end{aligned}$$

so that (4.9) and (4.11) entail

$$\begin{aligned} \oint_{S_j} G^2\theta\nabla\theta\cdot d\mathbf{S} &= \arcsin\left(\frac{n_j}{(m_j^2+n_j^2)^{1/2}}\right)\underbrace{\oint_{S_j} G^2\nabla\theta\cdot d\mathbf{S}}_{=0} \\ &\quad + \oint_{S_j} G^2 O(|x-x_j|)\nabla\theta\cdot d\mathbf{S} = O(|x-x_j|)\ , \end{aligned}$$

while,

$$\oint_{S_\infty} G^2\theta\nabla\theta\cdot d\mathbf{S} = O(|x|^{-3}).$$

It follows that the right-hand-side of (4.10) vanishes in the limit as the B_j shrink onto the x_j and S_∞ recedes to infinity. Therefore so does the left and θ is constant, but θ vanishes at infinity, so θ is everywhere zero and V is real. Thus the metric is a standard Majumdar-Papapetrou metric. This completes the proof of Theorem 1.1.

5 Solutions with maximal hypersurfaces

As first pointed out by Sudarsky and Wald [25, 26], maximal surfaces which are Cauchy for the domain of outer communications provide a powerful tool to study stationary black holes. The existence of such hypersurfaces in our context is an open question, which we will not address here, but some comments are in order.

We want to calculate the mean extrinsic curvature, say H_f, of the hypersurfaces $\{t = f(x^i)\}$. For this, we note that, from (3.45),

$$^4g^{\mu\nu}\partial_\mu\partial_\nu = ((r^2A)^{-1} - r^2A|\omega|^2_\delta)\partial_t^2 + 2r^2A\sum_i \omega_i\partial_i\partial_t - r^2A\sum_i \partial_i\partial_i\ , \quad (5.1)$$

$$\sqrt{-\det {}^4g_{\mu\nu}} = (r^2A)^{-1}\ , \quad (5.2)$$

where $|\omega|^2_\delta = \sum_i \omega_i^2$, so that the field of unit normals n to the level sets of $\{t = f(x^i)\}$ takes the form

$$\begin{aligned} n_\mu dx^\mu &= \frac{r\sqrt{A}}{\sqrt{1-r^4A^2|\omega+df|^2_\delta}}(dt-df)\ , \\ n^\mu\partial_\mu &= n^0\partial_t + \frac{r^3A^{3/2}}{\sqrt{1-r^4A^2|\omega+df|^2_\delta}}\sum_i(\omega_i+\partial_i f)\partial_i\ , \end{aligned} \quad (5.3)$$

where n^0 equals

$$n^0 = \frac{\Big(1 - r^4A^2\sum_i \omega_i(\omega_i + \partial_i f)\Big)}{r\sqrt{A}\sqrt{1 - r^4A^2|\omega + df|^2_\delta}} \;.$$

This leads to

$$\begin{aligned} H_f &= \nabla_\mu n^\mu = \frac{1}{\sqrt{-\det{}^4g_{\mu\nu}}}\partial_\alpha\Big(\sqrt{-\det{}^4g_{\mu\nu}}n^\alpha\Big) \\ &= r^2A\sum_i \partial_i\left(\frac{r\sqrt{A}(\omega_i + \partial_i f)}{\sqrt{1 - r^4A^2|\omega + df|^2_\delta}}\right) \;. \end{aligned} \tag{5.4}$$

The choice (3.46) leads to a one-form ω which, in the coordinates x^i, satisfies near $x = 0$

$$\omega_i = O(|x|^{-1}) \;, \quad \partial_j\omega_i = O(|x|^{-2}) \;.$$

Equation (5.4) with $f = 0$ leads to $H_0 = O(r)$, in particular H_0 vanishes in the limit $r \to 0$. A rough estimate leads one to expect that there should exist a class of solutions of the equation $H_f = 0$ with the asymptotic behavior, for small r,

$$f = O(\ln^2 r) \;. \tag{5.5}$$

(This can probably be improved to $O(\ln r)$ by a more careful inspection of the equation satisfied by ω, but we have not undertaken that analysis.) Solutions with such an asymptotic behavior will lead to a geometry of the level sets of f with decay rates worse than (5.10)-(5.11) below, but more than sufficient for (5.12) to hold.

