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Anti-self-dual conformal structures with null Killing vectors from projective structures

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Abstract

Using twistor methods, we explicitly construct all local four-dimensional analytic neutral signature anti-self-dual conformal structures $(M, [g])$ with a null conformal Killing vector K. We show that M is foliated by anti-self-dual null surfaces, and the two-dimensional leaf space U inherits a natural projective structure $[\Gamma]$. The twistor space of $(U, [\Gamma])$ is the space of trajectories of a one-dimensional distribution $\widehat{\mathcal{K}}$ on the twistor space of $(M, [g])$. We find a local classification of all $(M, [g], K)$, which branches according to whether or not K has is hyper-surface orthogonal. We give examples of conformal classes which contain Ricci–flat metrics on compact complex surfaces.

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1 Introduction

The anti–self–duality (ASD) condition in four dimensions seems to underlie the concept of integrability of ordinary and partial differential equations [27]. Many lower dimensional integrable models (KdV, NlS, Sine–Gordon, ...) arise as symmetry reductions of the ASD Yang–Mills equations on a flat background, and various solution generation techniques are reductions of the twistor correspondence [17]. Other integrable models (dispersionless Kadomtsev–Petviashvili, $SU(\infty)$ Toda, ...) are reductions of the ASD conformal equations which say that the self–dual Weyl tensor of a conformal class of metrics vanishes [28, 6]. Generalisations to ASD Yang–Mills on ASD conformal background are also possible [25, 3].

In all cases the main interest is in conformal structures of signature $++--$ which are called *neutral*, as the reductions can lead to interesting hyperbolic and parabolic equations. There are no non–trivial ASD structures in the Lorentzian signature $+++-$, and the reductions from Riemannian manifolds can only yield elliptic equations thus ruling out interesting soliton dynamics.

The main gap in the programme to classify the reductions of ASD neutral conformal structures was understanding the reductions by a null conformal Killing vector. We embarked on this project hoping to incorporate more integrable systems into the framework of of anti–self–duality, but we have found (Theorem 2) that the resulting geometry is a *completely solvable* system.

When studying conformal structures with non-null conformal Killing vectors, it is natural to look at the space of Killing vector trajectories, since this will inherit a non-degenerate conformal structure. In the case of a *null* conformal Killing vector, the situation is different. The space of trajectories inherits a *degenerate* conformal structure. We find that it is necessary to go down one dimension more, and consider a *two* dimensional space U of anti-self-dual totally null surfaces in M, called β–surfaces, containing K, which exist as a consequence of the conformal Killing equation. It turns out that there is a naturally defined projective structure $[\Gamma]$ on U. Moreover, we show that the twistor spaces of $(M, [g])$ and $(U, [\Gamma])$ are related by dimensional reduction. Specifically, the twistor space Z of $(U, [\Gamma])$ is the space of trajectories of a vector field on the twistor space $\mathcal{PT}$ of $(M, [g])$ corresponding to K. Projective structures are just equivalence classes of torsion-free connections, which do not need to satisfy any equations; this underlies the complete solvability of null reductions, and contrasts with the non-null case where one obtains Einstein-Weyl structures [11], and associated integrable systems [28, 6, 5, 3].

From now on we restrict to real analytic conformal structures since these can be complexified, and we can then make use of Penrose's twistor construction and its relatives. This article is structured as follows. In Section 2 we derive some elementary properties of null conformal Killing vectors. Section 3 is an introduction to projective structures. In Section 4 we prove the following:

Theorem 1. *Let $(M, [g])$ be a four dimensional real analytic neutral ASD conformal structure with a null conformal Killing vector K. Let U be the two dimensional space of β-surfaces containing K. Then there is a naturally defined projective structure on U, whose twistor space is the space of trajectories of a distribution $\widehat{\mathcal{K}}$ induced on $\mathcal{PT}$ by the action of K on M.*

In Section 5 we use the theorem to find a local coordinate form for any analytic $++--$ ASD conformal structure with null conformal Killing vector. This is expressed in the following:

Theorem 2. *Let $(M, [g], K)$ be as in Theorem 1. Then there exist local coordinates (t, x, y, z) and $g \in [g]$ such that $K = \partial_t$ and g has one of the following two forms, according to whether $\mathbb{K} \wedge d\mathbb{K}$ vanishes or not ($\mathbb{K} := g(K, .)$):*

1. $\mathbb{K} \wedge d\mathbb{K} = 0$.
$$g = (dt + z\Phi - \Omega)dy - dxdz, \tag{1.1}$$
where $\Phi = A_2(x, y)dx + A_3(x, y)dy$, $\Omega = P(x, y)dx + Q(x, y)dy$, and A_2, A_3, P, Q are arbitrary functions.

2. $\mathbb{K} \wedge d\mathbb{K} \neq 0$.
$$g = (dt - \Omega)\ (dy - zdx) - (\partial_z G)dx\ (dz - \Phi), \tag{1.2}$$
where $\Phi = A_0(x, y)dx + A_1(x, y)dy$, $\Omega = (\partial_y G + A_1 \partial_z G)dx$, and $G = G(x, y, z)$ satisfies the following PDE:
$$(\partial_x + z\partial_y + (A_0 + zA_1)\partial_z)\partial_z G = 0. \tag{1.3}$$

In (1.1) and (1.2), the respective Ωs are defined up to addition of dR where $R = R(x, y)$, since one can perform the coordinate change $t \to t + R(x, y)$ without changing the form of the metric. The one forms labelled Φ in the metrics (1.1) and (1.2) determine projective structures on the (x, y) space in the following way. A general projective structure corresponds to a second-order ODE

$$\frac{d^2y}{dx^2} = A_3(x, y)\Big(\frac{dy}{dx}\Big)^3 + A_2(x, y)\Big(\frac{dy}{dx}\Big)^2 + A_1(x, y)\Big(\frac{dy}{dx}\Big) + A_0(x, y). \tag{1.4}$$

The Φ in (1.1) corresponds to the ODE with A_0 and A_1 eliminated, and the Φ in (1.2) corresponds to the ODE with A_2 and A_3 eliminated. However, it is always possible to eliminate two of the four functions in (1.4) by coordinate transformation, so in fact these projective structures are the most general possible. If one interprets z as a fibre coordinate on the projective tangent bundle of the (x, y) space, then (1.3) says that $\partial_z G$ is constant along the projective structure spray, (compare formula 3.3).

Note that in both cases the Killing vector is ∂_t and is pure Killing (this comes from choosing a suitable $g \in [g]$). In the non-twisting case (1.1) the Killing vector K is covariantly constant, so this case is a natural conformal generalisation of Ricci–flat pp waves. In the twisting case (1.2) K is not covariantly constant, and (1.2) is a neutral analog of the Fefferman conformal class [9]. Restricting to the special form of G and Φ we recover some examples of [18], where neutral metrics were related to second order ODEs.

In Section 6 we examine some examples found by different means in the light of our results. If $d\Phi = 0$ then (1.1, 1.2) are (pseudo) hyper–complex and K is triholomorphic. The metric (1.1) with $\Phi = 0$ yields a compact example of a Ricci–flat metric on a Kodaira surface of a special type. We consider how to construct conformal structure twistor spaces from projective structure twistor spaces in Section 7. The more involved spinor calculations are moved to the Appendix

2 Null Conformal Killing Vectors

2.1 Spinors in neutral signature

We will denote by $(M, [g])$ a local patch of $\mathbb{R}^4$ endowed with a neutral signature conformal structure $[g]$. That is, $[g]$ is an equivalence class of neutral signature metrics with the equivalence relation $g \sim e^c g$ for a some function c on M.

Any neutral metric g on M can be put in the following form:

$$g = 2(\theta^{00'} \odot \theta^{11'} - \theta^{01'} \odot \theta^{10'}) = \epsilon_{AB}\epsilon_{A'B'}\theta^{AA'} \otimes \theta^{BB'}, \tag{2.1}$$

where $\epsilon_{AB}, \epsilon_{A'B'}$ are antisymmetric matrices with $\epsilon_{01} = \epsilon_{0'1'} = 1$. The four (real) basis one-forms $\theta^{AA'}$ for $A = 0, 1$, $A' = 0, 1$ are called a tetrad. The algebraic dual vector basis is denoted $\mathbf{e}_{AA'}$, and is defined by $\theta^{AA'}(\mathbf{e}_{BB'}) = \delta^A_B \delta^{A'}_{B'}$. Any vector V at a point can be written $V^{AA'}\mathbf{e}_{AA'}$, and this exhibits an isomorpism

$$TM \cong S \otimes S', \tag{2.2}$$

where S, S' are two-dimensional real vector bundles known as the *unprimed spin bundle* and the *primed spin bundle* respectively. For a general manifold M there is a topological obstruction to (2.2) but we are working locally so it always holds.

Using a particular choice of tetrad, a section μ of S is denoted μ^A, $A = 0, 1$. Similarly ν_A is a section of S^*, $\kappa^{A'}$ a section of S'^* and $\tau_{A'}$ a section of S'^*, where *

denotes the dual of a bundle. The natural pairing $S\times S^*\to\mathbb{R}$ is given by $\mu^A\nu_A$, using the summation convention, and similarly for primed spinors. We sometimes use the notation $\mu^{A'}\nu_{A'}=\mu.\nu$. This product is not commutative, we have $\mu.\nu=-\nu.\mu$.

It follows from (2.1) that $g(V,V)=\det V^{AA'}$. If V is null, then this gives $V^{AA'}=\mu^A\nu^{A'}$. Abstractly, if V is null then $V=\mu\otimes\nu$ under the isomorphism (2.2), where μ and ν are sections of S and S' respectively.

The relation (2.1) can be written abstractly as

$$g=\epsilon\otimes\epsilon'$$

under the isomorphism (2.2). ϵ and ϵ' are symplectic structures on S and S'. These give isomorphisms $S\cong S^*$ and $S'\cong S'^*$ by $\mu\to\epsilon(\mu,.)$, for μ a section of S, and similarly for S'. Given a choice of tetrad, the spinors ϵ and ϵ' are written ϵ_{AB} and $\epsilon_{A'B'}$, where we drop the prime on the latter because no confusion can arise due to the indices. Note these are anti-symmetric in AB and $A'B'$. Then the isomorphism $S\cong S^*$ is given in the trivialization by $\mu^A\to\mu^B\epsilon_{BA}:=\mu_A$ and similarly for primed spinors.

There are useful isomorphisms

$$\Lambda^2_+\cong \mathrm{Sym}(S^*\otimes S^*),\quad \Lambda^2_-\cong \mathrm{Sym}(S'^*\otimes S'^*),\tag{2.3}$$

where Λ_+, Λ_- are the bundles of self-dual and anti-self-dual two-forms, using an appropriate choice of volume form for the Hodge-$*$ operator. In the local trivialization, the isomorphisms (2.3) are expressed by the following formula for a two-form F in spinors:

$$F_{ab}=F_{AA'BB'}=\phi_{A'B'}\epsilon_{AB}+\psi_{AB}\epsilon_{A'B'},$$

where $\phi_{A'B'}$, ψ_{AB} are symmetric. The $\phi_{A'B'}$ term is the self-dual component of F and the ψ_{AB} is the anti-self-dual component.

The vector bundles S, S' and their duals inherit connections from the Levi-Civita connection of TM (see Appendix A). These are the unique torsion-free connections defined so that the sections ϵ and ϵ' are covariantly constant. Then covariant differentiation on either side of (2.2) is consistent.

A primed spinor $\mu^{A'}$ at a point corresponds to a totally null self-dual two-plane spanned by $\mu^{A'}\mathbf{e}_{AA'}$, $A=1,2$, whilst an unprimed spinor corresponds to an anti-self-dual two-plane in a similar way. In twistor theory, these two-planes are called α-planes and β-planes respectively.

2.2 Null conformal Killing vectors in neutral signature

Suppose g is a neutral metric with a conformal Killing vector K. That is, K satisfies

$$\mathfrak{L}_K g=\eta g\tag{2.4}$$

for a function η, where $\mathfrak{L}$ is the Lie-derivative. Then $\mathfrak{L}_K(e^c g) = (K(e^c) + e^c\eta)g$, so K is a conformal Killing vector for the conformally rescaled metric, and we can refer to K as a conformal Killing vector for the conformal structure $[g]$.

Now suppose g has a null conformal Killing vector K. We shall show (Lemma 1) that M is foliated in two different ways, by self-dual and anti-self-dual surfaces, whose leaves intersect tangent to K. This is a property of the conformal structure $[g]$, since the Hodge-$*$ acting on 2-forms is conformally invariant.

The spinor form of the conformal Killing equation is:

$$\nabla_a K_b = \phi_{A'B'}\epsilon_{AB} + \psi_{AB}\epsilon_{A'B'} + \frac{1}{2}\eta\epsilon_{AB}\epsilon_{A'B'}, \tag{2.5}$$

where $\phi_{A'B'}, \psi_{AB}$ are the self-dual and anti-self dual parts of the 2-form $\nabla_{[a}K_{b]}$, and η is a function on M.

Since K is null, we have $K = \iota \otimes o$, where ι is a section of S and o a section of S'. Choosing a null tetrad, and a trivialization of S and S', we have $K^{AA'} = \iota^A o^{A'}$. These spinors are defined up to multiplication by a non-zero function α, since $K^{AA'} = \iota^A o^{A'} = (\alpha\iota^A)(o^{A'}/\alpha)$.

Lemma 1. *Let $K = \iota^A o^{A'}\mathbf{e}_{AA'}$ be a null conformal Killing vector. Then*

1. *The following algebraic identities hold:*

$$\iota^A\iota^B\psi_{AB} = 0, \tag{2.6}$$
$$o^{A'}o^{B'}\phi_{A'B'} = 0. \tag{2.7}$$

2. *ι^A and $o^{A'}$ satisfy*

$$\iota^A\iota^B\nabla_{BB'}\iota_A = 0, \tag{2.8}$$
$$o^{A'}o^{B'}\nabla_{BB'}o_{A'} = 0. \tag{2.9}$$

Remark. The equations (2.8), (2.9) are equivalent to the statement that the distributions spanned by $\iota^A\mathbf{e}_{AA'}$ and $o^{A'}\mathbf{e}_{AA'}$ are Frobenius integrable (see Appendix). Equations of this type are often called 'geodesic shear free' equations, since in the Lorentzian case they result in shear-free congruences of null geodesics.