Now, we expect that the arguments in [11], concerning the global structure of globally hyperbolic domains of outer communications, remain valid in the current setting, but we have not checked this. Assuming this to be correct it is then standard, using [2], to construct an edgeless maximal surface in $\mathscr{M}$, as a limit of a sequence of solutions of the Dirichlet problem on compact sets. In view of the analysis in [1] it is not completely unreasonable to expect asymptotic flatness of $\mathscr{S}$, as well as "weakly cylindrical" behavior (as defined by (5.12) below) near the horizons, but this remains to be proved. We are planning to return to this question in a near future.

We continue with the analysis of the gravitational initial data near the horizons. Let n be the field of future directed unit normals to $\mathscr{S} = \{t = f\}$, and decompose the Killing vector field X as $X = Nn + Y$, where Y is tangent to $\mathscr{S}$. From (5.3) we obtain

$$\begin{aligned} X = \partial_t &= \underbrace{\frac{r\sqrt{A}\sqrt{1 - r^4A^2|\omega + df|^2_\delta}}{\Big(1 - r^4A^2\sum_i \omega_i(\omega_i + \partial_i f)\Big)}}_{N} n^\mu\partial_\mu \\ &\quad \underbrace{- \frac{r^4A^2}{\Big(1 - r^4A^2\sum_i \omega_i(\omega_i + \partial_i f)\Big)}\sum_i(\omega_i + \partial_i f)\,\partial_i}_{Y^i} \;. \end{aligned} \tag{5.6}$$

By inspection of (3.45) and (5.6), the gravitational initial data induced on $\{t=0\}$ behave, for small r, as

$$g_{ij} = O_1(r^{-2})\ , \quad g^{ij} = O_1(r^2)\ , \tag{5.7}$$
$$N = O_1(r)\ , \quad Y^i = O_1(r^3)\ , \quad Y_i = O_1(r)\ , \quad K_{ij} = O(1)\ , \tag{5.8}$$

where the equality $h = O_k(r^\sigma)$ means that $\partial_{i_1}\cdots\partial_{i_\ell} f = O(r^{\sigma-\ell})$ for $0 \le \ell \le k$. Here the estimate on K_{ij} can be obtained from the equation $NK_{ij} = -\frac{1}{2}\mathscr{L}_Y g_{ij}$.

To understand the geometry of the level sets of t it is convenient, near each puncture x_i, to return to spherical coordinates centred at x_i, replacing $|x-x_i|$ by a new radial coordinate ρ defined as

$$\rho = -\ln|x-x_i|\ .$$

We then obtain

$$g := g_{ij}dx^i dx^j = A^{-1}d\rho^2 + {}^2g_{ab}dy^a dy^b\ , \tag{5.9}$$

where the metrics ${}^2g_{ab}(\rho,\cdot)dy^a dy^b$ asymptote exponentially fast to the round metric on the sphere as ρ tends to infinity. This shows that for ρ large enough the space-metric is uniformly equivalent to a fixed, ρ–independent, product metric

$$\mathring{g} := d\rho^2 + \mathring{g}_{ab}dy^a dy^b\ .$$

Such metrics will be called *weakly cylindrical*.

Using the variable ρ, (5.8) can be rewritten as

$$N = O(e^{-\rho})\ , \quad |dN|_g = O(e^{-\rho})\ , \tag{5.10}$$
$$|Y|_g = O(e^{-2\rho})\ , \quad |DY|_g = O(e^{-2\rho})\ , \quad |K|_g = O(e^{-\rho})\ . \tag{5.11}$$

For the purposes of Theorem 1.2 we will need the following two hypotheses:

1. the metric induced on $\mathscr{S}$ is weakly cylindrical, and
2. $|K|_g|Y|_g$ approaches zero as ρ tends to infinity. (5.12)

The analysis above shows that this is clearly satisfied on the level sets of t.

Next, consider the Maxwell field. Equation (2.17) is equivalent to

$$\nabla_\mu(\Re V) = F_{\mu\nu}X^\nu\ , \quad \nabla_\mu(\Im V) = *_4F_{\mu\nu}X^\nu\ , \tag{5.13}$$

where $*_4$ is the space-time Hodge dual. As in Section 4.1, we choose V so that $V \to 1$ in the asymptotically flat region, in particular $\Im V$ approaches zero there.

Let $E = E_i dx^i$ be the electric field on $\mathscr{S}$ defined as $E_i = F_{i\mu}n^\mu$, similarly let the magnetic field $B = B_i dx^i$ on $\mathscr{S}$ be defined as $B_i = *_4F_{i\mu}n^\mu$. (As usual, when performing $3+1$ decompositions, the index i refers to a coordinate systems so that $\mathscr{S} = \{t=0\}$; this does not necessarily coincide with the t-coordinate of (2.13).) As already pointed out, we denote by g the *positive definite* metric induced by the space-time metric ${}^4g$ on $\mathscr{S}$ (so, in our current signature convention, $g_{ij} = -{}^4g_{ij}$). Furthermore, we let D denote the covariant derivative operator associated with g (not to be confused with D in Section 3.1). Then

$$B_i = \tfrac{1}{2}\epsilon_i{}^{jk}F_{jk} \quad \Longleftrightarrow \quad F_{ij} = \epsilon_{ijk}B^k \ ,$$
$$*_4F_{ij} = \epsilon_{ijk0}F^{k0} = \epsilon_{ijk}E^k \ ,$$

where ϵ_{ijk} is completely antisymmetric and equals $\sqrt{\det g}$ for $ijk = 123$. Further, indices on three dimensional objects are raised and lowered with g. From (5.13) we obtain

$$D_i(\Re V) = NE_i + F_{ij}Y^j \ , \quad D_i(\Im V) = NB_i + *_4F_{ij}Y^j \ . \tag{5.14}$$

Assume, now, that $\mathscr{S}$ is maximal. We then have the equations

$$D_iK^i{}_j = 2F_{jk}E^k = 2\epsilon_{jk\ell}E^kB^\ell \ , \quad D_{(i}Y_{j)} = -NK_{ij} \ , \quad D_iB^i = 0 \ ,$$

which together with the second equation in (5.14) lead to the divergence identity[8]

$$D_i\Big(K^i{}_jY^j - 2(\Im V)B^i\Big) = -(|K|^2 + 2|B|^2)N \ . \tag{5.15}$$

One integrates (5.15) over a set which consists of $\mathscr{S}$ from which coordinate balls $S_i(\epsilon)$ of radius ϵ around the punctures have been removed. The boundary integral at infinity vanishes by the asymptotic flatness conditions. Consider the integrals

$$Q_i^B = \oint_{S_i(\epsilon)} B^i dS_i \ ,$$

then Q_i^B does not depend upon ϵ, at least for ϵ small enough, since B has vanishing divergence. This implies that the boundary term involving $\Im V B^i$ gives a vanishing contribution in the limit $\epsilon \to 0$ (recall that V, and hence also $\Im V$, approaches zero at the punctures by Theorem 3.2). Similarly, the boundary contribution from the extrinsic curvature term vanishes in the limit by (5.11). Hence

$$\int_{\mathscr{S}} (|K|^2 + 2|B|^2)N = 0 \ .$$

Note that N is strictly positive on the interior of $\mathscr{S}$ since X is causal, which shows that $K \equiv B \equiv 0$. In particular the Killing development of $\mathscr{S}$ is static. We now have $\mathscr{L}_Y g = 0$, so that Y is a Killing vector of g which approaches zero in the asymptotic region. By [4] $Y \equiv 0$, so $X = Nn$, hence the Killing vector is strictly timelike in the domain of outer communications (understood as a subset of the Killing development) so that Theorem 1.1 applies.

6 Theorem 1.1 and sources

In this section we relax the hypothesis that the space-time is electrovacuum. It turns out to be straightforward to obtain a version of Theorem 1.1 assuming

[8] The calculations here allow one to simplify considerably the arguments in [25, 26]. This will be discussed elsewhere.

instead that *magnetic currents vanish*, so that $\Im V^{-1}$ is harmonic. We thus consider a space-time with a non-trivial super-covariantly constant spinor field ψ, and assume again that there are no null orbits of the associated Killing vector X.

Suppose, first, that there are no black holes. As already pointed out in Section 2.1, the (then globally defined) imaginary part of V^{-1} is harmonic in the (globally defined) metric $|V|^2\gamma$, where γ is the orbit space-metric (3.38), hence $\Im V = 0$, then curl $\omega = 0$ and the space-time is static. This leads to a Majumdar-Papapetrou solution, either non-empty or flat.