Proof. Using $K_{AA'} = \iota_A o_{A'}$, the Killing equation (2.5) becomes

$$o_{A'}\nabla_{BB'}\iota_A + \iota_A\nabla_{BB'}o_{A'} = \phi_{A'B'}\epsilon_{AB} + \psi_{AB}\epsilon_{A'B'} + \frac{1}{2}\eta\epsilon_{AB}\epsilon_{A'B'}. \tag{2.10}$$

Contracting both sides with $\iota^A o^{A'}$ gives

$$0 = o^{A'}\iota_B\phi_{A'B'} + \iota^A o_{B'}\psi_{AB} + \frac{1}{2}\eta\iota_B o_{B'}.$$

Multiplying by ι^B and $o^{B'}$ respectively leads to (2.6) and (2.7). To get (2.8) and (2.9), multiply (2.10) by $\iota^A \iota^B$ and $o^{A'} o^{B'}$. □

We have found that M is foliated in two different ways by totally null surfaces. Those determined by $o^{A'}$ are self-dual and are called α-surfaces, and those determined by ι^A are anti-self-dual and are called β-surfaces. It is clear that the α-surfaces and β-surfaces of Lemma 1 intersect on integral curves of K. Denote the β-surface distribution by $\mathcal{D}_\beta$; this will be used later.

It also follows from the Killing equations and the fact that K is null that

$$K^b \nabla_b K_a = \frac{1}{2} \eta K_a.$$

Thus K is automatically geodesic, and if it is pure then its trajectories are parameterized by an affine parameter.

3 Projective structures

Let $(U, [\Gamma])$ be a local two dimensional real projective structure. That is, U is a local patch of $\mathbb{R}^2$, and $[\Gamma]$ is an equivalence class of torsion-free connections whose unparameterized geodesics are the same. Then in a local trivialization, equivalent torsion-free connections are related in the following way:

$$\tilde{\Gamma}^i_{jk} - \Gamma^i_{jk} = a_j \delta^i_k + a_k \delta^i_j, \tag{3.1}$$

for functions a_i on U, and $i, j, k = 1, 2$. Note that this is a tensor equation since the difference between two connections is a tensor. The a_i on the RHS are the components of a one-form.

The geodesics satisfy the following ODE:

$$\frac{d^2 s^i}{dt^2} + \Gamma^i_{jk} \frac{ds^j}{dt} \frac{ds^k}{dt} = v \frac{ds^i}{dt},$$

where s^i are local coordinates of U, and t is a parameter, which is called affine if $v = 0$.

One can associate a second-order ODE to a projective structure by picking a connection in the equivalence class, choosing local coordinates $s^i = (x, y)$ say, and eliminating the parameter from the geodesic equations. The resulting equation determines the geodesics in terms of the local coordinates, without the parameter. The equation is as follows:

$$\frac{d^2 y}{dx^2} = \Gamma^x_{yy} \left(\frac{dy}{dx}\right)^3 + (2\Gamma^x_{xy} - \Gamma^y_{yy}) \left(\frac{dy}{dx}\right)^2 + (\Gamma^x_{xx} - 2\Gamma^x_{xy}) \frac{dy}{dx} - \Gamma^y_{xx}.$$

A general projective structure is therefore defined by a second-order ODE (1.4). In fact, two of the four functions A_0, A_1, A_2, A_3 can be eliminated by a choice of coordinates $(x, y) \to (\hat{x}(x, y), \hat{y}(x, y))$; we make use of this freedom in the text.

On TU, the horizontal lifts of $\partial/\partial s^i$ are defined by

$$S_i = \frac{\partial}{\partial s^i} - \Gamma^j_{ik}\mu^k \frac{\partial}{\partial \mu^j}.$$

The geodesics on U lift to integral curves of the following spray on TU:

$$\Theta = \mu^i S_i = \mu^i \frac{\partial}{\partial s^i} - \Gamma^i_{jk}\mu^j \mu^k \frac{\partial}{\partial \mu^i}, \tag{3.2}$$

where μ^i are the fibre coordinates of TU. Now Θ is homogeneous of degree 1 in the μ^i, so it projects to a section of a one dimensional distribution on PTU. PTU is the quotient of $TU - \{0\text{-section}\}$ by the vector field $\mu^i \frac{\partial}{\partial \mu^i}$. If λ is a standard coordinate on one patch of the $\mathbb{RP}^1$ factor [1], then the spray has the form

$$\Theta = \partial_x + \lambda \partial_y + (A_0(x, y) + \lambda A_1(x, y) + \lambda^2 A_2(x, y) + \lambda^3 A_3(x, y))\partial_\lambda. \tag{3.3}$$

There is a unique curve in any direction through a point in U, which is why the curves can be lifted to a foliation of the projective tangent bundle $U \times \mathbb{RP}^1$.

To obtain (3.1) we argue as follows. If $\tilde{\Theta}$ is the spray corresponding to a different connection $\tilde{\Gamma}$, then Γ and $\tilde{\Gamma}$ are in the same projective class if Θ and $\tilde{\Theta}$ push down to the same spray on PTU. This gives

$$\Theta - \tilde{\Theta} \propto \mu^i \frac{\partial}{\partial \mu^i},$$

from which (3.1) follows, using the fact that the connections are torsion-free (i.e. symmetric in their lower indices).

3.1 The twistor space of a projective structure

Now suppose we have a *holomorphic* projective structure on a local patch of $\mathbb{C}^2$, which we still denote U. All of the above is still valid, with real coordinates replaced by complex ones. The functions Γ^i_{jk} are now required to be holomorphic functions of the coordinates. Given a real-analytic projective structure, one can complexify by analytic continuation to obtain a holomorphic projective structure that will come equipped with a *reality structure* (see below).

[1] By standard coordinates $\lambda, \tilde{\lambda}$ on $\mathbb{RP}^1$ or $\mathbb{CP}^1$, we mean the usual coordinates a/b and b/a, where a, b are homogeneous coordinates.

The space PTU is obtained from TU on quotienting by $\mu^i \frac{\partial}{\partial \mu^i}$, which defines a tautological line bundle $\mathcal{O}(-1)$ over PTU.

As the S_i are weight zero in the μ^i coordinates, they push down to vector fields on PTU, giving a two-dimensional distribution $\mathcal{S}$. Since Θ is weight one in the μ^i, one must divide by a homogeneous polynomial of degree one in the μ^i to get something that pushes down to a vector field on PTU. The resulting vector field will have a singularity at a single point on each fibre, where the degree one polynomial vanishes. Different choices of polynomial will result in different vector fields on PTU, but they will always be in the same direction. In other words, Θ defines a one dimensional distribution which we shall call D_Θ. Restricting to a $\mathbb{CP}^1$ fibre, D_Θ defines a line bundle over $\mathbb{CP}^1$. A section of this line bundle corresponds to a vector field in D_Θ, i.e. a choice of polynomial as described above, and has a pole at a single point. Therefore by the classification of holomorphic line bundles over $\mathbb{CP}^1$, it must be $\mathcal{O}(-1)$ †.

Restricting to a $\mathbb{CP}^1$ fibre, one obtains the following exact sequence of vector bundles over $\mathbb{CP}^1$

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{S}/D_\Theta \to 0, \tag{3.4}$$

where the first bundle is D_Θ, the second is $\mathcal{S}$, and the last is the quotient. In fact, the quotient is $\mathcal{O}(1)$, for the following reason. Consider for instance the push down of S_0 to PTU. This defines a subbundle of $\mathcal{S}$ that is different to Θ everywhere except at a single point, the image of $\mu^1 = 0$. Hence it determines a section of $\mathcal{S}/\Theta$ which vanishes at a single point. Therefore, again using the classification of holomorphic line bundles over $\mathbb{CP}^1$, we have $\mathcal{S}/D_\Theta \cong \mathcal{O}(1)$.

The twistor space Z is the two dimensional quotient of PTU by D_Θ. A point $u \in U$ corresponds to a twistor line $\hat{u} \subset Z$ corresponding to all the geodesics through u. The normal bundle of an embedded $\hat{u} = \mathbb{CP}^1 \subset Z$ is given by the quotient bundle in the above sequence, i.e. $\mathcal{O}(1)$. This is summarized by a double fibration picture:

$$\begin{array}{ccc} & U \times \mathbb{CP}^1 & \\ \swarrow & & \searrow \\ U & & Z \end{array}$$

The left arrow denotes projection to U, and the right arrow denotes the quotient by D_Θ.

The converse is also valid:

† Coordinatize $\mathbb{CP}^1$ using two patches, $\mathcal{U}_0$ with coordinate $\lambda \in \mathbb{C}$, and $\mathcal{U}_1$ with coordinate $\eta \in \mathbb{C}$, and transition function $\lambda = 1/\eta$. The holomorphic line bundle $\mathcal{O}(n)$ over $\mathbb{CP}^1$ is defined by the transition function $a = \lambda^n b$ where $a \in \mathbb{C}$ is the fibre coordinate over $\mathcal{U}_0$ and $b(\eta) \in \mathbb{C}$ is the fibre coordinate over $\mathcal{U}_1$. The Birkhoff-Grothendieck theorem states that any holomorphic line bundle over $\mathbb{CP}^1$ is $\mathcal{O}(n)$ for some n. A global section of $\mathcal{O}(n)$ has $|n|$ zeroes or poles, for n positive or negative respectively.

Theorem. *Hitchin, LeBrun [10, 15] There is a 1-1 correspondence between local two dimensional holomorphic projective structures and complex surfaces containing an embedded* $\mathbb{CP}^1$ *with normal bundle* $\mathcal{O}(1)$.

A vector $V \in T_uU$ corresponds to a global section of the normal bundle $\mathcal{O}(1)$ of $\hat{u}$. Such a section vanishes at a single point $p \in Z$. The geodesic of the projective structure through this direction is given by points in U corresponding to twistor lines in Z that intersect $\hat{u}$ at p. That there is a one-parameter family of such lines can be shown by blowing up Z at the vanishing point and using Kodaira theory, see [10].

3.2 Flatness of projective structures

A projective structure is said to be *flat* if the corresponding second order ODE (1.4) can be transformed to the trivial ODE

$$\frac{d^2y}{dx^2} = 0 \tag{3.5}$$

by coordinate transformation $(x, y) \to (\hat{x}(x, y), \hat{y}(x, y))$. The terminology comes from the fact that given any second order ODE one can construct a Cartan connection on a certain G-structure [2], and when this connection is flat the equation can be transformed to the trivial ODE (3.5). It turns out that a second order ODE must be of the form (1.4) to be flat, and in addition the functions A_0, A_1, A_2, A_3 must satisfy some PDEs. Defining

$$F(x, y, \lambda) = A_0(x, y) + \lambda A_1(x, y) + \lambda^2 A_2(x, y) + \lambda^3 A_3(x, y),$$

the following must hold [2]:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{11} - 4\frac{\mathrm{d}}{\mathrm{d}x}F_{01} - F_1\frac{\mathrm{d}}{\mathrm{d}x}F_{11} + 4F_1F_{01} - 3F_0F_{11} + 6F_{00} = 0, \tag{3.6}$$

where

$$F_0 = \frac{\partial F}{\partial y}, \qquad F_1 = \frac{\partial F}{\partial \lambda}, \qquad \frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y} + F\frac{\partial}{\partial \lambda}$$

This is a set of PDEs for the functions A_0, A_1, A_2, A_3.

3.3 Reality conditions for projective structures

A reality structure for Z is an anti-holomorphic involution that leaves invariant a two real parameter family of twistor lines, and fixes an equator of each line. Given a line in this real family, all the sections pointing to nearby lines in the real family have a zero at some point, and the union of these points gives an equator of the line; this equator must be fixed by the reality structure. The real family of twistor lines then corresponds to a real manifold U with a projective structure.

In this paper all holomorphic projective structure have reality structures since they occur as complexifications of real projective structures.

4 Null Killing Vectors and Twistor Space

4.1 The twistor space of an ASD conformal structure

In the following and for the rest of the paper, $\tilde{\mathbf{e}}_{AA'}$ denote the horizontal lifts of $\mathbf{e}_{AA'}$ to S', or their push-down to PS'. Abstractly, the integral curves of these horizontal lifts define parallely transported primed spinors using the connection on S' (see Appendix A).

We can abstractly define the two-dimensional *twistor distribution* on S' as follows. A point $s \in S'$ is determined by a primed spinor π at a point $x \in M$. The null vectors $\pi \otimes \mu$ for all unprimed spinors μ span an α-plane at x. Define the twistor distribution at s to be the subspace of horizontal vectors at s whose push-down to the base lies in this α-plane.

Concretely, the twistor distribution is spanned by vectors L_A ($A = 0, 1$) on S', defined with a choice of tetrad by

$$L_A = \pi^{A'} \tilde{\mathbf{e}}_{AA'} = \pi^{A'} \big(\mathbf{e}_{AA'} - \Gamma_{AA'B'}{}^{C'} \pi^{B'} \frac{\partial}{\partial \pi^{C'}} \big), \tag{4.1}$$

where $\pi^{A'}$ are the local coordinates on the fibres of S'. In the Appendix it is shown that the twistor distribution is integrable for ASD conformal structures, which is a seminal result of Penrose [19]. In other words, given a neutral ASD conformal structure $[g]$, each self-dual two plane at a point is tangent to a unique α-surface through that point, which is the push down of a leaf of the twistor distribution. In the holomorphic case, the space of leaves of the twistor distribution (locally, over a suitably convex region of the base), is a three dimensional complex manifold $\mathcal{PT}$ called the twistor space [19, 10].