Both the proof and the statement of Proposition 3.1 remain unchanged. Indeed, the only difference is the need to analyse the supplementary contribution $\chi X_\mu X_\nu$ to the energy-momentum tensor. But

$$\chi X \otimes X = \chi \partial_u \otimes \partial_u = \epsilon^2 \chi \partial_{\hat{u}} \otimes \partial_{\hat{u}} \to_{\epsilon \to 0} 0 \ ,$$

which shows that the near-horizon space-time remains vacuum. The limit in (3.7) is not affected by source fields which are smooth functions in the physical space-time, and one concludes as before.

One finds by inspection that none of the equations of Section 3.2 is affected by a non-vanishing χ.

The non-harmonicity of $\Re V^{-1}$ might result in a somewhat different behavior of the function $\Re U_0$ appearing in Theorem 3.2, which now will be the sum of a constant and of a contribution of the form $r\chi$, for some function $\chi(r, y^a)$ which is smooth in spacetime (but not necessarily smooth with respect to the coordinates x^i). Note that $\Im U_0$ remains smooth, as before. This will lead to a correction $O(r)$ (with somewhat worse differentiability: bounded derivatives, and second derivatives $O(r^{-1})$) in the real part of V^{-1} in (4.5), the imaginary part of this last equation remaining of the same form as before.

The staticity argument in Section 4.3 applies without changes. As a consequence, one is led to a Majumdar-Papapetrou space-time with a potential V such that V^{-1} is a finite sum of monopoles and of a bounded function.

A An alternative proof of Proposition 3.1

Here we give an alternative spinor proof of Proposition 3.1. ¿From (3.9)-(3.13), we calculate the curvature components, setting the scalar curvature to zero. For the Weyl spinor, we find $\Psi_0 = O(r^2), \Psi_1 = O(r), \Psi_3 = O(1)$, and $\Psi_4 = O(1)$ together with two expressions for Ψ_2. One forces $\Psi_2 = O(r)$ while the other is

$$\Psi_2 = \frac{1}{4}(A - K) + O(r) \tag{A.1}$$

where $K = -\frac{1}{\mathring{Z}\overline{\mathring{Z}}}\partial_\zeta \partial_{\overline{\zeta}} \left(\log(\mathring{Z}\overline{\mathring{Z}})\right)$, which is the Gauss curvature of $\mathcal{S}$. Thus $A = K + O(r)$.

For the Ricci spinor, we find $\Phi_{00} = O(r^2), \Phi_{01} = O(r)$ and the remaining components are $O(1)$. In particular we have

$$\Phi_{11} = \frac{1}{4}(A + K) + O(r). \tag{A.2}$$

It follows from (2.26) that $\Phi_{00} = 2\phi_0\overline{\phi}_0$ and $\Phi_{11} = 2\phi_1\overline{\phi}_1$. Thus the component ϕ_0 of the Maxwell field is $O(r)$, while ϕ_1 is $O(1)$; ϕ_1 is also constrained by the Maxwell equations, specifically by (A.5b) of [20] which here becomes

$$\overline{\delta}\phi_1 = O(r)\ ,$$

or

$$\partial_{\overline{\zeta}}\phi_1 = O(r)\ .$$

This integrates at once to give $\phi_1 = Q + O(r)$ where Q is holomorphic in ζ on $\mathcal{S}$. It is also bounded (since it is the contraction of the self-dual part of the Maxwell field with the volume form of $\mathcal{S}$), and so it must be constant (the value of this constant is proportional to the charge of the black hole). Now from (A.1) and (A.2)

$$\mathring{A} = K = 4|Q|^2$$

which establishes (i) and (ii).

For (iii) return to (2.17). Taking components along the null tetrad, we find

$$\frac{\partial V}{\partial r} = 2\phi_1 \tag{A.3}$$

$$\delta V = -2\phi_0 \tag{A.4}$$

$$\overline{\delta} V = r^2 A\phi_2. \tag{A.5}$$

Thus δV and $\overline{\delta}V$ vanish at $\mathcal{S}$ while $\frac{\partial V}{\partial r} = 2Q$, so that

$$V = 2Qr + O(r^2). \tag{A.6}$$

□

B The mass-charge inequality

Consider the following spinor covariant derivative on $\mathscr{S}$:

$$\nabla_i = D_i + A_i\ , \tag{B.1}$$

where D_i is the standard spin connection for spinor fields, and

$$A_i = \frac{1}{2}K_{ij}\gamma^j\gamma_0 - \frac{1}{2}E^k\gamma_k\gamma_i\gamma_0 - \frac{1}{4}\epsilon_{jk\ell}B^j\gamma^k\gamma^\ell\gamma_i\ . \tag{B.2}$$

Here the γ^μ's are local sections of a bundle of spinor-endomorphisms which, in an ON-frame for the Riemannian metric g on $\mathscr{S}$, satisfy the usual relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\mathrm{diag}(1,-1,-1,-1)$. By construction, constancy of a spinor ψ in this connection is the projection into $\mathscr{S}$ of the equations (2.10, 2.11).

The identity which lies at the heart of the mass-charge inequality reads

$$D_i\langle\phi, (\nabla^i + \gamma^i\gamma^j\nabla_j)\phi\rangle = |\nabla\phi|^2 - |\gamma^i\nabla_i\phi|^2 + \frac{1}{4}\langle\phi, \Big\{\mu + (\nu_i\gamma^i + 4\mathrm{div}(E))\gamma_0 - 4\mathrm{div}(B)\gamma^1\gamma^2\gamma^3\Big\}\phi\rangle\ , \tag{B.3}$$

where

$$\begin{aligned} \mu &= R - |K|_g^2 + (\mathrm{tr}_g K)^2 - 2|E|_g^2 - 2|B|_g^2 \;, && \text{(B.4)} \\ \nu^j &= 2D_i(K^{ij} - \mathrm{tr}_g K g^{ij}) - 4\epsilon^j{}_{k\ell} E^k B^\ell \;. && \text{(B.5)} \end{aligned}$$

The matrix appearing in the second line of (B.3) is of the form

$$A := a^\mu \gamma_0 \gamma_\mu + b\gamma_0 + c\gamma_1\gamma_2\gamma_3 \;. \tag{B.6}$$

We note the inequality

$$\langle \psi, A\psi \rangle \geq \left(a^0 - \sqrt{|\vec{a}|_\delta^2 + b^2 + c^2} \right) |\psi|^2 \;, \tag{B.7}$$

with equality if and only if both sides vanish, in particular the quadratic form $\langle \psi, A\psi \rangle$ is non-negative if and only if

$$a^0 \geq \sqrt{|\vec{a}|_\delta^2 + b^2 + c^2} \;. \tag{B.8}$$

This shows that the second line of (B.3) will be non-negative when (2.6)-(2.7) hold. Furthermore, the vanishing of the left-hand-side of (B.7) with a non-trivial ψ implies equality in (B.8). Under the conditions of Theorem 2.1, standard arguments (see, e.g., [3, 15]) give existence of a solution of the equation $\gamma^i \nabla_i \psi = 0$ (with appropriate boundary conditions [12, 15] at a weakly trapped boundary, if relevant), which in turn leads to the inequality (2.9). The case of equality leads to the existence of a spinor field ψ satisfying

$$\nabla_i \psi = 0 \;. \tag{B.9}$$

We set

$$N = \langle \psi, \psi \rangle \;, \quad Y^i = \langle \psi, \gamma^0 \gamma^i \psi \rangle \;, \tag{B.10}$$

and we consider the Killing development $(\mathscr{M}_{\mathscr{K}}, g_{\mathscr{K}})$ of $(\mathscr{S}, g, N, Y)$ as in (4.3). The electromagnetic field $F_{\mu\nu}$ can be constructed out of E and B on $\mathscr{S}$ in the obvious way, and we extend F to $\mathscr{M}_{\mathscr{K}}$ by requiring $\mathscr{L}_X F = 0$.

We need to show that ψ extends to a super-covariantly constant Killing spinor on $\mathscr{M}_{\mathscr{K}}$.

Splitting the (Dirac) spinor ψ into a pair of 2-component spinors $(\alpha_A, \beta_{A'})$, we write (B.9) as the projection into $\mathcal{S}$ of (2.10, 2.11) as

$$\begin{aligned} P_\mu{}^\nu (\nabla_{NN'} \alpha_B + \sqrt{2} \phi_{NB} \beta_{N'}) &= 0 && \text{(B.11)} \\ P_\mu{}^\nu (\nabla_{NN'} \beta_{B'} - \sqrt{2} \overline{\phi}_{N'B'} \alpha_N) &= 0, && \text{(B.12)} \end{aligned}$$

where $P_\mu{}^\nu$ is the projection orthogonal to n (thus $P_{\mu\nu} = {}^4 g_{\mu\nu} - n_\mu n_\nu$).