The double fibration picture is very similar to the projective structure case discussed in Section 3.1. The projective primed spin bundle PS' is the quotient of S' by the vector field $\pi^{A'} \frac{\partial}{\partial \pi^{A'}}$. PS' is fundamental in the fibration picture, as each α-surfacein M has a unique lift, in the same way that each geodesic of a projective structure has a unique lift to the projective tangent bundle. The horizontal vectors $\tilde{\mathbf{e}}_{AA'}$ are weight zero in the $\pi^{A'}$ coordinates, so push down to vector fields on PS', giving a four-dimensional distribution Ξ on PS'. The L_A vectors (4.1) are weight one, so together define a two dimensional subdistribution $\mathcal{L}$ of Ξ, which restricts to

$\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ on a $\mathbb{CP}^1$ fibre; we also refer to this as the twistor distribution. Over a $\mathbb{CP}^1$ fibre, there is an exact sequence

$$0 \to \mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathcal{O}\otimes\mathbb{C}^4 \to \Xi/\mathcal{L} \to 0. \tag{4.2}$$

The first term is $\mathcal{L}$, the second is Ξ. As in the projective structure case, one can show that $\Xi/\mathcal{L}$ is $\mathcal{O}(1)\oplus\mathcal{O}(1)$. The twistor space $\mathcal{PT}$ is the quotient of PS' by $\mathcal{L}$. The image of a $\mathbb{CP}^1$ fibre over $x \in M$ is an embedded $\mathbb{CP}^1 \in \mathcal{PT}$, and has normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$, the quotient bundle in (4.2). It corresponds to all the α-surfaces through x.

The twistor correspondence is summarized by the double fibration:

$$\begin{array}{ccc} & PS' & \\ \swarrow & & \searrow \\ M & & \mathcal{PT} \end{array}$$

Here the left arrow denotes projection to M, and the right arrow denotes the quotient by $\mathcal{L}$.

Again, there is a converse:

Theorem. *(Penrose [19]) There is a 1-1 correspondence between local four dimensional holomorphic ASD conformal structures* $(M,[g])$ *and three dimensional complex manifolds* $\mathcal{PT}$ *with an embedded* $\mathbb{CP}^1 \subset \mathcal{PT}$*, with normal bundle* $\mathcal{O}(1)\oplus\mathcal{O}(1)$.

The essential fact is that an embedded $\mathbb{CP}^1$ with the above normal bundle belongs to a family of embedded $\mathbb{CP}^1$s parameterized by a complex 4-manifold M. Vectors at $x \in M$ correspond to sections of the normal bundle of $\hat{x}$, and null vectors are given by sections with a zero. This defines a conformal structure, because a global section of $\mathcal{O}(1)\oplus\mathcal{O}(1)$ is given by $(a\lambda+b, c\lambda+d)$ for affine coordinate $\lambda \in \mathbb{C}$, $(a,b,c,d) \in \mathbb{C}^4$, and this can only be $(0,0)$ when $ad-bc=0$, which is a quadratic condition. In this case there is a zero at a single point. The conformal structure is anti-self-dual, with α-surfaces defined by families of twistor lines through a fixed point in $\mathcal{PT}$.

In this picture, the α-surfaces are obtained as follows. Let $\hat{x} \subset \mathcal{PT}$ be the twistor line corresponding to a point $x \in M$. Let $V \in T_xM$ be a null vector. We want to show that V lies in a unique α-surface through x. The corresponding section of the normal bundle of $\hat{x}$ has a zero at some point $p \in \mathcal{PT}$ because V is null. The α-surface corresponds to all the twistor lines that intersect $\hat{x}$ at p. There is a two-parameter family of such lines. It is easy to see that there is a two-parameter family of sections that vanish at p. To show that these are tangent to a two-parameter family of lines one must blow-up $\mathcal{PT}$ at p and use Kodaira theory; see [10] for details.

4.1.1 Reality conditions for split signature

In order to obtain a real split signature metric from a twistor space, we must be able to distinguish a four *real* parameter family of twistor lines, whose parameter space will be the four real dimensional manifold. In addition we require that given a line in this real family, the sections of the normal bundle that point to others in the family inherit a split signature conformal structure. As described above, a section of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is defined by four complex numbers (a, b, c, d), with a quadratic form defined by $ad - bc$. If we restrict (a, b, c, d) to be real we obtain a real split signature conformal structure. The sections tangent to the real family are of this type.

The zero of such sections occurs for real λ, that is, on an equator of $\mathbb{CP}^1$. The conformal structure is thus invariant under an anti-holomorphic involution of the $\mathbb{CP}^1$ that has the equator fixed. A *split signature real structure* on $\mathcal{PT}$ is an anti-holomorphic involution that leaves invariant a four real parameter family of twistor lines, and when restricted to one of these fixes an equator.

Not all holomorphic metrics have real structures, but all the holomorphic metrics in this paper have obvious real 'slices' because they are complexifications of real metrics, obtained by letting the real coordinates be complex.

4.2 Lift of K to PS'

Now given a null conformal Killing vector for an ASD conformal structure, the fact that M is foliated by α-surfaces (Lemma 1) is not very illuminating, since they must already exist by anti-self-duality. The foliation by β-surfaces is more interesting, since these do not exist generically.

In this section we will prove that in the analytic case, the space of β-surfaces inherits a natural projective structure. We then explain how this arises geometrically, due to the presence of α-surfaces ensured by anti-self-duality.

Let K be a null conformal Killing vector for $(M, [g])$. We assume K is without fixed points, which can always be arranged by restricting M to a suitable open set.

Since K preserves the conformal structure, the corresponding diffeomorphism maps α-surfaces to α-surfaces, and hence it induces a vector field $\mathcal{K}$ on $\mathcal{PT}$. We now translate this fact into a statement on the projective primed spin bundle PS'. Each α-surface has a unique lift and these lifts foliate PS'. The following proposition shows how to lift K to PS', giving a vector field that is Lie-derived along the lifts of the α-surfaces.

Proposition 1. *Let $K = K^{AA'}\mathbf{e}_{AA'}$ be a conformal Killing vector for an ASD metric g. Define a vector field $\tilde{K}$ on S' by*

$$\tilde{K} := K^{AA'}\tilde{\mathbf{e}}_{AA'} + \pi_{A'}\phi^{A'B'}\frac{\partial}{\partial \pi^{B'}} + \frac{1}{2}\eta\pi^{A'}\frac{\partial}{\partial \pi^{A'}}. \tag{4.3}$$

Then this satisfies

$$[\tilde{K}, L_A] = (K^{BB'}\Gamma_{BB'A}{}^{D} - \psi_A{}^{D})L_D + \frac{3}{4}(\mathbf{e}_{AB'}\eta)\pi^{B'}\pi^{C'}\frac{\partial}{\partial \pi^{C'}}. \qquad (4.4)$$

Proof. See Appendix. □

Remark. Since $\tilde{K}$ is weight zero in the $\pi^{A'}$ coordinates, it defines a vector field on PS', which we will also refer to as $\tilde{K}$ by abuse of notation. The last term on the right hand side of (4.4) is proportional to the Euler vector field, so does not contribute to $\tilde{K}$ on PS'. Hence (4.4) shows that $\tilde{K}$ commutes with the twistor distribution $\mathcal{L}$ on PS'. The vector field $\mathcal{K}$ on $\mathcal{PT}$ is the push-forward of $\tilde{K}$ to $\mathcal{PT}$, which is well defined because $\tilde{K}$ is Lie-derived along $\mathcal{L}$.

4.3 Projective structure from a quotient

In this section we assume that $[g]$ is analytic, so we can complexify by analytic continuation. Thus we are now working on a local patch of $\mathbb{C}^4$, with a holomorphic conformal structure. We assume that we have restricted to a suitable open set on the base so that all the spaces of leaves involved are non-singular complex manifolds.

As in Section 2, write $K = \iota^A o^{A'}\mathbf{e}_{AA'}$, where now $\mathbf{e}_{AA'}$ is a holomorphic tetrad and ι^A, $o^{A'}$ are complex spinor fields that vary holomorphically.

When K is null, it is easy to see that $\mathcal{K}$, the induced vector field on twistor space $\mathcal{PT}$, will vanish on a hypersurface $\mathcal{H}$ in $\mathcal{PT}$, because K fixes a two-parameter family of α-surfaces, which are those to which it is tangent. These are the 'special' α-surfaces of Lemma 1. We now explain how this is seen from the lift $\tilde{K}$ to PS'.

On S', $\tilde{K}$ from Proposition 1 is given by:

$$\tilde{K} = \iota^A o^{A'}\tilde{\mathbf{e}}_{AA'} + \pi_{A'}\phi^{A'B'}\frac{\partial}{\partial \pi^{B'}} + \frac{1}{2}\eta\pi^{A'}\frac{\partial}{\partial \pi^{A'}}.$$

Now when $\pi^{A'} \propto o^{A'}$, one has $\pi_{A'}\phi^{A'B'} \propto o^{B'}$ from (2.6), so the second term on the RHS is proportional to the Euler vector field $\Upsilon = \pi^{A'}\frac{\partial}{\partial \pi^{A'}}$. The last term is everywhere proportional to the Euler vector field. To go from S' to PS' one quotients $S' - \{0\text{-section}\}$ by the integral curves of Υ. So we have shown that on the section $[\pi^{A'}] = [o^{A'}]$ of PS', $\tilde{K}$ is the push down of $\iota^A o^{A'}\tilde{\mathbf{e}}_{AA'}$ only. But this is in $\mathcal{L}$, so $\tilde{K}$ pushes down to the zero vector under the quotient of PS' by $\mathcal{L}$.

So there is a (complex) hypersurface in PS', defined by the section $[\pi^{A'}] = [o^{A'}]$, on which $\tilde{K}$ lies in the twistor distribution. One can also define this hypersurface as the image in PS' of the hypersurface $\pi.o = 0$ in S', under the quotient by Υ. We will refer to this hypersurface as H. It is easy to see by pushing down to the base that $\tilde{K}$ is linearly independent of the twistor distribution everywhere else on PS'.

Define a vector field

$$V = \iota^A L_A = \iota^A \pi^{A'} \tilde{\mathbf{e}}_{AA'}$$

on S'. This is weight one in the $\pi^{A'}$ coordinates, so gives a one dimensional distribution on PS' which restricts to $\mathcal{O}(-1)$ on fibres. Together with span$\{\tilde{K}\}$, we get a two dimensional distribution on $PS' - H$. On H, the distribution drops its rank from two to one.

The two dimensional distribution defined by $\{V, \tilde{K}\}$ on $PS' - H$ pushes down to the β-plane distribution $\mathcal{D}_\beta$ on the base.

Lemma 2. *The two dimensional distribution on $PS' - H$ determined by $\{V, \tilde{K}\}$ is integrable.*

Proof. We work on S' for convenience, and push down to PS' at the end. The distribution span$\{\tilde{K}, V\}$ on S' is two dimensional on S' when $\pi^{A'} o_{A'} \neq 0$. Multiples of the Euler field Υ are therefore irrelevant.

$$\begin{aligned}
[V, \tilde{K}] &= [\tilde{K}, \iota^C L_C] \\
&= \iota^C [\tilde{K}, L_C] + \tilde{K}(\iota^B) L_B \\
&= \iota^C ((K^{BB'} \Gamma_{BB'C}{}^D - \psi_C{}^D) L_D + \frac{3}{4}(\mathbf{e}_{CB'}\eta)\pi^{B'}\pi^{C'} \frac{\partial}{\partial \pi^{C'}}) \\
&\quad + K^{BB'} \mathbf{e}_{BB'}(\iota^C) L_C \\
&= (K^{BB'} \nabla_{BB'} \iota^C - \iota^D \psi_D{}^C) L_C + \#\Upsilon \\
&= (\iota^B o^{B'} \nabla_{BB'} \iota^C - \iota^D \psi_D{}^C) L_C + \#\Upsilon.
\end{aligned}$$

From (2.6) we have $\iota^D \psi_D^C \propto \iota^C$, and from (2.8) we have $\iota^B o^{B'} \nabla_{BB'} \iota^C \propto \iota^C$. Hence the RHS is proportional to V, ignoring the irrelevant Euler vector field part. □

Next we will show that it is possible to continue this distribution over the hypersurface H so it is rank two on the whole of PS', and that the resulting distribution commutes on the hypersurface. It will then be possible to quotient PS' by the leaves of this distribution.

Lemma 3. *There is a two-dimensional integrable distribution $\mathcal{D}$ over PS', which on $PS' - H$ is determined by $\{\tilde{K}, V\}$. Let ϱ be the projection $PS' \to M$. Then for every $p \in PS'$, we have $\varrho_*(\mathcal{D}\mid_p) = \mathcal{D}_\beta$.*

Remark. Intuitively one can think of $\mathcal{D}$ as a lift of the β-surfaces to PS', where each β-surface has a $\mathbb{CP}^1$ of lifts.

Proof. Choose a spinor $\iota^{A'}$ satisfying $o^{A'} \iota_{A'} = 1$. Define the following (singular) vector field on S':

$$W = \frac{1}{\pi^{C'} o_{C'}} (V - (\pi^{D'} \iota_{D'}) \tilde{K}). \tag{4.5}$$

This is weight zero in the $\pi^{A'}$, so defines a vector field on PS' by push-forward, which we shall also call W. We will now show that W is well defined even over $H \subset PS'$, despite the $1/(\pi^{C'}o_{C'})$ factor in (4.5).

Without loss of generality, choose a tetrad such that

$$K = \iota^A o^{A'} \mathbf{e}_{AA'} = \mathbf{e}_{00'}.$$

That is, $\iota^A = (1,0)$, $o^{A'} = (1,0)$. Define $\lambda = \pi^{1'}/\pi^{0'}$ to be the coordinate on the $\pi^{0'} \neq 0$ patch of $\mathbb{CP}^1$, and extend this to a patch of PS'; we call the patch $\mathcal{U}$. Then H lies entirely within $\mathcal{U}$ at $\lambda = 0$. We have the following expression for $\tilde{K}$, obtained by 'projectivizing' (4.3):

$$\begin{aligned} \tilde{K} &= \tilde{\mathbf{e}}_{00'} + ({\phi_{0'}}^{1'} + \lambda({\phi_{1'}}^{1'} - {\phi_{0'}}^{0'}) + \lambda^2 {\phi_{1'}}^{0'})\frac{\partial}{\partial \lambda} \\ &= \tilde{\mathbf{e}}_{00'} + (\lambda({\phi_{1'}}^{1'} - {\phi_{0'}}^{0'}) + \lambda^2 {\phi_{1'}}^{0'})\frac{\partial}{\partial \lambda}, \end{aligned}$$

where ${\phi_{0'}}^{1'} = 0$ due to (2.7).

In the above conventions, we have $V = \pi^{A'} \tilde{\mathbf{e}}_{0A'}$. On $\mathcal{U} \subset PS'$, the push forward of $\frac{1}{\pi^{C'}o_{C'}} V$ is

$$\frac{1}{\lambda}\tilde{\mathbf{e}}_{00'} + \tilde{\mathbf{e}}_{01'},$$

which is singular at H, corresponding to $\lambda = 0$. Choosing $\iota^{A'} = (0,-1)$, we then obtain the following expression for W on $\mathcal{U}$:

$$W = \frac{1}{\lambda}\tilde{\mathbf{e}}_{00'} + \tilde{\mathbf{e}}_{01'} - \frac{1}{\lambda}\tilde{K} = \tilde{\mathbf{e}}_{01'} - (({\phi_{1'}}^{1'} - {\phi_{0'}}^{0'}) + \lambda {\phi_{1'}}^{0'}))\frac{\partial}{\partial \lambda}.$$

This is a *non-singular* vector field on $\mathcal{U}$. By construction, away from H this lies in span$\{\tilde{K}, \tilde{V}\}$. Define $\mathcal{D}$ on $\mathcal{U}$ to be span$\{\tilde{K}, W\}$. This is clearly non-degenerate everywhere on $\mathcal{U}$. Note that W is also well defined over the other patch (i.e. at $\lambda = \infty$) so we can define $\mathcal{D}$ as span$\{\tilde{K}, W\}$ over the whole of PS'.

We now want to show that $\mathcal{D}$ is integrable over H. We know (Lemma 2) that $\mathcal{D}$ is integrable away from H. Therefore on $\mathcal{U}$ we have

$$[\tilde{K}, W] = f\tilde{K} + gW + Y,$$

where f, g are holomorphic functions on $\mathcal{U}$ and Y is a holomorphic vector field vanishing on $\mathcal{U} - H$. But such a vector field must vanish, otherwise it is not even continuous, so is certainly not holomorphic.

The last part of the lemma is obvious, just from inspecting the coordinate expressions of $\tilde{K}$, W. □

We now have a three dimensional integrable distribution $\mathcal{L} \cup \mathcal{D}$. It is three dimensional because at each point $\mathcal{L}$ and $\mathcal{D}$ have a direction in common, which is the one-dimensional distribution defined on PS' by the push-forward of V on S'. From Lemma 3, $\mathcal{D}$ is an integrable subdistribution. Note that $\mathcal{D}$ consists of a $\mathbb{CP}^1$ of lifts of each β-surface in the base. If we pick a suitably convex set on the base so that the space of β-surfaces U intersecting it is a Hausdorff complex manifold, then the quotient $PS'/\mathcal{D}$ will also be a Hausdorff complex manifold. A point in this quotient is a point in U together with a choice of lift.

In fact we can canonically identify $PS'/\mathcal{D}$ with PTU, the projective tangent bundle of U, as follows. Using the conventions of Lemma 3, the tangent planes to the β-surfaces in the base are spanned at each point by $\mathbf{e}_{00'}$, $\mathbf{e}_{01'}$. Now L_1 has the form $\mathbf{e}_{10'} + \lambda \mathbf{e}_{11'} + (\ldots)\partial_\lambda$, so at each point in the fibre above a point $x \in M$, L_1 pushes down to a different null direction transverse to the β-plane at x. Now suppose we take a lift of a β-surface Π, i.e. a leaf of $\mathcal{D}$ that projects down to Π. Push down L_1 at each point over this lift. This will give a vector field $\Theta = \mathbf{e}_{10'} + \lambda \mathbf{e}_{11'}$ over Π, where λ is now a function on the M.

We want to show that this determines a projective vector at the point $s \in U$ corresponding to S. This means we require $[\mathbf{e}_{00'}, \Theta] \propto \Theta \bmod\{\mathbf{e}_{00'}, \mathbf{e}_{01'}\}$, $[\mathbf{e}_{01'}, \Theta] \propto \Theta \bmod\{\mathbf{e}_{00'}, \mathbf{e}_{01'}\}$. But it is easy to show that this is satisfied, using the fact that the distribution spanned by $\tilde{K}, W, L_1$ commutes. Hence to determine the projective vector corresponding to a leaf of $\mathcal{D}$, just choose a point on the leaf and push down L_1. Because of the above considerations, this direction will be independent of the choice of point on the leaf.

Proof of Theorem 1. Define Z as the quotient of PS' by $\mathcal{L} \cup \mathcal{D}$. Equivalently, this is the quotient of $\mathcal{PT}$ by a one-dimensional distribution which on $\mathcal{PT} - \mathcal{H}$ is span$\{\mathcal{K}\}$. The image of a $\mathbb{CP}^1$ fibre of PS' under the quotient is a twistor line in Z.

On a $\mathbb{CP}^1$ fibre, the horizontal part of $\mathcal{D}$ defines a subbundle $\mathcal{O} \otimes \mathbb{C}^2$ of the horizontal distribution $\Xi = \mathcal{O} \otimes \mathbb{C}^4$, corresponding to the horizontal parts of the vectors $\tilde{K}$ and W. Choosing a spinor o^A such that $\iota^A o_A = 1$, we can form the vector field $o^A L_A$ on S', which pushes down to a horizontal distribution on PS' that is always linearly independent of $\mathcal{D}$. Since the L_A are weight one, this is $\mathcal{O}(-1)$ when restricted to a $\mathbb{CP}^1$ fibre. Because $\mathcal{L} \cup \mathcal{D}$ is integrable (Lemma 3), this distribution determines a one dimensional distribution D_Θ on $PTU = PS'/\mathcal{D}$. The spray Θ of a projective structure is a section of $\mathcal{D}_\Theta \otimes \mathcal{O}(1)$ where here $\mathcal{O}(1)$ is dual to the tautological line bundle over PTU. The situation is described by the following commuting diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}(-1) \oplus \mathcal{O}(-1) & \to & \mathcal{O} \otimes \mathbb{C}^4 & \to & \mathcal{O}(1) \oplus \mathcal{O}(1) & \to & 0 \\
 & & \downarrow_V & & \downarrow_{\mathcal{D}} & & \downarrow & & \\
0 & \to & \mathcal{O}(-1) & \to & \mathcal{O} \otimes \mathbb{C}^2 & \to & \mathcal{O}(1) & \to & 0
\end{array}$$

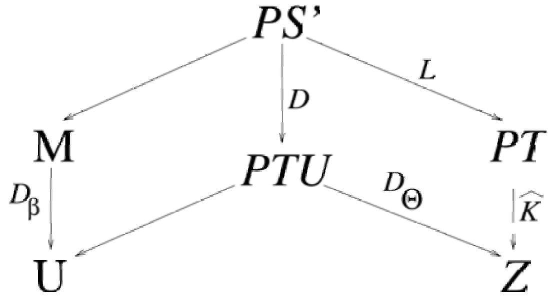


Figure 1: Relationship between foliation spaces.

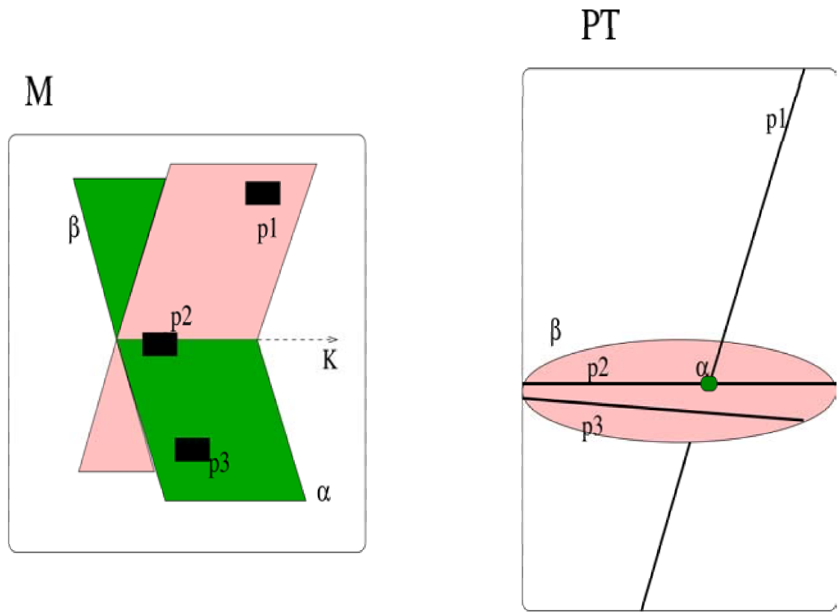
where these are bundles over a $\mathbb{CP}^1$ fibre of PS'. The vector field $o^A L_A$ on S' constructed above corresponds to the $\mathcal{O}(-1)$ in the bottom row after quotienting by V, and gives the projective structure spray. The bottom row is the sequence (3.4) on $PTU = PS'/\mathcal{D}$, where U is the space of β-surfaces in M. Thus there is a projective structure on U. □

Remark. The real space of β-surfaces has a system of curves that comes from the quotienting operations described above but with real spaces instead of complex. These real curves are described by the holomorphic projective structure with a reality structure.

Figure 1 illustrates the situation. Here p and q are the obvious projections. $\mathcal{D}_\beta$ represents the β-surface distribution on M. The $\widehat{\mathcal{K}}$ labelling the map from $\mathcal{PT}$ to Z requires some of explanation. The vector field $\tilde{K}$ over PS' commutes with the twistor distribution (Lemma 1), so determines a vector field $\mathcal{K}$ on $\mathcal{PT}$. This vector field vanishes on a hypersurface $\mathcal{H} \subset \mathcal{PT}$, corresponding to the α-surfaces to which K is tangent; these are the α-surfaces appearing in Lemma 1. Now $\mathcal{K}$ on $\mathcal{PT}$ only depends on $\tilde{K}$ modulo $\mathcal{L}$. Lemma 3 shows that we can multiply $\tilde{K}$ modulo $\mathcal{L}$ by a meromorphic function $(1/\lambda)$ and obtain a vector field W commuting with the twistor distribution. This means that there is a one-dimensional distribution $\widehat{\mathcal{K}}$ over the whole of $\mathcal{PT}$ that never degenerates, and which agrees with span $\{\mathcal{K}\}$ on $\mathcal{PT} - \mathcal{H}$. The quotient of $\mathcal{PT}$ by this distribution gives Z, as illustrated in the diagram.

One can rephrase this in terms of divisor line bundles. That is, there is a holomorphic line bundle $\mathcal{E}$ over $\mathcal{PT}$ defined by the property that it has a meromorphic section ζ with a pole of order one on $\mathcal{H}$. Then $\zeta \otimes \mathcal{K}$ defines a non-vanishing section of $\mathcal{E} \otimes T\mathcal{PT}$. This is equivalent to the one dimensional distribution $\widehat{\mathcal{K}}$ over $\mathcal{PT}$ described above. To obtain the distribution one simply finds trivializations of $\mathcal{E}$ and $T\mathcal{PT}$ over a patch, and expresses ζ in this trivialization. Its direction will be independent of the trivialization of $\mathcal{E}$, and defines the distribution over the patch.

Figure 2: The α and β surfaces in M intesect along a trajectory of K which is a null geodesic. This corresponds to a point α lying on a surface β in $\mathcal{PT}$. Points p_1, p_2, p_3 in M correspond to projective lines in $\mathcal{PT}$.



4.4 Relationship of the twistor spaces

Here we discuss the relationship between the twistor spaces without the foliation space picture. Incidence relation between various objects in M and $\mathcal{PT}$ is represented by (Fig. 2).

First one must understand what a β-surface corresponds to in $\mathcal{PT}$. The answer is a two-parameter family of twistor lines, each of which intersects any other at a single point. This is because all points on a β-surface are null separated. However, unlike the case of an α-surface, there is not just a single point of intersection of the whole family. To construct the family, pick a point on the β-surface, say x. Then $\hat{x}$ is a twistor line in $\mathcal{PT}$. Now $\mathcal{K}$ determines a section of the normal bundle with a zero. Twistor lines intersecting $\hat{x}$ at this zero are on the β-surface, and correspond to those along the trajectory of K through x. In fact this is a null geodesic, since null Killing vector fields have geodesic integral curves. Now pick another section of the normal bundle with a zero at a different point, such that all linear combinations of this with the section determined by $\mathcal{K}$ also have a zero. The resulting two dimensional distribution in M at x is a β-plane. Doing this for each $x \in M$ gives a β-plane distribution which is integrable.

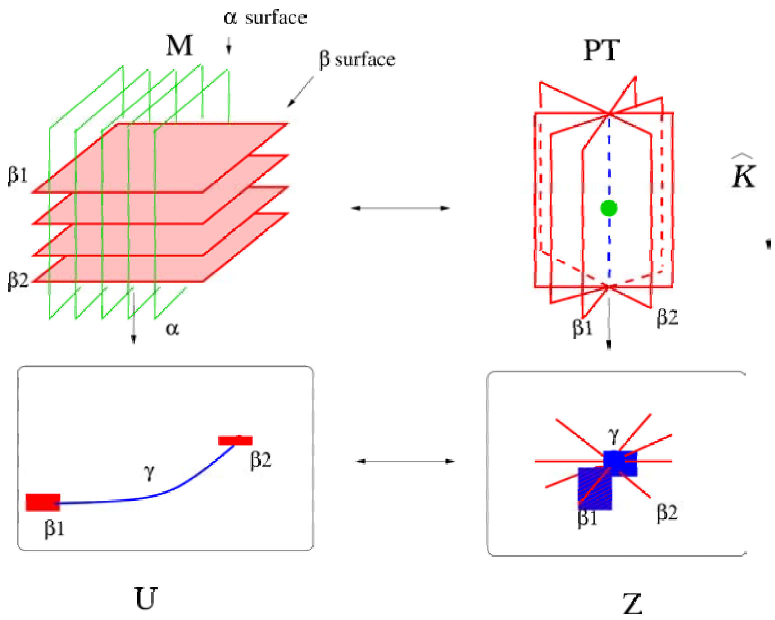


Figure 3: Relationship between M, U, $\mathcal{PT}$ and Z.