Following (B.10), the Killing vector $X = Nn + Y$ is given at $\mathcal{S}$ by

$$X = \frac{1}{\sqrt{2}} (\alpha^A \overline{\alpha}^{A'} + \overline{\beta}^A \beta^{A'}) \frac{\partial}{\partial x^{AA'}}, \tag{B.13}$$

compare (2.12). We extend the spinors $(\alpha_A, \beta_{A'})$ off of $\mathcal{S}$ by requiring their Lie-derivative along X to vanish. (Recall that the Lie-derivative of a spinor field α_A along a Killing vector X^a is defined as

$$\mathscr{L}_X \alpha_A := X^\mu \nabla_\mu \alpha_A + \Phi_A{}^M \alpha_M \tag{B.14}$$

where the symmetric spinor Φ_{MN} is defined by

$$\nabla_\mu X_\nu = \Phi_{MN}\epsilon_{M'N'} + \overline{\Phi}_{M'N'}\epsilon_{MN}.$$

See e.g. [16, p. 40]).

Then (B.13) holds throughout $\mathscr{M}_{\mathscr{K}}$. We define $V = \alpha_A \overline{\beta}^A$ as in Section 2.1 and then we can calculate the derivative of X at $\mathscr{S}$ and in directions tangent to $\mathscr{S}$ from (B.11) and (B.12) as

$$P_\mu{}^\nu \nabla_\nu X_\beta = P_\mu{}^\nu (\overline{V}\phi_{NB}\epsilon_{N'B'} + V\overline{\phi}_{N'B'}\epsilon_{NB}). \tag{B.15}$$

However, in $\mathscr{M}_{\mathscr{K}}$, X is a Killing vector so that $\nabla_{(\mu}X_{\nu)} = 0$. It follows that, at $\mathscr{S}$, we can omit the $P_\mu{}^\nu$ in (B.15). Then both sides have vanishing Lie-derivative along X and we recover equation (2.18) for the derivative of X throughout $\mathscr{M}_{\mathscr{K}}$.

By (B.13) we have

$$X^{AA'}\beta_{A'} = -\frac{\bar{V}}{\sqrt{2}}\alpha^A\ , \quad X^{AA'}\alpha_A = \frac{V}{\sqrt{2}}\beta^{A'}\ .$$

We may use (B.14) and the above to write the constancy of α_A and $\beta_{A'}$ along X in the form

$$\begin{aligned} X^\mu(\nabla_{MM'}\alpha_A + \sqrt{2}\phi_{AM}\beta_{M'}) &= 0 \\ X^\mu(\nabla_{MM'}\beta_{A'} - \sqrt{2}\overline{\phi}_{A'M'}\alpha_M) &= 0. \end{aligned}$$

Taken with (B.11) and (B.12), and since X has a nonzero component along n, this shows that (2.10) and (2.11) hold at $\mathscr{S}$.

To complete the proof we need a result from spinor calculus: for a Killing vector X and any spinor field χ

$$(\mathscr{L}_X\nabla_\mu - \nabla_\mu\mathscr{L}_X)\chi = 0.$$

To prove this, observe that it is true for any tensor in place of χ, so all that is necessary is to check it for the spinors ϵ_{AB}, $\epsilon_{A'B'}$; this property follows immediately from (B.14).

Now all quantities in (2.10) and (2.11) have vanishing Lie-derivative along X so that, by virtue of holding at $\mathscr{S}$, these equations hold throughout $\mathscr{M}_{\mathscr{K}}$: there is therefore a supercovariantly constant spinor in $\mathscr{M}_{\mathscr{K}}$.

The case $V = 0$ leads to metrics which, locally, are *pp*-waves [13, 27] (not necessarily electro-vacuum). Asymptotically flat *pp*-waves do not satisfy the regularity hypotheses set forth here, except if the space-time metric is flat [4]. As flat metrics belong to the IWP family, Theorem 2.1 is proved.

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References

[1] R. Bartnik, *The existence of maximal hypersurfaces in asymptotically flat space-times*, Comm. Math. Phys. **94** (1984), 155–175.

[2] ______, *Regularity of variational maximal surfaces*, Acta Math. **161** (1988), 145–181.

[3] R. Bartnik and P.T. Chruściel, *Boundary value problems for Dirac-type equations*, (2003), math.DG/0307278.

[4] R. Beig and P.T. Chruściel, *Killing vectors in asymptotically flat space–times: I. Asymptotically translational Killing vectors and the rigid positive energy theorem*, Jour. Math. Phys. **37** (1996), 1939–1961, gr-qc/9510015.

[5] P.T. Chruściel, *On the invariant mass conjecture in general relativity*, Commun. Math. Phys. **120** (1988), 233–248.

[6] ______, *"No Hair" Theorems – folklore, conjectures, results*, Differential Geometry and Mathematical Physics (J. Beem and K.L. Duggal, eds.), Cont. Math., vol. 170, AMS, Providence, 1994, pp. 23–49, gr–qc/9402032.

[7] ______, *The classification of static vacuum space–times containing an asymptotically flat spacelike hypersurface with compact interior*, Class. Quantum Grav. **16** (1999), 661–687, gr-qc/9809088.

[8] P.T. Chruściel and N.S. Nadirashvili, *All electrovacuum Majumdar–Papapetrou spacetimes with non–singular black holes*, Class. Quantum Grav. **12** (1995), L17–L23, gr-qc/9412044.

[9] P.T. Chruściel, H.S. Reall, and K.P. Tod, *On non-existence of static vacuum black holes with degenerate components of the event horizon*, Class. Quantum Grav. (2005), in press, gr-qc/0512041.

[10] P.T. Chruściel and K.P. Tod, *The classification of static electro-vacuum space–times containing an asymptotically flat spacelike hypersurface with compact interior*, gr-qc/0512043.

[11] P.T. Chruściel and R.M. Wald, *On the topology of stationary black holes*, Class. Quantum Grav. **11** (1994), L147–152.

[12] G.W. Gibbons, S.W. Hawking, G.T. Horowitz, and M.J. Perry, *Positive mass theorem for black holes*, Commun. Math. Phys. **88** (1983), 295–308.

[13] G.W. Gibbons and C.M. Hull, *A Bogomolny bound for general relativity and solitons in $N = 2$ supergravity*, Phys. Lett. **109B** (1982), 190–194.

[14] J.B. Hartle and S.W. Hawking, *Solutions of the Einstein–Maxwell equations with many black holes*, Commun. Math. Phys. **26** (1972), 87–101.

[15] M. Herzlich, *The positive mass theorem for black holes revisited*, Jour. Geom. Phys. **26** (1998), 97–111.

[16] S.A. Huggett and K.P. Tod, *An introduction to twistor theory*, second ed., London Mathematical Society Student Texts, vol. 4, Cambridge University Press, Cambridge, 1994.

[17] D. Kennefick and N. Ó Murchadha, *Weakly decaying asymptotically flat static and stationary solutions to the Einstein equations*, Class. Quantum Grav. **12** (1995), 149–158.

[18] J.M. Lee, *Riemannian manifolds*, Graduate Texts in Mathematics, vol. 176, Springer-Verlag, New York, 1997.

[19] S.D. Majumdar, *A class of exact solutions of Einstein's field equations*, Phys. Rev. **72** (1947), 390–398.

[20] E.T. Newman and K.P. Tod, *Asymptotically flat space–times*, General Relativity and Gravitation (A. Held, ed.), Plenum, New York and London, 1980, pp. 1–36.

[21] A. Papapetrou, *A static solution of the equations of the gravitational field for an arbitrary charge distribution*, Proc. Roy. Irish Acad. **A51** (1947), 191–204.

[22] H.S. Reall, *Higher dimensional black holes and supersymmetry*, Phys. Rev. **D68** (2003), 024024, hep-th/0211290.

[23] W. Simon, *Radiative Einstein-Maxwell spacetimes and 'no–hair' theorems*, Class. Quantum Grav. **9** (1992), 241–256.

[24] J. Stewart, *Advanced general relativity*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1990.

[25] D. Sudarsky and R. M. Wald, *Extrema of mass, stationarity, and staticity, and solutions to the einstein-yang-mills equations.*, Phys. Rev. D **46** (1992), no. 4, 1453–1474.

[26] D. Sudarsky and R.M. Wald, *Mass formulas for stationary Einstein Yang-Mills black holes and a simple proof of two staticity theorems*, Phys. Rev. **D47** (1993), 5209–5213, gr-qc/9305023.

[27] K.P. Tod, *All metrics admitting super–covariantly constant spinors*, Phys. Lett. **121B** (1983), 241–244.

[28] L. Véron, *Singularities of solutions of second order quasilinear equations*, Pitman Research Notes in Mathematics Series, vol. 353, Longman, Harlow, 1996.