The diagram (Fig. 4.4) illustrates the situation. In M, a one parameter family of β-surface is shown, each of which intersects a one parameter family of α-surfaces, also shown. The β-surfaces correspond to a projective structure geodesic in U, shown at the bottom left.

The β-surfaces in M correspond to surfaces in $\mathcal{PT}$, as discussed above. These surfaces intersect at the dotted line, which corresponds to the one parameter family of α-surfaces in M. When we quotient $\mathcal{PT}$ by $\mathcal{K}$ to get Z, the surfaces become twistor lines in Z, and the dotted line becomes a point at which the twistor lines intersect; this is shown on the bottom right. This family of twistor lines intersecting at a point corresponds to the geodesic of the projective structure.

5 Local classification

The second theorem stated in the Introduction gives a local expression for any analytic neutral signature ASD conformal structure. We now prove this theorem. In the proof we will use the following shorthand for coordinate transformations: $t \to F(t,x,y,z)$ means define a new coordinate $\tilde{t} = F(t,x,y,z)$ and then relabel it x again. This avoids having to introduce new symbols for new coordinates. We will denote partial derivatives by subscripts, for example $F_z := \partial_z F$.

Proof of Theorem 2. In what follows, we will use coordinates (x,y) for the two-dimensional space of β-surfaces U. We will always work on a single patch of PS', with λ a standard coordinate on one patch of the $\mathbb{CP}^1$ fibre. The projectivization of (4.1) is

$$L_0 = \mathbf{e}_{00'} + \lambda \mathbf{e}_{01'} + (f_0 + \lambda f_1 + \lambda^2 f_2 + \lambda^3 f_3)\partial_\lambda, \tag{5.1}$$
$$L_1 = \mathbf{e}_{01'} + \lambda \mathbf{e}_{11'} + (g_0 + \lambda g_1 + \lambda^2 g_2 + \lambda^3 g_3)\partial_\lambda, \tag{5.2}$$

where the f_i and g_i are functions on M derived from primed connection coefficients.

We can trivialize PTU by first choosing a two dimensional surface in M, transverse to the β-surfaces, and trivializing PS' over this, using the standard two patch coordinates for $\mathbb{CP}^1$. Then define a trivialization over the rest of PS' by requiring constant coordinate on each leaf of $\mathcal{D}$ (this will be a base dependent Möbius transformation of any other trivialization of PS' using a standard two patch trivialization of $\mathbb{CP}^1$, since any two standard trivializations of $\mathbb{CP}^1$ are related by a Möbius transformation). This gives a trivialization $PTU \cong U \times \mathbb{CP}^1$. The special feature of this particular trivialization is that $\tilde{K}$ and W will have no vertical terms, because it was defined by saying that the fibre coordinate is constant along them.

We will use the conventions of Lemma 3, that is we choose a tetrad with $K = \mathbf{e}_{00'}$, and the tangent planes to the β-surfaces are spanned by K and $\mathbf{e}_{01'}$. Now choose a coordinate system (t, x, y, z) such that $K = \partial_t$, and a conformal factor so that K is pure Killing. Any tetrad can then be written in these coordinates without any t dependence. Then $[\mathbf{e}_{00'}, \mathbf{e}_{01'}] = 0$ and we can in addition choose the z coordinate such that $\mathbf{e}_{01'} = \partial_z$. Then we have

$$\tilde{K} = \partial_t,$$
$$L_0 = \partial_t + \lambda \partial_z + f(x, y, z, \lambda)\partial_\lambda.$$

Note that f does not depend on t because it is composed from connection coefficients, which do not depend on t since it does not occur in the metric. Also note that $\tilde{K} = \partial_t$ because L_0, L_1 do not contain functions of t so it commutes with both. As vector fields on the base, ∂_x and ∂_y are transverse to the β-surfaces so are coordinates on U.

Now we will alter the λ coordinate, using a trivialization as described above, so that L_0 has no ∂_λ terms. This is achieved by a base-dependent Möbius transformation, $\lambda \to (\beta + \delta\lambda)/(\alpha + \gamma\lambda)$. Now the new λ coordinate satisfies $\tilde{K}(\lambda) = L_0(\lambda) = 0$. Therefore $\alpha, \ldots, \delta$ do not depend on t, from the first of these. This gives the following general form:

$$\tilde{K} = \partial_t,$$
$$L_0 = \alpha\, \partial_t + \beta\, \partial_z + \lambda(\gamma\, \partial_t + \delta\, \partial_z),$$

where

$$\alpha, \beta, \gamma, \delta$$

are functions of (x, y, z). Of course, there is an algebraic non-degeneracy condition on the $\alpha, \beta, \gamma, \delta$ from the Möbius transformation. Note that the choice of Möbius transformation is not unique, there is three dimensional space of solutions because we are finding a function constant on surfaces in a five dimensional manifold.

Now from Theorem 1, we know that L_1 must define a projective structure on U, the space of β-surfaces. Clearly U has coordinates (x, y), since the β-surfaces are spanned by (t, z). Also, λ is a fibre coordinate on PTU, since $\mathcal{D}$ is defined by constant λ. Therefore L_1 must have the following form:

$$L_1 := \partial_x + \lambda\partial_y + (A_0(x,y) + \lambda A_1(x,y) + \lambda^2 A_2(x,y) + \lambda^3 A_3(x,y))\partial_\lambda \\ + (C(x,y,z) + \lambda D(x,y,z))\partial_t + (E(x,y,z) + \lambda F(x,y,z))\partial_z. \quad (5.3)$$

Since $\mathcal{D}$ is spanned by ∂_t, ∂_z, the quotient of L_1 by $\mathcal{D}$ gives a projective structure. Note that the A_i do not depend on z because otherwise they would not define a projective structure on U. We have deduced that this form must exist using what we knew from Theorem 1.

We have found a general form that any $\tilde{K}, L_A$ can be put into. For it to give an ASD conformal structure, the L_A must commute modulo L_A. Imposing this gives equations for the unknown functions. Perhaps surprisingly, these equations can be solved, and lead to the two cases in Theorem 2. We will now show this.

First, it is convenient to change coordinates yet again. Using the coordinate changes $t \to t + J_1(x, y, z)$, $z \to z + J_2(x, y, z)$, as well as the fact that we can multiply L_0 by a non-zero function without changing the resulting conformal structure, we can put L_0 into the form

$$L_0 = \partial_t + \beta(x,y,z)\partial_z + \lambda\partial_z. \quad (5.4)$$

This doesn't change the general form of L_1.

One can read off a metric g corresponding to the twistor distribution given by (5.4) and (5.3), by comparing with (5.1) and (5.2) and reading off a null tetrad. One then finds that $\mathbb{K} \wedge d\mathbb{K} = \beta_z dx \wedge dy \wedge dz$, where $\mathbb{K} = g(\partial_t, .)$. Thus the twist of the Killing vector ∂_t vanishes iff β does not depend on z. Since existence of twist is a conformally invariant property, the cases $\beta_z = 0$ and $\beta_z \neq 0$ are genuinely distinct, not an artefact of our coordinate choices. We now analyse each in turn.

Twist-free case: $\beta_z = 0$. In this case, one can eliminate β altogether. Since $\beta = \beta(x, y)$, define a new Möbius transformed fibre coordinate $\lambda \to \lambda + \beta(x, y)$. Then we have

$$L_0 = \partial_t + \lambda\partial_z.$$

The ∂_x, ∂_y terms in L_1 will give rise to some extra ∂_λ terms, but these can be absorbed in the arbitrary functions. The same is true of the extra ∂_t and ∂_z terms. So the general form (5.3) of L_1 is unchanged. Note that if β had depended on z then we could not have done this, because L_0 would have inherited ∂_λ terms, taking it out of the standard form (5.4).

Next we will set $C(x,y,z)=0$ in L_1, which can be done simply by subtracting $C(x,y,z)L_0$ from L_1. Finally, we set $A_0(x,y)=A_1(x,y)=0$ by a choice of (x,y) coordinates. One can always do this since a two dimensional projective structure depends essentially on only two functions. Having made all these special choices, we calculate the commutator:

$$\begin{aligned}[L_0,L_1] &= [\partial_t+\lambda\partial_z, \partial_x+\lambda\partial_y+(\lambda^2A_2(x,y)+\lambda^3A_3(x,y))\partial_\lambda \\ &\quad +(\lambda D(x,y,z))\partial_t+(E(x,y,z)+\lambda F(x,y,z))\partial_z] \\ &= \lambda^2 D_z\,\partial_t+\lambda(E_z+\lambda F_z-\lambda A_2-\lambda^2A_3)\,\partial_z.\end{aligned}$$

Since this has no ∂_y, ∂_z terms, it must be proportional to L_0, as $\{L_0,L_1\}$ span an integrable distribution. From this, one obtains the following equations which can be trivially solved:

$$\begin{aligned}E_z &= 0 &&\Rightarrow && E=E(x,y),\\ F_z &= A_2 &&\Rightarrow && F=zA_2(x,y)-P(x,y),\\ D_z &= -A_3 &&\Rightarrow && D=-zA_3(x,y)-Q(x,y).\end{aligned}$$

One can now eliminate $E(x,y)$ from L_1 by a coordinate change $z\to z-\int E\,dx$. The commuting twistor distribution is now as follows:

$$\begin{aligned}L_0 &= \partial_t+\lambda\partial_z, \\ L_1 &= \partial_x+\lambda\partial_y+(\lambda^2A_2(x,y)+\lambda^3A_3(x,y))\partial_\lambda \\ &\quad +\lambda(zA_2(x,y)-P(x,y))\partial_z+\lambda(-zA_3(x,y)-Q(x,y))\partial_t.\end{aligned} \tag{5.5}$$

The corresponding ASD metric is

$$g=(dt+z\Phi-\Omega)dy-dxdz, \tag{5.6}$$

where $\Phi=A_2(x,y)dx+A_3(x,y)dy$, $\Omega=P(x,y)dx+Q(x,y)dy$. One of the functions P,Q can be eliminated by a coordinate change $t=t+R(x,y)$. This is the twist-free metric (1.1) in Theorem 2.

Twisting case: $\beta_z\neq 0$. Again, we start with and L_0 in the form (5.4). Since $\beta_z\neq 0$, we may perform a coordinate transformation $z\to -\beta(x,y,z)$. This does not affect the general form (5.3) of L_1. Performing the coordinate change and dividing by $-\beta_z$ gives the following form for L_0:

$$L_0=H_{zz}(x,y,z)\partial_t-z\partial_z-\lambda\partial_z,$$

where H_{zz} is a non-zero arbitrary function, written as a second derivative for later convenience. It is non-zero because β_z is non-zero; this is essential since otherwise L_0 would be degenerate. Now we perform the same sort of calculation as in non-twisting case. Here it is slightly more elaborate.

In the non-twisting case we set $A_0 = A_1 = 0$ by a choice of (x, y) coordinates; here we will set $A_2 = A_3 = 0$ instead. Neither is a loss of generality, since any two dimensional projective structure can be put into one of these forms.

We start with the following spray:

$$\begin{aligned} L_0 &= H_{zz}(x,y,z)\partial_t - z\partial_z + \lambda\partial_z, \\ L_1 &= \partial_x + \lambda\partial_y + (A_0(x,y) + \lambda A_1(x,y))\partial_\lambda \\ &\quad +(C(x,y,z) + \lambda D(x,y,z))\partial_x + (E(x,y,z) + \lambda F(x,y,z))\partial_z. \end{aligned}$$

The function $H_{zz}(x, y, z)$ is just an arbitrary function written as a second derivative for later convenience. As in the non-twisting case, we now find equations for the arbitrary functions from the requirement that L_0, L_1 commute modulo L_A. The commutator is as follows:

$$\begin{aligned} [L_0, L_1] = {} & (-H_{zzx} - zC_z - EH_{zzz} + \lambda(-H_{zzy} - zD_z + C_z - FH_{zzz}) + \lambda^2 D_z)\, \partial_t \\ & + (-zE_z + E - A_0 + \lambda(-zF_z + E_z + F - A_1) + \lambda^2 F_z)\, \partial_z. \end{aligned}$$

The RHS must be a linear combination of L_0, L_1. Since it has no ∂_y, ∂_x terms, it must be a multiple of L_0. Subtracting a suitable multiple of L_0 so that the ∂_t term vanishes, we obtain a ∂_z term only, which must vanish. This leads to several equations, one for each power of λ (since these must all vanish separately). It turns out that the λ^3 equation is simply $D_z = 0$. So we set $D = D(x, y)$. Having done this we start again. This time we get the following equation from the λ^2 term:

$$F_z H_{zz} + H_{zzz}F + H_{zzy} - C_z = 0.$$

This equation can be integrated to give

$$C = FH_{zz} + H_{zy} + \alpha(x, y),$$

for some arbitrary $\alpha(x, y)$. Plugging this back in and repeating the procedure, we get the following equation from the λ term:

$$zF_z H_{zz} + zFH_{zzz} + FH_{zz} + H_{zzx} + zH_{zzy} + E_z H_{zz} + EH_{zzz} - A_1 H_{zz} = 0.$$

Again, this can be integrated to give:

$$zFH_{zz} + H_{zx} + zH_{zy} - H_y + EH_{zz} - A_1 H_z - \omega(x, y) = 0,$$

for some arbitrary $\omega(x,y)$. This gives an expression for $F(x,y,z)$ in terms of the other functions, which we plug back into the spray. Then the commutator provides one final equation to solve, (from the λ^0 term), which turns out to only involve H. It is as follows:

$$-A_0 H_{zz} + A_1 H_z - zA_1 H_{zz} - H_{zx} - zH_{zy} + H_w + \omega = 0. \tag{5.7}$$

Differentiating this with respect to z gives

$$(\partial_x + z\partial_y + (A_0 + zA_1)\partial_z)H_{zz} = 0. \tag{5.8}$$

Writing out the spray, and using (5.7) gives

$$\begin{aligned} L_0 &= H_{zz}\partial_t - z\partial_z + \lambda\partial_z, \\ L_1 &= \partial_x + \lambda\partial_y + (A_0 + \lambda A_1)\,\partial_\lambda \\ &\quad +((-E + A_0 + zA_1)\frac{H_{zz}}{z} + H_{zy} + \alpha + \lambda D)\,\partial_t \\ &\quad +(E + \lambda(-E + A_0 + zA_1)\frac{1}{z})\,\partial_z. \end{aligned}$$

One can in fact eliminate $E(x,y,z)$, $\alpha(x,y)$ and $D(x,y)$. One can add multiples of L_0 to L_1 without changing the twistor distribution, since the conformal structure only depends on the distribution, this does not change the conformal stucture either. We eliminate E by changing L_1 as follows:

$$\begin{aligned} \tilde{L}_1 = L_1 - \frac{1}{z}(-E + A_0 - zA_1)L &= \partial_x + \lambda\partial_y + (A_0 + \lambda A_1)\,\partial_\lambda \\ &\quad +(H_{zw} + \alpha + \lambda D)\partial_t + (A_0 + zA_1)\partial_z. \end{aligned}$$

The commutator $[L_0, \tilde{L}_1]$ vanishes when (5.8) holds, so H_{zy} is determined up to an arbitrary function of (x,y), so $\alpha(x,y)$ can be absorbed into H_{zy}. Finally we can eliminate $D(x,y)$ by a coordinate change $t \to t + f(x,y)$. Our final spray is as follows (after setting $G_z := H_{zz}$ and removing the tilde on $\tilde{L}_1$):

$$\begin{aligned} L_0 &= G_z\partial_t - z\partial_z + \lambda\partial_z, \\ L_1 &= \partial_x + \lambda\partial_y + (A_0 + \lambda A_1)\partial_\lambda + G_y\partial_t + (A_0 + zA_1)\partial_z, \end{aligned} \tag{5.9}$$

which commutes when the following equation is satisfied:

$$(\partial_x + z\partial_y + (A_0 + zA_1)\partial_z)G_z = 0. \tag{5.10}$$

The conformal structure corresponding to (5.9) is (1.2) from Theorem 2. □

Remark. Given any projective structure (i.e. functions $A_0(x,y)$, $A_1(x,y)$) it is possible to find a metric of the form (1.2) without solving any equations, since $G_z = 1$ is a solution of (5.10). Then G_y is an arbitrary function of (x,y), and we obtain the following conformal structure:

$$g = (dt - \Psi)(dy - zdx) - \ dx\ (dz - \Phi), \tag{5.11}$$

for $\Phi = A_0 dx + A_1 dy$ and an arbitrary one–form Ψ on U.

6 Examples

6.1 Pseudo-hyper-Kähler

In this section we will find some examples of neutral ASD metrics with null Killing vectors by independent means, and interpret them using our results. We will use Plebański's method [21] adapted to neutral signature, which converts the problem of finding Ricci-flat ASD neutral metrics, or *pseudo-hyper-Kähler*, to the problem of solving a non-linear second order PDE. He showed that such metrics are locally of the form

$$g = dY(dT - \Theta_{XX}dY - \Theta_{TX}dZ) - dZ(dX + \Theta_{TT}dZ + \Theta_{TX}dY), \tag{6.12}$$

where $\Theta(T,X,Y,Z)$ satisfies the 'second Heavenly Equation':

$$\Theta_{YT} - \Theta_{ZX} + \Theta_{TT}\Theta_{XX} - \Theta_{XT}^2 = 0. \tag{6.13}$$

The primed connection coefficients vanish when using the tetrad indicated in (6.12), so there is a basis of covariantly constant primed spinors $o^{A'} = (1,0)$, $\iota^{A'} = (0,-1)$. There is therefore also a basis $\mathbf{\Sigma}^{A'B'}$ of covariantly constant null self-dual two forms, written in spinors as follows:

$$\begin{aligned}
\mathbf{\Sigma}^{0'0'} &= \frac{1}{2}\ \iota_{A'}\iota_{B'}\epsilon_{AB}\ \theta^{AA'} \wedge \theta^{BB'}, && (6.14)\\
\mathbf{\Sigma}^{0'1'} = \mathbf{\Sigma}^{1'0'} &= \frac{1}{2}\ o_{(A'}\iota_{B')}\epsilon_{AB}\theta^{AA'} \wedge \theta^{BB'}, && (6.15)\\
\mathbf{\Sigma}^{1'1'} &= \frac{1}{2}\ o_{A'}o_{B'}\epsilon_{AB}\ \ \theta^{AA'} \wedge \theta^{BB'}. && (6.16)
\end{aligned}$$

Using the identification between two-forms and endomorphisms given by g, we can write

$$R = \mathbf{\Sigma}^{0'0'} - \mathbf{\Sigma}^{1'1'}, \quad I = \mathbf{\Sigma}^{0'0} + \mathbf{\Sigma}^{1'1}, \quad S = \mathbf{\Sigma}^{0'1'}.$$

As endomorphisms, these satisfy

$$-I^2 = R^2 = S^2 = \mathrm{Id}, \quad IRS = \mathrm{Id}, \tag{6.17}$$

which is easy to check using their spinor forms. There is a hyperboloid's worth of almost complex structures, $aI + bR + cS$, where $a^2 - b^2 - c^2 = 1$, which are parallel and hence integrable. This is a *pseudo-hyper-Kähler* structure.

Now writing (A.2) using spinors by means of (2.5) and (A.1) gives

$$\iota^A o^{A'} C_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} = \nabla_{BB'}(\phi_{C'D'}\epsilon_{CD} + \psi_{CD}\epsilon_{C'D'} + \frac{1}{2}\eta\epsilon_{CD}\epsilon_{C'D'}),$$

where we have used Ricci flatness and anti-self-duality. For a pure Killing vector or a homothety (η constant), it follows that

$$\nabla_{AA'}\phi_{B'C'} = 0. \tag{6.18}$$

Therefore $\phi_{B'C'}$ is actually constant in the basis shown in (6.12). Now let us suppose we have a null Killing vector which preserves the α–plane distribution spanned by $o^{A'}\mathbf{e}_{AA'}$. Then $K = \iota^A o^{A'}\mathbf{e}_{AA'}$, for some ι^A and using (2.7) and (6.18) we get

$$\phi_{B'C'} = a_1\, o_{B'}o_{C'} + a_2\, o_{(B'}\iota_{C')},$$

for constant a_1, a_2. Consider the three distinct cases: $\phi_{B'C'}$ vanishing ($a = b = 0$), non-vanishing but degenerate ($a \neq 0, b = 0$), and non-degenerate ($a = 0, b \neq 0$). One can show that for $K = \partial_T$ we get the first case, $K = Y\partial_X + Z\partial_T$ the second, and $T\partial_T + X\partial_X$ the third. Moreover, these choices are canonical, in the sense that given a null Killing vector K preserving the $o^{A'}\mathbf{e}_{AA'}$ distribution, one can find a 'Plebański basis' such that it takes one of these forms [7]. In order for any of these to be Killing, an equation for Θ coming from the Killing equation must be satisfied. In fact we were only able to fully solve for the first two cases.

- $K = \partial_T$

Since ∂_T has no twist we expect this to be of the form (1.1). It is a neutral signature version of a tri-holomorphic Killing vector; i.e. it Lie-derives I, R, S. Solving the Killing equations in conjunction with (6.13) results in the following metric:

$$g = dYdT - dZdX - Q(X, Y)dY^2, \tag{6.19}$$

where Q is an arbitrary function. This is simply the split-signature pp-wave metric. The metric (1.1) is a generalization of this, with a non-trivial underlying projective structure. Here K is a self-dual Killing vector in the sense of Gibbons et al. [1].

The local expression (6.19) in this last example corresponds to a class of global neutral metrics on compact four–manifolds. To see this we compactify the flat projective space $\mathbb{R}^2$ to two–dimensional torus $U = T^2$ with the projective structure coming from the flat metric. Both T and Z in (6.19) are taken to be periodic, thus leading to $\hat{\pi}: M \longrightarrow U$, the holomorphic toric fibration over a torus. Assume the suitable periodicity on the function $Q: U \longrightarrow \mathbb{R}$. This leads to a commutative diagram

$$\begin{array}{ccc} & M & \\ T^2 & \downarrow & \searrow \hat{\pi}^* Q \\ & U & \xrightarrow{Q} \mathbb{R}. \end{array}$$

- $K = Y\partial_X + Z\partial_T$

Again, this is twist-free and we expect the metric to be of the form (1.1). Solving the Killing equations in conjunction with (6.13) results in the following metric:

$$g = dYdT - dZdX - \frac{H(\frac{Y}{YT-ZX}, \frac{Z}{YT-ZX})}{(YT-ZX)^3}(YdZ - ZdY)^2, \qquad (6.20)$$

where H is an arbitrary analytic function of two variables. This is a generalization of the Sparling-Tod metric [22]. It is easy to show that the arguments of H are in fact constant on the special β-surfaces, so serve as coordinates on U.

Using the following coordinate transformation

$$\begin{aligned} t &= -\frac{1}{2}(\frac{X}{Y} + \frac{T}{Z}), \\ z &= (YZ)^{-\frac{1}{2}}, \\ x &= \frac{YT - XZ}{(YZ)^{\frac{1}{2}}}, \\ y &= \log(\frac{Z}{Y}). \end{aligned}$$

the metric (6.20) takes the following form:

$$g = \frac{1}{z^2}(dydt - dzdx + zA_3(x,y)dy^2),$$

where now the Killing vector is ∂_t. Multiplying by the conformal factor z^2, we get a special case of (1.1) as expected. The projective structure is non-trivial, unlike for the pp-wave above. The projective structure is special in that it depends on only one arbitrary function instead of two.

- $T\partial_T + X\partial_X$

In this case we were not able to fully solve the Killing equations in conjunction with (6.13). This Killing vector is twisting, so the answer must be of the form (1.2).

6.2 Pseudo-hyper-hermitian

This is a generalization of the pseudo-hyper-Kähler case discussed in the last section. We will refer to a neutral metric g as pseudo-hyper-hermitian when there exist endomorphisms I, R, S satisfying the algebra (6.17), such that any complex structure $\mathcal{J}_{(a,b,c)} = aI + bR + cS$ is integrable for $a^2 - b^2 - c^2 = 1$, and g is hermitian with respect to any of these complex structures. For g to be hermitian with respect to a complex structure $\mathcal{J}$ means $g(\mathcal{J}X, \mathcal{J}Y) = g(X, Y)$. Note that for pseudo-hyper-Kähler, the endomorphisms I, R, S must also be covariantly constant with respect to the Levi-Civita connection of g.

In [4], it is shown that one can always find a tetrad for a pseudo-hyper-hermitian metric such that the twistor distribution has no ∂_λ terms. Equivalently, the twistor space fibres over $\mathbb{CP}^1$. Now let us suppose that we have a null conformal Killing that is tri-holomorphic, i.e. it preserves I, R and S and so it preserves the holomorphic fibration $\mathcal{PT} \to \mathbb{CP}^1$. All such cases are classified by the following

Proposition 2. *All real-analytic pseudo-hyper-hermitian ASD metrics with triholomorphic null conformal Killing vectors are of the form (1.1) or (1.2) with* $d\Phi = 0$.

Proof. When we put the twistor distribution into a form with no ∂_λ terms, the lift $\tilde{K}$ of K also contains no ∂_λ terms, because K is triholomorphic. Therefore the projective structure spray will also not contain ∂_λ terms, and must be of the following form:

$$\Theta = a\partial_x + b\partial_y + \lambda(c\partial_x + e\partial_y),$$

in some coordinates (x, y) on the space of β-surfaces U, where a, b, c, e are functions of (x, y) with $ae - bc \neq 0$. Coordinate freedom $(x, y) \to (\hat{x}(x, y), \hat{y}(x, y))$ and scaling freedom (the projective structure is unchanged if Θ is multiplied by a non-zero function) allows us to set $a = 1$, $c = 0$, $e = 1$, giving $\Theta = \partial_x + (b + \lambda)\partial_y$. Now define a fibre coordinate $\lambda \to b + \lambda$, which gives the following spray:

$$\partial_x + \lambda\partial_y + (b_x + \lambda b_y)\partial_\lambda. \tag{6.21}$$

This corresponds to the second-order ODE

$$\frac{d^2y}{dx^2} = A_1(x, y)\Big(\frac{dy}{dx}\Big) + A_0(x, y), \tag{6.22}$$

where $A_1 = \frac{\partial b}{\partial y}$, $A_0 = \frac{\partial b}{\partial x}$ for a function $b(x, y)$. This is the derivative of the general first-order ODE

$$\frac{dy}{dx} = b(x, y). \tag{6.23}$$

So a metric of the form (1.1) is pseudo-hyper-hermitian iff A_0 and A_1 are derivatives of a potential function b as above.

A similar argument can be applied to the case (1.2) to deduce that the functions A_2 and A_3 must be of the form $A_2 = \frac{\partial b}{\partial x}$, $A_3 = \frac{\partial b}{\partial x}$ for a function $b(x, y)$. To see it change λ to $1/\lambda$ and swap the role of x and y.

If we put a projective structure spray coming from the derivative of a first order ODE into either of the metrics (1.1) or (1.2), then we know that there is a coordinate transformation of (x, y) such that the resulting twistor distribution has no ∂_λ terms. Therefore the metric must be pseudo-hyper-hermitian by [4]. □

Note that if a (holomorphic) projective structure spray contains no ∂_λ terms, its twistor space fibres over $\mathbb{CP}^1$, since each integral curve can be labelled by the λ coordinate. So a by-product of the proof of the above proposition is the following

Proposition 3. *There is a one to one correspondence between holomorphic 2D projective structures s.t. the corresponding second order ODE is the derivative of a first order ODE, and projective structure twistor spaces which fibre over* $\mathbb{CP}^1$.

This is of interest purely as a statement about projective structures. Note that although all first order ODEs can be transformed to the trivial first order ODE $\partial y/\partial x = 0$ by coordinate transformation, this does not mean that the derivative of any such equation is flat, in the sense of Section 3.2. This can be shown by calculating the invariant (3.6) for (6.22) and showing that it does not necessarily vanish.

6.3 Conformal structures containing no Ricci-flat metrics

An interesting question is the following: Under what conditions do the conformal structures represented by (1.1) and (1.2) contain special types of metric, for example Ricci-flat metrics, or, more generally, Einstein metrics?

In this section we show that there are conformal structures of the form (1.1) which do not contain Ricci-flat metrics. We use the results of Szekeres [23]. Although these were derived for Lorentzian signature, they can also be applied to our ASD neutral signature case, essentially because the Weyl curvature is still made up of a single spinor $C_{abcd} = C_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}$ as in the Lorentzian case (of course in Lorentzian case it is complex hermitian, not real).

Consider the metric (1.1) in a spin frame where $K^a = \iota^A o^{A'}$. By direct calculation, one finds

$$\begin{aligned} C_{0000} &= -P_{xy} + Q_{xx} + z\big((A_2)_{xy} - (A_3)_{xx}\big), \\ C_{0001} &= -\frac{1}{4}(A_2)_x. \end{aligned}$$

In the standard Petrov-Penrose classification [20], C_{ABCD} is type N iff $(A_2)_x = 0$, otherwise it is type III. (The Petrov–Penrose classification of the conformal signature branches in the neutral signature which have to do with the principal null directions being complex or real [14]. These subtleties are not present for algebraic types N or III and need not concern us.)

Now suppose (1.1) is type III, i.e. $(A_2)_x \neq 0$. The reason for this is that we can apply a result of Szekeres to obtain an obstruction to Ricci-flatness. It is shown in [23] that for types I, II, D or III, a necessary condition for existence of a Ricci-flat metric in the conformal class is the following tensor equation

$$-\frac{1}{2}C_{pqfh}C_{rs}{}^{fh}C_{abc}{}^{d}{}_{;d} + (C_{pq}{}^{df}C_{rsf}{}^{h}{}_{;h} + C_{rs}{}^{df}C_{pqf}{}^{h}{}_{;h}) = 0.$$

This is just the tensor version of the spinor identity (3.1), page 209 [23]. Calculating this one finds that $(A_2)_{xx}$ is an obstruction to its vanishing (we used MAPLE for the calculation). Therefore we have a class of non-conformally vacuum type III neutral ASD conformal structures with non-twisting null conformal Killing vectors.

7 Twistor reconstruction

We have shown that when a conformal structure $[g]$ has a null conformal Killing vector, the twistor space $\mathcal{PT}$ fibres over the twistor space of a projective structure, and we have classified the possible local forms for such conformal structures.

The twistor lines in a projective structure twistor space Z have normal bundle $\mathcal{O}(1)$. The twistor lines in a conformal structure twistor space have normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$. Let $\mathcal{B}$ be a holomorphic fibre bundle over Z with one dimensional fibres. Let $\hat{u}$ be a twistor line in Z. Then if we want $\mathcal{B}$ to be a conformal structure twistor space, the normal bundle of $\hat{u}$ in $\mathcal{B}|_{\hat{u}}$ must be $\mathcal{O}(1)$. Given a projective structure twistor space, one way of forming a fibre bundle with the correct property is to take a power of the canonical bundle κ, which reduces to $\mathcal{O}(-3)$ on twistor lines. The bundle $\kappa^{-1/3}$ reduces to $\mathcal{O}(1)$ on twistor lines, and exists provided we take Z to be a suitably small neighbourhood of a twistor line. So the total space of $\kappa^{-1/3}$ is a conformal structure twistor space.

Consider the simplest possible case, where Z is the total space of $\mathcal{O}(1)$, corresponding to a flat projective structure. In this case $\kappa^{-1/3}$ is the total space of $\mathcal{O}(1)\oplus\mathcal{O}(1)$, the twistor space of the flat conformal structure. To go further, note that given a line bundle $\mathcal{A}$ over Z which reduces to $\mathcal{O}(1)$ on twistor lines, any affine bundle modelled on $\mathcal{A}$ will also have the correct property on twistor lines. In the simplest case described above, taking affine bundles modelled on $\kappa^{-1/3}$ results in the the twistor space of the pp-wave metric (6.19). In fact, this is precisely the first case discussed by Ward in [26], although he does not phrase it in this way. We will now show how this works.

7.1 Example

First we will give a twistorial demonstration of a fact shown in Section 6.1, namely that for a pseudo-hyper-Kähler metric with triholomorphic null Killing vector $K = \iota^A o^{A'} \mathbf{e}_{AA'}$ with $o^{A'}$ covariantly constant, the resulting projective structure is flat. The twistor space of an analytic pseudo-hyper-Kähler metric fibres over $\mathbb{CP}^1$, $\sigma : \mathcal{PT} \to \mathbb{CP}^1$ [19, 10]. There is a section ϖ of $\Lambda^2 \mathcal{PT} \times \sigma^* \mathcal{O}(2)$. This is a symplectic form of 'degree 2' on the fibres. In the spin bundle picture, ϖ is the push forward to $\mathcal{PT}$ of the symplectic form $\mathbf{\Sigma} = \mathbf{\Sigma}^{A'B'} \pi_{A'} \pi_{B'}$ on S', where $\mathbf{\Sigma}^{A'B'}$ are defined as in Section 6.1. This form is Lie-derived over the twistor distribution as a consequence of the $\mathbf{\Sigma}^{A'B'}$ being covariantly constant, and is homogeneous in the $\pi_{A'}$, so the push-forward is well defined.

As explained in Section 4, $\mathcal{K}$ vanishes on a hypersurface $\mathcal{H}$ in $\mathcal{PT}$, where $\mathcal{H}$ is the projection to $\mathcal{PT}$ of the hypersurface $\pi.o = 0$ in S'. For $o^{A'}$ covariantly constant, the function $1/(\pi.o)$ on S' gives a section ζ of $\sigma^* \mathcal{O}(-1)$ on $\mathcal{PT}$, which blows up on $\mathcal{H}$. Then $\zeta \otimes \mathcal{K}$ is a non-vanishing '$\sigma^* \mathcal{O}(1)$-valued' vector field. Now in a local trivialization, $\zeta \otimes \mathcal{K}$ Lie derives the symplectic form ϖ, so it is Hamiltonian,

$$\zeta \otimes \mathcal{K} = \frac{\partial h}{\partial \omega^A} \frac{\partial}{\partial \omega_A},$$

where $\varpi = \omega^0 \wedge \omega^1$. Now the ω^A should be regarded as coordinates of 'degree 1', that is they are coordinate functions multiplied by a section of $\sigma^* \mathcal{O}(1)$. Therefore for the weights to agree, h must be a section of $\kappa^* \mathcal{O}(1)$, rather than a bona fide function. This gives a projection $\mathcal{PT} \to Z = \mathcal{O}(1)$, with fibres the trajectories of $\zeta \otimes \mathcal{K}$, so the projective structure twistor space is the total space of $\mathcal{O}(1)$, which corresponds to the flat projective structure.

Now suppose we start with the total space of $\mathcal{O}(1)$ as the minitwistor space Z. The twistor lines are global holomorphic sections of $\mathcal{O}(1) \to \mathbb{CP}^1$.

We will use a homogeneous coordinate description of $Z = \mathcal{O}(1)$. Let $\pi_{A'}$ be homogeneous coordinates for the base $\mathbb{CP}^1$ of $Z = \mathcal{O}(1)$, and let ω be a homogeneous coordinate for the fibre of $Z = \mathcal{O}(1)$. That is, $\mathcal{O}(1) = \{[\pi_{0'}, \pi_{1'}, \omega] : [c\pi_{0'}, c\pi_{1'}, c\omega], c \in \mathbb{C}^*, [\pi_{0'}, \pi_{1'}] \neq [0, 0]\}$.

Now cover the base $\mathbb{CP}^1$ in $\mathcal{PT}$ with two open sets $(\mathcal{U}_0, \mathcal{U}_1)$, and lift this covering to $\mathcal{PT}$. Use homogeneous coordinates $(\pi_{A'}, \omega, \zeta_i)$ on $\mathcal{U}_i$.

The flat twistor space $\mathcal{O}(1) \oplus \mathcal{O}(1)$ can be formed as follows. Consider the projection $\tau : \mathcal{O}(1) \to \mathbb{CP}^1$. Then $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is the pull-back bundle $\tau^* \mathcal{O}(1)$ over the total space of $\mathcal{O}(1)$. It is easy to check that this is the same as taking $\kappa^{-1/3}$ where κ is the canonical bundle of $Z = \mathcal{O}(1)$. To obtain curved twistor spaces, we can take affine bundles over $\mathcal{O}(1)$ modelled on $\tau^* \mathcal{O}(1)$. To form these we use the following transition functions:

$$\zeta_0 = \zeta_1 + f(\pi_{A'}, \omega),$$

where $f \in [f] \in H^1(Z, \tau^*\mathcal{O}(1))$, where Z is $\mathcal{O}(1)$. The cohomology elements f classify affine bundles over Z modelled on $\tau^*\mathcal{O}(1)$.

Global holomorphic sections of $Z \to \mathbb{CP}^1$ are defined by $\omega = P(\pi_{A'}) = \pi_{A'}x^{A'}$, with $x^{A'} = (X, Y)$ say.

The sections of $\mathcal{PT} \to \mathbb{CP}^1$ are constructed by putting $\zeta_i = \pi_{A'}t^{A'} + f_i$, where $t^{A'} = -(T, Z)$ say, and $f = f_0 - f_1$. The reason f can be split in this way is that when restricted to a twistor line in Z, f becomes an element of $H^1(\mathbb{CP}^1, \mathcal{O}(1))$, and this group vanishes. To realise a splitting of f we divide it by $(\pi_{A'}o^{A'})^2$ for some constant $o^{A'}$, to get an element of $H^1(Z, \tau^*\mathcal{O}(-1))$. Then we can use the fact that $H^0(\mathbb{CP}^1, \mathcal{O}(-1)) = H^1(\mathbb{CP}^1, \mathcal{O}(-1)) = 0$, so any element can be written as a difference of coboundaries, and the splitting is unique. These sections are the $\mathbb{CP}^1$ twistor lines in $\mathcal{PT}$, we will refer to these as L_x, where x is the point in M with coordinates $t^{A'}$, $x^{A'}$).

Let $\rho_{A'}$ be homogeneous coordinates on $\mathbb{CP}^1$. The splitting is given by the Sparling formula:

$$\frac{f(\pi, P)}{(\pi.o)^2} = \oint_{\Gamma_0} \frac{f(\rho, P)}{(\rho.o)^2\pi.\rho}\rho.d\rho - \oint_{\Gamma_1} \frac{f(\rho, P)}{(\rho.o)^2\pi.\rho}\rho d\rho,$$

where we are using Cauchy's integral formula, and $\Gamma_i \subset L_x \cong \mathbb{CP}^1$ are contours that bound a region containing the point $\rho_{A'} = \pi_{A'}$. The measure $\rho.d\rho$ means $\epsilon_{A'B'}\rho^{A'}d\rho^{B'}$.

Therefore

$$f_i = \oint_{\Gamma_i} \frac{(\pi.o)^2}{(\rho.o)^2}\frac{f(\rho, P)}{\pi.\rho}\rho.d\rho.$$

The symplectic form ϖ discussed above is given by by $\varpi = d\omega \wedge d\zeta_i$ on $\mathcal{U}_i$. Restricting ϖ to a section and taking exterior derivatives keeping $\pi_{A'}$ constant, we obtain a formula for $\mathbf{\Sigma}$, the pull-back of ϖ to S':

$$\begin{aligned} \mathbf{\Sigma} &= d(\pi_{A'}x^{A'}) \wedge d(\pi_{B'}t^{B'} + f_0) \\ &= \pi_{A'}\pi_{B'}dx^{A'} \wedge dt^{B'} + \pi_{A'}dx^{A'} \wedge df_0, \end{aligned}$$

where we are working over $\mathcal{U}_0$. Now

$$\begin{aligned} df_0 &= dx^{B'} \otimes \frac{\partial}{\partial x^{B'}} \oint_{\Gamma_0} \frac{(\pi.o)^2}{(\rho.o)^2}\frac{f(\rho, \rho_{A'}x^{A'})}{\pi.\rho}\rho.d\rho \\ &= dx^{B'} \oint_{\Gamma_0} \frac{\rho'_B(\pi.o)^2}{(\rho.o)^2(\pi.\rho)}\frac{\partial f}{\partial P}\rho.d\rho. \end{aligned}$$

Where we have used $\frac{\partial}{\partial x^{A'}} \to \rho_{A'}\frac{\partial}{\partial P}$. Using this we get

$$\begin{aligned}\boldsymbol{\Sigma} &= \pi_{A'}\pi_{B'}dx^{A'}\wedge dt^{B'} + \Big(\oint_{\Gamma_0}\frac{\pi_{A'}\rho_{B'}(o.\pi)^2}{(o.\pi)^2(\pi.\rho)}\frac{\partial f}{\partial P}\rho.d\rho\Big)dx^{A'}\wedge dx^{B'}\\ &= \pi_{A'}\pi_{B'}dx^{A'}\wedge dt^{B'} + \frac{1}{2}\Big(\oint_{\Gamma_0}\frac{(o.\pi)^2}{(o.\rho)^2}\frac{\partial f}{\partial P}\rho.d\rho\Big)dY\wedge dX\\ &= \pi_{A'}\pi_{B'}dx^{A'}\wedge dt^{B'} + (o.\pi)^2 Q(X,Y)dY\wedge dX,\end{aligned}$$

where

$$Q(X,Y) = \frac{1}{2}\oint_{\Gamma_0}\frac{1}{(o.\rho)^2}\frac{\partial f}{\partial P}\rho.d\rho.$$

Putting $o^{A'} = (1,0)$, we get the following formula for $\boldsymbol{\Sigma}$ pulled back to $M\times\mathbb{C}^2$:

$$\boldsymbol{\Sigma} = \pi_{0'}^2(dT\wedge dX + Q(X,Y)dY\wedge dX) + \pi_{0'}\pi_{1'}(dT\wedge dY - dX\wedge dZ) + \pi_{1'}^2 dZ\wedge dY. \quad (7.1)$$

Calculating $\boldsymbol{\Sigma}$ in the Plebanski formalism from (6.14), (6.15) and (6.16) gives

$$\begin{aligned}\boldsymbol{\Sigma} &= \pi_{0'}^2(dT - \Theta_{XX}dY - \Theta_{TX}dZ)\wedge(dX + \Theta_{TT}dZ + \Theta_{TX}dY) + \\ &\quad \pi_{0'}\pi_{1'}(dT\wedge dY - dX\wedge dZ) + \pi_{1'}^2 dZ\wedge dY.\end{aligned}$$

Comparing gives the forms $\boldsymbol{\Sigma}^{A'B'}$ and hence the metric (6.19). The arbitrary function Q corresponds to some arbitrary cohomology element f.

7.2 Flat conformal structure → flat projective structure

Here we show that given a conformal Killing vector for the flat conformal structure, the underlying projective structure is also flat. By the results of [24], we need only consider the conformal Killing vectors ∂_t (non-twisting) and $t\partial_t + x\partial_x$ (twisting), where the flat metric is

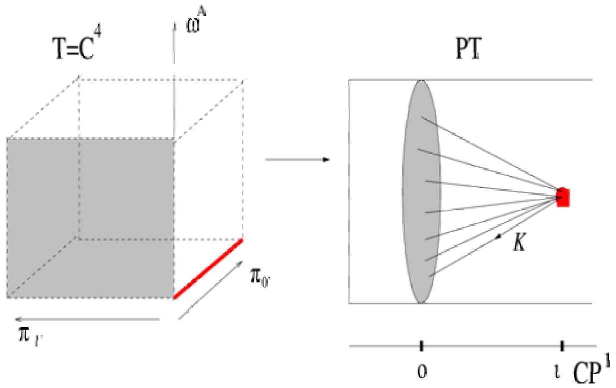
$$g = dtdy - dxdz.$$

The non-twisting case is covered by the example of the last section, with $Q(x,y) = 0$, so we know the projective structure is flat and $Z = \mathcal{O}(1)$.

The twisting case is slightly more complicated. One can use the spray picture, but instead we will analyse the twistor space $\mathcal{PT}$ and show that the space of trajectories of $\mathcal{K}$ is the flat projective structure twistor space $\mathbb{CP}^2$ We work on the non–projective twistor space $\mathcal{T} = \mathbb{C}^4$ with coordinates $(\omega^A, \pi_{A'})$. The projective twistor space $\mathcal{PT}$ is a quotient of $\mathcal{T}$ be the Euler homogeneity vector field $\Upsilon = \omega^A/\partial\omega^A + \pi_{A'}/\partial\pi_{A'}$. The flat conformal class on the complexified $\mathbb{R}^{2,2}$ and the conformal twisting Killing vector are represented by

$$g = \varepsilon_{AB}dp^B dq^A, \qquad K = p^A\frac{\partial}{\partial p^A}$$

Figure 4: Quotient of the non–projective twistor space by the Euler vector field showing the singular set of $\mathcal{K}$.



where $x^{AA'} := p^A o^{A'} + q^A \iota^{A'}$ are coordinares on M. The point (p^A, q^A) corresponds to a two–plane in $\mathcal{T}$ given by solutions to the twistor equation $\omega^A = x^{AA'}\pi_{A'}$. The lift of K to S' is

$$\tilde{K} = K + \pi_{1'}\frac{\partial}{\partial \pi_{1'}},$$

and the orbits of the induced group action on the non–projective twistor space are

$$\omega^A \rightarrow c\omega^A, \qquad \pi_{1'} \longrightarrow c\pi_{1'}, \qquad \pi_{0'} \longrightarrow \pi_{0'}.$$

The holomorphic vector field on $\mathcal{T}$

$$\mathcal{K} = \omega^A \frac{\partial}{\partial \omega^A} + \pi_{1'}\frac{\partial}{\partial \pi_{1'}},$$

vanishes on the projective twistor space when it is proportional to the Euler vector field. It happens on a set $B = \{\{\omega^A = 0, \pi_{1'} = 0\} \cup \{\pi_{0'} = 0\}\} \subset \mathcal{T}$ which is a union of the line and a hyperplane $\mathbb{C}^3 \subset \mathcal{T}$. The set B descents to a union of a hypersurface and a point in the projective twistor space (Fig. 4). The minitwistor space Z corresponding to the projective structure U is the factor space of $\mathcal{PT}/B$ by the trajectories of $\mathcal{K}$. Each trajectory in $\mathcal{T}$ is parametrised by its intersection with the singular surface $\mathbb{C}^3$ given by $\pi \cdot o = 0$ in $\mathcal{T}$ so the space of trajectories in $\mathcal{PT}$ is $Z = \mathbb{CP}^2$. Two $\mathbb{CP}^1$s in $\mathbb{CP}^2$ intersect in a point so the normal bundle of each $\mathbb{CP}^1$ is $\mathcal{O}(1)$ and we have a projective structure. To obtain the explicit parametrisation of these $\mathbb{CP}^1$s eliminate $\pi_{0'}$ from the twistor equation to get $\pi_{1'} = \omega^A u_A$ where $u_A := p_A/(p_B q^B)$ parametrise the twistor lines in Z and are coordinates on U.

8 Outlook

We have locally classified analytic neutral signature ASD conformal structures with null Killing vectors. Some of these are defined on compact manifolds. It would be interesting to investigate the global properties of other conformal structures we have found.

By taking the arbitrary functions in the local classification to be smooth, we obtain a large class of smooth examples. It is beyond our reach at this point to show whether or not the local classification extends to the smooth case. It might be possible to use the new twistor methods of [16] for this purpose.

It would be also be interesting to understand in more detail which conformal structures admit special types of metric, for example Ricci-flat or Einstein (in this case the pure Killing vectors must be twist–free [12]). So far the only results we have in this direction are given in Section 6.3. One expects existence of particular types of metric to be related to invariants of the projective structure, but we have not demonstrated this yet.

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A Appendix

Here we summarise the required spinor notation and present the calculations leading to a proof of (4.3). We use similar conventions to Penrose and Rindler [20] adapted to neutral signature, but our indices are concrete.

Spin connection and curvature decomposition. As usual, we denote the Levi-Civita connection of the metric by ∇. The 'spin connection coefficients' are defined by

$$\nabla(\mathbf{e}_{CC'}) = \theta^{DD'} \otimes ({\Gamma_{DD'C}}^{E}\mathbf{e}_{EC'} + {\Gamma_{DD'C'}}^{E'}\mathbf{e}_{CE'}),$$

together with the symmetry requirement $\Gamma_{DD'CE} = \Gamma_{DD'EC}$, $\Gamma_{DD'C'E'} = \Gamma_{DD'E'C'}$. These conventions result in the following expressions for differentiation of spinor *components*, where ι^A is a two-component spinor field over the manifold:

$$\nabla_{BB'}\iota^A = \mathbf{e}_{BB'}(\iota^A) + {\Gamma_{BB'C}}^{A}\iota^C,$$
$$\nabla_{BB'}\iota_A = \mathbf{e}_{BB'}(\iota_A) - {\Gamma_{BB'A}}^{C}\iota_C,$$

and similarly for a primed spinor field. These are the concrete expressions for the covariant differentiation of spinors using the connections on S and S' inherited from the Levi-Civita connection, mentioned in Section 2.1. One can extend the above expressions to multi-component objects in the obvious way, allowing covariant differentiation of tensors, which agrees with covariant differentiation using the Levi-Civita connection.

The Riemann tensor has the following spinor decomposition ([20], pg. 236):

$$\begin{aligned} R_{abcd} &= C_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \tilde{C}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\ &\quad +\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \Phi_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \\ &\quad +2\Lambda(\epsilon_{AC}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}). \end{aligned} \tag{A.1}$$

Here $C_{ABCD}, \tilde{C}_{A'B'C'D'}$ are completely symmetric, and $\Phi_{ABC'D'}$ is symmetric on each pair of indices. The $C, \tilde{C}$ spinors make up the self-dual and anti-self dual parts of the Weyl tensor. In the language of representation theory, this is the decomposition of R_{abcd} into irrreducible representations under the action of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ (with $\mathbb{R}$ replaced by $\mathbb{C}$ for the holomorphic case).

Note that in $++--$, spinor components are real. For analytic metrics, we can analytically continue which amounts to allowing the spinors to be complex. The remaining calculations in this appendix are valid in both cases.

Integrability of α and β surfaces. We now show that (2.8) and (2.9) are equivalent to the fact that the two-plane distributions defined by $o^{A'}$ and ι^A are integrable. The leaves are called α-surfaces and β-surfaces respectively. The argument is well-known in twistor theory. We will do the calculation for the $o^{A'}$ case; the ι^A case is identitical.

Let $X = \alpha^A o^{A'} \mathbf{e}_{AA'}$, $Y = \beta^A o^{A'} \mathbf{e}_{AA'}$ be vector fields, which by definition are in the α-planes determined by $o^{A'}$. Then if they commute we have:

$$[X,Y]_{AA'} = (f\alpha_A + g\beta_A)o_{A'},$$

for some functions f, g. Multiplying by $o^{A'}$ gives

$$o^{A'}[X,Y]_{AA'} = o^{A'}(X^{BB'}\nabla_{BB'}Y_{AA'} - Y^{BB'}\nabla_{BB'}X_{AA'}) = 0.$$

Substituting the spinor expressions for $X^{AA'}$ and $Y^{AA'}$ results in

$$o^{A'}o^{B'}\nabla_{BB'}o_{A'} = 0,$$

which is (2.9), and it is easy to show this is sufficient as well as necessary.

Twistor distribution and ASD. Locally, the primed spin bundle S' is isomorphic to $M \times \mathbb{C}^2$. We choose the coordinates on the $\mathbb{C}^2$ to be $\pi^{A'}$ for $A' = 0, 1$. This vector bundle has a connection inherited from the Levi-Civita connection of the metric, and therefore we can find the horizonal lifts $\tilde{\mathbf{e}}_{AA'}$ of the $\mathbf{e}_{AA'}$, defined by covariantly constant sections. These lifts are as follows:

$$\tilde{\mathbf{e}}_{AA'} = \mathbf{e}_{AA'} - \Gamma_{AA'B'}{}^{C'}\pi^{B'}\frac{\partial}{\partial \pi^{C'}}.$$

Using the following formula ([20], pg. 247) relating curvature quantities to the derivatives of $\Gamma_{AA'C'}{}^{D'}$: and the spinor decomposition of the curvature (A.1) we find

$$\begin{aligned}[][\pi^{A'}\tilde{\mathbf{e}}_{AA'}, \pi^{B'}\tilde{\mathbf{e}}_{BB'}] &= (\Gamma_{AA'B}{}^{D} - \Gamma_{BA'A}{}^{D})\pi^{A'}\pi^{B'}\tilde{\mathbf{e}}_{DB'} \\ &\quad + \pi^{A'}\pi^{B'}\epsilon_{AB}\epsilon^{F'Q'}\tilde{C}_{A'B'E'Q'}\pi^{E'}\frac{\partial}{\partial \pi^{F'}}.\end{aligned}$$

One can see from this that if $\tilde{C}_{A'B'C'D'} = 0$ then $\pi^{A'}\tilde{\mathbf{e}}_{AA'}$, $A = 0, 1$, forms an integrable distribution. The projection of a leaf of this distribution to M gives an α-surface. We have demonstrated that if the metric is anti-self-dual, then given any point $p \in M$ and an α-plane at p, there is a unique α-surface through p tangent to this α-plane . This was first shown by Penrose [19], although without using the primed spin bundle. For our purposes the above formulation will be most useful.

Proof of Proposition 1. The spinor for of the following identity:

$$K^a R_{abcd} = \nabla_b \nabla_c K_d - \frac{1}{2}(\eta_{,b} g_{cd} - \eta_{,c} g_{bd} + \eta_{,d} g_{bc}), \tag{A.2}$$

where η is the conformal factor appearing in (2.5). and the curvature decomposition (A.1) one can calculate

$$\begin{aligned}&[K^{AA'}\tilde{\mathbf{e}}_{AA'}, \pi^{B'}\tilde{\mathbf{e}}_{BB'}] = (K^{AA'}\Gamma_{AA'B}{}^{D} - \psi_B{}^{D})L_D \\ &-\pi^{B'}(\phi_{B'}{}^{A'}\epsilon_B{}^{A} + \frac{1}{2}\eta\epsilon_{B'}{}^{A'}\epsilon_B{}^{A})\,\tilde{\mathbf{e}}_{AA'} \\ &+\pi^{B'}\pi^{E'}(\mathbf{e}_{BB'}\phi_{E'}{}^{F'} - \Gamma_{BB'E'}{}^{G'}\phi_{G'}{}^{F'} + \Gamma_{BB'G'}{}^{F'}\phi_{E'}{}^{G'} - \frac{1}{4}(\mathbf{e}_{BE'}\eta)\epsilon^{F'}{}_{B'})\frac{\partial}{\partial \pi^{F'}}.\end{aligned} \tag{A.3}$$

We wish to add a vertical term to $K^{AA'}\tilde{\mathbf{e}}_{AA'}$ which will cancel all the non-L_A terms on the RHS of (A.3). We don't mind about multiples of the Euler vector field since this gets quotiented out on projectivizing. A simple calculation shows that $\tilde{K}$ as defined in (4.4) does the trick. □

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